Carter subgroups of finite almost simple groups
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Annotation

In the paper we complete the classification of Carter subgroups in finite almost simple groups. In particular, we prove that Carter subgroups of every finite almost simple group are conjugate. Together with previous results by the author and F. Dalla Volta, A. Lucchini, and M. C. Tamburini, as a corollary, it follows that Carter subgroups of every finite group are conjugate.

1 Introduction

We recall that a subgroup of a finite group is called a Carter subgroup if it is nilpotent and self-normalizing. By a well-known result, any finite solvable group contains exactly one conjugacy class of Carter subgroups (cf. [1]), and it is reasonable to conjecture that a finite group contains at most one conjugacy class of Carter subgroups. The evidence for this conjecture is based on extensive investigation, by several authors, of classes of finite groups which are close to be simple. In particular it has been shown that the conjecture holds for the symmetric and alternating groups (cf. [2]); denoting by $p^t$ a power of a prime $p$, for any group $A$ such that $\text{SL}_n(p^t) \leq A \leq \text{GL}_n(p^t)$ (cf. [3] and [4]), for the symplectic groups $\text{Sp}_{2n}(p^t)$, the full unitary groups $\text{GU}_n(p^{2t})$ and, when $p$ is odd, the full orthogonal groups $\text{GO}^\pm_n(p^t)$ (cf. [5]). Later in [6] results of [5] were extended to any group $G$ with $O^p(S) \leq G \leq S$, where $S$ is a full classical matrix group. Also some of the sporadic simple groups were investigated (cf. [7], for example). In the non-solvable cases, when Carter subgroups exist, they always turned out to be the normalizers of Sylow 2-subgroups.

In the paper we consider the following problem, which we refer later as the conjugacy problem.

**Problem.** Are any two Carter subgroups of a finite group conjugate?

In [8] it is proven that the minimal counterexample $A$ to this problem should be almost simple. Later in [9] a stronger result was obtained.

**Definition 1.1.** A finite group $G$ is said to satisfy condition (C) if, for every non-Abelian composition factor $S$ of every composition series of $G$ and for every its nilpotent subgroup $N$, Carter subgroups of $\langle \text{Aut}_N(S), S \rangle$ are conjugate (definition of Aut$_N(S)$ one can find below).

Since $S$ is simple, it is isomorphic to the group of its inner automorphisms Inn$(S)$ and we identify $S$ with the subgroup Inn$(S)$ of Aut$(S)$. In [9] the following theorem was proven.

**Theorem 1.2.** If a finite group $G$ satisfies condition (C), then Carter subgroups of $G$ are conjugate.

Thus our goal here is to prove that for every known finite simple group $S$ and every nilpotent subgroup $N$ of Aut$(S)$, Carter subgroups of $\langle S, N \rangle$ are conjugate. Some classes of almost simple groups which can not be minimal counter example to the conjugacy problem are found in [6].
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and [10]. The table of almost simple groups, for which the conjugacy problem has an affirmative answer, is given in [9].

Our notations is standard. If $G$ is a finite group, we denote by $PG$ the factor group $G/Z(G) \simeq \text{Inn}(G)$. If $\pi$ is a set of primes then we denote by $\pi'$ its complement in the set of all primes. For a positive integer $n$ the set of prime divisors of $n$ is denoted by $\pi(n)$, and by $n_\pi$ the maximal divisor $t$ of $n$ with $\pi(t) \subseteq \pi$ is denoted. As usual we denote by $O_\pi(G)$ the maximal normal $\pi$-subgroup of $G$, by $O^{\pi'}(G)$ the subgroup generated by all $\pi$-elements of $G$ is denoted. If $\pi = \{2\}$ is the set of all odd primes, then $O_\pi(G) = O_2(G)$ is denoted by $O(G)$. If $g \in G$, then we denote by $g_\pi$ the $\pi$-part of $g$, i.e., $g_\pi = g^{|g|_\pi}$.

We denote by $F(G)$ the Fitting subgroup of $G$ and by $F^*(G)$ the generalized Fitting subgroup of $G$. A central product of groups $G$ and $H$ is denoted by $G \cdot H$. For a finite group $G$ we denote by $\text{Aut}(G)$ the group of automorphisms of $G$. If $\lambda \in \text{Aut}(G)$, then we denote by $G_\lambda$ the set of $\lambda$-stable points, i.e., $G_\lambda = \{g \in G | g^\lambda = g\}$.

If $Z(G) = \{e\}$, then $G$ is isomorphic to the group of its inner automorphisms and we may suppose that $G \leq \text{Aut}(G)$. A finite group $G$ is said to be almost simple if there is a simple group $S$ with $S \leq G \leq \text{Aut}(S)$, i.e., if $F^*(G)$ is a simple group.

If $G$ is a group, $A, B, H$ are subgroups of $G$ and $B$ is normal in $A$ ($B \trianglelefteq A$), then $N_H(A/B) = N_H(A) \cap N_H(B)$. If $x \in N_H(A/B)$, then $x$ induces an automorphism $Ba \mapsto Bx^{-1}ax$ of $A/B$. Thus, there is a homomorphism of $N_H(A/B)$ into $\text{Aut}(A/B)$. The image of this homomorphism is denoted by $\text{Aut}_H(A/B)$ while its kernel is denoted by $C_H(A/B)$. In particular, if $S = A/B$ is a composition factor of $G$, then for any $H \leq G$ the group $\text{Aut}_H(S)$ is denoted. If $A, H$ are subgroups of $G$, then $\text{Aut}_H(A) = \text{Aut}_H(A/\{e\})$ by definition.

2 Preliminary results

Lemma 2.1. Let $G$ be a finite group, $H$ be a normal subgroup of $G$, $S = (A/H)/(B/H)$ be a composition factor of $G/H$ and $L$ be a subgroup of $G$. Then $\text{Aut}_L(A/B) \simeq \text{Aut}_{LH/H}(\langle A/H \rangle/(B/H))$.

Proof. Since $H \leq B$, then $H \leq C_G(A/B)$, so we may assume that $L = LH$. Further more we may assume that $L \leq N_G(A/B)$ and $G = LA$. Then the action on $A/B$ given by $x : Ba \mapsto Bx^{-1}ax$ coincides with the action on $(A/H)/(B/H)$ given by $xH : BaH \mapsto Bx^{-1}axH$, and the lemma follows.\qed

The following lemma is known.

Lemma 2.2. Let $G$ be a finite group, $H$ be a normal subgroup of $G$ and $\overline{N}$ be a nilpotent subgroup of $\overline{G} = G/H$. Then there exists a nilpotent subgroup $N$ of $G$ such that $NH/H = \overline{N}$.

Proof. Clearly we may assume that $G/H = \overline{N}$. There exists a subgroup $U$ of $G$ such that $UH = G$. Choose a subgroup of minimal order with this property. Then $U \cap H \leq \Phi(U)$, where $\Phi(U)$ is a Frattini subgroup of $U$. Indeed, if there exists a maximal subgroup $M$ of $U$, not containing $U \cap H$, then clearly $MH = G$, which contradicts the minimality of $U$. Thus the group $U/\Phi(U)$ is nilpotent, hence $U$ is nilpotent and $N = U$.\qed

By Lemmas 2.1 and 2.2 it follows that, if a finite group $G$ satisfies (C), then for every its normal subgroup $N$ and solvable subgroup $H$, groups $G/N$ and $HN$ satisfy (C).

Lemma 2.3. Let $G$ be a finite group, let $K$ be a Carter subgroup of $G$ and assume that $N$ is a normal subgroup of $G$. Assume that $KN$ satisfies (C) (this is always true if $G$ satisfies (C) or $N$ is solvable) or $KN = G$. Then $KN/N$ is a Carter subgroup of $G/N$. 
Let us consider the implications of these results. Assume that $KN = G$ then the statement is clear. Assume that $KN \neq G$, i.e., $KN$ satisfies (C). Consider $x \in G$ and assume that $xN \leq N_{G/N}(KN/N)$. It follows that $x \in N_G(KN)$. We have that $K^x$ is a Carter subgroup of $KN$. Since $KN$ satisfies (C), we have that its Carter subgroups are conjugate. Thus there exists $y \in KN$ such that $K^y = K^x$. Since $K$ is a Carter subgroup of $G$, it follows that $xy^{-1} \in N_G(K) = K$ and $x \in KN$.

**Lemma 2.4.** [9, Lemma 5] Let $K$ be a Carter subgroup of a finite group $G$, a nonidentical element $z$ is in $Z(K)$ and $C_G(z)$ satisfies (C). Then

(a) every subgroup $Y$ which contains $K$ and satisfies (C) is self-normalizing in $G$;

(b) no conjugate of $z$ in $G$, except $z$, lies in $Z(K)$;

(c) if $H$ is a Carter subgroup of $G$, non-conjugate to $K$, then $z$ is not conjugate to any element in the center of $H$.

In particular the centralizer $C_G(z)$ is self-normalizing in $G$, and $z$ is not conjugate to any power $z^k \neq z$.

**Lemma 2.5.** Let $G$ be a finite group and $Q$ be a Sylow 2-subgroup of $G$. Then a Carter subgroup $K$ of $G$, containing $Q$, exists if and only if $N_G(Q) = QC_G(Q)$.

**Proof.** Assume that there exists a Carter subgroup $K$ of $G$ containing $Q$. Since $K$ is nilpotent, it follows that $Q$ is normal in $K$ and $K \leq QC_G(Q) \leq N_G(Q)$. By Feit-Thompson Theorem (see [11]) we obtain that $N_G(Q)$ is solvable. Thus, by Lemma 2.4(a) we have that $QC_G(Q)$ is self-normalizing in $G$, therefore $N_G(Q) = QC_G(Q)$.

Assume now that $N_G(Q) = QC_G(Q)$, i.e., $N_G(Q) = Q \times O(C_G(Q))$. Since $O(C_G(Q))$ is of odd order, it is solvable. Hence it contains a Carter subgroup $K_1$. Consider a nilpotent subgroup $K = Q \times K_1$ of $G$. Clearly $N_G(K) \leq N_G(Q)$. But $K$ is a Carter subgroup of $N_G(Q)$, hence $K$ is a Carter subgroup of $G$.

**Definition 2.6.** Keeping in mind Lemma 2.5 we say that a finite group $G$ satisfies condition (ESyl2) if, for a Sylow 2-subgroup $Q$ of $G$ the equality $N_G(Q) = QC_G(Q)$ holds. In other words, a group $G$ satisfies (ESyl2) if each element of odd order, normalizing $Q$, centralizes $Q$.

**Lemma 2.7.** Let $G$ be a finite group, let $Q$ be a Sylow 2-subgroup of $G$ and $x$ be an element of odd order of $N_G(Q)$. Assume that there exist normal subgroups $G_1, \ldots, G_k$ of $G$ such that $G_1 \cap \ldots \cap G_k \cap Q \leq Z(N_G(Q))$ and $x$ centralizes $Q$ modulo $G_i$ for all $i$. Then $x$ centralizes $Q$. In particular, if $G/G_i$ satisfies (ESyl2) for all $i$, then $G$ satisfies (ESyl2).

**Proof.** Consider the normal series $Q \rightharpoonup Q_1 \rightharpoonup \ldots \rightharpoonup Q_k \rightharpoonup Q_{k+1} = \{e\}$, where $Q_i = Q \cap (G_1 \cap \ldots \cap G_i)$. The conditions of the lemma implies that $x$ centralizes each factor $Q_{i-1}/Q_i$. Since $x$ is of odd order this imply that $x$ centralizes $Q$.

**Lemma 2.8.** [9, Lemma 3] Let $K$ be a Carter subgroup of a finite group $G$. Assume that there exists a normal subgroup $B = T_1 \times \ldots \times T_k$ of $G$ such that $G = BK$, $Z(T_i) = \{e\}$ and $T_i$ is indecomposable into a direct product of its proper subgroups for all $i$. Then $\text{Aut}_H(T_i)$ is a Carter subgroup of $\langle \text{Aut}_H(T_i), T_i \rangle$.

**Lemma 2.9.** Let $H$ be a subgroup of a finite group $G$ such that $|G : H| = 2^t$. $H$ satisfies (ESyl2), and each element of odd order of $G$ is in $H$ (this property is clearly equivalent to the subnormality of $H$). Then $G$ satisfies (ESyl2).
3 Groups of Lie type

Our notations for groups of Lie type agrees with [12] and for linear algebraic groups agrees with [13]. If \( G \) is a canonical \( \sigma \)-finite group of Lie type (the definition is given below) with trivial center (we do not exclude non-simple groups of Lie type, such as \( A_1(2) \), all exceptions are given in [12, Theorems 11.1.2 and 14.4.1]), then \( \hat{G} \) denotes the group of inner-diagonal automorphisms of \( G \). In view of [14, 3.2] we have that \( \text{Aut}(G) \) is generated by inner-diagonal, \( \sigma \)-eld and graph automorphisms. Since we are assuming that \( Z(G) \) is trivial, we have that \( G \simeq \text{Inn}(G) \) and hence we may suppose that \( G \leq \hat{G} \leq \text{Aut}(G) \).

Let \( \overline{G} \) be a simple connected linear algebraic group over an algebraically closed \( \sigma \)-eld \( \overline{\mathbb{F}}_p \) of positive characteristic \( p \). It is possible here that \( Z(\overline{G}) \) is nontrivial. An endomorphism \( \sigma \) of \( \overline{G} \) is called a Frobenius map if \( \overline{G}_\sigma \) is \( \sigma \)-nite and \( \sigma \) is an automorphism of \( \overline{G} \) as an abstract group. Groups \( O^\sigma(\overline{G}_\sigma) \) are called canonical \( \sigma \)-nite groups of Lie type and every group \( G \) satisfying \( O^\sigma(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma \) is called a \( \sigma \)-nite group of Lie type. If \( \overline{G} \) is a simple algebraic group of adjoint type then we say that \( G \) is also of adjoint type. Note that in [12] only groups \( O^\sigma(\overline{G}) \) are called groups of Lie type. But later in [15] R.Carter said that every group \( \overline{G}_\sigma \) is a \( \sigma \)-nite group of Lie type for an arbitrary connected reductive group \( \overline{G} \). More over, in [16] and [17] without any explanation every group \( G \) with \( O^\sigma(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma \) is called a \( \sigma \)-nite group of Lie type. Thus, giving the \( \sigma \)-end of \( \sigma \)-nite groups of Lie type and canonical \( \sigma \)-nite groups of Lie type we intend to clarify the situation here. For example, \( \text{PSL}_2(3) \) is a canonical \( \sigma \)-nite group of Lie type and \( \text{PGL}_2(3) \) is a \( \sigma \)-nite group of Lie type. Note that an element of order 3 is not conjugate to its inverse in \( \text{PSL}_2(3) \) and is conjugate to its inverse in \( \text{PGL}_2(3) \). Since such information about conjugation is important in many cases (and is very important and useful in this paper), we say it is nontivial to use such notation. By \( \Phi(\overline{G}) \) the root system of \( \overline{G} \) is denoted and by \( \Phi \) or \( \Phi(G) \) the root system of \( O^\sigma(G) \) is denoted. We denote by \( \Delta(\overline{G}) \) the fundamental group of \( \overline{G} \) and by \( \Delta(\Phi) \) the quotient of the lattice generated by all fundamental weights of a root system \( \Phi \) by the lattice generated by all roots of \( \Phi \). Note that \( \Delta(\overline{G}) \) is always a quotient of \( \Delta(\Phi(\overline{G})) \) and for each root system \( \Phi \) distinct from \( D_{2n} \), the group \( \Delta(\Phi) \) is cyclic, while \( \Delta(D_{2n}) \) is elementary Abelian of order 4. The Weyl group of \( \overline{G} \) is denoted by \( W(\overline{G}) \) and the Weyl group of \( \Phi \) is denoted by \( W(\Phi) \). If \( W(\Phi) \) is the Weyl group of a root system \( \Phi \), then by \( w_0 \) we denote a unique element mapping all positive roots onto negative ones.

We say that groups of Lie type \( G \) such that \( O^\sigma(G) \) is equal to one of the groups \( 2A_n(q^2) \), \( 2D_n(q^2) \), \( 2E_6(q^2) \) are denoted over \( GF(q^2) \), groups of Lie type \( 3D_4(q^3) \) are denoted over \( GF(q^3) \) and over groups of Lie type are denoted over \( GF(q) \). The \( \sigma \)-end \( GF(q) \) in all cases is called the base \( \sigma \)-eld. In view of [18, Lemma 2.5.8] we have that if \( G \) is of adjoint type then \( \overline{G}_\sigma \) is a group of inner-diagonal automorphisms of \( O^\sigma(\overline{G}_\sigma) \). If \( \overline{G} \) is simply connected, then \( \overline{G}_\sigma = O^\sigma(\overline{G}_\sigma) \) (cf. [19, 12.4]). Any way by [18, Theorem 2.2.6(g)] \( \overline{G}_\sigma = T_\sigma O^\sigma(\overline{G}_\sigma) \) for each \( \sigma \)-stable maximal torus \( T \) of \( \overline{G} \). In general for given \( \sigma \)-nite group of Lie type \( G \) (if we consider it as an abstract group) the corresponding algebraic group is not uniquely determined. For example, if \( G = \text{PSL}_2(5) \simeq \text{PSL}_2(3) \).
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$SL_2(4)$, then $G$ can be obtained either as $(SL_2(\mathbb{F}_2))_\sigma$, or as $O^\nu((PSL_2(\mathbb{F}_3))_\sigma)$ (for appropriate $\sigma$). So, for every finite group of Lie type $G$, we fix (in some way) corresponding algebraic group $\mathcal{G}$ and a Frobenius map $\sigma$ such that $O^\nu(\mathcal{G}_\sigma) \leq G \leq \mathcal{G}_\sigma$. Let $U = \langle X_i|r \in \Phi^+ \rangle$ be the maximal unipotent subgroup of $G$. If we fix an order on $\Phi(\mathcal{G})$, then every $u \in U$ can be uniquely written as

$$u = \prod_{r \in \Phi^+} x_r(t_r),$$

where roots are taken in given order and $t_r$ are from the $\sigma$-elde of $G$. If $O^\nu(G)$ coincides with $2A_n(q^2)$, $2B_2(2^{2n+1})$, $2D_n(q^2)$, $3D_4(q^3)$, $2E_6(q^2)$, $2G_2(3^{2n+1})$, or $2F_4(2^{2n+1})$, then we say that $G$ is twisted, otherwise $G$ is split. If $O^\nu(\mathcal{G}_\sigma) \leq G \leq \mathcal{G}_\sigma$ is a twisted group of Lie type and $r \in \Phi(\mathcal{G})$, then by $\bar{r}$ we always denote the image of $r$ under the symmetry of the root system corresponding to a graph automorphism, which is used for the construction of $G$. Sometimes we use notation $\Phi^\varepsilon(q)$, where $\varepsilon \in \{+,-\}$, and $\Phi^+(q) = \Phi(q)$ is a split group of Lie type with the base $\sigma$-eld $GF(q)$, $\Phi^-(q) = 2\Phi(q^2)$ is a twisted group of Lie type defined over a $\sigma$-eld $GF(q^2)$ (with the base $\sigma$-eld $GF(q)$).

Now let $\bar{\mathcal{G}}$ be a closed $\sigma$-stable subgroup of $\mathcal{G}$. Let $R = G \cap \bar{\mathcal{G}}$ and $N(G, R) = G \cap N_{\mathcal{G}}(\bar{\mathcal{R}})$. Note that $N(G, R) \neq N_G(R)$ in general and we call $N(G, R)$ an algebraic normalizer of $R$. For example, if we consider $G = SL_n(2)$, then the group of diagonal matrices $H$ of $G$ is trivial, hence $N_G(H) = G$. But $G = (SL_n(\mathbb{F}_2))_\sigma$, where $\sigma$ is the Frobenius map $\sigma : (a_{ij}) \mapsto (a_{ij}^2)$. So $H = \bar{\mathcal{H}}_\sigma$, where $\bar{\mathcal{H}}$ is the subgroup of diagonal matrices in $SL_n(\mathbb{F}_2)$. Thus $N(G, H)$ is the group of monomial matrices of $G$. We use term algebraic normalizer in order to avoid such difficulties, and to make our proofs universal. A group $R$ is said to be a torus (resp. a reductive subgroup, a parabolic subgroup, a maximal torus, a reductive subgroup of maximal rank) if $\bar{R}$ is a torus (resp. a reductive subgroup, a parabolic subgroup, a maximal torus, a reductive subgroup of maximal rank) of $\mathcal{G}$. A maximal $\sigma$-stable torus $\bar{T}$ such that $\bar{T}_\sigma$ is a Cartan subgroup of $\mathcal{G}$, is called a maximal split torus. If $\bar{\mathcal{G}}$ is a connected reductive subgroup of maximal rank of $\mathcal{G}_1 \ast \ldots \ast \mathcal{G}_k \ast \bar{Z}$, where $\mathcal{G}_i$ is a simple connected linear algebraic group and $\bar{Z} = Z(\bar{\mathcal{G}})^0$ (see [13, Theorem 27.5]). Moreover, if $\Phi_1, \ldots, \Phi_k$ are root systems of $\mathcal{G}_1, \ldots, \mathcal{G}_k$ respectively, then $\Phi_1 \cup \ldots \cup \Phi_k$ is a subsystem of $\Phi(\bar{\mathcal{G}})$. There is a nice algorithm due to Borel and de Siebental [20] and independently Dynkin [21] of determining subsystems of $\Phi$. One has to remove some nodes from the extended Dynkin diagram of $\Phi$ and repeat the procedure for the remaining connected components. The connected components obtained in this way are Dynkin diagrams of indecomposable subsystems and Dynkin diagram of every irreducible subsystem can be derived in this way.

Now assume that a reductive subgroup $\bar{\mathcal{G}}$ is $\sigma$-stable. In view of [19, 10.10] there exists a $\sigma$-stable maximal torus $\bar{T}$ of $\bar{\mathcal{G}}$. Let $\bar{\mathcal{G}}_{i_1} \ast \ldots \ast \bar{\mathcal{G}}_{i_l}$ be the $\sigma$-orbit of $\bar{\mathcal{G}}_{i_1}$. Consider the induced action of $\sigma$ on the factor group

$$(\bar{\mathcal{G}}_{i_1} \ast \ldots \ast \bar{\mathcal{G}}_{i_l})/Z(\bar{\mathcal{G}}_{i_1} \ast \ldots \ast \bar{\mathcal{G}}_{i_l}) \simeq P\bar{\mathcal{G}}_{i_1} \times \ldots \times P\bar{\mathcal{G}}_{i_l}.$$

Since $P\bar{\mathcal{G}}_{i_1} \simeq \ldots \simeq P\bar{\mathcal{G}}_{i_l}$ are simple (as abstract groups), then $\sigma$ induces a cyclic permutation on $P\bar{\mathcal{G}}_{i_1}, \ldots, P\bar{\mathcal{G}}_{i_l}$, and we may assume that the numbering is chosen so that $P\bar{\mathcal{G}}_{i_1}^\sigma = P\bar{\mathcal{G}}_{i_2}$, $P\bar{\mathcal{G}}_{i_2} = P\bar{\mathcal{G}}_{i_3}$, $\ldots$, $P\bar{\mathcal{G}}_{i_{l-1}} = P\bar{\mathcal{G}}_{i_l}$. Thus the equality

$$(P\bar{\mathcal{G}}_{i_1} \times \ldots \times P\bar{\mathcal{G}}_{i_l})_\sigma = \{x \mid x = g \cdot g^\sigma \cdot \ldots \cdot g^{\sigma_{i_l-1}} \text{ for some } g \in P\bar{\mathcal{G}}_{i_1}\}_\sigma \simeq (P\bar{\mathcal{G}}_{i_1})_{\sigma_{i_1}}$$
holds. In view of [19, 10.15] the group $\mathbf{P}\mathcal{G}_{\sigma_j}$ is finite, so $O''((\mathbf{P}\mathcal{G}_{\sigma_j})_{\sigma_j})$ is a finite canonical group of Lie type, probably with the base older, than the base older of $O''(\mathcal{G}_\sigma)$.

Let $\mathcal{B}_{\sigma_j}$ be a preimage of a $\sigma^j$-stable Borel subgroup of $\mathbf{P}\mathcal{G}_{\sigma_j}$ in $\mathcal{G}_{\sigma_j}$ under the natural epimorphism, and $\mathcal{T}_{\sigma_j}$ be a $\sigma^j$-stable maximal torus of $\mathcal{G}_{\sigma_j}$, contained in $\mathcal{B}_{\sigma_j}$ (their existence follows from [19, 10.10]). Then, from the note in the beginning of section 11 from [19] it follows that subgroups $\mathcal{U}_{\sigma_j}$ and $\mathcal{U}_{\sigma_j}^-$ generated by $\mathcal{T}_{\sigma_j}$-stable root subgroups taken by all positive and negative roots respectively are also $\sigma^j$-stable. Since $\mathcal{G}_{\sigma_j}$ is a simple algebraic group, then $\mathcal{G}_{\sigma_j}$ is generated by subgroups $\mathcal{U}_{\sigma_j}$ and $\mathcal{U}_{\sigma_j}^-$. Now $Z(\mathcal{G}_{\sigma_j}) \ldots Z(\mathcal{G}_{\sigma_j})$ consists of semisimple elements, so the restriction of the natural epimorphism $\mathcal{G}_{\sigma_j} \rightarrow \mathbf{P}\mathcal{G}_{\sigma_j}$ on $\mathcal{U}_{\sigma_j}$ and $\mathcal{U}_{\sigma_j}^-$ is an isomorphism. So, for every $k$ subgroups $(\mathcal{U}_{\sigma_j})^{\sigma_k}$ and $(\mathcal{U}_{\sigma_j}^-)^{\sigma_k}$ are maximal $\sigma^j$-stable connected unipotent subgroups of $\mathcal{G}_{\sigma_j}$ and generate $\mathcal{G}_{\sigma_j}$. Thus, $\mathcal{U}_{\sigma_j} \times (\mathcal{U}_{\sigma_j})^{\sigma} \times \ldots \times (\mathcal{U}_{\sigma_j})^{\sigma_{j-1}}$ and $\mathcal{U}_{\sigma_j}^- \times (\mathcal{U}_{\sigma_j}^-)^{\sigma} \times \ldots \times (\mathcal{U}_{\sigma_j}^-)^{\sigma_{j-1}}$ are maximal $\sigma$-stable connected unipotent subgroups of $\mathcal{G}_{\sigma_j} \ldots \mathcal{G}_{\sigma_j}$ and generate the group $\mathcal{G}_{\sigma_j} \ldots \mathcal{G}_{\sigma_j}$. By [19, Corollary 12.3(a)], we have

$$O''((\mathcal{G}_{\sigma_j} \ldots \mathcal{G}_{\sigma_j})_{\sigma_j}) = \langle (\mathcal{U}_{\sigma_j} \times (\mathcal{U}_{\sigma_j})^{\sigma} \times \ldots \times (\mathcal{U}_{\sigma_j})^{\sigma_{j-1}})_{\sigma_j}, (\mathcal{U}_{\sigma_j}^- \times (\mathcal{U}_{\sigma_j}^-)^{\sigma} \times \ldots \times (\mathcal{U}_{\sigma_j}^-)^{\sigma_{j-1}})_{\sigma_j} \rangle \simeq \langle (\mathcal{U}_{\sigma_j})_{\sigma_j}, (\mathcal{U}_{\sigma_j}^-)_{\sigma_j} \rangle = O''((\mathcal{G}_{\sigma_j})_{\sigma_j}).$$

By [19, 11.6 and Corollary 12.3], the group $\langle (\mathcal{U}_{\sigma_j})_{\sigma_j}, (\mathcal{U}_{\sigma_j}^-)_{\sigma_j} \rangle$ is a canonical finite group of Lie type. Moreover, from the above arguments it follows that $\langle (\mathcal{U}_{\sigma_j})_{\sigma_j}, (\mathcal{U}_{\sigma_j}^-)_{\sigma_j} \rangle \simeq Z((\mathcal{U}_{\sigma_j})_{\sigma_j}, (\mathcal{U}_{\sigma_j}^-)_{\sigma_j})$ and $O''((\mathbf{P}\mathcal{G}_{\sigma_j})_{\sigma_j})$ are isomorphic. Denoting $O''((\mathcal{G}_{\sigma_j} \ldots \mathcal{G}_{\sigma_j})_{\sigma_j})$ by $G_i$, we obtain that $G_i$ is a finite group of Lie type for all $i$. Subgroups $G_i$ of $O''(\mathcal{G}_\sigma)$, appearing in this way are called subsystem subgroups of $O''(\mathcal{G}_\sigma)$.

For a $\sigma$-orbit $\{\mathcal{G}_{\sigma_j} \ldots \mathcal{G}_{\sigma_j}\}$ of $\mathcal{G}_{\sigma_j}$, with $G_i = O''((\mathcal{G}_{\sigma_j})_{\sigma_j})$, consider $\text{Aut}_{\pi_\sigma}(G_i)$. Since $G_1 \ldots G_{i-1} \times G_{i+1} \ldots G_k \times \mathcal{Z}_{\pi_\sigma} \leq C_{\mathbf{P}\mathcal{G}_\sigma}(G_i)$, we have that $\text{Aut}_{\pi_\sigma}(G_i) \simeq (\mathbf{T}_{\pi_\sigma})_{\mathcal{G}_{\sigma_j}} \simeq (\mathbf{T}_{\pi_\sigma}G_i)/Z(\mathbf{T}_{\pi_\sigma}G_i).$ By [18, Proposition 2.6.2] it follows that automorphisms, induced by $\mathbf{T}_{\pi_\sigma}$ on $G_i$ are diagonal. Therefore the following inclusions $PG_i \leq \text{Aut}_{\pi_\sigma}(G_i) \leq \mathbf{P}\mathcal{G}_\sigma$ hold, in particular, $\text{Aut}_{\pi_\sigma}(G_i)$ is a finite group of Lie type.

Let $\mathcal{R}$ be a $\sigma$-stable connected reductive subgroup of maximal rank (in particular, $\mathcal{R}$ can be a maximal torus) of $G$. Since $N_{\mathcal{G}_\sigma}(\mathcal{R})/\mathcal{R}$ and $N_W(W_{\mathcal{R}})/W_{\mathcal{R}}$ are isomorphic, we obtain the induced action of $\sigma$ on $N_W(W_{\mathcal{R}})/W_{\mathcal{R}}$ and we say that $w_1 \equiv w_2$, for $w_1, w_2 \in N_W(W_{\mathcal{R}})/W_{\mathcal{R}}$ if there exists $w \in N_W(W_{\mathcal{R}})/W_{\mathcal{R}}$ with $w_1 = w^{-1}w_2w$. Let $Cl(\mathcal{G}_{\sigma_j}, \mathcal{R})$ be the set of $\mathcal{G}_{\sigma_j}$-conjugacy classes of $\sigma$-stable subgroups $\mathcal{R}^\sigma$, where $g \in \mathcal{G}_{\sigma_j}$. Then $Cl(\mathcal{G}_{\sigma_j}, \mathcal{R})$ is in $1 - 1$ correspondence with the set of $\sigma$-conjugacy classes $Cl(N_W(W_{\mathcal{R}})/W_{\mathcal{R}}, \sigma)$, where $W$ is the Weyl group of $\mathcal{G}$, $W_{\mathcal{R}}$ is the Weyl group of $\mathcal{R}$ (and it is a subgroup of $W$). If $w$ is an element of $N_W(W_{\mathcal{R}})/W_{\mathcal{R}}$, and $(\mathcal{R}^\sigma)_w$ corresponds to the $\sigma$-conjugated class of $w$, then $(\mathcal{R}^\sigma)_w$ is said to obtained by twisting by $\mathcal{R}$ with $w\sigma$. Moreover $(\mathcal{R}^\sigma)_w \simeq (\mathcal{R}_{\mathcal{G}_{\sigma_j}})_{w\sigma}$. For more details see [23].

**Lemma 3.1.** Let $\mathcal{G}$ be a simple connected linear algebraic group over a field of characteristic $p$, $t$ be an element of order $r$ of $\mathcal{G}$ not divisible by $p$. Then $C_{\mathcal{G}_{\sigma_j}}(t)/(C_{\mathcal{G}_{\sigma_j}}(t))_0$ is a $\pi(r)$-group.

**Proof.** Since $p$ does not divide $r$, then $t$ is semisimple. Hence, $C_{\mathcal{G}_{\sigma_j}}(t)_0$ is a connected reductive subgroup of maximal rank of $\mathcal{G}$ and every $p$-element of $C_{\mathcal{G}_{\sigma_j}}(t)$ is contained in $C_{\mathcal{G}_{\sigma_j}}(t)_0$ (see [22, Theorem 2.2]). Assume that some prime $s \not\equiv \pi(r)$ divides $|C_{\mathcal{G}_{\sigma_j}}(t)/(C_{\mathcal{G}_{\sigma_j}}(t))_0|$. Then $s \neq p$ and,
for some natural $k > 0$, the centralizer $C_{\mathcal{G}}(t)$ contains an element $x$ of order $s^k$ such that $x \notin C_{\mathcal{G}}(t)^0$. Since $x$ and $t$ commute we have that $x \cdot t$ is a semisimple element of $\mathcal{G}$. Therefore there exists a maximal torus $T$ of $\mathcal{G}$ with $x \cdot t \in T$. Then $(xt)^r = x^t \in T$. Since $(s, r) = 1$ we have that there exists $m$ such that $rm \equiv 1 \pmod{s^k}$, thus $(x^r)^m = x \in \mathcal{T}$. Since $xt, x \in \mathcal{T}$, then $t \in \mathcal{T}$, so $\mathcal{T} \leq C_{\mathcal{G}}(t)^0$, hence $x \in C_{\mathcal{G}}(t)^0$; a contradiction.

Recall that an element $x$ of a linear algebraic group $\mathcal{G}$ is called regular, if its centralizer has the minimal possible dimension. In particular, if an element $x$ is semisimple and the group $\mathcal{G}$ is connected and reductive, then the element $x$ is called regular, if the connected component of its centralizer is a maximal torus of $\mathcal{G}$.

Assume now that $\mathcal{R}$ is a $\sigma$-stable parabolic subgroup of $\mathcal{G}$ and $\mathcal{U}$ is its unipotent radical. Then it contains a connected reductive subgroup $\mathcal{L}$ such that $\mathcal{R}/\mathcal{U} \simeq \mathcal{L}$. The subgroup $\mathcal{L}$ is called a Levi factor of $\mathcal{R}$. Moreover, if $\mathcal{Z} = Z(\mathcal{L})^0$, then $\mathcal{L} = C_{\mathcal{G}}(\mathcal{Z})$. Let $\text{Rad}(\mathcal{R})$ be the radical of $\mathcal{R}$. Then it is a $\sigma$-stable connected solvable subgroup, hence, by [19, 10.10] it contains a $\sigma$-stable torus $\mathcal{Z}$. Now $C_{\mathcal{G}}(\mathcal{Z}) = C_{\mathcal{R}}(\mathcal{Z})$ is a $\sigma$-stable Levi factor of $\mathcal{R}$, i.e., every $\sigma$-stable parabolic subgroup of $\mathcal{G}$ contains a $\sigma$-stable Levi factor $\mathcal{L}$ and $\mathcal{L}$ is a connected reductive subgroup of maximal rank of $\mathcal{G}$.

Lemma 3.2. (Hartley-Shute Lemma [24, Lemma 2.2]) Let $G = \text{O}^\nu(\mathcal{G}_\sigma)$ be a finite canonical adjoint group of Lie type with the definition \textit{r}eld $GF(q)$. Let $H$ be a Cartan subgroup of $G$ and $s \in GF(q)$. If $r = \bar{r}$ and the group $G$ is twisted, then assume also that $s$ is contained in the base \textit{r}eld of $G$. Then there exists an element $h(\chi) \in H$, such that $\chi(r) = s$, except the following case, where $h(\chi)$ can be chose so that $\chi(r)$ has given values:

(a) $G = A_1(q)$, $\chi(r) = s^2$;
(b) $G = C_n(q)$, $r$ is a long root, $\chi(r) = s^2$;
(c) $G = 2A_2(q^2)$, $r \neq \bar{r}$, $\chi(r) = s^3$;
(d) $G = 2A_3(q^2)$, $r \neq \bar{r}$, $\chi(r) = s^2$;
(e) $G = 2D_n(q^2)$, $r \neq \bar{r}$, $\chi(r) = s^2$;
(f) $G = 2G_2(3^{2n+1})$, $r \neq a$ or $r = 3a + b$, where $a$ is a short, $b$ is a long fundamental roots, $\chi(r) = s^2$.

Lemma 3.3. Let $O^\nu(\mathcal{G}_\sigma) \leq G \leq \mathcal{G}_\sigma$ be a finite adjoint group of Lie type over a \textit{r}eld of odd characteristic $p$ and the root system $\Phi$ of $\mathcal{G}$ be one of the following: $A_n$ ($n \geq 2$), $D_n$ ($n \geq 3$), $B_n$ ($n \geq 3$), $E_6$, $E_7$, or $E_8$ and $G \neq 2G_2(3^{2n+1})$. Let $U$ be a maximal unipotent subgroup of $G$, $H$ be a Cartan subgroup of $G$ which normalizes $U$, and $Q$ be a Sylow 2-subgroup of $H$. Then $C_U(Q) = \{e\}$.

Proof. Clearly we need to prove the lemma only for the case $G = O^\nu(\mathcal{G}_\sigma) = O^\nu(G)$, i.e., we may assume that $G$ is a canonical adjoint group of Lie type. If $G$ is split or $G \simeq 2D_n(q^2)$, then the lemma follows from [10, Lemma 2.8]. Assume that $G \simeq 2A_n(q^2)$ or $G \simeq 2E_6(q^2)$, then $\Phi(\mathcal{G})$ is equal to $A_n$ and $E_6$ respectively. Denote by $\bar{r}$ the image of the root $r$ of $\Phi$ under the corresponding symmetry. In terms of [12], the root system $\Phi(\mathcal{G})$ is a union of the equivalency classes $\Psi_i$, where each $\Psi_i$ has either type $A_1$, or $A_1 \times A_1$, or $A_2$. By [12, Proposition 13.6.1], the equality $U = \prod_i X_{\Psi_i}$ holds, where

$$X_{\Psi_i} = \{x_r(t) \mid t \in GF(q)\}.$$
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if $\Psi_i = \{r\}$ has type $A_1$ (here $r = \bar{r}$):

$$X_{\Psi_i} = \{x_r(t)x_r(t^q) \mid t \in GF(q^2)\},$$

if $\Psi_i = \{r, \bar{r}\}$ has type $A_1 \times A_1$ (here $r \neq \bar{r}$ and $r + \bar{r} \notin \Phi(\mathbb{G})$):

$$X_{\Psi_i} = \{x_r(t)x_r(t^q)x_{r+\bar{r}}(u) \mid t \in GF(q^2), u + u^q = -N_rtt^q\},$$

if $\Psi_i = \{r, \bar{r}, r + \bar{r}\}$ has type $A_2$ (here $r \neq \bar{r}$ and $r + \bar{r} \in \Phi(\mathbb{G})$). Now if $h(\chi)$ is an element of $H$, then the following equalities hold (see [12, p. 263]):

$$h(\chi)x_r(t)h(\chi)^{-1} = x_r(\chi(r)t),$$

if $r = \bar{r}$ and $\Psi_i = \{r\}$ has type $A_1$;

$$h(\chi)x_r(t)x_r(t^q)h(\chi)^{-1} = x_r(\chi(r)t)x_r(\chi(\bar{r})t^q),$$

if $r \neq \bar{r}$, $r + \bar{r} \notin \Phi(\mathbb{G})$ and $\Psi_i = \{r, \bar{r}\}$ has type $A_1 \times A_1$;

$$h(\chi)x_r(t)x_r(t^q)x_{r+\bar{r}}(u)h(\chi)^{-1} = x_r(\chi(r)t)x_r(\chi(\bar{r})t^q)x_{r+\bar{r}}(\chi(r + \bar{r})u),$$

if $r \neq \bar{r}$, $r + \bar{r} \in \Phi(\mathbb{G})$ and $\Psi_i = \{r, \bar{r}, r + \bar{r}\}$ has type $A_2$.

Let $u$ be a nontrivial element of $C_U(Q)$. Then $u$ contains a nontrivial multiplier from $X_{\Psi_i}$ for some $i$. In view of the uniqueness of decomposition into the product $\prod_i X_{\Psi_i}$ (see. [12, Proposition 13.6.1]) we may assume that $u \in X_{\Psi_i}$.

Assume that $\Psi$ has type $A_1$, i. e. $u = x_r(t)$, $t \in GF(q)$, $r = \bar{r}$. By Hartley-Shute Lemma 3.2 for every $s \in GF(q)$ there exists $h(\chi) \in H$ such that $\chi(r) = s$. Take $s = -1$. Then there exists $h(\chi) \in H$ such that $\chi(r) = -1$. Since $h(\chi)^2 = h(\chi^2)$ (see the formula on p. 98 of [12]), then we have that $\chi^2(r) = 1$, i. e. $|h(\chi)^2| < |h(\chi)|$. So the order $|h(\chi)|$ is odd and we may write $h(\chi) = h_1 \cdot h_2 = h(\chi_1) \cdot h(\chi_2)$ to be the decomposition of $h(\chi)$ into the product of its $2$- and $2'$- parts. Now $\chi(r) = \chi_1(r) \cdot \chi_2(r)$, therefore, $\chi_1(r) = -1$ and $\chi_2(r) = 1$. Thus $h(\chi_1)x_r(t)h(\chi_1)^{-1} = x_r(-t) \neq x_r(t)$. So the case $u = x_r(t)$ and $\Psi = \{r\}$ has type $A_1$ is impossible.

Assume that $\Psi = \{r, \bar{r}\}$ has type $A_1 \times A_1$. By Hartley-Shute Lemma 3.2 for every $s \in GF(q^2)$ there exists $h(\chi) \in H$, such that $\chi(r) = s^2$. Since there exists $s \in GF(q^2)$ such that $s^2 = -1$, then there exists $h(\chi) \in H$ such that $\chi(r) = -1$. As above $h(\chi)$ can be written as $h(\chi_1) \cdot h(\chi_2)$, a product of its $2$- and $2'$- parts. Then $\chi_1(r) = -1$, so

$$h(\chi_1)x_r(t)x_r(t^q)h(\chi_1)^{-1} = x_r(-t)x_r(-t^q) \neq x_r(t)x_r(t^q).$$

Thus the case $u = x_r(t)x_r(t^q)$ and $\Psi = \{r, \bar{r}\}$ has type $A_1 \times A_1$ is impossible.

Now assume that $\Psi = \{r, \bar{r}, r + \bar{r}\}$ has type $A_2$. By Hartley-Shute Lemma 3.2 for every $s \in GF(q^2)$ there exists $h(\chi) \in H$ such that $\chi(r) = s^3$. Choose $s = -1$, then there exists $h(\chi) \in H$ such that $\chi(r) = -1$. Again $h(\chi) = h(\chi_1) \cdot h(\chi_2)$ can be written as the product of its $2$- and $2'$- parts and $\chi_1(r) \neq 1$. Then

$$h(\chi_1)x_r(t)x_r(t^q)x_{r+\bar{r}}(u)h(\chi_1)^{-1} = x_r(-t)x_r(\chi_1(-t^q)x_{r+\bar{r}}(\chi_1(r + \bar{r})u) \neq x_r(t)x_r(t^q)x_{r+\bar{r}}(u).$$
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if \( t \neq 0 \). If \( t = 0 \), choose \( s \) so that \( s^2 = -1 \). Then \( \chi_1(r + \bar{r}) = -1 \) and as above we obtain inequality. So this last case is impossible.

By using Hartley-Shute Lemma 3.2, the similar arguments prove the lemma in the remaining cases \( G \simeq 3D_4(q^3), G \simeq G_2(q) \) and \( G \simeq F_4(q) \). We shall not need lemma 3.3 for these groups, so we do not give a complete proof for them.

Lemma 3.4. Let \( O^p(\overline{G}_\sigma) = G \) be a canonical \( \sigma \)-finite adjoint group of Lie type over a \( \sigma \)-field of odd characteristic \( p \) and \( -1 \) is not a square in the base \( \sigma \)-field of \( G \). Assume that the root system \( \Phi \) of \( G \) is equal to \( C_n \). Let \( U \) be a maximal unipotent subgroup of \( G \), \( H \) be a Cartan subgroup of \( G \) which normalizes \( U \), and \( Q \) be a Sylow \( 2 \)-subgroup of \( H \). Then \( C_U(Q) = \langle x_r \mid r \text{ is a long root} \rangle \), where \( \Omega(H) = \{ h \in H \mid h^2 = e \} \).

Proof. If \( r \) is a short root, then there exists a root \( s \) with \( < s, r > = 1 \). Thus \( x_r(t)^{h_s(-1)} = x_r((-1)^{<s,r>}t) = x_r(-t) \) (cf. [12, Proposition 6.4.1]). Therefore, if \( x \in C_U(Q) \) and \( x_r(t) \) is a nontrivial multiplier in decomposition (1) of \( x \), then \( r \) is a long root. Now if \( r \) is a long root, then, for every root \( s \), either \( |< s, r >| = 2 \), or \( < s, r > = 0 \), i. e., \( x_r(t)^{h_s(-1)} = x_r(t) \). Under our conditions \( (h_s(-1) \mid s \in \Phi) = Q \), and the lemma follows.

Lemma 3.5. Let \( G = \PSp_{2n}(q) \) be a simple canonical group of Lie type, \( J \) be a subset of the set of fundamental roots, containing the long fundamental root \( r_n \), \( P_J \) be a parabolic subgroup, generated by a Borel subgroup \( B \) and by groups \( X_r \) with \( -r \in J \), \( L \) be a Levi factor of \( P_J \). Denote by \( S \) the quasisimple normal subgroup of \( L \) isomorphic \( \Sp_{2k}(q) \) (it always exists, since \( r_n \in J \)). Then \( \Aut_L(S/Z(S)) = S/Z(S) \).

Proof. This statement is known, it is proven in an unpublished paper by N.A.Vavilov. We give a proof here for the completeness. As we noted above, \( L \) is a reductive subgroup of maximal rank of \( G \), and so the following inclusions hold \( S/Z(S) \leq \Aut_L(S/Z(S)) \leq S/Z(S) \).

Since \( [\overline{C_n}(q) : C_n(q)] = (2, q - 1) \), then for \( q \) even the statement is evident. If \( q \) is odd, then for \( \Aut_L(S/Z(S)) \) there can be only two possibilities: either \( \Aut_L(S/Z(S)) = S/Z(S) \), or \( \Aut_L(S/Z(S)) = S/Z(S) \). We shall show that the second equality is impossible.

In our notations fundamental roots of the root system of \( S \) are \( r_{n-k+1}, \ldots, r_n \). If the equality \( \Aut_L(S/Z(S)) = S/Z(S) \) holds, then there exist elements \( s_1, \ldots, s_k \) of \( Z\Phi = ZC_n \) such that

\[
< s_i, r_{n-k+j} > = \frac{(s_i, r_{n-k+j})}{(s_i, s_i)} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}
\]

(They generate the lattice of fundamental weights, thus allow to obtain all diagonal automorphisms of \( S \)). But for each root \( s \) of \( C_n \) we have that either \( < s, r_n > = 0 \), or \( < s, r_n > = \pm 2 \), i. e., for each element \( s \in Z\Phi \) the number \( < s, r_n > > 0 \), in particular is distinct from 1. Therefore such a set of elements \( s_1, \ldots, s_k \) does not exists.

Lemma 3.6. Let \( G \) be a \( \sigma \)-finite group of Lie type over a \( \sigma \)-field of odd characteristic and \( \overline{G} \), \( \sigma \) are chosen so that \( O^p(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma \). If \( G \) satisfies (ESy12), then every subgroup \( L \) with \( G \leq L \leq O^p(\overline{G}_\sigma) \) satisfies (ESy12).

Proof. Let \( Q \) be a Sylow 2-subgroup of \( \overline{G}_\sigma \) and \( Q^0 = O^p(\overline{G}_\sigma) \cap Q \) be a Sylow 2-subgroup of \( O^p(\overline{G}_\sigma) \). If \( N_{\overline{G}_\sigma}(Q^0) = QC_{\overline{G}_\sigma}(Q) \), then the statement of the lemma is clearly true. In view of [25, Theorem 1] for a classical group \( \overline{G}_\sigma \) the equality \( N_{\overline{G}_\sigma}(Q^0) = QC_{\overline{G}_\sigma}(Q) \) can fail to be true.
only if the root system of $\mathcal{G}$ has type $A_1$ or $C_n$. If the root system of $\mathcal{G}$ has type $A_1$ or $C_n$, then $|\mathcal{G}_\sigma : O^{p'}(\mathcal{G}_\sigma)| = 2$ and the statement of the lemma follows from Lemma 2.9.

Assume now that $G$ is a group of exceptional type. If $\mathcal{G}_\sigma = O^{p'}(\mathcal{G}_\sigma)$, then the statement of the lemma is clearly true. The equality $N_{\mathcal{G}_\sigma}(Q^{p'}) = QC_{\mathcal{G}_\sigma}(Q)$ might fail to be true only if the root system of $\mathcal{G}$ has type $E_6$ or $E_7$. If the root system of $\mathcal{G}$ has type $E_7$, then $|\mathcal{G}_\sigma : O^{p'}(\mathcal{G}_\sigma)| = 2$ and the statement of the lemma follows from Lemma 2.9.

Assume that the root system of $\mathcal{G}$ has type $E_6$. Then either $\mathcal{G}_\sigma = O^{p'}(\mathcal{G}_\sigma)$ or $|\mathcal{G}_\sigma : O^{p'}(\mathcal{G}_\sigma)| = 3$. In the first case we have nothing to prove, so assume that $|\mathcal{G}_\sigma : O^{p'}(\mathcal{G}_\sigma)| = 3$. Since the group $\mathcal{G}$ coincides with $\mathcal{G}_\sigma$, or with $O^{p'}(\mathcal{G}_\sigma)$, and since in case $G = \mathcal{G}_\sigma$ there is nothing to prove, we may assume that $G = O^{p'}(\mathcal{G}_\sigma)$. By [18, Theorem 4.10.2] there exists a maximal torus $T$ of $\mathcal{G}_\sigma$ such that $Q$ is contained in $N(\mathcal{G}_\sigma, T)$. Since $|\mathcal{G}_\sigma : G| = 3$, then $Q = Q^0 \leq N(G, T \cap G)$. By [26, Theorem 6] the equality $N_{\mathcal{G}_\sigma}(Q) = Q \times R^0$ holds, where $R^0 \leq T$ is a cyclic group of odd order. Now since $\mathcal{G}_\sigma = TG$, then $N_{\mathcal{G}_\sigma}(Q) = (N_T(Q), N_G(Q))$. Indeed, $N(G, T \cap G)/(T \cap G) \simeq N(G, T)/T$. Hence, a Sylow 2-subgroup $QT/T$ of $N(G, T)/T$ coincides with its normalizer. Since the factor group $\mathcal{G}_\sigma/G$ is cyclic of order 3, then $N_{\mathcal{G}_\sigma}(Q) = (tg, N_G(Q))$, where $t \in T$ and $g \in G$. Moreover, since $|\mathcal{G}_\sigma : G| = 3$, we may assume that $tg$ is an element of order $3^k$ for some $k > 0$. Since $t \in T \leq N(\mathcal{G}_\sigma, T)$, then $Q^t \leq N(G, T \cap G)$. So there exists an element $1 \leq Q^t \leq N(G, T \cap G)$ such that $Q^t = Q^{3n}$. Therefore we may assume that $tg = t_{g1} \in N(\mathcal{G}_\sigma, T)$. Under the natural epimorphism $\pi : N(\mathcal{G}_\sigma, T) \to N(\mathcal{G}_\sigma, T)/T$ the image of $N_{\mathcal{G}_\sigma}(Q)$ coincides with $Q$. Hence, $(tg)^n = e$, so $tg \in T$. Thus each element of odd order of $\mathcal{G}_\sigma$ normalizing $Q$ lies in $T$. Since $T$ is a torus, then $T$ is Abelian, hence the set of elements of odd order of $N_{\mathcal{G}_\sigma}(Q)$ forms a normal subgroup $R \leq T$. Therefore $N_{\mathcal{G}_\sigma}(Q) = Q \times R$, i.e., $\mathcal{G}_\sigma$ satisfies (ESy12).

The following lemma is immediate from [25, Theorem 1].

**Lemma 3.7.** Let $O^{p'}(\mathcal{G}_\sigma) = G$ be a canonical $p$-nilpotent group of Lie type and $\mathcal{G}$ is either of type $A_1$ or of type $C_n$, $p$ is odd, $q = p^n$ is the order of the base field of $G$. Then $G$ satisfies (ESy12) if and only if $q \equiv \pm 1 \pmod{8}$.

Note that Lemma 3.6 together with [25, Theorem 1] [26, Theorem 6] implies that every group of Lie type over a field of odd characteristic, distinct from a Ree group and groups from Lemma 3.7, satisfies (ESy12).

**Lemma 3.8.** Let $O^{p'}(\mathcal{G}_\sigma) \leq G \leq \mathcal{G}_\sigma$ be a finite adjoint group of Lie type with the base field of characteristic $p$ and order $q$. Assume also that $O^{p'}(G)$ is not isomorphic to $2D_{2n}(q^2)$, $3D_4(q^3)$, $2B_2(2^{2n+1})$, $2G_2(3^{2n+1})$, $2F_4(2^{2n+1})$. Then there exists a maximal $\sigma$-stable torus $\mathcal{T}$ of $\mathcal{G}$ such that

(a) $(N_{\mathcal{G}}(\mathcal{T})/\mathcal{T})_\sigma \simeq (N_{\mathcal{G}}(\mathcal{T})_\sigma)/\mathcal{T}_\sigma = N(\mathcal{G}_\sigma, \mathcal{T}_\sigma)/\mathcal{T}_\sigma \simeq W$, where $W$ is the Weyl group of $\mathcal{G}$;

(b) if $r$ is an odd primitive divisor of $q - (\varepsilon 1)$, where $\varepsilon = +$, if $G$ is split and $\varepsilon = -$ if $G$ is twisted, then $N(\mathcal{G}_\sigma, \mathcal{T}_\sigma)$ contains a Sylow $r$-subgroup of $\mathcal{G}_\sigma$;

(c) if $r$ is a prime divisor of $q - (\varepsilon 1)$, and $s$ is an element of order $r$ of $G$ such that $C_{\mathcal{G}}(s)$ is connected, then, up to conjugation by an element of $G$, the element $s$ is contained in $T = \mathcal{T}_\sigma \cap G$.

The torus $\mathcal{T}$ is unique, up to conjugation in $O^{p'}(\mathcal{G}_\sigma)$ and $|\mathcal{T}_\sigma| = (q - \varepsilon 1)^n$, where $n$ is the rank of $\mathcal{G}$. 


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Proof. Since for every maximal torus $T$ of $\mathcal{G}_\sigma$, we have that $\mathcal{G}_\sigma = TO^\sigma(\mathcal{G}_\sigma)$, without lost we may assume that $G = \mathcal{G}_\sigma$. If $G$ is split then the lemma can be easily proven. In this case $T$ is a maximal torus such that $T_\sigma$ is a Cartan subgroup of $\mathcal{G}_\sigma$ (i.e., $T$ is a maximal split torus) and (a) is clear. Point (b) follows from [29, (10.1)]. Moreover, from [29, (10.2)] it follows that the order of $T_\sigma$ is uniquely determined and is equal $(q - 1)^n$, where $n$ is the rank of $\mathcal{G}$. By [27, F, §6] we have that every element of order $r$ of $T$ is contained in $T_\sigma$. Now there exists $g \in \mathcal{G}$ such that $s^g \in T$, hence $s^g \in G$. Since the centralizer of $s$ is connected, elements $s$ and $s^g$ are conjugate in $\mathcal{G}$ if and only if they are conjugate in $G$ and (c) follows. By information about the classes of maximal tori given in [27, G] and [28] we have that up to conjugation in $G$ there exists a unique torus $T$ such that $|T_\sigma| = (q - 1)^n$.

Assume that $G \simeq 2A_n(q^2)$. Then $T$ is a maximal torus such that $|T_\sigma| = (q + 1)^n$. Note that $T_\sigma$ can be obtained from a maximal split torus by twisting with $w_0\sigma$. The uniqueness follows from [16, Proposition 8]. Direct calculations by using [15, Proposition 3.3.6] show that $N(\mathcal{G}_\sigma, T_\sigma)/T_\sigma$ is isomorphic to $\cong \mathbb{Z}(G)$ that is isomorphic to $\text{Sym}_{n+1}$. The uniqueness follows from [16, Proposition 8]. Point (b) follows from [29, (10.1)]. To prove point (c) we first show that every element of order $r$ of $T$ is contained in $G$. Assume that $t$ is an element of order $r$ in $T$ (recall that in this case $r$ divides $q + 1$). Let $H$ be a $\sigma$-stable maximal split torus of $\mathcal{G}$. The torus $T_\sigma$ is obtained from $H$ by twisting with an element $w_0$, where $w_0 \in W(G)$ is a unique element mapping all positive roots onto negative ones and $T_\sigma \simeq H_{\sigma w_0}$. Let $r_1, \ldots, r_n$ be the set of fundamental roots of $A_n$. Then $t$, as an element of $H$, can be written as $h_{r_1}(\zeta_1) \cdot \ldots \cdot h_{r_n}(\zeta_n)$. Now for every $i$ we have that $\sigma w_0 : h_{r_i}(\lambda) \mapsto h_{-r_i}(\lambda^q) = h_{r_i}(\lambda^{-q})$, i.e., $t^{\sigma w_0} = t^{-q}$. Since $r$ divides $q + 1$ we obtain that $t^{q+1} = t$, i.e., $t = t^{-q}$. Hence $t^{\sigma w_0} = t$ and $t \in T_\sigma$. Now as in the untwisted case, there exists an element $g \in \mathcal{G}$ such that $s^g \in T$, hence $s^g \in T_\sigma$. Since $C_{\mathcal{G}}(s)$ is connected, elements $s$ and $s^g$ are conjugate in $G$.

For $G = 2D_{2n+1}(q^2)$ we take $T$ to be a unique (up to conjugation in $G$) maximal torus of order $|T_\sigma| = (q + 1)^{2n+1}$ (the uniqueness follows from [16, Proposition 10]) and for $G = 2E_6(q^2)$ we take $T$ to be a unique (again up to conjugation in $G$) maximal torus of order $|T_\sigma| = (q + 1)^6$ (the uniqueness follows from [17, Table 1, p. 128]). Like in case $G = 2A_n(q^2)$ it is easy to show that $T$ satisfies (a), (b), and (c) of the lemma.

Lemma 3.9. Let $G$ be a finite group of Lie type and $\mathcal{G}$, $\sigma$ are chosen so that $O^\sigma(\mathcal{G}_\sigma) \leq G \leq \mathcal{G}_\sigma$. Let $s$ be a regular semisimple element of odd prime order $r$ of $G$. Then $N_G(C_{\mathcal{G}}(s)) \neq C_{\mathcal{G}}(s)$.

Proof. In view of [22, Proposition 2.10] we have that $C_{\mathcal{G}}(s)/C_{\mathcal{G}}(s)^0$ is isomorphic to a subgroup of $\Delta(\mathcal{G})$. Now, if the root system $\Phi$ of $\mathcal{G}$ is not equal to either $A_n$, or $E_6$, then $|\Delta(\Phi)|$ is a power of 2. Since $\Delta(\mathcal{G})$ is a quotient of $\Delta(\Phi(\mathcal{G}))$, then Lemma 3.1 implies that $C_{\mathcal{G}}(s)/C_{\mathcal{G}}(s)^0 = T$ is a maximal torus and $C_{\mathcal{G}}(s) = C_{\mathcal{G}}(s) \cap G = T$. Since $N_G(T) \geq N(G, T) \neq T$ we obtain the statement of the lemma in this case. Thus we may assume that either $\Phi = A_n$, or $\Phi = E_6$.

Assume $\sigma$-stabilized that $\Phi = A_n$, i.e., $O^\sigma(G) = A_n^\epsilon(q)$, where $\epsilon \in \{+, -\}$. Clearly $T = C_{\mathcal{G}}(s)^0 \cap G$ is a normal subgroup of $C_{\mathcal{G}}(s)$, hence $C_{\mathcal{G}}(s) \leq N(G, T)$. Assume that $N_G(C_{\mathcal{G}}(s)) = C_{\mathcal{G}}(s)$. Then $C_{\mathcal{G}}(s) = N_{N(G, T)}(C_{\mathcal{G}}(s))$ and $C_{\mathcal{G}}(s)/T$ is a self-normalizing subgroup of $N(G, T)/T$. As we noted above $C_{\mathcal{G}}(s)/T$ is isomorphic to a subgroup of $\Delta(A_n)$, i.e., it is cyclic. By Lemma 3.1, we also have that $C_{\mathcal{G}}(s)/T$ is a $r$-group, thus $C_{\mathcal{G}}(s)/T = \langle x \rangle$ for some $r$-element $x \in N(G, T)/T$. Thus $\langle x \rangle$ is a Carter subgroup of $N(G, T)/T$. Now, in view of [15, Proposition 3.3.6], we have that $N(G, T)/T \simeq C_{W(\mathcal{G})}(y)$ for some $y \in W(\mathcal{G}) \simeq \text{Sym}_{n+1}$. Clearly $C_{C_{W(\mathcal{G})}(y)}(\langle x \rangle)$ contains $y$, thus $y$ must be an $r$-element, otherwise $C_{C_{W(\mathcal{G})}(y)}(\langle x \rangle)$ would contain an element of order coprime to $r$, i.e., $N_{C_{W(\mathcal{G})}(y)}(\langle x \rangle) \neq \langle x \rangle$. A contradiction with the fact that $\langle x \rangle$ is a Carter subgroup of $C_{W(\mathcal{G})}(y)$.
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Now let \( y = \tau_1 \ldots \) be the decomposition of \( y \) into the product of independent cycles and \( l_1, \ldots \) be the lengths of \( \tau_1, \ldots \) respectively. Assume \( \ast\ast \)rst that \( m_1 \) cycles has the same length \( l_1 \), \( m_2 \) cycles has the length \( l_2 \) etc. Let \( m_0 = n + 1 - (l_1 m_1 + \ldots + l_k m_k) \). Then

\[
C_{\text{Sym}_{n+1}}(y) \simeq (Z_{l_1} \wr \text{Sym}_{m_1}) \times \ldots \times (Z_{l_k} \wr \text{Sym}_{m_k}) \times \text{Sym}_{m_0},
\]

where \( Z_{l_i} \) is a cyclic group of order \( l_i \). If \( m_j > 1 \) for some \( j \geq 0 \), then there exists a normal subgroup \( N \) of \( C_{\text{Sym}_{n+1}}(y) \) such that \( C_{\text{Sym}_{n+1}}(y)/N \simeq \text{Sym}_{m_j} \neq \{e\} \). In view of [9, Table] and [10, Table] we obtain that Carter subgroup in a group \( S \) satisfying \( \text{Alt} \ell \leq S \leq \text{Sym} \ell \) are conjugate for all \( \ell \geq 1 \). Thus \( C_{\text{W}(T)}(y) \) and \( N \) satisfy (C) and \( \langle x \rangle \) is the unique, up to conjugation, Carter subgroup of \( C_{\text{W}(T)}(y) \). By Lemma 2.3 we obtain that \( \langle x \rangle \) maps onto a Carter subgroup of \( C_{\text{W}(T)}(y)/N \simeq \text{Sym}_{m_j} \). In view of [2] we have that only a Sylow 2-subgroup of \( \text{Sym}_{m_j} \) can be a Carter subgroup of \( \text{Sym}_{m_j} \). A contradiction with the fact that \( x \) is an \( r \)-element and \( r \) is odd.

Thus we may assume that \( C_{\text{W}(T)}(y) \simeq (Z_{l_1} \times \ldots \times Z_{l_k}) \) and \( l_i \neq l_j \) if \( i \neq j \). From the known structure of maximal tori and their normalizers in \( A_n(q) \) (cf. [16, Propositions 7,8], for example) we obtain the structure of \( T \) and \( N(G,T) \), which we explain by using matrices. Below a group \( GL_n^\varepsilon(q) \) is isomorphic to \( GL_n(q) \) if \( \varepsilon = + \) and is isomorphic to \( GU_n(q) \) if \( \varepsilon = - \). For the decomposition \( l_1 + l_2 + \ldots + l_k = n + 1 \) in \( GL_{n+1}^\varepsilon(q) \) consider a subgroup \( L \), consisting of block-diagonal matrices of view

\[
\begin{pmatrix}
A_1 & 0 & \ldots & 0 \\
0 & A_1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & A_k
\end{pmatrix},
\]

where \( A_i \in GL_{l_i}^\varepsilon(q) \). Then \( L \simeq GL_{l_1}^\varepsilon(q) \times \ldots \times GL_{l_k}^\varepsilon(q) \). Denote, for brevity, \( GL_{l_i}^\varepsilon(q) \) by \( G_i \). In every group \( G_i \) consider a Singer cycle \( T_i \). \( N_{G_i}(T_i)/T_i \) is known to be a cyclic group of order \( l_i \) and \( N(G_i,T_i) = N_{G_i}(T_i) \). There exists a subgroup \( Z \) of \( Z(SL_{n+1}^\varepsilon(q)) \) such that \( O^p(G) \simeq SL_{n+1}^\varepsilon(q)/Z \). Then \( T \simeq ((T_1 \times \ldots \times T_k) \cap SL_{n+1}^\varepsilon(q))/Z \) and \( N(G,T) \simeq ((N(G_1,T_1) \times \ldots \times N(G_k,T_k)) \cap SL_{n+1}^\varepsilon(q))/Z \). Since for every Singer cycle \( T_i \) the group \( N(G_i,T_i)/T_i \) is cyclic, we may assume that \( N(G,T) = C_G(s) \) and \( T \) is a Singer cycle, i. e., is a cyclic group of order \( q^{n+1-\varepsilon(n+1)/q-\varepsilon} \) and \( n + 1 = r k \) for some \( k \geq 1 \) (the last equality holds, since \( N(G,T)/T \) is an \( r \)-group). But \( q^{rk} \equiv q \) (mod \( r \)), hence, \( r \) divides \( q - (\varepsilon) \). By Lemma 3.8 we obtain that \( s \) is in \( N(G,H) \), where \( H \) is a maximal torus such that the factor group \( N(G,H)/H \) is isomorphic to \( \text{Sym}_{n+1} \) and \( |H| = (q-\varepsilon)^n \). In particular, \( H \) is not a Singer cycle. If \( s \in H \), this immediately implies a contradiction with the choice of \( s \). If \( s \not\in H \), then, since the order of \( s \) is prime, the intersection \( \langle s \rangle \cap H \) is trivial. Hence, under the natural homomorphism \( N(G,H) \rightarrow N(G,H)/H \simeq \text{Sym}_{n+1} \) the element \( s \) maps on an element of order \( r \). But in \( \text{Sym}_{n+1} \) every element of odd order is conjugate to its inverse. Thus, there exists a 2-element \( z \) of \( G \), which normalizes, but not centralizes \( \langle s \rangle \). Therefore, \( z \leq N_{\Phi}(C_{\Phi}(s)) \leq N_{\Phi}(C_{\Phi}(s)^{0}) \) and \( |N(G,T)/T| \) is divisible by 2, that contradicts the above proven statement that \( N(G,T)/T \) is an \( r \)-group. This \( \ast \)nal contradiction \( \ast \)nish the case \( \Phi(G) = A_n \).

In the remaining case \( \Phi = E_6 \) it is easy to see, that for every \( y \in W(E_6) \), the group \( C_{W(E_6)}(y) \) does not contain Carter subgroup of order 3. Indeed, if \( C_{W(E_6)}(y) \) has a Carter subgroup of order 3, then it is generated by \( y \). But it is known (and can be easily checked with [28, Table 9]), that in \( W(E_6) \) there is no elements of order 3, which centralizer has order 3. Since \( |C_G(s)/T| \) divides 3 and the group \( C_G(s)/T \) is a Carter subgroup of \( C_{W(E_6)}(y) \) for some \( y \), we get a contradiction.

\( \square \)
4 Semilinear groups of Lie type

Now we define some overgroups of finite groups of Lie type. First we give precise description of a Frobenius map $\sigma$. Note that all maps in this section are automorphisms, if $\mathcal{G}$ is considered as an abstract group, and are endomorphisms, if $\mathcal{G}$ is considered as an algebraic group. Since we use these maps to construct respective automorphisms of finite groups and of groups over an algebraically closed field, we need it reasonable to call all maps automorphisms. Let $\mathcal{G}$ be a simple connected linear algebraic group over an algebraically closed field $\mathbb{F}_p$ of positive characteristic $p$. Below, if nothing contrary is said, we shall consider groups of adjoint type. Choose a Borel subgroup $B$ of $\mathcal{G}$, let $U = R_u(B)$ be the unipotent radical of $B$. There exists a Borel subgroup $B^-$ with $B \cap B^- = T$, where $T$ is a maximal torus of $B$ (hence of $\mathcal{G}$). Let $\Phi$ be the root system of $\mathcal{G}$ and let $\{X_r | r \in \Phi^+\}$ be the set of $T$-invariant 1-dimensional root subgroups of $U$. Every $X_r$ is isomorphic to the additive group of $\mathbb{F}_p$, so every element of $X_r$ can be written as $x_r(t)$, where $t$ is the image of $x_r(t)$ under this isomorphism. Denote by $U^- = R_u(B^-)$ the unipotent radical of $B^-$. As above define $T$-invariant 1-dimensional subgroups $\{X_r | r \in \Phi^-\}$ of $U^-$. Then $\mathcal{G} = (U, U^-)$. Let $\bar{\varphi}$ be an automorphism of $\mathcal{G}$ (considered as an abstract group) and $\bar{\gamma}$ be a graph automorphism of $\mathcal{G}$. It is known that $\bar{\varphi}$ can be chosen so that it acts by $x_r(t) \mapsto x_r(\bar{\varphi}t)$ (see [12, 12.2] and [15, 1.7], for example). In view of [12, Proposition 12.2.3 and Proposition 12.3.3] we can choose $\bar{\gamma}$ so that it acts by $x_r(t) \mapsto x_r(t)$ if $\Phi$ has no root of distinct length or by $x_r(t) \mapsto x_r(t^{\lambda_r})$ for appropriate $\lambda_r \in \{1, 2, 3\}$ if $\Phi$ has roots of distinct length. Here $\bar{\varphi}$ is the image of $\varphi$ under the symmetry $p$ (corresponding to $\bar{\gamma}$) of the root system $\Phi$. In both cases we can write $x_r(t)^{\bar{\varphi}} = x_r(t^{\lambda_r})$, where $\lambda_r \in \{1, 2, 3\}$. From these formula it is evident that $\bar{\varphi} \cdot \bar{\gamma} = \bar{\gamma} \cdot \bar{\varphi}$. Let $n_r(t) = x_r(t)x_r(-t^{-1})x_r(t)$ and $N_r = (n_r(t) | t \in \mathbb{F}_p)$. Let $h_r(t) = n_r(t)n_r(-1)$ and $H_r = (h_r(t) | t \in \mathbb{F}_p)$. In view of [12, Chapters 6 and 7], $H_r$ is a maximal torus of $\mathcal{G}$, $N_r = N_{\sigma_r}(H_r)$ and $X_r$ are root subgroups with respect to $H_r$. So we can substitute $T$ by $H_r$ and suppose that under our choice $T$ is $\varphi$- and $\bar{\gamma}$- invariant. Moreover $\bar{\varphi}$ induces the trivial automorphism of $N/T$.

An automorphism $\varphi^k, k \in \mathbb{N}$ is called a classical Frobenius automorphism. We shall call an automorphism $\sigma$ a Frobenius automorphism if $\sigma$ is conjugate under $\mathcal{G}$ to $\bar{\gamma}^e \varphi^k, e \in \{0, 1\}$. It follows from Lang-Steinberg theorem [19, Theorem 10.1] that for any $\bar{g} \in \mathcal{G}$, elements $\sigma$ and $\sigma \bar{g}$ are conjugate under $\mathcal{G}$. Thus, in view of [19, 11.6], we have that a Frobenius map, denoted in the previous section, coincides with a Frobenius automorphism denoted here.

Now we choose $G, \varphi, \bar{\gamma}$, and $\sigma = \bar{\gamma}^e \varphi^k$; and assume that $|\bar{\gamma}| \leq 2$, i.e., we do not consider the triality automorphism of a group with root system $\Phi(\mathcal{G}) = D_4$. Let $B = B_\sigma, H = H_\sigma$, and $U = U_\sigma$. Since $B, H, \bar{\gamma}$ are $\varphi$- and $\bar{\gamma}$- invariant, they give us a Borel subgroup, a Cartan subgroup, and a maximal unipotent subgroup (a Sylow $p$-subgroup of $\mathcal{G}_\sigma$ (see [15, 1.7]{uni04B71.9} or [18, Chapter 2] for details).

Assume that $e = 0$, i.e., $O^p(\mathcal{G}_{\sigma})$ is not twisted (is split). Then $U = (X_r | r \in \Phi^+)$, where $X_r$ is isomorphic to the additive group of $GF(p^k) = GF(q)$ and every element of $X_r$ can be written in the form $x_r(t), t \in GF(q)$. Let also $U^- = U_{\sigma}^-$. As for $U$ we can write $U^- = (X_r | r \in \Phi^-)$ and every element of $X_r$ can be written in the form $x_r(t), t \in GF(q)$. Now we can define an automorphism $\varphi$ by the restriction of $\bar{\varphi}$ on $\mathcal{G}_{\sigma}$ and automorphism $\gamma$ by the restriction of $\bar{\gamma}$ on $\mathcal{G}_{\sigma}$. By definition we have that $x_r(t) = x_r(\bar{\varphi}t)$ and $x_r(t) = x_r(t^{\lambda_r})$ (see the definition of $\bar{\gamma}$ above) for all $r \in \Phi$. Define $\zeta = \gamma^e \varphi^k, \varphi^k \neq e, e \in \{0, 1\}$ to be an automorphism of $\mathcal{G}_{\sigma}$ and $\bar{\zeta} = \bar{\gamma}^e \varphi^k$ to be an automorphism of $\mathcal{G}$. Choose a $\zeta$-invariant subgroup $G$ with $G^\varphi(\mathcal{G}_{\sigma}) \leq G \leq \mathcal{G}_{\sigma}$. Note that if the root system $\Phi$ of $\mathcal{G}$ is not $D_{2n}$, then $\mathcal{G}_{\sigma}/(O^p(\mathcal{G}_{\sigma}))$ is
cyclic. Thus for most groups and automorphisms, except groups of type $D_{2n}$ over a $\sigma$-eld of odd characteristic, any subgroup $G$ of $\mathcal{G}_\sigma$ satisfying $O^\ell(\mathcal{G}_\sigma) \leq G \leq \mathcal{G}_\sigma$ is $\gamma$- and $\varphi$- invariant. Define $\Gamma G$ as the set of subgroups of type $\langle G; \zeta g \rangle \leq \mathcal{G}_\sigma \ltimes \langle \zeta \rangle$, where $g \in \mathcal{G}_\sigma$, $\langle \zeta g \rangle \cap \mathcal{G}_\sigma \leq G$, and $\Gamma \mathcal{G}$ as the set of subgroups of type $\mathcal{G} \ltimes \langle \zeta \rangle$. Following [18, Definition 2.5.13], we shall call $\zeta$ a $\sigma$-eld automorphism, if $\varepsilon = 0$, i. e. $\zeta = \varphi^\ell$ and a graph-$\sigma$eld automorphism in other cases (recall that we are assuming $\varphi^\ell \neq e$).

Assume now that $\varepsilon = 1$, i. e. $O^\ell(\mathcal{G}_\sigma) = O^\ell(\mathcal{G}_\sigma)$ is twisted. Then $U = \overline{U}_\sigma$ and $U^- = \overline{U}_\sigma$ Define $\varphi$ on $U^\pm$ to be the restriction of $\varphi$ on $U^\pm$. Since $O^\ell(\mathcal{G}_\sigma) = \langle U^+, U^- \rangle$ we obtain the automorphism $\varphi$ of $O^\ell(\mathcal{G}_\sigma)$. Consider $\zeta = \varphi^\ell \neq e$ and let $G$ be a $\zeta$-invariant group with $O^\ell(\mathcal{G}_\sigma) \leq G \leq \mathcal{G}_\sigma$. Then $\bar{\zeta} = \varphi^\ell$ is an automorphism of $\mathcal{G}$. Define $\Gamma G$ as the set of subgroups of type $\langle G, \zeta g \rangle \leq \mathcal{G}_\sigma \ltimes \langle \zeta \rangle$, where $g \in \mathcal{G}_\sigma$, $\langle \zeta g \rangle \cap \mathcal{G}_\sigma \leq G$, and $\Gamma \mathcal{G}$ as the set of subgroups of type $\mathcal{G} \ltimes \langle \zeta \rangle$. Following [18, Definition 2.5.13] we shall say that $\zeta$ is a $\sigma$-eld automorphism if $|\zeta|$ is not divisible by $|\gamma|$ (this definition we shall use also in the case $|\gamma| = 3$ and $\mathcal{G}_\sigma \simeq D_4(q^3)$, and is a graph automorphism in other cases.

Groups from the set $\Gamma G$ defined above are called semilinear finite group of Lie type (they are called semilinear canonical finite group of Lie type if $G = O^\ell(\mathcal{G}_\sigma)$) and groups from the set $\Gamma \mathcal{G}$ are called semilinear algebraic groups. Note that $\Gamma \mathcal{G}$ can not be defined without $\Gamma G$, since we need to know that $\varphi^\ell \neq e$. If $G$ is written in notations of [12], i. e. $O^\ell(G) = G = A_n(q)$ or $O^\ell(G) = G = A_n(q^2)$ etc., then we shall write $G \Gamma$ by $\Gamma A_n(q)$, $\Gamma^2 A_n(q^2)$, etc.

Consider $A \in \Gamma G$ and $x \in A \setminus G$. Then $x = \zeta^k y$ for some $k \in \mathbb{N}$ and $y \in \mathcal{G}_\sigma$. Define $\bar{x}$ to be $\bar{\zeta}^k y$. Conversely, if $\bar{x} = \bar{\zeta}^k y$ for some $y \in \mathcal{G}_\sigma$, $\zeta^k \neq e$ and $\langle \zeta^k y \rangle \cap \mathcal{G}_\sigma \leq G$, define $x$ to be equal to $\zeta^k y$. Note that we do not need to suppose that $\bar{x} \notin \mathcal{G}$$ since $|\zeta| = \infty$. If $x \in G$ we define $\bar{x} = x$.

**Lemma 4.1.** In the above notations let $X$ be a subgroup of $G$. Then $x$ normalizes $X$ if and only if $\bar{x}$ normalizes $X$ as a subgroup of $\mathcal{G}$.

**Proof.** Since $\zeta$ is the restriction of $\bar{\zeta}$ on $G$ our statement is trivial. \hfill $\Box$

Let $X_1$ be a subgroup of $A \in \Gamma G$. Then $X_1$ is generated by a normal subgroup $X = X_1 \cap G$ and an element $x = \zeta^k y$. In view of Lemma 4.1 we can consider the subgroup $\overline{X}_1 = \langle \bar{x}, X \rangle$ of $\mathcal{G} \ltimes \langle \zeta \rangle$. Now we need it reasonable to explain, why we use such complicated notations and definitions. We have that $\zeta$ is always of finite order, but $\zeta$ is always of infinite order. Thus, even if $Z(G)$ is trivial, we can not consider $G \ltimes \langle \zeta \rangle$ as a subgroup of $\text{Aut}(G)$. Hence, we need to define in some way (one possible way is just given) the connection between elements of $\text{Aut}(G)$ and elements of $\text{Aut}(\mathcal{G})$ in order to use the machinery of linear algebraic groups.

Let $\overline{R}$ be a $\sigma$-stable maximal torus (resp. reductive subgroup of maximal rank, parabolic subgroup) of $\mathcal{G}$, and $y \in N_{\mathcal{G} \ltimes \langle \zeta \rangle}(\overline{R})$ be chosen so that there exists $x \in \langle G, \zeta g \rangle$ with $y = \bar{x}$. Then $R_X = \langle x, \overline{R} \cap G \rangle$ is called a maximal torus (resp. a reductive subgroup of maximal rank, a parabolic subgroup) of $\langle G, \zeta g \rangle$.

**Lemma 4.2.** Let $M = \langle x, X \rangle$, where $X = M \cap G \unlhd M$ be a subgroup of $\langle G, \zeta g \rangle$ such that $O_p(X)$ is nontrivial. Then there exists $\sigma$- and $\bar{x}$- stable parabolic subgroup $\overline{P}$ of $G$ such that $X \leq \overline{P}$ and $O_p(X) \leq R_\sigma(\overline{P})$.

**Proof.** Define $U_0 = O_p(X)$, $N_0 = N_{\mathcal{G}}(U_0)$. Then $U_i = U_0 R_{\sigma}(N_{i-1})$ and $N_i = N_{\mathcal{G}}(U_i)$. Clearly $U_i, N_i$ are $\bar{x}$- and $\sigma$- stable for all $i$. In view of [13, Proposition 30.3], the chain of subgroups $N_0 \leq N_1 \leq \ldots \leq N_k \leq \ldots$ is finite and $\overline{P} = \cup_i N_i$ is a proper parabolic subgroup of $\mathcal{G}$. Clearly $\overline{P}$ is $\sigma$- and $\bar{x}$- stable. \hfill $\Box$
Lemma 4.3. Let $G$ be a finite group of Lie type over a field of odd characteristic $p$. Assume that $\overline{G}$ and $\sigma$ are chosen so that $O^p(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$. Let $\psi$ be a $\sigma$-automorphism of odd order of $O^p(\overline{G}_\sigma)$.

Then $\psi$ centralizes a Sylow 2-subgroup of $G$, and there exists a $\psi$-stable Cartan subgroup $H$ such that $\psi$ centralizes a Sylow 2-subgroup of $H$. Moreover, if $G \not\simeq 2G_2(3^{2n+1})$, $3D_4(q^3)$, $2D_{2n}(q^2)$, then there exists a $\psi$-stable torus $T$ of $G$ such that $\psi$ centralizes a Sylow 2-subgroup of $T$ and the factor group $N(G, T)/T$ is isomorphic to $N_{\overline{G}(T)}/T$.

Proof. Clearly we need to prove the lemma only for the case $G = \overline{G}_\sigma$. Assume that $|\psi| = k$. Let $GF(q)$ be the base field of $G$. Then $q = p^k$ and $\alpha = k \cdot m$. Now $|G|$ can be written as $|G| = q^N(q^{m_1} + \varepsilon_1) \cdots (q^{m_n} + \varepsilon_n)$ for some $N$, where $n$ is the rank of $G$, $\varepsilon_i = \pm$. (cf. [12, Theorems 9.4.10 and 14.3.1]). Similarly we have that $|G_\psi| = (p^m)^N((p^m)^{m_1} + \varepsilon_1) \cdots ((p^m)^{m_n} + \varepsilon_n)$, i.e., $|G_\psi|_2 = |G_\psi|_2$ and a Sylow 2-subgroup of $G_\psi$ is a Sylow 2-subgroup of $G$. By [18, Proposition 2.5.17] there exists an automorphism $\psi_1$ of $\overline{G}$ such that $\sigma = \psi_1^k$ and $\psi$ coincides with the restriction of $\psi_1$ on $\overline{G}_\sigma$. Note that $\psi_1$, in general, is not equal to $\psi$ defined above. Consider a maximal split torus $\overline{T}_{\psi_1}$ of $\overline{G}_\psi$. Then $H = \overline{H}_\sigma$ is a $\psi$-stable Cartan subgroup of $G$.

Now assume that $G \not\simeq 2G_2(3^{2n+1})$, $3D_4(q^3)$, $2D_{2n}(q^2)$. By Lemma 3.8, there exists a maximal torus $T$ of $G_\psi$ such that $N(G_\psi, T)/T \simeq N_{\overline{G}}(T)/T$ and $|T_\psi| = (p^m - \varepsilon_1)^n$. Clearly $|T_\sigma| = (q - \varepsilon_1)^n$. Since $|\psi|$ is odd and $\overline{T}_{\psi_1}$ is obtained by twisting a maximal split torus $\overline{T}$ with an element $w_0$, then $\overline{T}_{\psi}$ is also obtained by from the maximal split torus $\overline{T}$ by twisting with an element $w_0$ (see the proof of Lemma 3.8). Therefore $|T_\sigma| = (q - \varepsilon_1)^n$, $|T_{\psi_1}| = (p^m - \varepsilon_1)^n$, so $|T_\sigma|_2 = |T|_2 = |T_{\psi_1}|_2$.

Lemma 4.4. [29, (7-2)] Let $\overline{G}$ be a connected simple linear algebraic group of adjoint type over a field of characteristic $p$, $\sigma$ be a Frobenius map of $\overline{G}$ and $G = \overline{G}_\sigma$ be a finite group of Lie type. Let $\varphi$ be a $\sigma$-endomorphism or a graph $\sigma$-endomorphism of $G$ and let $\varphi'$ be an element of $(G \times \langle \varphi \rangle) \backslash G$ such that $|\varphi'| = |\varphi|$ and $G \times \langle \varphi \rangle = \langle G, \varphi' \rangle = G \times \langle \varphi' \rangle$. Then there exists an element $g \in G$ such that $(\varphi)^g = \langle \varphi' \rangle$. In particular, if $G/O^p(G)$ is a 2-group and $\varphi$ is of odd order, then such $g$ can be chosen in $O^p(G)$.

Lemma 4.5. Let $G$ be a $\sigma$-admissible group of Lie type over a field of odd characteristic, $G \not\simeq 3D_4(q^3)$, and $\overline{G}$, $\sigma$ are chosen so that $O^p(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$. Let $A$ be a subgroup of $\text{Aut}(O^p(\overline{G}_\sigma))$ such that $A \cap \overline{G}_\sigma = G$. If $O^p(G) \simeq D_4(q)$, assume also that $A$ is contained in the group generated by inner-diagonal $\sigma$-endomorphisms and a graph automorphism of order 2. Then $A$ satisfies $\text{(ESyl2)}$ if and only if $G$ satisfies $\text{(ESyl2)}$.

Proof. Assume that $G$ satisfies $\text{(ESyl2)}$. In the conditions of the lemma we have that the factor group $A/G$ is Abelian, so $A/G = \overline{A}_1 \times \overline{A}_2$, where $\overline{A}_1$ is a Hall 2'-subgroup of $A/G$ and $\overline{A}_2$ is a Sylow 2-subgroup of $A/G$. Denote by $A_1$ the complete preimage of $\overline{A}_1$ in $A$. If $A_1$ satisfies $\text{(ESyl2)}$, then by Lemma 2.9 $A$ satisfies $\text{(ESyl2)}$ as well. Thus we may assume that the order $|A/G|$ is odd. Since we are assuming that a graph automorphism of order 3 is not contained in $A$, then $A/G$ is cyclic, hence $A = \langle G, \psi g \rangle$, where $\psi$ is a $\sigma$-endomorphism of odd order and $g \in \overline{G}_\sigma$. Since $|A : G| = |\psi|$ is odd, we may assume that $|\psi| = 1$ is odd. By Lemma 4.3, $\psi$ centralizes a Sylow 2-subgroup of $\overline{G}_\sigma$, therefore $g$ is of odd order. Now the quotient $\overline{G}_\sigma/G$ is Abelian and can be written as $\overline{L} \times \overline{Q}$, where $\overline{L}$ is a Hall 2'-subgroup of $\overline{G}_\sigma/G$ and $\overline{Q}$ is a Sylow 2-subgroup of $\overline{G}_\sigma/G$. Let $L$ be the complete preimage of $\overline{L}$ in $\overline{G}_\sigma$, under the natural homomorphism. Then $g \in L$. Consider $L \times \langle \psi \rangle \geq A$. By construction, $|L \times \langle \psi \rangle : A| = |L : G|$. 
is odd. By Lemma 3.6 the group $L$ satisfies (ESyl12). By Lemma 4.3 the sylow automorphism \( \psi \) centralizes a Sylow 2-subgroup $Q$ of $L$. Thus

\[
N_{L \times \langle \psi \rangle}(Q) = N_L(Q) \times \langle \psi \rangle = QC_L(Q) \times \langle \psi \rangle = QC_{L \times \langle \psi \rangle}(Q),
\]

i.e., the group $L \times \langle \psi \rangle$ satisfies (ESyl12). Since $|L \times \langle \psi \rangle : A|$ is odd, then $A$ satisfies (ESyl12).

Now assume that $A$ satisfies (ESyl12). If $G$ does not satisfies (ESyl12), then [25, Theorem 1] and [26, Theorem 6] imply that the root system of $\overline{G}$ either has type $A_1$, or has type $C_n$. In both cases the factor group $\text{Aut}(O^p(\overline{G})) / \text{Aut}G$ is an element of Lemma 4.6.

\[\phi\text{ and [26, Theorem 6] imply that the root system of } i.e., the group } \psi\text{ centralizes a Sylow } q\text{ of } G, \text{ thus } \langle n, t \rangle \not= 0 \text{ and } h_{r_i}(t_i)^{t} = h_{r_i}(t_i). \text{ Denote by } W \text{ the Weyl group of } \overline{G}. \text{ Let } w_0 \text{ be the unique element of } W \text{ mapping all positive roots onto negative roots and let } n_0 \text{ be its preimage in } N_{\overline{G}}(T) \text{ under the natural homomorphism } N_{\overline{G}}(T) \to N_{\overline{G}}(T) / T \simeq W. \text{ Since } \sigma \text{ acts trivially on } W = N(G, T) / T \text{ (see Lemma 3.8), we can take } n_0 \in G, \text{ i.e., } n_0^\sigma = n_0. \text{ Then for all } r_i \text{ and } t \text{ we have that }
\[
h_{r_i}(t)^{n_0\tau} = h_{r_i}^{n_0\tau}(t) = h_{-r_i}(t) = h_{r_i}(t^{t}).
\]
Thus $x^{n_0\tau} = x^{-1}$ for all $x \in T$.

Now let $s$ be a semisimple element of $G$. Then there exists a maximal $\sigma$-stable torus $S$ of $\overline{G}$ containing $s$. Since all maximal tori of $\overline{G}$ are conjugate, we have that there exists $g \in \overline{G}$ such that $S = \overline{T}$. Since $\overline{G}_\sigma = O^p(\overline{G}) / \overline{T}$, we may assume that $a \in \overline{T}_\sigma$. Therefore $s^{g_0\tau g^{-1}} = s^{-1}$. Since $n_0^\sigma = n_0$ and $\tau^\sigma = \tau$, we have that $(g_0\tau g^{-1})^\sigma = g^{2}n_0\tau a(g^{-1})^\sigma$. Moreover, since $S$ is $\sigma$-stable, then for every $x \in S$ we have that $x^{g_0\tau g^{-1}} = x^{g_0\tau a(g^{-1})^\sigma} = x^{-1}$, i.e., $g_0\tau a(g^{-1})^\sigma = g^{2}n_0\tau a(g^{-1})^\sigma$. In particular, there exists $t \in S$ such that $g_0\tau a(g^{-1})^\sigma = g^{2}n_0\tau a(g^{-1})^\sigma$. In view of Lang-Steinberg Theorem [19, Theorem 10.1] there exists $y \in S$ such that $t = y \cdot (y^{-1})^\sigma$. Therefore, $g_0\tau a(g^{-1})^\sigma = (g_0\tau a(g^{-1})^\sigma)^\sigma$, i.e., $g_0\tau a(g^{-1})^\sigma \in T_\sigma \times \langle \tau \rangle$, and $s^{g_0\tau g^{-1}} = s^{-1}$. Since $O^p(\overline{G}) S = \overline{G}_\sigma$, and $\overline{S}$ is Abelian, we may send $z \in \overline{S}$ such that $g_0\tau a(g^{-1})^\sigma yz \in (O^p(\overline{G}), \tau a).$  

**Lemma 4.7.** Let $(G, \zeta G)$ be a finite semilinear group of Lie type and $\overline{G}$, $\sigma$ are chosen so that $O^p(\overline{G}) \leq G \leq \overline{G}_\sigma$. Let $s$ be a regular semisimple element of odd order of $G$. Then $N_{G, \zeta G}(S_{G, \zeta G}(s)) \neq S_{G, \zeta G}(s)$. 

\[\text{Proof.} \]
Proof. Since $s$ is semisimple, there exists $\sigma$-stable maximal torus $T$ of $G$ containing $s$. Since $\overline{G}_\sigma = O^\sigma(\overline{G})/\overline{T}$ we may assume that $g \in \overline{G}_\sigma$, i.e. elements $g$ and $s$ commutes. If $C_{(G,\zeta g)}(s)G \not\subseteq \langle G, \zeta g \rangle$, then we can substitute $\langle G, \zeta g \rangle$ by $C_{(G,\zeta g)}(s)G$ and prove the lemma for this group. Moreover, if $C_{(G,\zeta g)}(s) = C_G(s)$, then the lemma follows from Lemma 3.9, so we may assume that $\zeta$ centralizes $s$. If either $G$ is not twisted, or $|\zeta|$ is odd, then by [18, Proposition 2.5.17] it follows that we may assume $\sigma = \tilde{\chi}^k$ for some $k > 0$. By Lemma 3.9 there exists an element of $N_{G,\zeta g}(C_{\overline{G}}(s))$, not contained in $C_{G,\zeta g}(s)$, and the lemma follows.

Assume that $G$ is twisted and $|\zeta|$ is even. Then $\sigma = \tilde{\gamma}^\varphi^k$, $\tilde{\zeta} = \varphi^k$, where $k$ divides $\ell$. Therefore $s$ is in $\overline{G}_\gamma$. Depending on the root system $\Phi(\overline{G})$, we obtain that $\overline{G}_\gamma$ is isomorphic to a simple algebraic group with root system equal to $B_n$ (for some $m > 1$), $C_m$ (for some $m > 2$), or $F_4$. By [10, Lemma 2.2], the element $s$ is conjugate with its inverse under $O^\theta((\overline{G}_\gamma)_{\zeta g}) \leq G_{\zeta g}$, so $N_{G,\zeta g}(C_{(G,\zeta g)}(s)) \neq C_{(G,\zeta g)}(s)$.

Lemma 4.8. Let $\langle G, \zeta g \rangle$ be a finite semilinear group of Lie type over a field of characteristic $p$ (we do not exclude the case $\langle G, \zeta g \rangle = G$) and $G$ is of adjoint type (recall that $g \in \overline{G}_\sigma$, but not necessary $g \in G$). Assume that $B = U \gamma H$, where $H$ is a Cartan subgroup of $G$, is a $\zeta g$-invariant Borel subgroup of $G$ and $(B, \zeta g)$ contains a Carter subgroup $K$ of $\langle G, \zeta g \rangle$. Assume that $K \cap U \neq \{e\}$. Then one of the following statements holds:

(a) either $\langle G, \zeta g \rangle = \langle \tilde{2}A_2(2^n) \rangle$, or $\langle G, \zeta g \rangle = \tilde{2}A_2(2^n) \zeta \langle \zeta \rangle$; the order $|\zeta| = t$ is odd and is not divisible by $3$, $C_G(\zeta) \simeq \tilde{2}A_2(2^n)$, $K \cap G$ is Abelian and has order $2 \cdot 3$;
(b) $\langle G, \zeta g \rangle = \langle \tilde{2}A_2(2^n) \rangle$, the order $|\zeta| = t$ is odd, $C_G(\zeta) \simeq \tilde{2}A_2(2^n)$, the subgroup $K \cap G$ is a Sylow 2-subgroup of $G$;
(c) either $\langle G, \zeta g \rangle = \langle \tilde{2}A_2(2^n) \rangle$, or $\langle G, \zeta g \rangle = \tilde{2}A_2(2^n) \zeta \langle \zeta \rangle$; $\zeta$ is a graph-elder automorphism of order $2t$, $t$ is not divisible by $3$, and $C_G(\zeta) \simeq \tilde{2}A_2(2^n)$, the subgroup $K \cap G$ is Abelian and has order $2^{2e+1} \cdot 3$;
(d) $\langle G, \zeta g \rangle = \langle \tilde{2}A_2(2^n) \rangle$, $\zeta$ is a graph-elder automorphism and $C_G(\zeta) \simeq \tilde{2}A_2(2^n)$, the subgroup $K \cap G$ is a Sylow 2-subgroup of $G_{\zeta g}$;
(e) $G$ is de\-\v{s}ned over $GF(2^n)$, $(G, \zeta g) = G \zeta \langle \zeta g \rangle$, $\zeta$ is either a graph-elder automorphism of order $t$ of $O^\sigma(2^n) (G)$, if $O^\sigma(2^n)$ is split, or a graph automorphism of order $t$, if $O^\sigma(2^n)$ is twisted, and, up to conjugation in $G$, $K = Q \times \langle \zeta g \rangle$, where $Q$ is a Sylow 2-subgroup of $G_{(\zeta g)_{\varphi}}$;
(f) $G$ is de\-\v{s}ned over $GF(2^n)$, $(G, \zeta g) = G \times \langle \zeta g \rangle$, $\zeta$ is a product of a graph-elder automorphism of odd order $t$ of $O^\sigma(2^n)$ and a graph automorphism of order $2$, $\zeta$ and $\zeta g$ are conjugate under $\overline{G}_\sigma$, and, up to a conjugation in $G$, $K = Q \times \langle \zeta g \rangle$, where $Q$ is a Sylow 2-subgroup of $G_{(\zeta g)_{\varphi}}$;
(g) $G/Z(G) \simeq PSL_2(3^t)$, the order $|\zeta| = t$ is odd (in particular $\zeta \in (G, \zeta g)$), and $K$ contains a Sylow 3-subgroup of $G_{\zeta g}$;
(h) $\langle G, \zeta g \rangle = \tilde{2}G_2(2^{3^{n+1}})$, $|\zeta| = 2n + 1$, $K \cap \tilde{2}G_2(2^{3^{n+1}}) = Q \times P$, where $Q$ is of order $2$ and $|P| = 3^{\ell+n}$.

Note that in all points (a)-(h) of the lemma Carter subgroups, having given structure, do exist. The existence of Carter subgroups in points (a) and (c) follows from the existence of a
Carter subgroup of order 6 in $\mathbf{PGU}_3(2)$ (see [5]). The existence of Carter subgroups in points (b), (d)-(f) follows from the fact that a Sylow 2-subgroup in a group of Lie type descends over a $\varphi$-eld of order 2, coincides with its normalizer. The existence of Carter subgroups in point (g) follows from the fact that a Sylow 3-subgroup of $\mathbf{PSL}_2(3)$ coincides with its normalizer. The existence of a Carter subgroup, satisfying point (h) of the lemma, follows from the existence of a Carter subgroup $K$ of order 6 in a (non simple) group $^2G_2(3)$. The existence of a Carter subgroup $K$ of order 6 in $^2G_2(3)$ follows from the results given in [31] and [32].

Proof. If $G$ is one of the groups $A_1(q), G_2(q), F_4(q), 2B_2(2^{2n+1}),$ or $2^2F_4(2^{2n+1})$, then the lemma follows from [9, Table] or [10, Table]. If $\langle G, \zeta_g \rangle = G$ then our result follows from [6], [9], and [10]. So we may assume that $\langle G, \zeta_g \rangle \neq G$, i.e., that $\zeta$ is a nontrivial $\varphi$-eld, graph-$\varphi$-eld, or graph automorphism. If $\Phi(G) = C_n$, the lemma follows from Theorem 5.1 below, that does not use Lemma 4.8, so we assume that $\Phi(G) \neq C_n$. If $\Phi(G) = D_4$ and either a graph-$\varphi$-eld automorphism $\zeta$ is a product of a $\varphi$-eld automorphism and a graph automorphism of order 3, or $G \simeq 3D_4(q^3)$, then the lemma follows from Theorem 6.1 below, that does not use Lemma 4.8, so we assume that $\langle G, \zeta_g \rangle$ is contained in the group $A_4$ descended in Theorem 6.1, and $G \neq 3D_4(q^3)$. Since we shall use Lemma 4.8 in the proof of Theorem 7.1, after Theorems 5.1 and 6.1, it is possible to make such additional assumptions.

Assume that $q$ is odd and $\Phi(G)$ is one of the following types: $A_n$ $(n \geq 2)$, $D_n$ $(n \geq 4)$, $B_n$ $(n \geq 3)$, $E_6$, $E_7$ or $E_8$. By Lemma 2.3 we have that $KU/U$ is a Carter subgroup of $\langle B, \zeta_g \rangle/U \simeq \langle H, \zeta_g \rangle$. Since $\overline{G}_\sigma = G\overline{\varphi}_\sigma$, where $\overline{\varphi}$ is a maximal split torus of $G$ and $\overline{\varphi}_\sigma \cap G = H$, then we may assume that $g \in H$, and $g$ centralizes $H$. So $H_\zeta \leq Z(\langle H, \zeta_g \rangle)$, and we obtain, up to conjugation in $B$, that $H_\zeta \leq K$. By Lemma 4.3, the automorphism $\zeta_g$ centralizes a Sylow 2-subgroup $Q$ of $H$. Thus, each element of odd order of $\langle H, \zeta_g \rangle$ centralizes $Q$ and Lemma 2.5 implies, that up to conjugation in $B$, the inclusion $Q \leq K$ holds. By Lemma 3.3 it follows that $C_U(Q) = \{e\}$, a contradiction with the fact that $K \cap U$ is nontrivial.

Assume that $G \simeq ^2G_2(3^{2n+1})$ and $\langle G, \zeta_g \rangle = G \ltimes \langle \zeta \rangle$ (in this case $O^{2^r}(\overline{G}_\sigma) = \overline{G}_\sigma$). Again by Lemma 2.3 we have that $KU/U$ is a Carter subgroup of $\langle B \ltimes \zeta \rangle/U \simeq H \ltimes \zeta$. By [10, Lemma 2.2] every semisimple element of $G$ is conjugate to its inverse. Since non-Abelian composition factors of every semisimple element of $G$ can be isomorphic only to groups $A_1(q)$, by [9, Table] it follows that the centralizer of every semisimple element of $G$ satisfies condition (C). So Lemma 2.4 implies that $KU/U \cap B/U$ is a 2-group. On the other hand, $|H|_2 = 2$ and $KU/U \geq Z(B/U) \geq H_\zeta$, hence $|H_\zeta| = 2$ and $|\zeta| = 2n + 1$. Thus $K \cap G = (K \cap U \times t)$, where $t$ is an involution. It follows that $K \cap U = C_G(t) \cap G_\zeta$. Now the structure results from [31] and [32, Theorem 1] imply point (h) of the lemma.

Assume now that $q = 2^r$. Assume first that $\Phi(G)$ has one of the types $A_n$ $(n \geq 2)$, $D_n$ $(n \geq 4)$, $B_n$ $(n \geq 3)$, $E_6$, $E_7$ or $E_8$, $G$ is split, and $\zeta$ is a $\varphi$-eld automorphism. Like above we obtain that $H_\zeta \leq K$, and $O^{2^r}(G_\zeta)$ is a split group of Lie type with descent $\varphi$-eld of order $q = 2^{r|\zeta|}$. By Hartley-Shute Lemma 3.2, for every $r \in \Phi(G)$ and for every $s \in GF(2^{r|\zeta|})$ there exists $h(\chi) \in H_\zeta \cap O^{2^r}(G_\zeta)$ such that $\chi^{-1}(\chi) = s$. The same arguments as in Lemma 3.3 imply that if $\frac{r}{|\zeta|} \neq 1$, inequality $K \cap U \leq C_U(H_\zeta)$ holds, a contradiction. So $|\zeta| = t$ and $H_\zeta = \{e\}$. Since $g$ can be chosen in $H_\zeta \cap Q$ and $\langle g \rangle \cap \overline{G}_\sigma \leq \langle g \rangle \cap \overline{\varphi}_\sigma \leq H_\zeta = \{e\}$, then $\langle g \rangle \cap \overline{G}_\sigma = \{e\}$. By Lemma 4.4 elements $\zeta g$ and $\zeta$ are conjugate under $\overline{G}_\sigma$, and point (e) of the lemma follows.

Now assume that $\Phi(G)$ is of type $A_n$ $(n \geq 3)$, $D_n$ $(n \geq 4)$, or $E_6$; and either $\zeta$ is a graph-$\varphi$-eld automorphism and $G$ is split, or $G$ is twisted. Let $\rho$ be the symmetry of the Dynkin diagram of $\Phi(G)$ corresponding to $\gamma$ (recall that $\zeta = \gamma^\rho \varphi^\rho$ by descent), and $\bar{r}$ denotes $r^\rho$ for $r \in \Phi(G)$. 4 SEMILINEAR GROUPS OF LIE TYPE 18
Like above it is possible to prove that, up to conjugation, \( H_\zeta \leq K \). If \( |\zeta| = 2t \), then \( H_\zeta \neq \{e\} \), then by Hartley-Shute Lemma 3.2 we obtain that \( C_U(H_\zeta) = \{e\} \) that contradicts the condition \( K \cap U \neq \{e\} \). If \( H_\zeta = \{e\} \), then either \( G \) is twisted and \( |\zeta| = t \), that implies statement (e) of the lemma; or \( G \) is twisted, \( |\zeta| = 2t \), in particular, \( t \) is odd, that implies point (f) of the lemma.

Assume that \( O^2(G) \simeq A_2(2^t) \) and \( \zeta \) is a graph-\( \ast \)eld automorphism and \( t \) is odd. If \( |\zeta| \neq 2t \), then arguments, using Hartley-Shute Lemma 3.2, similar to the proof of Lemma 2.4 show that \( C_U(H_\zeta) = \{e\} \), that contradicts to the condition \( K \cap U \neq \{e\} \). If \( |\zeta| = 2t \), then we obtain point (f) of the lemma.

Assume now that \( O^2(G) \simeq A_2(2^t) \) and \( \zeta \) is a graph-\( \ast \)eld automorphism. Again for \( |\zeta| \neq 2t \) from Hartley-Shute Lemma 3.2 it follows that \( C_U(H_\zeta) = \{e\} \), that contradicts to the condition \( K \cap U \neq \{e\} \). If \( |\zeta| = 2t \), then either \( G_\zeta \simeq A_2(2^t) \), or \( G_\zeta \simeq A_2(2^t) \). If \( G_\zeta \simeq A_2(2^t) \), then \( H_\zeta = \{e\} \) and we obtain the statement (d) of the lemma. If \( G_\zeta \simeq A_2(2^t) \), then \( |H_\zeta| = 3 \), and so \( KU/U \cap HU/U \) is a cyclic group \( \langle y \rangle \) of order \( (2^t + 1)3 = 3^k \), where \( 3^{k-1} = t_3 \). If \( k > 1 \), then Hartley-Shute Lemma 3.2 implies that \( C_U(y) = \{e\} \), that is impossible. Thus \( t \) is not divisible by 3 and \( K \cap U \) is contained in the centralizer of an element \( x_0 \), generating \( H_\zeta \). Consider the homomorphism \( GL_3(2^t) \to PGL_3(2^t) \). Then some preimage of \( x_0 \) is similar to the matrix

\[
\begin{pmatrix}
\lambda & 0 & 0 \\
0 & \lambda^2 & 0 \\
0 & 0 & \lambda
\end{pmatrix},
\]

where \( \lambda \) is the generating element of the multiplicative group of \( GF(2^t) \). The preimage of \( U \) is similar with the set of upper triangular matrices with the same elements on the diagonal. Direct calculations show that \( C_U(x) \) is isomorphic to the additive group of \( GF(2^t) \). The nilpotency of \( K \) implies that \( K \cap U = (C_U(x))_{\zeta^t} \), and point (c) of the lemma follows.

Assume now that \( O^2(G) \simeq 2A_2(2^t) \). By Lemma 2.3 \( KU/U \) is a Carter subgroup of \( \langle B, \zeta \rangle/ U \simeq \langle H, \zeta \rangle \) and, as above, we may assume that \( H_\zeta \leq K \). If \( |\zeta| = 2t \), then \( G_\zeta \simeq SL_2(2) \) and \( H_\zeta = \{e\} \), and point (e) of the lemma follows. Assume that \( t \) is even and \( |\zeta| \leq t \). Then either \( O^2(G_\zeta) \simeq SL_2(2^{t/|\zeta|}) \) (if the order \( |\zeta| \) is even), or \( O^2(G_\zeta) \simeq 2A_2(2^{t/|\zeta|}) \) (if the order \( |\zeta| \) is odd, hence \( |\zeta| < t \)). Clearly \( H_\zeta \) contains an element \( x \) such that \( K \cap U \leq C_U(H_\zeta) = \{e\} \), and this gives a contradiction with the condition \( K \cap U \neq \{e\} \). If \( t \) is odd and \( t \neq |\zeta| \), then \( O^2(G_\zeta) \simeq 2A_2(2^{t/|\zeta|}) \), and it follows that \( H_\zeta \) contains an element \( x \) such that \( C_U(x) = \{e\} \). If \( |\zeta| = t \) and \( t \) is odd, then the order \( |\mathcal{K}/U \cap B/U| \) can be divisible only by 3 (otherwise by Hartley-Shute Lemma 3.2 it again follows that \( C_U(H_\zeta) = \{e\} \)). If \( G_\zeta \simeq 2A_2(2^{t/|\zeta|}) \), then \( H_\zeta = \{e\} \) and we obtain point (b) of the lemma. If \( G_\zeta \simeq 2A_2(2^{t/|\zeta|}) \), then \( KU/U \cap HU/U \) is a cyclic group \( \langle y \rangle \) of order \( (2^t + 1)3 = 3^k \), where \( 3^{k-1} = t_3 \). If \( k > 1 \), then Hartley-Shute Lemma 3.2 implies, that \( C_U(y) = \{e\} \), that is impossible. Thus \( t \) is not divisible by 3 and \( K \cap U \) is contained in the centralizer of an element \( x \), generating \( H_\zeta \). As in the non-twisted case above, we obtain that \( C_U(x) \) is isomorphic to the additive group of \( GF(2^t) \). The nilpotency of \( K \) implies that \( K \cap U = (C_U(x))_{\zeta^t} \), and point (a) of the lemma follows.

5 Carter subgroups in symplectic groups

Consider a set \( \mathcal{A} \) of almost simple groups \( A \) such that a unique non-Abelian composition factor \( S = F^\ast(A) \) is a canonical simple group of Lie type and \( A \) contains nonconjugate Carter subgroups. If the set \( \mathcal{A} \) is not empty, denote by \( \mathbf{C} \) the minimal possible order of \( F^\ast(A) \).
with $A \in \mathcal{A}$. If the set $\mathcal{A}$ is empty, then let $\text{Cmin} = \infty$. We shall prove that $\text{Cmin} = \infty$, i.e., that $\mathcal{A} = \emptyset$. Note that if $A \in \mathcal{A}$ and $G = F^*(A)$, then there exists a subgroup $A_1$ of $A$ such that $A_1 \in \mathcal{A}$ and $A_1 = KG$ for a Carter subgroup $K$ of $A$. Indeed, if for every nilpotent subgroup $N$ of $A$ Carter subgroups of $NG$ are conjugate, then $A$ satisfies (C), hence Carter subgroups of $A$ are conjugate, that contradicts to the choice of $A$. So there exists a nilpotent subgroup $N$ of $A$ such that Carter subgroups of $NG$ are not conjugate. Let $K$ be a Carter subgroup of $NG$. Then clearly $KG/G$ is a Carter subgroup of $NG/G$, i.e., coincides with $NG/G$. Therefore Carter subgroups of $KG$ are not conjugate and $KG = A_1 \in \mathcal{A}$. So the condition $A = KG$ in Theorems 5.1, 6.1, and 7.1 is not a restriction and is used only to simplify arguments.

In this section we consider Carter subgroups in an almost simple group $A$ with simple socle $G = F^*(A) \simeq \mathbf{PSp}_{2n}(q)$. We consider such groups in the separate section, since for groups of type $\mathbf{PSp}_{2n}(q)$ Lemma 3.3 is not true and we use arguments slightly different from those that we use in the proof of Theorem 7.1.

**Theorem 5.1.** Let $G$ be a finite adjoint group of Lie type over a field of characteristic $p$, and $\overline{G}$, $\sigma$ are chosen so that $\mathbf{PSp}_{2n}(p') \simeq \mathsf{O}^\sigma(p) \leq G \leq \overline{G}_\sigma$. Choose a subgroup $A$ of $\text{Aut}(\mathbf{PSp}_{2n}(p'))$ so that $A \cap \overline{G}_\sigma = G$. Let $K$ be a Carter subgroup of $A$. Assume also that $|\mathbf{PSp}_{2n}(p')| \leq \text{Cmin}$ and $A = KG$. Then exactly one of the following statements holds:

(a) $G$ is defined over $GF(2^t)$, a field automorphism $\zeta$ is in $A$, $|\zeta| = t$, and, up to conjugation in $G$, the equality $K = Q \langle \zeta \rangle$ holds, where $Q$ is a Sylow 2-subgroup of $G_{\zeta'}$;

(b) $G \simeq \mathbf{PSL}_2(3^t) \simeq \mathbf{PSp}_2(3^t)$, a field automorphism $\zeta$ is in $A$, $|\zeta| = t$ is odd, and, up to conjugation in $G$, the equality $K = Q \langle \zeta \rangle$ holds, where $Q$ is a Sylow 3-subgroup of $G_{\zeta'}$;

(c) $p$ does not divide $|K \cap G|$ and $K$ contains a Sylow 2-subgroup of $A$.

In particular, Carter subgroups of $A$ are conjugate, i.e., if $A_1 \in \mathcal{A}$ and $F^*(A_1) = \text{Cmin}$, then $F^*(A_1) \not\simeq \mathbf{PSp}_{2n}(p')$.

**Proof.** Assume that the theorem is not true and $A$ is a counter example such that $|F^*(A)|$ is minimal. Note that no more than one statement of the theorem can be fulfilled, since if statement (b) holds, then, by Lemmas 3.7 and 4.5, for a Sylow 2-subgroup $Q$ of $A$ the condition $N_G(S) = SC_G(S)$ is true, i.e., statement (c) of the theorem does not hold. Moreover, if $A_1$ is an almost simple group with $F^*(A_1)$ being a simple group of Lie type of order less, than $|F^*(A)|$, then Carter subgroups of $A_1$ are conjugate. In view of the main theorem of [6] we may assume that $A \neq G$. Moreover, by [10, Theorem 3.5], we may assume that $q$ is odd, i.e., that $\text{Aut}(\mathbf{PSp}_{2n}(q))$ does not contain a graph automorphism. Thus we may assume that $A = \langle G, \zeta \rangle$.

Assume that $K$ is a Carter subgroup of $\langle G, \zeta \rangle$ and $K$ does not satisfy to the statement of the theorem. Write $K = \langle x, K \cap G \rangle$. If either $p \neq 3$ or $t$ is even, then the theorem follows from [10, Theorem 3.5]. Thus we may assume that $q = 3^t$ and $t$ is odd. Since $|\overline{G}_\sigma : \mathsf{O}^\sigma(\overline{G}_\sigma)| = 2$ and the order $|\zeta|$ is odd, we may assume that the order $|\zeta|$ is odd and so $\zeta \in \langle G, \zeta \rangle$, i.e., $A = G \langle \zeta \rangle$. By [10, Lemma 2.2] every semisimple element of odd order is conjugate to its inverse in $G$. Now, for every semisimple element $t \in G$, each non-Abelian composition factor of $C_G(t)$ is a simple group of Lie type (cf. [23]) of order less, than $\text{Cmin}$. Therefore, for every non-Abelian composition factor $S$ of $C_A(t)$ and every nilpotent subgroup $N \leq C_A(t)$, Carter subgroups of $\langle \text{Aut}_N(S), S \rangle$ are conjugate. It follows that $C_A(t)$ satisfies (C). Hence, by Lemma 2.4, $|K \cap G| = 2^\alpha \cdot 3^\beta$ for some $\alpha, \beta \geq 0$. 
If $G = \mathbf{PSp}_{2n}(q)$, then by [34, Theorem 2] every unipotent element is conjugate to its inverse. Since 3 is a good prime for $G$, then [35, Theorems 1.2 and 1.4] imply that, for any element $u \in G$ of order 3, all composition factors of $C_G(u)$ are simple groups of Lie type of order less, than Cmin. Thus $C_A(u)$ satisfies (C), hence, by Lemma 2.4, we obtain that $K \cap G$ is a 2-group. By Lemmas 4.3 and 4.4 every element $x \in A \setminus G$ of odd order with $\langle x \rangle \cap G = \{e\}$ centralizes some Sylow 2-subgroup of $G$. Hence $K$ contains a Sylow 2-subgroup of $G$, and hence of $A$, i.e., $K$ satisfies statement (c) of the theorem.

Thus we may assume that $G = \mathbf{PSp}_{2n}(q)$ and $\beta \geq 1$, i.e., a Sylow 3-subgroup $O_3(K \cap G)$ of $K \cap G$ is nontrivial. By Lemma 4.2 we obtain that $K \cap G$ is contained in some $K$-invariant parabolic subgroup $P$ of $G$ with a Levi factor $L$ and, up to conjugation in $P$, a Sylow 2-subgroup $O_2(K \cap G)$ of $K \cap G$ is contained in $L$. Note that all non-Abelian composition factors of $P$ are simple groups of Lie type of order less, than Cmin, so $P$ and each its homomorphic image satisfy (C). The group $\tilde{K} = KO_3(P)/O_3(P)$ is isomorphic to $K/O_3(K \cap G)$ and, by Lemma 2.3, $\tilde{K}$ is a Carter subgroup of $\langle K, P/O_3(P) \rangle$. Now $K \cap P/O_3(P) \simeq O_3(K \cap G)$ is a 2-group and every element $x \in \langle \tilde{K}, P/O_3(P) \rangle \setminus O_3(P/O_3(P))$ of odd order with $\langle x \rangle \cap P/O_3(P) = \{e\}$ centralizes a Sylow 2-subgroup of $P/O_3(P) \simeq L$ (see Lemmas 4.3 and 4.4). Therefore $O_2(K \cap G)$ contains a Sylow 2-subgroup of $L$, in particular, contains a Sylow 2-subgroup $H_2$ of $H$. Since $K$ is nilpotent, Lemma 3.4 implies that $O_3(K \cap G) \leq C_L(H_2) = \langle X_r | r \rangle$ is a long root of $\Phi(G)^+$. Since for every two long positive roots $r, s$ in $\Phi(G)^+$ we have that $r + s \notin \Phi(G)$, Chevalley commutator formula [12, Theorem 5.2.2] implies that $\langle X_r | r \rangle$ is a long root of $\Phi(G)^+$ is Abelian.

Since $\zeta$ is a Sylow root, it normalizes each parabolic subgroup of $G$ containing a $\zeta$-stable Borel subgroup. Thus for every subset $J$ of the set of fundamental roots $\Pi = \{r_1, \ldots, r_n\}$ of $\Phi = \Phi(G)$ the parabolic subgroup $P_J$ is $\zeta$-stable. Therefore we may suppose that $P = P_J$, where $J$ is a proper subset of the set of fundamental roots $\Pi$ of $\Phi$. Choose the numbering of fundamental roots so that $r_n$ is a long fundamental root, while the remaining fundamental roots $r_i$ are short. If $r_n \in J$, then one of the components of the Levi factor, $G_1$ for example, is isomorphic to $Sp_{2k}(q)$ for some $k < n$ (note that since $A \neq G$ then $q \neq 3$). By Lemma 3.5 we obtain that $L/C_L(G_1) = \text{Aut}_L(G_1/Z(G_1)) = G_1/Z(G_1)$. By Lemma 2.3 $K_1 = K C_L(G_1) O_3(P)/C_L(G_1) O_3(P)$ is a Carter subgroup of $(P \times \langle \zeta \rangle)/C_L(G_1) O_3(P)$.

Since $|K_1 \cap P/C_L(G_1) O_3(P)|$ is not divisible by 3, and $\zeta$ centralizes a Sylow 2-subgroup of $G_1/Z(G_1)$ (see Lemma 4.3), then $K_1$ contains a Sylow 2-subgroup of $P/C_L(G_1) O_3(P) \simeq G_1/Z(G_1) \simeq \mathbf{PSp}_{2k}(q)$. Moreover by Lemma 4.3 a Sylow 2-subgroup of $(P/C_L(G_1) O_3(P))_\zeta$ is a Sylow 2-subgroup of $P/C_L(G_1) O_3(P)$. Thus $K_1 \cap P/C_L(G_1) O_3(P)$ is a Sylow 2-subgroup of $(P/C_L(G_1) O_3(P))_\zeta \simeq \mathbf{PSp}_{2k}(3)$. By Lemma 3.7 there exists an element $x$ of odd order of $\mathbf{PSp}_{2k}(3)$ that normalizes but not centralizes a Sylow 2-subgroup; a contradiction with the fact that $K_1$ is a Carter subgroup of $(P \times \langle \zeta \rangle)/C_L(G_1) O_3(P)$. Thus we may assume that $r_n \notin J$.

Consider the set $J_n = \Pi \setminus \{r_n\}$ and the parabolic subgroup $P_{J_n}$. From the above arguments it follows that $K \leq P_J \times \langle \zeta \rangle \leq P_{J_n} \times \langle \zeta \rangle$. Now the subgroup $\langle X_{r} | r \rangle$ is a long root of $\Phi(G)^+$ is contained in $O_3(P_{J_n})$ and $O_3(K \cap G)$ is contained in $\langle X_{r} | r \rangle$ is a long root of $\Phi(G)^+$, so $N_G(O_3(K \cap G)) \leq O_3(P_{J_n})$ and we may assume that $P = P_{J_n}$. By Lemma 2.3, $\tilde{K} = KO_3(P)/O_3(P)$ is a Carter subgroup of $(P \times \langle \zeta \rangle)/O_3(P)$. Note that a unique non-Abelian composition factor of $P \times \langle \zeta \rangle$ is isomorphic to $A_{n-1}(q) \simeq \mathbf{PSL}_n(q)$. By [25, Theorem 1] and [26, theorem 4] we obtain that $\tilde{K} = R \times \langle \zeta \rangle$, where $R$ is a Sylow 2-subgroup of $P$ centralized by $\zeta$. Thus $O_3(K \cap G) \leq C_R(R)$. Consider $Q = O_3(K \cap G) \cap P_2$. Since $O_3(K \cap G)$ is nontrivial and $K$ is nilpotent, then $Q = O_3(K \cap G) \cap P_2 = Z(K) \cap O_3(K \cap G)$ is nontrivial. Therefore $N_G(Q)$ is a proper subgroup of $G$ and by Lemma 4.2 $N_G(Q)$ is contained in a proper parabolic
6 Carter subgroups in groups with triality automorphism

Theorem 6.1. Let $G$ be a finite adjoint group of Lie type over a field of characteristic $p$, $\overline{G}$, $\sigma$ are chosen so that $O^p(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$, and $O^p(\overline{G}_\sigma)$ is isomorphic to either $D_4(q)$, or $3D_4(q^3)$. Assume that $\tau$ is a graph automorphism of order 3 of $O^p(G)$ (recall that for $G \simeq 3D_4(q^3)$ $\tau$ is an automorphism such that the set of its stable points is isomorphic to $G_2(q)$). Denote by $A_1$ the subgroup of $\text{Aut}(D_4(q))$ generated by inner-diagonal and field automorphisms, and also by a graph automorphism of order 2. Let $A \leq \text{Aut}(G)$ be such that $A \not\leq A_1$ (if $O^p(G) \simeq D_4(q)$), and $K$ be a Carter subgroup of $A$. Assume also that $|O^p(G)| \leq \text{Cmin}$, $G = A \cap \overline{G}_\sigma$ and $A = KG$. Then one of the following statements holds:

(a) $G \simeq 3D_4(q^3)$, $(|A : G|, 3) = 1$, $q$ is odd and $K$ contains a Sylow 2-subgroup of $A$;

(b) $(|A : G|, 3) = 3$, $q$ is odd, $\tau \in A$ and, up to conjugation by an element of $G$, the subgroup $K$ contains a Sylow 2-subgroup of $C_{\overline{G}}(\tau) \in \Gamma G_2(q)$, and $\tau \in K$;

(c) $(|A : G|, 3) = 3$, $q = 2^t$, $|A : G| = 3t$, $A = G \times \langle \tau, \varphi \rangle$, where $\varphi$ is a field automorphism of order $t$ commuting with $\tau$ and, up to conjugation by an element of $G$, the subgroup $K$ contains a Sylow 2-subgroup of $C_{\overline{G}}((\tau, \varphi)^{-1}) \simeq G_2(2^{tx})$ and $\tau \in K$;

(d) $O^p(G) \simeq D_4(p^{3t})$, $p$ is odd, the factor group $A/G$ is cyclic, $\tau \not\in A$, $A = G \times \langle \zeta \rangle$, where for some natural $m$, $\zeta = \tau^{p^m}$ is a graph-field automorphism, and, up to conjugation by an element of $G$, $K = Q \times \langle \zeta \rangle$, where $Q$ is a Sylow 2-subgroup of $C_{\overline{G}}(\zeta^q) \simeq 3D_4(p^{3t}/[q, q])$.

In particular, Carter subgroups of $A$ are conjugate, i.e., if $A_2 \in A$ and $|F^*(A_2)| = \text{Cmin}$, then $A_2$ does not satisfy to the conditions of the theorem, so $F^*(A_2) \not\simeq 3D_4(q^3)$.

Proof. Assume that the theorem is not true and $A$ is a counter example such that $|O^p(G)|$ is minimal. In view of [36, Theorem 1.2(vi)] we have that every element of $G$ is conjugate to its inverse. By [23] and [35, Theorems 1.2 and 1.4] we obtain that for every element $t \in G$ of odd prime order, all non-Abelian composition factors of $C_G(t)$ are simple groups of Lie type of order
less, than $\text{Cmin}$. Thus, $C_A(t)$ satisfies (C) and Lemma 2.4 implies that $K_G = K \cap G$ is a 2-group. Now Lemma 4.4 implies that all cyclic groups, generated by semeld automorphisms of the same odd order of $G$, are conjugate under $G$. Since the centralizer of every semeld automorphism in $G$ is a group of Lie type of order less than $\text{Cmin}$, we again use Lemma 2.4 and obtain the statement of the theorem by induction. Lemma 4.4 implies also that if $O^p(G) \simeq D_4(q)$, then all cyclic groups generated by graph-semeld automorphisms are conjugate. Since the centralizers of each graph-semeld automorphism in $G$ is a group of Lie type of order less than $\text{Cmin}$, we again use Lemma 2.4 and obtain statement (d) of the theorem by induction. Therefore we may assume $A$ does not contain a semeld automorphism or a graph-semeld automorphism of odd order. Therefore either $G \simeq 3D_4(q^3)$ and $A/G$ is a 2-group, or $K$ contains an element $s$ of order 3 such that $\langle s \rangle \cap A_1 = \{e\}$ (for groups $3D_4(q^3)$ the equality $\langle s \rangle \cap G = \{e\}$ holds), $G \times \langle s \rangle = G \times \langle \tau \rangle$ and $K \cap G$ is a 2-group.

In the semeld case we obtain the statement (a) of the theorem with condition $|A : G|, 3 = 1$. In the second case there exists two non-conjugate cyclic subgroups $\langle \tau \rangle$ and $\langle x \rangle$ of order 3 of $A$ such that $\langle \tau \rangle \cap A_1 = \langle x \rangle \cap A_1 = \{e\}$ and $G \times \langle x \rangle = G \times \langle \tau \rangle$ (see [29, (9-1)]). Hence, either $s = \tau \in K$, or $s = x \in K$. Assume that $q \neq 3^i$. In the semeld case from the known structure of Carter subgroups in a group $\Gamma G_2(q)$, obtained in [10], the statement (b) or (c) of the theorem follows, in the second case we have that $K \leq C_A(x)$. By [29, (9-1)] $C_G(x) \simeq \text{PGL}_3(q)$, where $q \equiv \varepsilon 1$ (mod 3), $\varepsilon = \pm$ and $\text{PGL}_3(q) = \text{PGL}_3(q), \text{PGL}_3(q) = \text{PGU}_3(q)$. Then $K = (K \cap G) \times \langle y, \varphi \rangle$, where $\varphi$ is a semeld automorphism of $O^p(G)$ of order equal to a power of 2 and $y$ is a graph automorphism such that its order is a power of 3 and $x \in \langle y \rangle$. By nilpotency of $K$ we obtain that $y \varphi = \varphi y$, it follows that $C_{C_G(x)}(x) = C_{C_G(x)}(\varphi)$. Now we have that

$$C_G(\varphi) = \begin{cases} D_4(q^{1/|\varphi|}), & \text{if } O^p(G) \simeq D_4(q), \\ 3D_4(q^{3/|\varphi|}), & \text{if } G \simeq 3D_4(q^3). \end{cases}$$

Hence $C_{C_G(x)}(\varphi) = C_{C_G(x)}(x) \simeq \text{PGL}_3(q^{1/|\varphi|})$, with $q^{1/|\varphi|} \equiv \mu 1$ (mod 3), where $\mu = \pm$ (note that $\varepsilon$ and $\mu$ can be different). As we noted above, $K \cap G$ is a 2-group. On the other hand, by [26, Theorem 4] there exists an element $y$ of order 3 centralizing a Sylow 2-subgroup of $C_G(x) = \text{PGL}_3(q)$ and belonging to $C_{C_G(x)}(\varphi) \simeq \text{PGL}_3(q^{1/|\varphi|})$. Thus $y$ centralizes $K$, hence is in $K$. But $K \cap G$ does not contain elements of odd order, therefore this second case is impossible.

Assume now that $q = 3^i$. Then $C_G(\tau) \simeq G_2(q)$ and we obtain the theorem. In the second case $C_G(x) \simeq SL_2(q) \times U$, where $U$ is a 3-group and $Z(C_G(x)) \cap U \neq \{e\}$, a contradiction with Lemma 2.4. \hfill $\Box$

7. Carter subgroups in semilinear groups of Lie type

**Theorem 7.1.** Let $G$ be a semeld adjoint group of Lie type ($G$ is not necessary simple) over a semeld of characteristic $p$ and $\overline{C}$, $\sigma$ are chosen so that $O^p(\overline{C}_\sigma) \leq G \leq \overline{C}_\sigma$. Assume also that $G \neq 3D_4(q^3)$. Choose a subgroup $A$ of $\text{Aut}(O^p(\overline{C}_\sigma))$ with $A \cap \overline{C}_\sigma = G$ and, if $O^p(G) = D_4(q)$, assume that $A$ is contained in the subgroup $A_1$ defined in Theorem 6.1. Let $K$ be a Carter subgroup of $A$ and assume that $A = KG$.

Then exactly one of the following statements holds:

(a) $G$ is defined over a semeld of characteristic 2, $A = \langle G, \zeta g, t \rangle$, where $t$ is a 2-element, $K$ is contained in the normalizer of a $t$-stable Borel subgroup of $G$ $K \cap \langle G, \zeta g \rangle$ satisfies to one of the statements (a)-(f) of Lemma 4.8;
8 Carter subgroups of order divisible by the characteristic

Denote $K \cap G$ by $K_G$. For every group $A$, satisfying conditions of Theorem 7.1, the factor group $A/G$ is Abelian and, for some natural $t$ is isomorphic to a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_t$, where $\mathbb{Z}_t$ denotes a cyclic group of order $t$. If the factor group $A/G$ is not cyclic, then the group $O^\theta(G)$ is split and $A$ contains an element $\tau a$, where $\tau$ is a graph automorphism of $O^\theta(G)$ and $a \in G$. Then every semisimple element of odd order is conjugate to its inverse in $A$ (cf. Lemma 4.6).
By Lemma 2.4 we obtain that $|K_G|$ is divisible only by 2 and $p$. If $p = 2$, then we obtain that $K_G$ is a 2-group, i.e., it is contained in a proper $K$-invariant parabolic subgroup $P$ of $G$ and by Lemma 2.3 $K_O(P)/O_2(P)$ is a Carter subgroup of $KP/O_2(P)$. Since $K_G \leq O_2(P)$, then $(K_O(P)/O_2(P)) \cap (P/O_2(P)) = \{e\}$. Hence $P$ is a Borel subgroup of $G$, otherwise we would have $C_P(O_2(P))(K_O(P)/O_2(P)) \neq \{e\}$, a contradiction with the fact that $K_O(P)/O_2(P)$ is a Carter subgroup of $KP/O_2(P)$. Thus $P$ is a Borel subgroup and the theorem follows from Lemma 4.8. Now if $p \neq 2$, then again $K_G$ is contained in a proper parabolic subgroup $P$ of $G$ such that $O_p(K_G) \leq O_p(P)$ and $O_2(K_G) \leq L$. Then Lemmas 4.3 and 4.4 imply that $H_2 \leq O_2(K \cap G) \leq K$. Now Lemma 3.3 implies that $O_p(K_G) \leq C_U(H_2) = \{e\}$. Therefore $K \cap G$ is a 2-group. By Lemmas 4.3 and 4.4 every element $x \in A \setminus G$ of odd order such that $(x) \cap G = \{e\}$ centralizes some Sylow 2-subgroup of $G$. Hence $K$ contains a Sylow 2-subgroup of $A$, i.e., $K$ satisfies statement (d) of the theorem. Therefore $A/G$ is cyclic and we may assume that $A = \langle G, \zeta_g \rangle \in \Gamma_G$.

Recall that we are in the conditions of Theorem 7.1, $A = \langle G, \zeta_g \rangle$ is supposed to be a counter example to the theorem with $|O^\rho(G)|$ and $|A|$ minimal, and $K$ is a Carter subgroup of $\langle G, \zeta_g \rangle$ such that $p$ divides $|K_G|$. We have that $K = \langle \zeta^g, K_G \rangle$. Since $|O^\rho(G)| \leq C_{\text{min}}$, Lemma 2.3 implies that $K_G$ is a Carter subgroup of $\langle G, \zeta_g \rangle / G$. Therefore $|\zeta^k| = |\zeta|$, and we may assume that $k = 1$ and $K = \langle K_G, \zeta_g \rangle$.

In view of Lemma 4.2 there exists a proper $\sigma$- and $\tilde{\zeta}_g$-invariant parabolic subgroup $\overline{P}$ of $\overline{G}$ such that $O_p(K_G) \leq R_a(\overline{P})$ and $K_G \leq \overline{P}$. In particular, $\overline{P}$ and $\overline{P}^\epsilon$ are conjugate in $\overline{G}$. Let $\Phi$ be the root system of $\overline{G}$ and $\Pi$ be a set of fundamental roots of $\Phi$. In view of [12, Proposition 8.3.1] $\overline{P}$ is conjugate to some $\overline{P}_J = \overline{B} \cdot \overline{N}_J \cdot \overline{B}$, where $J$ is a subset of $\Pi$ and $\overline{N}_J$ is a complete preimage of $W_J$ in $\overline{N}$ under the natural homomorphism $\overline{N} / T \to W$. Now $\overline{P}_J$ is $\overline{\varphi}$-invariant, hence $\overline{P}_J^\epsilon = \overline{P}_J^{\epsilon_\Phi}$ (recall that $\tilde{\zeta} = \overline{\varphi}^\epsilon$ by definition). Consider the symmetry $\rho$ of the Dynkin diagram of $\Phi$ corresponding to $\tilde{\gamma}$. Let $\overline{J}$ be the image of $J$ under $\rho$. Clearly $\overline{P}_J^\epsilon = \overline{P}_J^{\overline{\gamma}}$. Since $\overline{P}$ and $\overline{P}^\epsilon$ are conjugate in $\overline{G}$ we obtain that $\overline{P}_J$ and $\overline{P}_J^\epsilon$ are conjugate in $\overline{G}$.

In view of [12, Theorem 8.3.3] it follows that either $\epsilon = 0$, or $J = \overline{J}$; i.e., $\overline{P}_J$ is $\tilde{\zeta}$-invariant.

Now we have that $\overline{P}^\overline{\gamma} = \overline{P}_J$ for some $\overline{y} \in \overline{G}$. So $\langle \tilde{\zeta}_g, \overline{P}^\overline{\gamma} \rangle = \langle \tilde{\zeta}_g^\overline{\gamma}, \overline{P}_J \rangle$ and $\overline{P}_J^\langle \tilde{\zeta}_g \rangle^\overline{\gamma} = \overline{P}_J$. It follows

$$\langle \tilde{\zeta}_g \rangle^\overline{\gamma} = \overline{y}^{-1} \tilde{\zeta}_g \overline{y} = \tilde{\zeta} \left( \overline{y}^{-1} \overline{y} \tilde{\zeta}_g \overline{y} \right) = \tilde{\zeta} \cdot h,$$

where $h = \langle \overline{y}^{-1} \tilde{\zeta}_g \overline{y} \rangle \in \overline{G}$. Since $\overline{P}_J^\epsilon = \overline{P}_J = \overline{P}_J^{\epsilon^{-1}}$ we obtain that $h \in N_{\overline{G}}(\overline{P}_J)$. By [12, Theorem 8.3.3], $N_{\overline{G}}(\overline{P}_J) = \overline{P}_J$, thus $\langle \tilde{\zeta}_g, \overline{P}^\overline{\gamma} \rangle = \langle \tilde{\zeta}_g, \overline{P}_J \rangle$. Now both $\overline{P}$ and $\overline{P}_J$ are $\sigma$-invariant. Hence $\overline{y} \sigma(\overline{y}^{-1}) \in N_{\overline{G}}(\overline{P}) = \overline{P}$. Therefore, by Lang-Steinberg Theorem [19, Theorem 10.1] we may assume that $\overline{y} = \sigma(\overline{y})$, i.e., $\overline{y} \in \overline{G}_\sigma$. Since $\overline{G}_\sigma = T_\sigma \cdot O^\rho(\overline{G}_\sigma)$ and $\overline{T} \leq \overline{P}_J$, then we may assume that $\overline{y} \in O^\rho(\overline{G}_\sigma)$. Thus, up to conjugation in $\overline{G}$, we may assume that $\overline{K} \leq \langle \tilde{\zeta}, \overline{P}_J \rangle$ and

$$K \leq \langle \overline{P}_J \cap G, \zeta_g \rangle = \langle P_J, \zeta_g \rangle,$$

in particular, $g \in \langle P_J \rangle$. Further if $T_J = \langle T, X_r \rangle | r \in J \cup -J \rangle$, then $T_J$ is a $\sigma$- and $\tilde{\zeta}_g$-invariant Levi factor of $\overline{P}_J$ and $L_J = \overline{L}_J \cap G$ is a $\zeta$-invariant Levi factor of $P_J$. Then $L_J^\epsilon$ is a $\zeta_g$-stable factor Levi of $P_J$. Since all Levi factors are conjugate under $O_p(P_J)$, we may assume that $L_J$ is a $\zeta_g$-stable Levi factor. Lemma 2.3 implies that

$$K O_p(P_J) / O_p(P_J) = X$$

is a Carter subgroup of $\langle P_J, \zeta_g \rangle / O_p(P_J)$ and

$$K Z(L_J) O_p(P_J) / Z(L_J) O_p(P_J) = \tilde{X}$$
is a Carter subgroup of $\langle P_J, \zeta g \rangle / Z(L_J)O_p(P_J)$. Recall that $K = \langle \zeta g, K_G \rangle$, hence, if $v$ and $\tilde{v}$ are the images of $g$ under the natural homomorphisms

$$\omega : \langle P_J, \zeta g \rangle \to \langle L_J, \zeta g \rangle \simeq \langle P_J, \zeta g \rangle / O_p(P_J),$$

$$\tilde{\omega} : \langle P_J, \zeta g \rangle \to \langle P_J, \zeta g \rangle / Z(L_J)O_p(P_J) \simeq \langle L_J, \zeta g \rangle / Z(L_J),$$

then $X = \langle \zeta v, K_G \rangle$ and $\tilde{X} = \langle \zeta \tilde{v}, K_G \rangle$. Note that $O_p(P)$ and $Z(L_J)$ are characteristic subgroups of $P$ and $L_J$ respectively, hence we may consider $\zeta$ as an automorphism of $L_J \simeq P/O_p(P)$ and $\tilde{L} = L_J/Z(L_J)$. Note also that all non-Abelian composition factors of $P$ are simple groups of Lie type of order less than $\mathbf{C_{min}}$, hence $\langle P, \zeta g \rangle$ satisfies (C). Thus we may apply Lemma 2.3 to $\tilde{\tilde{L}}, \langle \tilde{\tilde{L}}, \zeta \rangle, \langle L, \zeta \rangle, \langle L, \zeta g \rangle, \langle P, \zeta g \rangle$.

If $P_J$ is a Borel subgroup of $G$, then the statement of the theorem follows from Lemma 4.8. So we may assume that $L_J \neq Z(L_J)$, i.e., that $P_J$ is not a Borel subgroup of $G$. Then $L_J = H(G_1 \ast \ldots \ast G_k)$, where $G_i$ are subsystem subgroups of $G$, $k \geq 1$, and $H$ is a Cartan subgroup of $G$. Let $\zeta g = (\zeta g_1) \cdot (\zeta g_2)$ be the product of $2$- and $2'$-parts of $\zeta g$ (with $g_2,g_2' \in (P_J)\zeta$). Now $\zeta g_2 = \varphi^k$, for some $k$, is a $2'$-automorphism (recall that we do not consider the triality automorphism) and it normalizes each $G_i$, since $\varphi$ normalizes each $G_i$. Moreover, in view of Lemma 4.3, we have that $\zeta g_2$ centralizes a Sylow $2$-subgroup of $H$. In particular, it centralizes a Sylow $2$-subgroup of $Z(L_J) \leq H$. Therefore, every element of odd order of $\langle L_J, \zeta g_2 \rangle$ centralizes a Sylow $2$-subgroup of $Z(L_J)$ (here $g_2$ is the image of $g_2'$ under $\omega$).

Now $\tilde{L} = (P_G \times \ldots \times P_{G_k})\tilde{H}$, where $\tilde{H} = H^\omega$ and $P_G,\ldots,P_{G_k}$ are canonical finite groups of Lie type with trivial center. Set $\tilde{M}_i = C_{\tilde{L}}(P_{G_i})$, clearly $\tilde{M}_i = (P_G \times \ldots \times P_{G_{i-1}} \times P_{G_{i+1}} \times \ldots \times P_{G_k})C_{\tilde{H}}(P_{G_i})$; denote by $L_i$ the factor group $\tilde{L}/\tilde{M}_i$ and by $\pi_i$ corresponding natural homomorphism. Then $L_i$ is a finite group of Lie type and $P_{G_i} \leq L_i \leq \tilde{P}_{G_i}$.

Set $\tilde{M}_{i,j} = C_{\tilde{L}}(P_{G_i} \times P_{G_j})$, then

$$\tilde{M}_{i,j} = (P_G \times \ldots \times P_{G_{i-1}} \times P_{G_{i+1}} \times \ldots \times P_{G_{j-1}} \times P_{G_{j+1}} \times \ldots \times P_{G_k})C_{\tilde{H}}(P_{G_i} \times P_{G_j});$$

denote by $\pi_{i,j}$ corresponding natural homomorphism $\tilde{L} \to \tilde{L}/\tilde{M}_{i,j}$. If $M_i$ (respectively $M_{i,j}$) is $\zeta$-invariant, then $M_i$ (resp. $M_{i,j}$) is normal in $\langle \tilde{L}, \zeta \rangle$ and we denote by $\pi_i$ (resp. $\pi_{i,j}$) the natural homomorphism $\pi_i : \langle \tilde{L}, \zeta \rangle \to \langle \tilde{L}, \zeta \rangle/M_i$ and $\pi_{i,j} : \langle \tilde{L}, \zeta \rangle \to \langle \tilde{L}, \zeta \rangle/M_{i,j}$.

Now consider $\zeta$. Since $\zeta^2$ is a $2'$-automorphism, there can be two cases: either $\zeta$ normalizes $P_{G_i}$, or $\zeta^2$ normalizes $P_{G_i}$ and $\tilde{P}_{G_i} = P_{G_i}$ for some $j \neq i$. Consider these two cases separately.

Let $\zeta$ normalizes $P_{G_i}$. Then $\zeta$ normalizes $M_i$, and Lemma 2.3 implies that $\tilde{X}^\pi_i = K_i$ is a Carter subgroup of $\langle L_i, (\zeta\tilde{v})^{\pi_i} \rangle$. Since $\langle L_i, (\zeta\tilde{v})^{\pi_i} \rangle$ is a semilinear group of Lie type satisfying the conditions of Theorem 7.1 (by definition, $\zeta^2$ is a $2'$-automorphism, so we are not in the conditions of Theorem 6.1), $|L_i| < |G_i|$, and $p$ does not divide $|K_i|$, we have that $K_i$ contains a Sylow $2$-subgroup $Q_i$ of $\langle L_i, (\zeta\tilde{v})^{\pi_i} \rangle$ (in particular, $p \neq 2$) and, by Lemma 2.5, the group $\langle L_i, (\zeta\tilde{v})^{\pi_i} \rangle$ satisfies (ESyl2).

Let $\zeta^2$ normalizes $P_{G_i}$ and $\tilde{P}_{G_i} = P_{G_i}$. Then $M_{i,j}$ is normal in $\langle \tilde{L}, \zeta \rangle$. We want to show that $\langle \tilde{L}, \zeta \rangle^{\pi_{i,j}}$ satisfies (ESyl2). Since $M_{i,j}$ is a normal subgroup of $\langle \tilde{L}, \zeta \rangle$, then, by Lemma 2.3, $\langle X \rangle^{\pi_{i,j}}$ is a Carter subgroup of $\langle \tilde{L}, \zeta \rangle$. Consider the subgroup

$$\langle (P_{G_i})^{\pi_{i,j}} \times (P_{G_j})^{\pi_{i,j}}, \tilde{X}^{\pi_{i,j}} \rangle$$

of $\langle \tilde{L}, \zeta \rangle^{\pi_{i,j}}$ (note that $(P_{G_i})^{\pi_{i,j}} \simeq P_{G_i}$ and $(P_{G_j})^{\pi_{i,j}} \simeq P_{G_i}$, and till the end of this paragraph for brevity we shall identify these groups). Now we are in the conditions of Lemma 2.8, namely
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We have a finite group $\tilde{G} = (\tilde{X})^{x_{i,j}}(PG_i \times PG_j)$, where $PG_i \cong PG_j$ has trivial center. Then $\text{Aut}(\tilde{X})^{x_{i,j}}(PG_i) \cong \text{Aut}_{\tilde{X}}(PG_i)$ is a Carter subgroup of $\text{Aut}(PG_i)$. Now $PG_i$ is a canonical finite group of Lie type and

$$PG_i \leq \text{Aut}(PG_i) \leq \text{Aut}(PG_i),$$

i.e., $\text{Aut}(PG_i)$ satisfies the conditions of Theorem 7.1 (by construction $\zeta^2$ is a field automorphism and so we are not in the conditions of Theorem 6.1) and $(\tilde{X})^{x_{i,j}} \cap (PG_i \times PG_j)$ is not divisible by the characteristic. By induction, $\text{Aut}(\tilde{X})^{x_{i,j}}(PG_i)$ contains a Sylow 2-subgroup of $\text{Aut}(PG_i)$ (in particular, $p \neq 2$). The same arguments show that $\text{Aut}(\tilde{X})^{x_{i,j}}(PG_i)$ contains a Sylow 2-subgroup of $\text{Aut}(PG_j)$. Therefore, $\text{Aut}(PG_i)$ and $\text{Aut}(PG_j)$ satisfy (ESy12). Since $\text{Aut}(PG_i) \leq \text{Aut}_{(\tilde{L},\tilde{\zeta})}^{x_{i,j}}(PG_i)$ and $\text{Aut}(PG_j) \leq \text{Aut}_{(\tilde{L},\tilde{\zeta})}^{x_{i,j}}(PG_j)$, Lemmas 3.6 and 4.5 imply that groups of induced automorphisms $\text{Aut}_{(\tilde{L},\tilde{\zeta})}^{x_{i,j}}(PG_i)$ and $\text{Aut}_{(\tilde{L},\tilde{\zeta})}^{x_{i,j}}(PG_j)$ satisfy (ESy12). Consider $N_{(\tilde{L},\tilde{\zeta})}^{x_{i,j}}(PG_i)$ and $N_{(\tilde{L},\tilde{\zeta})}^{x_{i,j}}(PG_j)$. Since

$$|{(\tilde{L},\tilde{\zeta})}^{x_{i,j}} : N_{(\tilde{L},\tilde{\zeta})}^{x_{i,j}}(PG_i)| = |{(\tilde{L},\tilde{\zeta})}^{x_{i,j}} : N_{(\tilde{L},\tilde{\zeta})}^{x_{i,j}}(PG_j)| = 2,$$

it is easy to see that for every element $h$ of ${(\tilde{L},\tilde{\zeta})}^{x_{i,j}}$ the equality of cosets $hN_{(\tilde{L},\tilde{\zeta})}^{x_{i,j}}(PG_i) = hN_{(\tilde{L},\tilde{\zeta})}^{x_{i,j}}(PG_j)$ holds, it follows that $N_{(\tilde{L},\tilde{\zeta})}^{x_{i,j}}(PG_i) = N_{(\tilde{L},\tilde{\zeta})}^{x_{i,j}}(PG_j)$. By construction $C_{(\tilde{L},\tilde{\zeta})}^{x_{i,j}}(PG_i) \cap C_{(\tilde{L},\tilde{\zeta})}^{x_{i,j}}(PG_j) = \{e\}$, so Lemma 2.7 (with $C_{(\tilde{L},\tilde{\zeta})}^{x_{i,j}}(PG_i)$ and $C_{(\tilde{L},\tilde{\zeta})}^{x_{i,j}}(PG_j)$ as normal subgroups) implies that the normalizer $N_{(\tilde{L},\tilde{\zeta})}^{x_{i,j}}(PG_i)$ satisfies (ESy12). Now

$$|{(\tilde{L},\tilde{\zeta})}^{x_{i,j}} : N_{(\tilde{L},\tilde{\zeta})}^{x_{i,j}}(PG_j)| = 2,$$

thus Lemma 2.9 implies that ${(\tilde{L},\tilde{\zeta})}^{x_{i,j}}$ satisfies (ESy12).

Now we shall show that $L_j,\zeta^v$ satisfies (ESy12). Since $\tilde{L} \neq \{e\}$, then, as we noted above, $p \neq 2$. Let $Q$ be a Sylow 2-subgroup of $L_j,\zeta^v$. Consider an element $x \in N_{(\tilde{L},\tilde{\zeta})}^{x_{i,j}}(Q)$ of odd order. We need to prove that $x$ centralizes $Q$. As we noted above, every element of odd order of $L_j,\zeta^v$ centralizes $Q \cap Z(L_j)$, hence, if $\tilde{x} = x^2$ centralizes $Q = Q^v \cong Q/(Q \cap Z(L_j))$, then $x$ centralizes $Q$. Now either $M_i$ is normal in $L_j,\zeta^v$, or $M_{i,j}$ is normal in $L_j,\zeta^v$ and $(\cap_i M_i) \cap (\cap_j M_{i,j}) = \{e\}$. Moreover, as we proved above, $x^{z_i}$ centralizes $QM_i/M_i$, and $x^{z_{i,j}}$ centralizes $QM_{i,j}/M_{i,j}$. By Lemma 2.7 (with normal subgroup $M_i$ and $M_{i,j}$) we obtain that $\tilde{x}$ centralizes $Q$.

Thus $L_j,\zeta^v$ satisfies (ESy12) and by Lemma 2.5 there exists a Carter subgroup $F$ of $L_j,\zeta^v$ containing $Q$. Since $L_j,\zeta^v$ satisfies (C), Theorem 1.2 implies that $X = K^v$ and $F$ are conjugate, i.e., $X$ contains a Sylow 2-subgroup of $(L_j,\zeta^v)$ and, up to conjugation in $P_j,\zeta^v$, $K$ contains a Sylow 2-subgroup of $(P_j,\zeta^v)$. In particular, a Sylow 2-subgroup $Q_1$ of a Cartan subgroup $H$ in $K$ and $Q_1$ centralizes $\cap_{O_2(P_j)} \neq \{e\}$; a contradiction with Lemma 3.3.

9 Carter subgroups of order not divisible by the characteristic

Again we are in the conditions of Theorem 7.1. As we noted in the previous section, for every group $A$ satisfying conditions of Theorem 7.1, the factor group $A/G$ is Abelian and, for some natural $t$ is isomorphic to a subgroup of $Z_2 \times Z_t$. If the factor group $A/G$ is not cyclic, then $O^{\prime}\Phi(G)$ is split and $A$ contains an element $\tau a$, where $\tau$ is a graph automorphism of $O^{\prime}\Phi(G)$ and $a \in \overline{G}_\sigma$. Thus, if $A/G$ is not cyclic, or $\Phi(G) \neq A_n, D_{2n+1}, E_6$, then by Lemma 4.6 and [10,
Lemma 2.2] every semisimple element of \( G \) is conjugate to its inverse. By Lemma 2.4 we obtain that \( K_G = K \cap G \) is a 2-group. In the conditions of Theorem 7.1 the group \( A/G \) is Abelian and, if \( \bar{A}_1 \) is a Hall 2'-subgroup of \( A/G \), then \( \bar{A}_1 \) is cyclic. Let \( x \) be the preimage of the generating element of \( \bar{A}_1 \) taken in \( K \). Then \( \langle x \rangle \cap G \leq \langle x \rangle \cap \bar{G}_v \leq K \cap \bar{G}_v = K \cap (A \cap \bar{G}_v) = K \cap G \).

As we noted above, \( K \cap G \) is a 2-group, hence \( \langle x \rangle \cap \bar{G}_v = \{e\} \). By Lemma 4.4, the element \( x \) under \( \bar{G}_v \) is conjugate to a self-automorphism of odd order and by Lemma 4.3, the element \( x \) centralizes a Sylow 2-subgroup of \( G \) (in particular, \( p \neq 2 \)) and, since \( A/G \) is Abelian, Lemma 2.7 implies that \( K \) contains a Sylow 2-subgroup of \( A \). Thus Theorem 7.1 is true in this case. So we may assume that \( A = \langle G, \zeta g \rangle \) is a semilinear group of Lie type, \( K = \langle \zeta^k g, K_G \rangle \) is a Carter subgroup of \( A \), and \( \Phi(\bar{G}) = \{A_n, D_{2n+1}, E_8\} \). Like in the previous section we may assume that \( k = 1 \). Since \( G_\zeta \) is nontrivial, then the centralizer \( C_G(\zeta g) \) is also nontrivial, we have that \( K_G \) is also nontrivial. Since \( G_\zeta \) is nontrivial, then the centralizer \( C_G(\zeta g) \) is nontrivial, so \( K_G \) is also nontrivial. Therefore \( Z(K) \cap K_G \) is nontrivial. Consider an element \( x \in Z(K) \cap K_G \) of prime order. Then \( K \leq C_A(x) = \langle \zeta g, C_G(x) \rangle \). Now \( C_\sigma(x)^0 = \bar{C} \) is a connected \( \sigma \)-stable subgroup of maximal rank of \( \bar{G} \). Moreover \( \bar{C} \) is a characteristic subgroup of \( C_\sigma(x) \) and \( C_\sigma(x) / \bar{C} \) is isomorphic to a subgroup of \( \Delta \) (see [22, Proposition 2.10]). Thus \( K \) is contained in \( \langle K, C \rangle \), where \( C = \bar{C} \cap G \). Moreover, by Lemma 4.1, the subgroup \( C = \bar{C} \cap G = T(G_1 \ast \ldots \ast G_m) \) is normal in \( C_A(x) \) and \( K_G/C/C \) is isomorphic to a subgroup of \( \Delta \). Assume that \( |K_G| \) is not divisible by 2.

If \( m = 0 \), then \( C = T = Z(C) \) is a maximal torus. Then \( T \) is \( \zeta g \)-stable. In view of Lemma 4.7 we obtain that \( N_A(C_A(x)) \neq C_A(x) \). Since \( C_A(x) \) is solvable in this case this gives a contradiction with Lemma 2.4.

If \( m \geq 1 \), then \( Z(C) \) and \( G_1 \ast \ldots \ast G_m \) are normal subgroups of \( \langle K, C \rangle \). Hence we may consider \( \tilde{G} = \langle K, G_1 \ast \ldots \ast G_m \ast Z(C) / Z(C) \rangle / Z(C) \leq \langle K, C \rangle / Z(C) \). Then \( \tilde{G} = \tilde{K}(\mathbf{P}G_1 \ast \ldots \ast \mathbf{P}G_m) \), where \( \tilde{K} = KZ(C) / Z(C) \) is a Carter subgroup of \( \bar{G} \) (cf. Lemma 2.3) and \( Z(\mathbf{P}G_i) \) is trivial. Now \( \tilde{K} \) acts by conjugation on \( \{\mathbf{P}G_1, \ldots, \mathbf{P}G_m\} \) and without lost of generality we may assume that \( \{\mathbf{P}G_1, \ldots, \mathbf{P}G_m\} \) is a \( \tilde{K} \)-orbit. Thus we are in the condition of Lemma 2.8 and \( \text{Aut}_{\tilde{K}}(\mathbf{P}G_1) \) is a Carter subgroup of \( \text{Aut}_{\tilde{G}}(\mathbf{P}G_1) \). Moreover \( |\tilde{K} \cap \mathbf{P}G_1 \ast \ldots \ast \mathbf{P}G_m| \) is not divisible by the characteristic. By induction we have that either \( \text{Aut}_{\tilde{K}}(\mathbf{P}G_1) \) contains a Sylow 2-subgroup of \( \text{Aut}_{\tilde{G}}(\mathbf{P}G_1) \), or \( \text{Aut}_{\tilde{G}}(\mathbf{P}G_1) \) satisfies the conditions of Theorem 6.1 and \( \text{Aut}_{\tilde{G}}(\mathbf{P}G_1) \cap \text{PG}_1 \) is a nontrivial 2-group, in particular \( p \) is odd. In any case \( |K \cap G| \) is divisible by 2 that contradicts our assumption. Therefore the order \( |K_G| \) is even and we may assume that \( x \in Z(K) \cap K_G \) is an involution.

Write \( \zeta g = \zeta_2 g_1 \cdot \zeta_2 g_2 \), where \( \zeta_2 g_1 \) is the 2-part and \( \zeta_2 g_2 \) is the 2'-part of \( \zeta g \). By Lemma 4.3 the element \( \zeta_2 \) centralizes a Sylow 2-subgroup \( Q \) of \( G \), so we may assume that the order of \( g_2 \) is odd. Up to conjugation in \( G \) we may assume that \( \zeta_2 \) centralizes a Sylow 2-subgroup of \( K_G \).

In particular, \( \zeta_2 \) centralizes \( x \). Let \( Q \) be a Sylow 2-subgroup of \( C_G(x) \). Then there exists \( y \in G \) such that \( Q^y \leq Q_g \). Substituting the subgroup \( K \) by its conjugate \( K^y \), we may assume that \( \zeta_2 \) centralizes a Sylow 2-subgroup of \( C_G(x) \). Since \( \zeta_2 g_2 \) centralizes \( x \), we obtain that \( g_2 \in C_\sigma(x) \). Moreover, by Lemma 3.1 it follows that \( g_2 \in C_\sigma(x)^0 \). In particular, \( g_2 \) normalizes each \( G_1 \) and centralizes \( Z(C) \) and \( Z(C_G(x)) \).

Note that \( \zeta_2 \) normalizes each \( G_i \) and centralizes a Sylow 2-subgroup of \( Z(C_G(x)) \) (recall that \( \zeta_2 \) centralizes a Sylow 2-subgroup of \( C_G(x) \)). Indeed, \( \zeta_2 \) normalizes \( C \), hence normalizes characteristic subgroups \( O_{p'}(C) = G_1 \ast \ldots \ast G_m \) and \( Z(C) \) of \( C \). So we may consider the induced automorphism \( \zeta_2 \) of

\[
O_{p'}(C) / (Z(C) \cap O_{p'}(C)) = \mathbf{P}G_1 \times \ldots \times \mathbf{P}G_m.
\]
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Since each $P_G_i$ has trivial center and can not be written as a direct product of proper subgroups, corollary from Krull-Remak-Schmidt Theorem [37, 3.3.10] implies that $\zeta_{2^i}$ permutes distinct $P_G_i$. Since $\zeta_{2^i}$ centralizes a Sylow 2-subgroup of $C_G(x)$ and $C \leq C_G(x)$, then $\zeta_{2^i}$ centralizes a Sylow 2-subgroup of $C$, hence centralizes a Sylow 2-subgroup $Q_1 \times \ldots \times Q_m$ of $P_G_1 \times \ldots \times P_G_m$, where $Q_i$ is a Sylow 2-subgroup of $P_G_i$. If $\zeta_{2^i}$ would induce a nontrivial permutation on the set $\{P_G_1, \ldots, P_G_m\}$, then in would induce a nontrivial permutation on $\{Q_1, \ldots, Q_m\}$. Since each $Q_i$ is nontrivial, this is impossible. Thus every element of odd order of $\langle K, C \rangle$ centralizes a Sylow 2-subgroup of $Z(C)$ and normalizes each $G_i$.

If $\Phi(\overline{G}) = \overline{e}$, then by Lemma 3.1 the centralizer of every involution of $G$ in $\overline{G}$ is connected. By Lemma 3.8 every involution of $G$ is contained in a maximal torus $T$ such that $N(G, T)/T \simeq W$, where $W$ is a Weyl group of $\overline{G}$. $\overline{G}$ is wellknown to be generated by the torus $\overline{T}$ and $\overline{T}$-root subgroups. Write $\overline{G} = \overline{T}(\overline{G}_1 \times \ldots \times \overline{G}_k)$. Since $\overline{T}_i$ either is obtained from a maximal split torus $\overline{H}$ by twisting with an element $w_0$ of order 2, or is equal to $\overline{H}$, and each $\overline{e}$-autormophism acts trivially on the factor group $N_G(\overline{H})/\overline{H}$, then $\overline{e}_i$ normalizes every subgroup $\overline{G}_i$. So, if $\Phi(\overline{G}_i) = \overline{D}_i$, then $\overline{e}_i$ induces a $\overline{e}$-eld (but not a graph or a graph-$\overline{e}$ld) automorphism of $\overline{G}_i$. Moreover, since $\sigma$ acts trivially on the factor group $N_G(\overline{T})/\overline{T}$ (see Lemma 4.3), then [23, Proposition 6] implies that $\sigma$ normalizes each $\overline{G}_i$. Therefore, none of $\overline{G}_i$ is isomorphic to $\overline{D}_i(q^3)$. If $\Phi(\overline{G})$ coincides with $\overline{A}_n$ or $\overline{D}_n$, then [15, Propositions 7, 8, 10] imply that none of $\overline{G}_i$ is isomorphic to $\overline{D}_i(q^3)$. Therefore in any case none of $\overline{G}_i$ is isomorphic to $\overline{D}_i(q^3)$. Moreover Lemma 3.1 implies that $|K_G : (K_G \cap C)|$ divides $|C_G(x)/C_G(x)^0|$ and $C_G(x)/C_G(x)^0$ is a 2-group. In [16] it is proven that if a root system $\Phi$ has type $\overline{D}_n$ and $\Psi$ is its subsystem of type $\overline{D}_4$, then no element from $N_{\Psi}(\Phi)(W(\Psi))$ induces a symmetry of order 3 of the Dynkin diagram of $\Psi$. Since $\zeta^2$ is a $\overline{e}$-ld automorphism, lack of a symmetry of order 3 together with [23, Proposition 6] implies that for each $\overline{G}_i$ the automorphism $\zeta_{2^i}$ is $\overline{e}$-ld (but not graph $\overline{e}$graph-$\overline{e}$ld). Therefore the group of induced automorphisms $\langle \text{Aut}_{\overline{G}}(P_G_i), \overline{G}_i \rangle$ satisfies to the conditions of Theorem 7.1 for all $\overline{G}_i$.

Now consider $G = \overline{K}P_G_1 \times \ldots \times P_G_m) \leq (K, C)/Z(C)$ (probably, $m = 0$), where $\overline{K} = KZ(C)/Z(C)$ is a Carter subgroup of $\overline{G}$ (see Lemma 2.3) and, for all $i$, $Z(P_G_i) = \{e\}$. By Lemma 2.8 we have that $\text{Aut}_{\overline{G}}(P_G_i)$ is a Carter subgroup of $\text{Aut}_{\overline{G}}(P_G_i)$. Since $P_G_1$ is a $\star$nite group of Lie type satisfying Theorem 7.1, by induction we obtain that $\text{Aut}_{\overline{G}}(P_G_i)$ satisfies (ESy12). Similarly we have that $\text{Aut}_{\overline{G}}(P_G_i)$ satisfies (ESy12) for all $i$. Since

$$\text{Aut}_{(K,C)/Z(C)}(P_G_i) \geq \text{Aut}_{\overline{G}}(P_G_i),$$

Lemmas 3.6 and 4.5 imply that $\text{Aut}_{(K,C)/Z(C)}(P_G_i)$ satisfies (ESy12). Since $C_{(K,C)/Z(C)}(P_G_1 \times \ldots \times P_G_m) = \{e\}$, Lemma 2.7 with normal subgroups $C_{(K,C)/Z(C)}(P_G_1) \cap N_{(K,C)/Z(C)}(P_G_1), \ldots, C_{(K,C)/Z(C)}(P_G_m) \cap N_{(K,C)/Z(C)}(P_G_1)$ implies that $N_{(K,C)/Z(C)}(P_G_1)$ satisfies (ESy12). Now

$$|\langle (K,C)/Z(C) : N_{(K,C)/Z(C)}(P_G_1) \rangle| = 2^r,$$

and each element of odd order of $\langle K, C \rangle/Z(C)$ normalizes $P_G_1$, thus, by Lemma 2.9, we obtain that the factor group $\langle K, C \rangle/Z(C)$ satisfies (ESy12) and, by Lemma 2.7 $\langle K, C \rangle$ satisfies (ESy12). Since $|P_G_1| < C_{\text{min}}$, then $\langle K, C \rangle$ satisfy (C). By Lemma 2.5 we obtain that there exists a Carter subgroup $F$ of $\langle K, C \rangle$ containing a Sylow 2-subgroup of $\langle K, C \rangle$. By Theorem 1.2, subgroups $F$ and $K$ are conjugate in $\langle K, C \rangle$, thus $K$ contains a Sylow 2-subgroup $Q$ of $\langle K, C \rangle$. Since $|C_G(x) : C|$ is a power of 2 and $\langle K, C \rangle$ normalizes $C_G(x)$, we obtain that $|\langle K, C_G(x) \rangle : \langle K, C \rangle|$ is a power of 2. Moreover by construction each element of odd order of $\langle K, C_G(x) \rangle$ is in $\langle K, C \rangle$. Thus by Lemma 2.9 $\langle K, C_G(x) \rangle$ satisfies (ESy12) and $K$ contains a Sylow 2-subgroup $Q$ of $\langle K, C_G(x) \rangle$. 
Let $\Gamma Q$ be a Sylow 2-subgroup of $\langle G, \zeta g \rangle$ containing $Q$ and $t \in \text{Z}(\Gamma Q) \cap G$. Then $t \in C_G(x)$, hence, $t \in \text{Z}(Q)$ and $t \in \text{Z}(K)$. Thus we may substitute $x$ by $t$ in arguments above and obtain that $Q = \Gamma Q$, i.e., $K$ contains a Sylow 2-subgroup of $\langle G, \zeta g \rangle$, which completes the proof of Theorem 7.1.

10 Carter subgroups of finite groups are conjugate

Before we formulate the main theorem, note a corollary of Theorem 7.1.

**Corollary 10.1.** $C_{\text{min}} = \infty$, i.e., $A = \emptyset$.

**Proof.** Indeed, let $A \neq \emptyset$ and $A \in A$ is such that $|F^*(A)| = C_{\text{min}}$. Since $F^*(A) = O_{p'}(G_\sigma)$ for an adjoint simple connected algebraic group $G$, and a Frobenius map $\sigma$, denote the intersection $A \cap G_\sigma$ by $G$. As we noted in the beginning of Section 5, we may assume that $A = KF^*(A) = KG$. Therefore the group $A$ either satisfies to the conditions of Theorem 6.1, or satisfies to the conditions of Theorem 7.1. In both cases we have proven that Carter subgroups of $A$ are conjugate, that contradicts to the choice of $A$. \hfill \Box

In order to state the main theorem without using of the classification of finite simple groups, we give the following definition. A finite group is said to be a $K$-group if all its non-Abelian composition factors are known simple groups.

**Theorem 10.2.** (Main Theorem) Let $G$ be a finite $K$-group. Then Carter subgroups of $G$ are conjugate.

**Proof.** B [6, Theorem 1.1], [9, Table], and [10, Theorems 3.3 and 3.5]; and Theorems 5.1, 6.1, and 7.1 from the present paper we obtain that for every known simple group $S$ and every nilpotent subgroup $N$ of group of its automorphisms, Carter subgroups of $\langle N, S \rangle$ are conjugate. Thus $G$ satisfies (C). Hence, by Theorem 1.2, Carter subgroups of $G$ are conjugate. \hfill \Box

From Lemma 2.3 and Main Theorem 10.2 it follows that a homomorphic image of a Carter subgroup is a Carter subgroup.

**Theorem 10.3.** Let $G$ be a finite $K$-group, $H$ be a Carter subgroup of $G$ and $N$ be a normal subgroup of $G$, then $HN/N$ is a Carter subgroup of $G/N$.

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