The spectrum of the Laplacian in a domain bounded by a flexible polyhedron in $\mathbb{R}^d$ does not always remain unaltered during the flex

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Abstract

Being motivated by the theory of flexible polyhedra, we study the Dirichlet and Neumann eigenvalues for the Laplace operator in special bounded domains of Euclidean d-space. The boundary of such a domain is an embedded simplicial complex which allows a continuous deformation (a flex), under which each simplex of the complex moves as a solid body and the change in the spatial shape of the domain is achieved through a change of the dihedral angles only. The main result of this article is that both the Dirichlet and Neumann spectra of the Laplace operator in such a domain do not necessarily remain unaltered during the flex of its boundary.

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1 Introduction

In this article, a *polyhedron* is either a continuous map $f : K \to \mathbb{R}^d$ of a connected $(d-1)$-dimensional simplicial complex K which is affine on every simplex or the image $f(K) \subset \mathbb{R}^d$ of K under the action of f. A polyhedron is called *embedded* if f is injective. A polyhedron $P_0 = f(K)$ is called *flexible* if its spatial shape can be changed continuously only by changing its dihedral angles attached to its $(d-2)$-dimensional faces, i. e. by means of a continuous deformation, which does not change the dimensions of the faces. In other words, P_0 is said to be flexible if there exists a continuous family $\{P_t\}_{t\in[0,1]}$ of polyhedra such that, for every $t \in (0,1]$, P_0 and P_t are combinatorially equivalent to each other and every two corresponding faces of

P_0 and P_t are congruent, while P_0 and P_t themselves are not congruent. Wherein, the family $\{P_t\}_{t\in[0,1]}$ is called a *flex* of P_0.

The above definitions are standard in the theory of flexible polyhedra. However, in this article, we will also use the following notation, which is not commonly used: by $[\![P_0]\!]$ we denote the bounded domain in $\mathbb{R}^d$, whose boundary is an embedded boundary-free polyhedron $P_0 \subset \mathbb{R}^d$.

The fact that embedded boundary-free flexible polyhedra do exist in $\mathbb{R}^3$ is non-trivial and was established only in 1977 by Robert Connelly [6] (see also [7], [22]). For $d \geqslant 4$, the question about the existence of an embedded boundary-free flexible polyhedron in $\mathbb{R}^d$ remains open.

Over the last 40 years, through the efforts of many geometers, it was shown that flexible polyhedra (not necessarily embedded) have a number of remarkable properties. Below we mention some of them:

(i) *In $\mathbb{R}^3$, there exists an embedded sphere-homeomorphic flexible polyhedron with nine vertices only.* This polyhedron was built by Klaus Steffen around 1980 and bears his name. Although Steffen published no text about his polyhedron, its description can be found in many books and articles (see, e. g., [9, Section 23.2.3] or [3]). The question about the existence of an embedded boundary-free flexible polyhedron in $\mathbb{R}^3$ with eight vertices remains open, see [23].

(ii) *In $\mathbb{R}^3$, there exist embedded boundary-free flexible polyhedra of an arbitrary genus, both orientable and non-orientable.* Explicit examples of such polyhedra can be found in [34].

(iii) With every oriented boundary-free polyhedron in $\mathbb{R}^d$ one can associate a quantity called its *oriented volume*. For an embedded boundary-free polyhedron P_0 in $\mathbb{R}^d$, its oriented volume coincides, up to a sign, with the d-dimensional volume of the domain $[\![P_0]\!]$. So, *for any $d \geqslant 3$ and any oriented boundary-free flexible polyhedron P_0 in $\mathbb{R}^d$, the oriented volume of P_0 remains constant during the flex.* For several decades, this statement was known as the Bellows Conjecture. In $\mathbb{R}^3$, it was first proved by I.Kh. Sabitov in 1995–1996 in [26], [27] (see also [28]). Another proof, which is also limited to the case $d = 3$, was published in 1997 in [8] (see also an expository paper [30]). For $d \geqslant 4$, the Bellows Conjecture was proved by A.A. Gaifullin in 2014 in [12] and [13] (see also an expository paper [17]).

(iv) With every oriented polyhedron in $\mathbb{R}^d$ one can associate a quantity called its *integral mean curvature*, see, e. g., [29, Chapter 13]. In the case of an oriented polyhedron in $\mathbb{R}^3$, its integral mean curvature is given by $\frac{1}{2}\sum_i(\pi - \varphi_i)\ell_i$, where the sum is taken over all edges of the polyhedron, ℓ_i is the length of the i-th edge and φ_i is the dihedral angle along the edge. So, *for every $d \geqslant 3$, the integral mean curvature of an oriented flexible polyhedron $P_0 \subset \mathbb{R}^d$ remains constant during the flex.* This statement was proved by Ralph Alexander in 1985 in [1].

(v) *For every $d \geqslant 3$, if an embedded boundary-free polyhedron $P_1 \subset \mathbb{R}^d$ is obtained from an embedded boundary-free polyhedron $P_2 \subset \mathbb{R}^d$ by a flex then $[\![P_1]\!]$ and $[\![P_2]\!]$*

have the same Dehn invariants. This statement was proved by A.A. Gaifullin and L.S. Ignashchenko in 2017 in [19]. Since it is known that domains with polyhedral boundaries in $\mathbb{R}^3$ are scissors congruent if and only if they have the same volume and the same Dehn invariants, then from the property under discussion it follows immediately that *if an embedded boundary-free polyhedron $P_1 \subset \mathbb{R}^3$ is obtained from an embedded boundary-free polyhedron $P_2 \subset \mathbb{R}^3$ by a flex then the domains $[\![P_1]\!] \subset \mathbb{R}^3$ and $[\![P_2]\!] \subset \mathbb{R}^3$ are scissors congruent.* Since 1980s, the latter statement was known as the Strong Bellows Conjecture.

Note that, for every $d \geqslant 3$, the notion of the flexible polyhedron can be defined in any d-dimensional space of constant curvature (i. e., not only in Euclidean space, but also in spherical and hyperbolic spaces), as well as in Minkowski space. The reader, interested in properties of flexible polyhedra in these spaces, which are similar to the properties (i)–(v), is referred to [2], [14]–[16], [18], [32], and [33].

Being motivated by the properties (i)–(v), we would like to find new invariants of flexible polyhedra in $\mathbb{R}^d$, $d \geqslant 3$, which are preserved during the flex. In our opinion, it is natural to try the Dirichlet and Neumann eigenvalues of the Laplace equation in the domain $[\![P_0]\!]$ on the role of such invariants. The statement that the Dirichlet and Neumann spectra of the Laplacian are not altered during the flex seems natural to us because it agrees with the Weyl law on the asymptotics of eigenvalues of the Laplacian. In fact, the coefficients of the first and second terms of the Weyl asymptotics are expressed in terms of the volume and the surface area of the boundary of the domain and, thus, remain unaltered during the flex (see the formula (3) in Section 2, where we recall the Weyl law in more detail).

The main result of this article is the following theorem showing that the above conjecture on the invariance of the Dirichlet and Neumann spectra of the Laplacian during the flex of the boundary of a domain is false:

Theorem 1 *For every $d \geqslant 3$, $\varepsilon > 0$, and every embedded flexible polyhedron $P_0 \subset \mathbb{R}^d$ there is an embedded flexible polyhedron $\widetilde{P}_0 \subset \mathbb{R}^d$ and its flex $\widetilde{\mathfrak{F}} = \{\widetilde{P}_s\}_{s\in[0,1)}$ such that*

(α_d) *the combinatorial structure of $\widetilde{P}_0$ is a subdivision of the combinatorial structure of P_0;*

(β_d) *the Hausdorff distance between the sets $\widetilde{P}_0$ and P_0 is less than ε;*

(γ_d) *both Dirichlet and Neumann spectra of the d-dimensional Laplacian in the domain $[\![\widetilde{P}_s]\!] \subset \mathbb{R}^d$ do not remain unaltered when s changes in the interval* $[0, 1)$.

Remark 1 For $d = 2$, the following statement, similar to Theorem 1, holds true: *For every $\varepsilon > 0$ and every closed embedded polygon $P_0 \subset \mathbb{R}^2$, there is a closed embedded polygon $\widetilde{P}_0 \subset \mathbb{R}^2$ and its flex $\widetilde{\mathfrak{F}} = \{\widetilde{P}_s\}_{s\in[0,1)}$ such that*

(α_2) *the number of vertices of $\widetilde{P}_0$ exceeds the number of vertices of P_0 by no more than four*;

(β_2) *the Hausdorff distance between the polygons* $\widetilde{P}_0$ *and* P_0 *is less than* ε;

(γ_2) *both Dirichlet and Neumann spectra of the* 2*-dimensional Laplacian in the domain* $[\![\widetilde{P}_s]\!] \subset \mathbb{R}^2$ *do not remain unaltered when* s *changes in the interval* $[0, 1)$.

This statement does not belong to the theory of flexible polyhedra. For this reason, we prefer to formulate it separately from Theorem 1. The proof of the statement under discussion is left to the reader. It can be obtained by simplifying the proof of Theorem 1 which is given in Section 4.

Remark 2 It would be interesting to obtain an analogue of Theorem 1 for spherical and hyperbolic spaces of dimension $d \geqslant 2$. We cannot do this in this paper, because our proof of Theorem 1, presented in Section 4, relies on an asymptotic formula for the eigenvalues of the Laplace operator given below in Lemma 1, while we are not aware about any analogue of Lemma 1 for spherical or hyperbolic spaces.

2 Weyl's law and Fedosov's asymptotic formula

Let $d \geqslant 2$, Ω be a bounded domain in $\mathbb{R}^d$, and $\Delta = \sum_{i=1}^{d} \partial^2/\partial x_i^2$ be the Laplace operator in $\mathbb{R}^d$. Consider the following boundary problems:

$$\begin{cases} \Delta u &= -\nu^2 u \qquad \text{in } \Omega, \\ u|_{\partial\Omega} &= 0 \end{cases} \tag{1}$$

and

$$\begin{cases} \Delta u &= -\nu^2 u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}}|_{\partial\Omega} &= 0, \end{cases} \tag{2}$$

where $\partial u/\partial \boldsymbol{n}$ is the directional derivative of the function u in the direction $\boldsymbol{n}$ and $\boldsymbol{n}$ is the exterior unit normal to $\partial\Omega$.

As usual, we call the number ν^2 for which there exists a nonzero the solution u of the problem (1) (respectively, (2)) the *Dirichlet eigenvalue* (respectively, the *Neumann eigenvalue*) of the Laplace operator in Ω, For each of the problems (1) and (2), we denote by $\mathcal{N}(k)$ the number of eigenvalues, which do not exceed k^2 (repeating each eigenvalue according to its multiplicity). We call $\mathcal{N}(k)$ the *eigenvalue counting function* of the corresponding problem.

Under certain assumptions about the boundary $\partial\Omega$ of a bounded domain $\Omega \subset \mathbb{R}^d$, the following asymptotic formula holds true for $k \to \infty$:

$$\mathcal{N}(k) = \frac{\mathrm{vol}_d(\Omega)}{\Gamma\big(\frac{d+2}{2}\big)} \left(\frac{k}{2\sqrt{\pi}}\right)^d \mp \frac{\mathrm{vol}_{d-1}(\partial\Omega)}{4\Gamma\big(\frac{d+1}{2}\big)} \left(\frac{k}{2\sqrt{\pi}}\right)^{d-1} + o(k^{d-1}). \tag{3}$$

Here and below, vol_p denotes the p-dimensional volume of a set and Γ denotes the Euler gamma function. The minus sign corresponds to the problem (1), while the plus sign corresponds to the problem (2).

The formula (3) is known as Weyl's law or the Weyl asymptotic formula. The reader interested in more details about this formula and its influence on mathematics and physics is referred to [4], [21], and literature mentioned there.

If the boundary $\partial\Omega$ of the domain $\Omega \subset \mathbb{R}^d$ is a flexible polyhedron, then the coefficients of k^d and k^{d-1} in the formula (3) remain constant during the flex of $\partial\Omega$. This observation is our main argument supporting the conjecture on the invariance of the Dirichlet and Neumann spectra of the Laplacian formulated in Section 1.

Direct calculation of the eigenvalues of the Laplacian for domains bounded by flexible polyhedra is hardly possible due to the complex geometry of such domains. The use of the formula (3) is also hardly possible because finding every new term in this asymptotics is a difficult problem (see, e. g., [5], [31]) and the nonsmoothness of the boundary of the domain gives rise to additional difficulties (see, e. g., [20], [25]). Nevertheless, we refute the conjecture under discussion. For this we use the following version of Weyl's law for the Riesz means of the eigenvalues of the Laplacian in a domain bounded by a polyhedron in Euclidean d-space, $d \geqslant 2$:

Lemma 1 (B.V. Fedosov [11]) *Let $d \geqslant 2$, $0 \leqslant p \leqslant d-1$, and let a bounded domain $D \subset \mathbb{R}^d$ be such that its boundary ∂D is a polyhedron. Let $\{F_i^{d-2}\}_i$ be the set of all $(d-2)$-dimensional faces of ∂D, and let φ_i stand for the value of the dihedral angle of D at F_i^{d-2}. Then the following asymptotic formula, involving the eigenvalue counting function $\mathcal{N}(k)$, holds true for each of the problems* (1) *and* (2) :

$$\frac{1}{\Gamma(p+1)}\int\limits_0^k (k-t)^p\, d\mathcal{N}(t) = \sum_{l=1}^{d} a_l \frac{\Gamma(l+1)}{\Gamma(p+l+1)} k^{p+l} + O(k^{d-1})$$

as $k \to \infty$. Here

$$a_d = \frac{\mathrm{vol}_d(D)}{2^d \pi^{d/2}\Gamma\big(\frac{d}{2}+1\big)},$$

$$a_{d-1} = \mp\frac{\mathrm{vol}_{d-1}(\partial D)}{2^{d+1}\pi^{(d-1)/2}\Gamma\big(\frac{d+1}{2}\big)}, \tag{4}$$

$$a_{d-2} = \frac{1}{2^{d+1}\pi^{d/2}\Gamma\big(\frac{d}{2}\big)}\sum_i \frac{\varphi_i^2-\pi^2}{3\varphi_i}\mathrm{vol}_{d-2}\big(F_i^{d-2}\big). \tag{5}$$

In the formula (4), *the minus sign corresponds to the problem* (1), *while the plus sign corresponds to the problem* (2).

Lemma 1 was proved by B.V. Fedosov in 1964 in [11], where he used the same method as in [10], where he proved Lemma 1 for the case $d = 2$. From our point of view, the article [10] is an inalienable part of the article [11], because the latter does not contain details which are common for the cases $d = 2$ and $d \geqslant 3$.

Note also that, for the case $d = 2$, the formula (5) is discussed in [24].

In Section 4 we show that Theorem 1 can be deduced from the formula (5).

3 Special flexible polyhedron

To prove Theorem 1, we need not only the formula (5), but also some special flexible polyhedron $\widetilde{P}_0$, whose existence is established by the following lemma:

Lemma 2 *Let $d \geqslant 3$, $\varepsilon > 0$, and let P_0 be an embedded flexible polyhedron in $\mathbb{R}^d$. Then there is an embedded flexible polyhedron $\widetilde{P}_0 \subset \mathbb{R}^d$ and its flex $\widetilde{\mathcal{F}} = \{\widetilde{P}_s\}_{s\in[0,1)}$ such that*

$(\widetilde{\alpha})$ the combinatorial structure of $\widetilde{P}_0$ is a subdivision of the combinatorial structure of P_0;

$(\widetilde{\beta})$ the Hausdorff distance between $\widetilde{P}_0$ and P_0 is less than ε;

$(\widetilde{\gamma})$ for every $s \in (0,1)$, $\widetilde{P}_s$ is an embedded polyhedron; moreover, there is a $(d-2)$-dimensional face $\widetilde{x}_0$ of $\widetilde{P}_0$ such that the $(d-2)$-dimensional face $\widetilde{x}_s$ of $\widetilde{P}_s$, which corresponds to $\widetilde{x}_0$ according to the property $(\widetilde{\alpha})$, possesses the following properties:

$(\widetilde{\gamma}_1)$ the value of the interior dihedral angle $\varphi(\widetilde{x}_s)$ of $\widetilde{P}_s$ at $\widetilde{x}_s$ tends to zero as $s \to 1$;

$(\widetilde{\gamma}_2)$ there exist constants $\widetilde{m}$ and $\widetilde{M}$ such that $0 < \widetilde{m} < \widetilde{M} < 2\pi$ and, for every $(d-2)$-dimensional face $\widetilde{y}_0$ of $\widetilde{P}_0$, which is different from $\widetilde{x}_0$, and for every $s \in [0,1)$ the inequality $\widetilde{m} < \varphi(\widetilde{y}_s) < \widetilde{M}$ holds true.

Proof Since the polyhedron P_0 is assumed to be flexible, there exists a continuous family $\mathcal{F} = \{P_t\}_{t\in[0,1]}$ of polyhedra such that, for every $t \in (0,1]$, P_0 and P_t are combinatorially equivalent, every two of their corresponding faces are congruent to each other, but the polytopes P_0 and P_t themselves are not congruent.

In the process of proving Lemma 2, we will construct the polyhedron $\widetilde{P}_0$ and the family $\widetilde{\mathcal{F}} = \{\widetilde{P}_s\}_{s\in[0,1)}$ by modifying P_0 and $\mathcal{F}$ in a special way. We will implement this modification in several steps.

Step 1. Since P_0 is embedded, none of it dihedral angles is equal to 0 or 2π. Hence, there exist constants m and M such that $0 < m < M < 2\pi$ and, for every $(d-2)$-dimensional face $x_0^{(d-2)}$ of P_0, the dihedral angle $\varphi(x_0^{(d-2)})$ of P_0 at $x_0^{(d-2)}$ satisfies the following inequalities: $m < \varphi(x_0^{(d-2)}) < M$.

Since the family $\mathcal{F} = \{P_t\}_{t\in[0,1]}$ is continuous, there exists $\omega > 0$ such that, for all $t \in [0,\omega]$, the following properties are valid:

(ω_1) P_t is an embedded polyhedron; and

(ω_2) for every $(d-2)$-dimensional face $x_t^{(d-2)}$ of P_t, the following inequality holds true: $0 < m/2 < \varphi(x_t^{(d-2)}) < M/2 + \pi < 2\pi$.

Substituting $t \mapsto t/\omega$, we may assume without loss of generality that $\omega = 1$, i. e., that the properties (ω_1) and (ω_2) are valid for all $t \in [0,1]$.

Step 2. Let's choose a $(d-2)$-dimensional face $X_0^{(d-2)}$ of P_0 such that, for some $\delta > 0$, the dihedral angle $\varphi(X_t^{(d-2)})$ is not constant on the interval $[0,\delta]$. Such a face $X_0^{(d-2)}$ does exist because P_0 it is assumed to be flexible. By definition, put $T = \{t \in (0,\delta) | \varphi(X_t^{(d-2)}) > \varphi(X_0^{(d-2)})\}$. Since the function $t \mapsto \varphi(X_t^{(d-2)})$ is continuous, T is an open subset of $\mathbb{R}$.

By definition, put $t_* = 0$, if $t = 0$ is a limit point of T. If $t = 0$ is not a limit point of T, choose $t_* > 0$ such that t_* does not lie in T and is so small that the Hausdorff distance between P_0 and P_{t_*} is less than $\varepsilon/3$. In the both cases, put by definition

$$\varphi_* = \varphi(X_{t_*}^{(d-2)}) \tag{6}$$

and $T_* = \{t \in (0,\delta) | \varphi(X_t^{(d-2)}) > \varphi_*\}$.

Choose an interval $(t_1, t_2) \subset T_*$ such that only one of the points t_1, t_2 belongs to the boundary of T_*, i. e., such that the equality $\varphi(X_t^{(d-2)}) = \varphi_*$ is fulfilled for $t = t_1$ and is violated for $t = t_2$, or it is fulfilled for $t = t_2$ and is violated for $t = t_1$.

Let us introduce a new parameter s (which is linearly expressed in terms of the parameter t) such that the segment $[t_1, t_2]$ (within which changes t) corresponds to the segment $[0, 1]$ (within which changes s) and the equality $\varphi(X_s^{(d-2)}) = \varphi_*$ holds true for $s = 1$. For the corresponding values of t and s, we put $P'_s = P_t$. Denote by $\mathcal{F}' = \{P'_s\}_{s\in[0,1]}$ the family of polyhedra obtained in this way.

Step 3. Denote by ${X'_0}^{(d-2)}$ the $(d-2)$-dimensional face of P'_0, which corresponds to the $(d-2)$-dimensional face ${X_0}^{(d-2)}$ of P_0 which was chosen in Step 2. Then the dihedral angle $\varphi({X'_s}^{(d-2)})$ is not constant for $s \in [0, 1]$, moreover $\varphi({X'_s}^{(d-2)}) > \varphi_*$ for all $s \in [0, 1)$ and $\varphi({X'_s}^{(d-2)}) = \varphi_*$ for $s = 1$.

Denote by ${Y'_0}^{(d-1)}$ and ${Z'_0}^{(d-1)}$ the $(d-1)$-dimensional faces of P'_0 incident to ${X'_0}^{(d-2)}$. Let ξ' be a triangulation of ${X'_0}^{(d-2)}$, η' be a triangulation of ${Y'_0}^{(d-1)}$ and ζ' be a triangulation of ${Z'_0}^{(d-1)}$ such that the following properties hold true:

(ξ_1) the restriction of η' on ${X'_0}^{(d-2)}$ coincides with ξ';

(ξ_2) the restriction of ζ' on ${X'_0}^{(d-2)}$ coincides with ξ';

(ξ_3) there exist simplicies ${y'_0}^{(d-1)} \in \eta'$ and ${z'_0}^{(d-1)} \in \zeta'$ such that ${y'_0}^{(d-1)} \cap {X'_0}^{(d-2)} = {z'_0}^{(d-1)} \cap {X'_0}^{(d-2)}$ and the Hausdorff distance between the sets ${y'_0}^{(d-1)} \cup {z'_0}^{(d-1)}$ and $P'_0 \diagdown ({Y'_0}^{(d-1)} \cup {Z'_0}^{(d-1)})$ is positive.

Denote by ${x'_0}^{(d-2)}$ the simplex ${y'_0}^{(d-1)} \cap {X'_0}^{(d-2)}$. Obviously, ${x'_0}^{(d-2)} \in \xi'$.

In Figure 1 we show schematically the cross-section of the domain $[\![P'_0]\!] \subset \mathbb{R}^d$ by a two-dimensional plane, which passes through an interior point of the simplex ${x'_0}^{(d-2)}$ and which is orthogonal to the affine hull of this simplex.

Step 4. Choose continuous families $\{v'_s\}_{s\in[0,1]}$ and $\{w'_s\}_{s\in[0,1]}$ of points in $\mathbb{R}^d$ such that the following properties hold true for all $s \in [0, 1]$:

(σ_1) $v'_s \in [\![P'_s]\!]$ and $w'_s \in [\![P'_s]\!]$;

(σ_2) there exists an isometry $f_s : \mathbb{R}^d \to \mathbb{R}^d$ such that $f_s({y'_0}^{(d-1)}) = {y'_s}^{(d-1)}$ and $f_s(v'_0) = v'_s$;

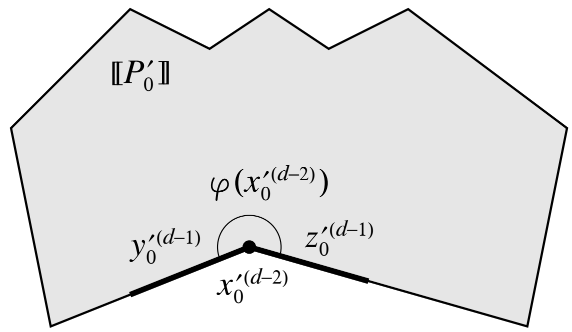

Figure 1: The cross-section of the domain $[\![P_0']\!] \subset \mathbb{R}^d$ by a two-dimensional plane, which passes through an interior point of the simplex ${x_0'}^{(d-2)}$ and which is orthogonal to the affine hull of ${x_0'}^{(d-2)}$. Intersections with the simplicies ${x_0'}^{(d-2)}$, ${y_0'}^{(d-1)}$ and ${z_0'}^{(d-1)}$ are shown in bold

(σ_3) there exists an isometry $g_s : \mathbb{R}^d \to \mathbb{R}^d$ such that $g_s({z_0'}^{(d-1)}) = {z_s'}^{(d-1)}$ and $g_s(w_0') = w_s'$;

(σ_4) the set $\partial(\operatorname{conv}({y_s'}^{(d-1)} \cup \{v_s'\}))\diagdown {y_s'}^{(d-1)}$ has no common points with P_s';

(σ_5) the set $\partial(\operatorname{conv}({z_s'}^{(d-1)} \cup \{w_s'\}))\diagdown {z_s'}^{(d-1)}$ has no common points with P_s';

(σ_6) the sum of the internal dihedral angles of the simplicies $\operatorname{conv}({y_s'}^{(d-1)} \cup \{v_s'\})$ and $\operatorname{conv}({z_s'}^{(d-1)} \cup \{w_s'\})$ at ${x_s'}^{(d-2)}$ is equal to φ_*, where φ_* was determined by the formula (6);

(σ_7) the Hausdorff distance between the sets $P_0' \cup \operatorname{conv}({y_0'}^{(d-1)} \cup \{v_0'\}) \cup \operatorname{conv}({z_0'}^{(d-1)} \cup \{w_0'\})$ and P_0' is less than $\varepsilon/3$.

Obviously, the property (σ_6) can be satisfied for $s = 0$ because, by a suitable choice of the points v_0' and w_0', the dihedral angle at ${x_0'}^{(d-2)}$ in each of the simplicies $\operatorname{conv}({y_0'}^{(d-1)} \cup \{v_0'\})$ and $\operatorname{conv}({z_0'}^{(d-1)} \cup \{w_0'\})$ can be made equal to any number in the interval $(0, \pi)$, while $0 < \varphi_* < 2\pi$.

It is just as simple to verify that the properties (σ_4), (σ_5), and (σ_7) can also be fulfilled for $s = 0$. Hence, there exists $\lambda > 0$ such that the properties $(\sigma_4) - (\sigma_7)$ are fulfilled for all $s \in [0, \lambda]$. Substituting $s \mapsto s/\lambda$, we may assume without loss of generality that $\lambda = 1$, i. e., that the properties $(\sigma_1) - (\sigma_7)$ are valid for all $s \in [0, 1]$.

Step 5. Let $\operatorname{relint}({y_s'}^{(d-1)})$ denote the relative interior of the simplex ${y_s'}^{(d-1)}$, i. e., the interior of ${y_s'}^{(d-1)}$ within its affine hull. For every $s \in [0, 1)$, replace the simplicies ${y_s'}^{(d-1)}$ and ${z_s'}^{(d-1)}$ in P_s' by the polyhedra $\partial(\operatorname{conv}({y_s'}^{(d-1)} \cup \{v_s'\}))\diagdown \operatorname{relint}({y_s'}^{(d-1)})$ and $\partial(\operatorname{conv}({z_s'}^{(d-1)} \cup \{w_s'\}))\diagdown \operatorname{relint}({z_s'}^{(d-1)})$ respectively (see Figure 2). Denote the resulting polyhedron by $\widetilde{P}_s$.

Immediately from the construction of $\widetilde{P}_s$ it follows that the polyhedron $\widetilde{P}_0$ and family $\widetilde{\mathcal{F}} = \{\widetilde{P}_s\}_{s\in[0,1)}$ possess the properties $(\widetilde{\alpha}) - (\widetilde{\gamma})$. $\square$

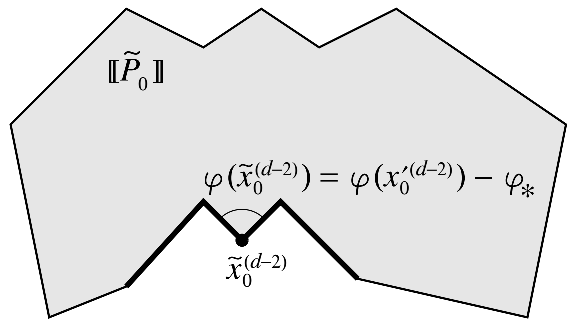

Figure 2: The cross-section of the domain $[\![\widetilde{P}_0]\!] \subset \mathbb{R}^d$ by a two-dimensional plane, which passes through an interior point of the simplex $\widetilde{x}_0^{(d-2)}$ and which is orthogonal to the the affine hull of $\widetilde{x}_0^{(d-2)}$. Intersections with $\widetilde{x}_0^{(d-2)}$ and the sets $\partial(\operatorname{conv}(y_0'^{(d-1)} \cup \{v_0'\}))\diagdown y_0'^{(d-1)}$ and $\partial(\operatorname{conv}(z_0'^{(d-1)} \cup \{w_0'\}))\diagdown z_0'^{(d-1)}$ are shown in bold

4 Proof of Theorem 1

Proof Let P_0 be an embedded flexible polyhedron in $\mathbb{R}^d$, $d \geqslant 3$, and let $\varepsilon > 0$. According to Lemma 2, there exists an embedded flexible polyhedron $\widetilde{P}_0 \subset \mathbb{R}^d$ and its flex $\widetilde{\mathcal{F}} = \{\widetilde{P}_s\}_{s\in[0,1)}$ such that the properties $(\widetilde{\alpha}) - (\widetilde{\gamma})$ are fulfilled.

For $j = 1, 2$ and $s \in [0, 1)$, denote by $\mathcal{N}_s^{(j)}(k)$ the eigenvalue counting function of the problem (1) (for $j = 1$) or problem (2) (for $j = 2$) in the domain $\Omega = [\![\widetilde{P}_s]\!]$.

Putting $p = 2$ and applying Lemma 1 for $[\![\widetilde{P}_s]\!]$, we get

$$\int_0^k (k-t)^2\, d\mathcal{N}_s^{(j)}(t) = \frac{2a_d^s}{(d+1)(d+2)}k^{d+2} + \frac{2a_{d-1}^s}{d(d+1)}k^{d+1} + \frac{2a_{d-2}^s}{(d-1)d}k^d + O(k^{d-1}) \quad (7)$$

as $k \to \infty$, where

$$\begin{aligned} a_d^s &= \frac{\mathrm{vol}_d([\![\widetilde{P}_s]\!])}{2^d\pi^{d/2}\Gamma\left(\frac{d}{2}+1\right)}, \\ a_{d-1}^s &= \mp\frac{\mathrm{vol}_{d-1}(\widetilde{P}_s)}{2^{d+1}\pi^{(d-1)/2}\Gamma\left(\frac{d+1}{2}\right)}, \quad (8) \\ a_{d-2}^s &= \frac{1}{2^{d+1}\pi^{d/2}\Gamma\left(\frac{d}{2}\right)} \sum_{\widetilde{x}_s^{(d-2)}} \frac{[\varphi(\widetilde{x}_s^{(d-2)})]^2 - \pi^2}{3\varphi(\widetilde{x}_s^{(d-2)})}\mathrm{vol}_{d-2}\big(\widetilde{x}_s^{(d-2)}\big). \quad (9) \end{aligned}$$

In the formula (8), the minus sign corresponds to the problem (1), while the plus sign corresponds to the problem (2). In the formula (9), the sum is taken over all $(d-2)$-dimensional faces $\widetilde{x}_s^{(d-2)}$ of $\widetilde{P}_s$ and $\varphi(\widetilde{x}_s^{(d-2)})$ stands for the value of the dihedral angle of $[\![\widetilde{P}_s]\!]$ at $\widetilde{x}_s^{(d-2)}$.

Assuming that the Dirichlet or Neumann spectrum of the Laplacian in $[\![\widetilde{P}_s]\!]$ does not change when s changes in the interval $[0,1)$, we get that, for $j=1$ or $j=2$, the eigenvalue counting function $\mathcal{N}_s^{(j)}(k)$ is independent of s. Hence, for $j=1$ or $j=2$, the left-hand side of the formula (7) is independent of s, i. e., the formula (7) is an asymptotic expansion of the function

$$k \mapsto \int\limits_0^k (k-t)^2\, d\mathcal{N}_s^{(j)}(t)$$

which is independent of s. Since the asymptotic expansion of a function is determined uniquely, it follows that the coefficients at k^{d+2}, k^{d+1} and k^d on the right-hand side of the formula (7) are also independent of s.

However, the latter statement is false for the coefficient at k^d, since a_{d-2}^s is not constant with respect to s. To prove this, we recall that, according to Lemma 2, there exists a $(d-2)$-dimensional face $\widetilde{x}_0$ of $\widetilde{P}_0$ such that, for the corresponding $(d-2)$-dimensional face $\widetilde{x}_s$ of $\widetilde{P}_s$, we have

- $\varphi(\widetilde{x}_s)\to 0$ as $s\to 1$; and
- the value of the dihedral angle of $[\![\widetilde{P}_s]\!]$ at every $(d-2)$-dimensional face $\widetilde{y}_s$ of $\widetilde{P}_s$, different from $\widetilde{x}_s$, is uniformly separated from 0 and 2π: $\widetilde{m} < \varphi(\widetilde{y}_s) < \widetilde{M}$.

Therefore, in the sum on the right-hand side of the formula (9), exactly one term (namely, the term corresponding to the face $\widetilde{x}_s^{(d-2)}=\widetilde{x}_s$), tends to infinity as $s\to 1$, while each of the remaining terms is uniformly bounded for $s\in[0,1)$.

Hence, choosing s sufficiently close to 1, we can to make a_{d-2}^s as large as we want. Thus, a_{d-2}^s is not constant in s, and both Dirichlet and Neumann spectra of the d-dimensional Laplace operator in the domain $[\![\widetilde{P}_s]\!]\subset\mathbb{R}^d$ do not remain unaltered when s changes in the interval $[0,1)$. □

Remark 3 Observe that the coefficients a_d^s and a_{d-1}^s in the formula (7) are constant in s. This fact underlines the non-triviality of the conjecture on the invariance of the Dirichlet and Neumann spectra of the Laplacian during the flex of the boundary of a domain, flipped by Theorem 1.

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