Why there is no an existence theorem for a convex polytope with prescribed directions and perimeters of the faces?

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Abstract

We choose some special unit vectors $\boldsymbol{n}_1, \dots, \boldsymbol{n}_5$ in $\mathbb{R}^3$ and denote by $\mathfrak{L} \subset \mathbb{R}^5$ the set of all points $(L_1, \dots, L_5) \in \mathbb{R}^5$ with the following property: there exists a compact convex polytope $P \subset \mathbb{R}^3$ such that the vectors $\boldsymbol{n}_1, \dots, \boldsymbol{n}_5$ (and no other vector) are unit outward normals to the faces of P and the perimeter of the face with the outward normal $\boldsymbol{n}_k$ is equal to L_k for all $k = 1, \dots, 5$. Our main result reads that $\mathfrak{L}$ is not a locally-analytic set, i.e., we prove that, for some point $(L_1, \dots, L_5) \in \mathfrak{L}$, it is not possible to find a neighborhood $U \subset \mathbb{R}^5$ and an analytic set $A \subset \mathbb{R}^5$ such that $\mathfrak{L} \cap U = A \cap U$. We interpret this result as an obstacle for finding an existence theorem for a compact convex polytope with prescribed directions and perimeters of the faces.

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1. Introduction and the statement of the main result. In 1897, Hermann Minkowski proved the following uniqueness theorem:

Theorem 1 (H. Minkowski, [5] and [6, p. 103–121]). *A convex polytope is uniquely determined, up to translations, by the directions and the areas of its faces.*

Here and below a convex polytope is the convex hull of a finite number of points. By the direction of a face, we mean the direction of the outward normal to the face.

Theorem 1 has numerous applications and generalizations. In order to discuss some of them, we will use the following notation.

Let P be a compact convex polytope in $\mathbb{R}^3$ and $\boldsymbol{n} \in \mathbb{R}^3$ be a unit vector. By $P^{\boldsymbol{n}}$ we denote the intersection of P and its support plane with the outward normal $\boldsymbol{n}$. Note that $P^{\boldsymbol{n}}$ is either a vertex, or an edge, or a face of P. Accordingly, we say that $P^{\boldsymbol{n}}$ has dimension 0, 1, or 2.

In 1937, A.D. Alexandrov proved several generalizations of the above uniqueness theorem of Minkowski, including the following

Theorem 2 (A.D. Alexandrov, [1] and [2, p. 19–29]). *Let P_1 and P_2 be convex polytopes in $\mathbb{R}^3$. Then one of the following mutually exclusive possibilities realizes:*

(i) *P_1 is obtained from P_2 by a parallel translation*;

(ii) *there exist $k=1,2$ and a unit vector $\boldsymbol{n}\in\mathbb{R}^3$ such that $P_k^{\boldsymbol{n}}$ has dimension 2 and, for some translation $T:\mathbb{R}^3\to\mathbb{R}^3$, the formula $T(P_j^{\boldsymbol{n}})\subsetneq P_k^{\boldsymbol{n}}$ holds true, where $j\in\{1,2\}\diagdown\{k\}$.*

Note that the formula $T(P_j^{\boldsymbol{n}})\subsetneq P_k^{\boldsymbol{n}}$ means that the face $P_j^{\boldsymbol{n}}$ can be embedded inside the face $P_k^{\boldsymbol{n}}$ by translation T as a proper subset.

For more details about Theorems 1 and 2, the reader is referred to [3].

For us, it is important that Theorem 1 is a special case of Theorem 2. In fact, suppose the conditions of Theorem 1 are fulfilled. Then, using the notation of Theorem 2, we observe that, for every unit vector $\boldsymbol{n}$ such that $P_k^{\boldsymbol{n}}$ has dimension 2, $P_j^{\boldsymbol{n}}$ also has dimension 2 and its area is equal to the area of $P_k^{\boldsymbol{n}}$. Hence, there is no translation T such that $T(P_j^{\boldsymbol{n}})\subsetneq P_k^{\boldsymbol{n}}$. This means that the possibility (ii) in Theorem 2 is not realized. Therefore, the possibility (i) in Theorem 2 is realized and Theorem 1 is a consequence of Theorem 2.

In fact, Theorem 2 has many other consequences, including the following

Theorem 3 (A.D. Alexandrov, [3, Chapter II, § 4]). *A convex polytope in $\mathbb{R}^3$ is uniquely determined, up to translations, by the directions and the perimeters of its faces.*

For the sake of completeness, we mention that a direct analog of Theorem 1 is valid in $\mathbb{R}^d$ for all $d\geqslant 4$; a direct analog of Theorem 2 is not valid in $\mathbb{R}^d$ for every $d\geqslant 4$; in $\mathbb{R}^3$, a refinement of Theorem 2 was found by G.Yu. Panina [7] in 2008.

We explained above that uniqueness Theorems 1 and 3 are similar to each other and both follow from Theorem 2. In the rest part of this section we explain the difference that appear when we are interested in existence results corresponding to uniqueness Theorems 1 and 3.

In 1897, Hermann Minkowski also proved the following existence theorem:

Theorem 4 (H. Minkowski, [5] and [6, p. 103–121]). *Let unit vectors $\boldsymbol{n}_1,\dots,\boldsymbol{n}_m$ in $\mathbb{R}^3$ and real numbers $F_1,\dots,F_m$ satisfy the following conditions:*

(i) *$\boldsymbol{n}_1,\dots,\boldsymbol{n}_m$ are not coplanar and no two of them coinside with each other*;

(ii) *F_k is positive for every $k=1,\dots,m$;*

(iii) $\sum_{k=1}^m F_k\boldsymbol{n}_k=0$.

Then there exists a convex polytope $P\subset\mathbb{R}^3$ such that $\boldsymbol{n}_1,\dots,\boldsymbol{n}_m$ (and no other vector) are outward face normals for P and F_k is the area of the face with outward normal $\boldsymbol{n}_k$ for every $k=1,\dots,m$.

For the sake of completeness, we mention that a direct analog of Theorem 4 is valid in $\mathbb{R}^d$ for all $d\geqslant 4$.

For more details about Theorem 4, the reader is referred to [3].

Recall that a set $A\subset\mathbb{R}^d$ is said to be algebraic if $A=\{x\in\mathbb{R}^d : p(x)=0\}$ for some polynomial $p:\mathbb{R}^d\to\mathbb{R}$, and A is said to be locally-algebraic if, for every

$x \in A$, there is a neighborhood $U \subset \mathbb{R}^d$ and an algebraic set $A_0 \subset \mathbb{R}^d$ such that $A \cap U = A_0 \cap U$.

For a given set $\{\boldsymbol{n}_1, \dots, \boldsymbol{n}_m\}$ of vectors satisfying the condition (i) of Theorem 4, denote by $\mathcal{F}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_m)$ the set of all points $(F_1, \dots, F_m) \subset \mathbb{R}^m$ such that there exists a convex polytope $P \subset \mathbb{R}^3$ for which $\boldsymbol{n}_1, \dots, \boldsymbol{n}_m$ (and no other vector) are the outward face normals for P, and F_k is the area of the face with the outward normal $\boldsymbol{n}_k$ for every $k = 1, \dots, m$. The set $\mathcal{F}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_m) \subset \mathbb{R}^m$ can be referred to as a natural configuration space of convex polytopes (treated up to translations) with prescribed set $\{\boldsymbol{n}_1, \dots, \boldsymbol{n}_m\}$ of outward unit normals when a polytope is determined by the areas $F_1, \dots, F_m$ of its faces.

From Theorem 4, it follows immediately that *the set* $\mathcal{F}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_m) \subset \mathbb{R}^m$ *is locally-algebraic for every set* $\{\boldsymbol{n}_1, \dots, \boldsymbol{n}_m\}$ *of vectors satisfying the condition* (i) *of Theorem 4.* In fact, we can define the algebraic set A_0 as the zero set of the quadratic polynomial

$$\sum_{j=1}^{3}\left(\sum_{k=1}^{m}(\boldsymbol{n}_k, \boldsymbol{e}_j)F_k\right)^2,$$

where $(\boldsymbol{n}_k, \boldsymbol{e}_j)$ stands for the standard scalar product in $\mathbb{R}^3$ and $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is the standard orthonormal basis in $\mathbb{R}^3$.

Recall that a set $B \subset \mathbb{R}^d$ is said to be analytic if $B = \{x \in \mathbb{R}^d : \varphi(x) = 0\}$ for some real-analytic function $\varphi : \mathbb{R}^d \to \mathbb{R}$, and B is said to be locally-analytic if, for every $x \in B$, there is a neighborhood $U \subset \mathbb{R}^d$ and an analytic set $B_0 \subset \mathbb{R}^d$ such that $B \cap U = B_0 \cap U$.

Obviously, every locally-algebraic set is locally analytic.

Let unit vectors $\boldsymbol{n}_1, \dots, \boldsymbol{n}_5$ in $\mathbb{R}^3$ be defined by the formulas

$$\begin{aligned} \boldsymbol{n}_1 = (-1, 0, 0), \quad \boldsymbol{n}_2 &= \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \quad \boldsymbol{n}_3 = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \\ \boldsymbol{n}_4 &= \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad \boldsymbol{n}_5 = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right). \end{aligned} \tag{1}$$

For convenience of the reader, the vectors $\boldsymbol{n}_1, \dots, \boldsymbol{n}_5$ are shown schematically in Figure 1. By $\mathcal{L} = \mathcal{L}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5) \subset \mathbb{R}^5$ we denote the set of all points $(L_1, \dots, L_5) \in \mathbb{R}^5$ with the following property: there exists a convex polytope $P \subset \mathbb{R}^3$ such that the vectors $\boldsymbol{n}_1, \dots, \boldsymbol{n}_5$ (and no other vector) are the unit outward normals to the faces of P, and L_k is the perimeter of the face with the outward normal $\boldsymbol{n}_k$ for every $k = 1, \dots, 5$. The set $\mathcal{L}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5) \subset \mathbb{R}^5$ can be referred to as a natural configuration space of convex polytopes (treated up to translations) with prescribed set $\{\boldsymbol{n}_1, \dots, \boldsymbol{n}_5\}$ of outward unit normals, when a polytope is determined by the perimeters $L_1, \dots, L_5$ of its faces.

The main result of this article reads as follows:

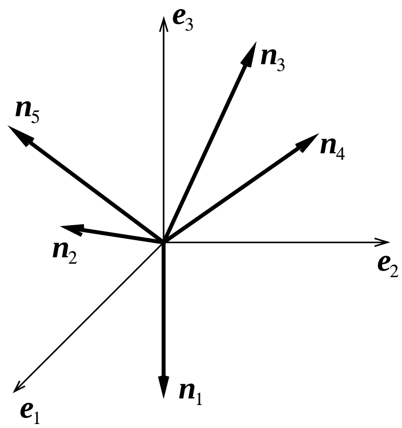

Figure 1: Unit vectors $\boldsymbol{n}_1, \dots, \boldsymbol{n}_5$ defined by the formulas (1)

Theorem 5. *Let the vectors $\boldsymbol{n}_1, \dots, \boldsymbol{n}_5$ be given by the formulas* (1)*. Then the set $\mathfrak{L}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5) \subset \mathbb{R}^5$ is not locally-analytic.*

From our point of view, Theorem 5 explains why a general existence theorem is not known which determines a convex polytope in $\mathbb{R}^3$ via unit normals and perimeters of its faces. The reason is that no analytic condition, similar to the condition (iii) in Theorem 4, does exist.

2. Auxiliary constructions and preliminary results. Let $P \subset \mathbb{R}^3$ be a convex polytope such that the vectors $\boldsymbol{n}_1, \dots, \boldsymbol{n}_5$ defined by the formulas (1) (and no other vector) are the unit outward normals to the faces of P. For $k = 1, \dots, 5$, denote by π_k the 2-dimensional plane in $\mathbb{R}^3$ containing the face of P with the outward normal $\boldsymbol{n}_k$.

The straight lines $\pi_1 \cap \pi_2$ and $\pi_1 \cap \pi_3$ are parallel to the vector $\boldsymbol{e}_2 = (0, 1, 0)$, and the straight lines $\pi_1 \cap \pi_4$ and $\pi_1 \cap \pi_5$ are parallel to the vector $\boldsymbol{e}_1 = (1, 0, 0)$. Hence, the face $P \cap \pi_1$ is a rectangle. Computing the angles between the vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_k$ for $k = 2, \dots, 5$, we conclude that the dihedral angle attached to any edge of the face $P \cap \pi_1$ is equal to $\pi/4$. Now it is clear that the polytope P can be of one of the three types schematically shown in Figure 2.

In order to be more specific, we put by definition $A = \pi_1 \cap \pi_2 \cap \pi_5$, $B = \pi_1 \cap \pi_2 \cap \pi_4$, $C = \pi_1 \cap \pi_3 \cap \pi_4$, and $D = \pi_1 \cap \pi_3 \cap \pi_5$. Denote by $2x$ the length of the straight line segment AB, and by $2y$ the length of the straight line segment BC. We say that the polytope P is of Type I, if $x < y$; is of Type II, if $x = y$; and is of Type III, if $x > y$. Polytopes P of Types I–III are shown schematically in Figure 2.

Denote by $\mathfrak{L}_I(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5)$ (respectively, by $\mathfrak{L}_{II}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5)$ and $\mathfrak{L}_{III}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5)$) the set of all points $(L_1, \dots, L_5) \in \mathbb{R}^5$ such that there exists a convex polytope $P \subset \mathbb{R}^3$ of Type I (respectively, of Type II or Type III) such that the vectors $\boldsymbol{n}_1, \dots, \boldsymbol{n}_5$ (and no other vector) are outward unit normals to the faces of P, and L_k

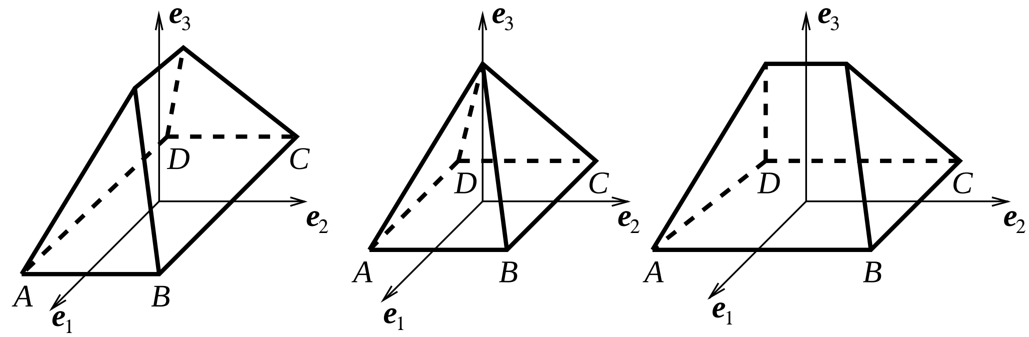

Figure 2: Three types of polytopes with outward normals $\boldsymbol{n}_1, \dots, \boldsymbol{n}_5$ defined by the formulas (1): Type I (left), Type II (center), and Type III (right)

is the perimeter of the face of P with the outward normal $\boldsymbol{n}_k$ for every $k = 1, \dots, 5$. Below, we use also the following notation

$$\begin{aligned}
\boldsymbol{v}_I &= (2, -(3+2\sqrt{3}), -(3+2\sqrt{3}), 5, 5),\\
\boldsymbol{v}_{II} &= (2(\sqrt{3}-1), 1, 1, 1, 1),\\
\boldsymbol{v}_{III} &= (2, 5, 5, -(3+2\sqrt{3}), -(3+2\sqrt{3})).
\end{aligned}$$

Lemma 1. *Let the set $\{\boldsymbol{n}_1, \dots, \boldsymbol{n}_5\}$ of unit vectors in $\mathbb{R}^3$ be defined by the formulas* (1)*. Then the following three statements are equivalent to each other:*

(i) $(L_1, \dots, L_5) \in \mathfrak{L}_I(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5)$;

(ii) $L_1 = (2\sqrt{3}-3)L_2 + L_4$, $L_2 = L_3$, $L_4 = L_5$, $L_4 > L_2 > 0$;

(iii) $(L_1, \dots, L_5) = \alpha \boldsymbol{v}_I + \beta \boldsymbol{v}_{II}$ *for some* $\alpha, \beta \in \mathbb{R}$ *such that* $\beta > (3+2\sqrt{3})\alpha > 0$.

Proof: Suppose the statement (i) of Lemma 1 holds true. In addition to the notation introduced above in Section 2, let $E = \pi_2 \cap \pi_4 \cap \pi_5$, $F = \pi_3 \cap \pi_4 \cap \pi_5$, G be the base of the perpendicular dropped from E on the edge AB, H be the base of the perpendicular dropped from E on the face $ABCD$, and K be the base of the perpendicular dropped from E on the edge BC, see Figure 3. Using this notation, we obtain easily

$$\begin{aligned}
BG &= GH = EH = BK = x,\\
BH &= x\sqrt{2},\\
BE &= x\sqrt{3},\\
EF &= BC - 2BK = 2y - 2x,\\
L_1 &= 2AB + 2BC = 4x + 4y, && (2)\\
L_2 &= L_3 = AB + 2BE = 2(1+\sqrt{3})x, && (3)\\
L_4 &= L_5 = BC + EF + 2BE = 2(\sqrt{3}-1)x + 4y. && (4)
\end{aligned}$$

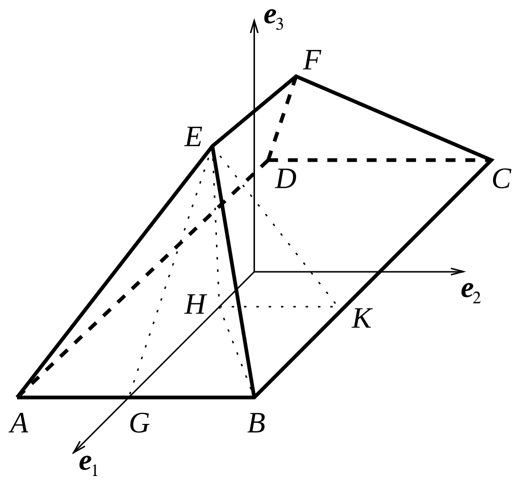

Figure 3: Elements of a polyhedron P of Type I, needed for finding the relations between the perimeters $L_1, \dots, L_5$ of the faces

Eliminating x and y from the last three formulas, we get $L_1 = (2\sqrt{3}-3)L_2 + L_4$. Since the relations $L_2 = L_3$, $L_4 = L_5$, and $L_4 > L_2 > 0$ are obvious for every polytope P of Type I, we conclude that the statement (i) implies the statement (ii).

Now suppose the statement (ii) of Lemma 1 holds true. We, first, find the numbers x and y that satisfy the relations (2)–(4) and, second, construct a convex polyhedron P of Type I for which $L_1, \dots, L_5$ are the perimeters of the faces.

In accordance with (3), we put by definition $x = L_2(\sqrt{3}-1)/4$. Since $L_2 > 0$, $x > 0$. In accordance with (4), we put by definition $y = L_2(\sqrt{3}-2)/4 + L_4/4$. Since $L_1 = (2\sqrt{3}-3)L_2 + L_4$, the condition (2) is satisfied: $4x+4y = (2\sqrt{3}-3)L_2 + L_4 = L_1$. Using the inequalities $L_4 > L_2 > 0$, we obtain $4y = L_4 + (\sqrt{3}-2)L_2 > L_2 + (\sqrt{3}-2)L_2 = (\sqrt{3}-1)L_2 > 0$. Hence, there exists a rectangle $ABCD$ in $\mathbb{R}^3$ such that the edge AB is parallel to the vector $\boldsymbol{e}_2 = (0,1,0)$ and its length is equal to $2x$, and the edge BC is parallel to the vector $\boldsymbol{e}_1 = (1,0,0)$ and its length is equal to $2y$.

Denote by π_1 the plane that contains $ABCD$. Obviously, π_1 is perpendicular to $\boldsymbol{n}_1$. Denote by π_2 the plane perpendicular to $\boldsymbol{n}_2$ and containing AB. Denote by π_3 the plane perpendicular to $\boldsymbol{n}_3$ and containing CD. Denote by π_4 the plane perpendicular to $\boldsymbol{n}_4$ and containing BC. And denote by π_5 the plane perpendicular to $\boldsymbol{n}_5$ and containing AD. The five planes $\pi_1, \dots, \pi_5$ determine a compact convex polyhedron with outward normals $\boldsymbol{n}_1, \dots, \boldsymbol{n}_5$. Denote it by P. According to the statement (ii), $L_4 > L_2$. Hence, $4y = L_4 + (\sqrt{3}-2)L_2 > L_2 + (\sqrt{3}-2)L_2 = (\sqrt{3}-1)L_2 = 4x$. Thus, $P \in \mathfrak{L}_I$ and the statement (ii) implies the statement (i).

So, we have proved that the statements (i) and (ii) are equivalent to each other.

In order to prove that the statements (ii) and (iii) are equivalent to each other, we observe that the equations $L_1 = (2\sqrt{3}-3)L_2 + L_4$, $L_2 = L_3$, and $L_4 = L_5$

from the statement (ii) define a 2-dimensional plane in $\mathbb{R}^5$. Denote this plane by λ_I. More over, the vectors $\boldsymbol{v}_I$ and $\boldsymbol{v}_{II}$ constitute an orthogonal basis in λ_I. This means that every vector $(L_1, \dots, L_5) \in \lambda_I$ can be uniquely written in the form $\alpha \boldsymbol{v}_I + \beta \boldsymbol{v}_{II}$. Direct calculations show that the inequalities $L_4 > L_2 > 0$ from the statement (ii) are equivalent to the inequalities $\beta > (3+2\sqrt{3})\alpha > 0$ from the statement (iii). □

Lemma 2. *Let the set* $\{\boldsymbol{n}_1, \dots, \boldsymbol{n}_5\}$ *of unit vectors in* $\mathbb{R}^3$ *be defined by the formulas* (1). *Then the following three statements are equivalent to each other:*

(i) $(L_1, \dots, L_5) \in \mathcal{L}_{II}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5)$;

(ii) $L_1 = 2(\sqrt{3}-1)L_2$, $L_2 = L_3 = L_4 = L_5 > 0$;

(iii) $(L_1, \dots, L_5) = \gamma \boldsymbol{v}_{II}$ *for some* $\gamma \in \mathbb{R}$ *such that* $\gamma > 0$.

Proof is left to the reader. It can be obtained by arguments similar to those used above in the proof of Lemma 1. But in fact, it is sufficient to observe that Lemma 2 is the limit case of Lemma 1 as L_4 approaches L_2.

Lemma 3. *Let the set* $\{\boldsymbol{n}_1, \dots, \boldsymbol{n}_5\}$ *of unit vectors in* $\mathbb{R}^3$ *be defined by the formulas* (1). *Then the following three statements are equivalent to each other:*

(i) $(L_1, \dots, L_5) \in \mathcal{L}_{III}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5)$;

(ii) $L_1 = L_2 + (2\sqrt{3}-3)L_4$, $L_2 = L_3$, $L_4 = L_5$, $L_2 > L_4 > 0$;

(iii) $(L_1, \dots, L_5) = \delta \boldsymbol{v}_{III} + \varepsilon \boldsymbol{v}_{II}$ *for some* $\delta, \varepsilon \in \mathbb{R}$ *such that* $\varepsilon > (3+2\sqrt{3})\delta > 0$.

Proof is left to the reader. It can be obtained by arguments similar to those used above in the proof of Lemma 1. But in fact, it is sufficient to observe that if we rotate a polytope of Type III around the vector $\boldsymbol{e}_3 = (0,0,1)$ to the angle $\pi/2$, we get a polytope of Type I and can apply Lemma 1 to it.

In the proof of Lemma 1, we denoted by λ_I the 2-dimensional subspace in $\mathbb{R}^5$ which is spanned by the vectors $\boldsymbol{v}_I$ and $\boldsymbol{v}_{II}$. Now we denote by λ_{II} the 1-dimensional subspace in $\mathbb{R}^5$ spanned by $\boldsymbol{v}_{II}$ and denote by λ_{III} the 2-dimensional subspace spanned by $\boldsymbol{v}_{II}$ and $\boldsymbol{v}_{III}$.

Lemma 4. $\lambda_{II} = \lambda_I \cap \lambda_{III}$.

Proof : Each subspace λ_I, λ_{II}, and λ_{III} contains $\boldsymbol{v}_{II}$. Hence, $\dim(\lambda_I \cap \lambda_{III}) \geqslant 1$. On the other hand, $\dim \lambda_I = \dim \lambda_{III} = 2$. Hence, $\dim(\lambda_I \cap \lambda_{III})$ is equal to either 1 or 2.

Suppose $\dim(\lambda_I \cap \lambda_{III}) = 2$. Then $\lambda_I = \lambda_{III}$. Hence, the vectors $\boldsymbol{v}_I$, $\boldsymbol{v}_{II}$, and $\boldsymbol{v}_{III}$ are linearly dependant. But this is not the case because the 3×3 minor composed of the first, third and fifth columns of the matrix

$$\begin{pmatrix} \boldsymbol{v}_I \\ \boldsymbol{v}_{II} \\ \boldsymbol{v}_{III} \end{pmatrix} = \begin{pmatrix} 2 & -(3+2\sqrt{3}) & -(3+2\sqrt{3}) & 5 & 5 \\ 2(\sqrt{3}-1) & 1 & 1 & 1 & 1 \\ 2 & 5 & 5 & -(3+2\sqrt{3}) & -(3+2\sqrt{3}) \end{pmatrix}$$

is non-zero. Hence, $\dim(\lambda_I \cap \lambda_{III}) = 1$, and $\lambda_{II} = \lambda_I \cap \lambda_{III}$. □

3. Half-branches of analytic sets and the proof of Theorem 5. Let A be a one-dimensional analytic set, and $x \in A$. For every sufficiently small open ball U with center x, $A \cap (U \setminus \{x\})$ has a finite number of connected components $A_1, \dots, A_k$

such that x belongs to the closure of A_j for every $j = 1, \dots, k$. These A_j are called the half-branches of A centered at x. It is known that *the number of half-branches of a one-dimensional analytic set centered at a point is even,* see, e. g. [8].

For completeness, we mention that a comprehensive exposition of a similar result for algebraic sets of dimension 1 may be found in [4, Section 9.5].

Proof of Theorem 5 : Let Λ be the straight line in $\mathbb{R}^5$ defined by the formula $\Lambda = \{x \in \mathbb{R}^5 | x = \boldsymbol{v}_{II} + t\boldsymbol{v}_I \text{ for some } t \in \mathbb{R}\}$.

Our proof is by contradiction. Suppose the set $\mathcal{L}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5) \subset \mathbb{R}^5$ is locally-analytic. Then $\Lambda \cap \mathcal{L}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5)$ is also locally-analytic. Moreover, it is one-dimensional, contains the point $\boldsymbol{v}_{II}$, and has only one half-branch centered at $\boldsymbol{v}_{II}$.

Let us explain the last statements in more details. From the definition of polytopes of Types I–III we know that

$$\mathcal{L}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5) = \mathcal{L}_I(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5) \cup \mathcal{L}_{II}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5) \cup \mathcal{L}_{III}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5).$$

From Lemma 1 we know that $\mathcal{L}_I(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5)$ is an angle on the 2-dimensional plane $\lambda_I \subset \mathbb{R}^5$. From Lemma 3 we know that $\mathcal{L}_{III}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5)$ is an angle on the 2-dimensional plane $\lambda_{III} \subset \mathbb{R}^5$. These angles are glued together along the ray $\mathcal{L}_{II}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5)$ (see Lemma 2), and no 2-dimensional plane contains the both of them (see Lemma 4). The line Λ lies in the plane λ_I and passes through the point $\boldsymbol{v}_{II}$. Hence, for every sufficiently small open ball $U \subset \mathbb{R}^5$ with center $\boldsymbol{v}_{II}$,

$$U \cap \Lambda \cap \mathcal{L}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5) = \{\boldsymbol{v}_{II}\} \cup (U \cap \Lambda \cap \mathcal{L}_I(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5)).$$

This formula means that we may obtain $U \cap \Lambda \cap \mathcal{L}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5)$ in the following way: first, we divide the straight line Λ into two rays by the point $\boldsymbol{v}_{II}$; then we observe that only one of these rays has at least one common point with the angle $\mathcal{L}_I(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5)$ and select that ray; at last, we intersect the ray selected with U.

From this description, it is clear that $U \cap \Lambda \cap \mathcal{L}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5)$ is the half-branch of the locally-analytic set $\mathcal{L}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5) \subset \mathbb{R}^5$ centered at $\boldsymbol{v}_{II}$. Moreover, this is the only half-branch centered at $\boldsymbol{v}_{II}$. This contradicts to the fact that the number of half-branches of a locally-analytic set centered at a point is even, see [8]. □

Remark : The proof of Theorem 5 provides us with a new, more technical, answer to the question of the title of this article. As a part of the proof of Theorem 5, we demonstrated that the set $\mathcal{L}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5) \subset \mathbb{R}^5$ is not convex. In Section 1, we mentioned that $\mathcal{L}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_5)$ can be considered as a natural configuration space of convex polytopes (treated up to translations) with prescribed outward unit normals and perimeters of its faces. The reader, familiar with the proof of Theorem 4 given in [3, Chapter VII, §1], may remember that convexity of the analogous 'natural configuration space' $\mathcal{F}(\boldsymbol{n}_1, \dots, \boldsymbol{n}_m) \subset \mathbb{R}^m$ plays an important role in that proof.

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