

An analytical approach to the Rational Simplex Problem

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Abstract

In 1973, J. Cheeger and J. Simons raised the following question that still remains open and is known as the Rational Simplex Problem: Given a geodesic simplex in the spherical 3-space so that all of its interior dihedral angles are rational multiples of π , is it true that its volume is a rational multiple of the volume of the 3-sphere? We propose an analytical approach to the Rational Simplex Problem by deriving a function $f(t)$, defined as an integral of an elementary function, such that if there is a rational t , close enough to zero, such that the value $f(t)$ is an irrational number then the answer to the Rational Simplex Problem is negative.

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1 Introduction

At first glance, the problem of finding the volume of a polyhedron in the spherical or hyperbolic 3-space is both elementary and classical, and nobody can expect new deep results here. In fact this first impression is completely wrong. This problem is interesting not only by itself [9], [16], but it is important also for many problems of the geometry and topology of 3-manifolds [1], theory of flexible polyhedra [2], [12], etc. that attract attention of modern geometers. Several problems remain open even if we restrict our study by spherical 3-simplices only. One of such problems (that still remains open and is known as the Rational Simplex Problem) was raised by Jeff Cheeger and James Simons in a conference held at Stanford in 1973 [5]: *Given a geodesic simplex in the spherical 3-space so that all of its interior dihedral angles are rational multiples of π , is it true that its volume is a rational multiple of the volume of the 3-sphere?*

Cheeger and Simons conjectured that the answer is 'usually' no, although the answer is positive in all known cases. In order to demonstrate the depth of the Rational Simplex Problem we can mention that, in [5, Theorem 2], Johan Dupont and Sah Chih-Han proved, among other results, the following theorem: *Let Δ denote a spherical simplex with all of its dihedral angles in $\mathbb{Q}\pi$. Then, we have the following alternatives:*

- (a) Δ is scissors congruent to a lune so that its volume is in $\mathbb{Q}\pi^2$.
- (b) Δ is not scissors congruent to a lune and exactly one of the following possibilities holds:
 - (b₁) Δ leads to a negative answer of the generalized Hilbert Third Problem for spherical space; or
 - (b₂) Δ does not have volume in $\mathbb{Q}\pi^2$, so it is an example desired by Cheeger–Simons.

Furthermore, for an orthoscheme Δ , case (a) and (b) are distinguished by an explicit algorithm.

In this theorem, as usual, two polyhedra P and Q are said to be scissors congruent if P and Q can each be decomposed into finite pairwise interior disjoint union of polyhedra $P_i, Q_i, 1 \leq i \leq n$, so that P_i is congruent to Q_i for each i . Recall also that a *lune* is defined to be the orthogonal suspension to the two poles of a geodesic spherical triangle lying on the equator of spherical space and that the generalized

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Hilbert Third Problem still remains open and reads as follows: *Is it true that two geodesic polyhedra in spherical (respectively hyperbolic) 3-space are scissors congruent if and only if they have the same volume and the same Dehn invariant?*

In the present paper we propose an analytical approach to the Rational Simplex Problem. More precisely, we introduce a function $f(t)$, defined as an integral of an elementary function, such that if there is a rational t , close enough to zero, such that the value $f(t)$ is an irrational number then the answer to the Rational Simplex Problem is negative.

2 The main result

DEFINITION. For all $|t| < 1/10$, put

$$f(t) = \frac{1}{\pi^2} \int_0^{\pi t} \arccos \left(\frac{\sin s}{1 + 2 \sin s} \right) ds. \quad (1)$$

THEOREM. *If for some rational t , such that $|t| < 1/10$, the number $f(t)$, given by the formula (1), is not rational then the answer to the Rational Simplex Problem is negative, i.e., there exists a geodesic simplex in the spherical 3-space so that all of its interior dihedral angles are rational multiples of π , while its volume is not a rational multiple of π^2 .*

Proof: Let \mathbb{S}^3 be the spherical 3-space of curvature +1. For example, we can treat \mathbb{S}^3 as the unit sphere centered at the origin of the Euclidean space \mathbb{R}^4 .

Consider a family of regular spherical simplices $\sigma(x) \subset \mathbb{S}^3$ with the edge lengths x varying from 0 to $\arccos(-1/3)$. Observe, that

(i) for $x_* = \arccos(-1/3)$, the simplex $\sigma(x_*)$ is equal to a hemisphere and, thus (i_1) in \mathbb{S}^3 , there is no regular spherical simplex whose edge length is greater than x_* and (i_2) the dihedral angles φ_* of the simplex $\sigma(x_*)$ are equal to π ;

(ii) the value $x_* = \arccos(-1/3)$ is approximately equal to $3\pi/5$ (in fact it is approximately 1.5% larger than $3\pi/5$).

Equivalently, we can treat the family of regular spherical simplices $\sigma(x) \subset \mathbb{S}^3$ as being parameterized by their dihedral angles φ . In this case we write $\sigma(\varphi)$ instead of $\sigma(x)$. Observe that $\varphi = \pi/3$ corresponds to the edge length $x = 0$ and φ increases from $\pi/3$ to π as x increases from 0 to x_* .

Denote by α the plane angle of a face of $\sigma(x)$. A median of the face divides it into two mutually congruent right-angled spherical triangles; the spherical law of sines applied to any of them yields

$$\frac{\sin x}{\sin(\pi/2)} = \frac{\sin(x/2)}{\sin(\alpha/2)} \quad \text{or} \quad \cos \frac{x}{2} = \frac{1}{2 \sin(\alpha/2)}. \quad (2)$$

From (2), it follows that $\alpha \rightarrow \pi/3$ as $x \rightarrow 0$ and $\alpha \rightarrow 2\pi/3$ as $x \rightarrow x_* = \arccos(-1/3)$.

A small sphere centered at a vertex of $\sigma(\varphi)$ intersects the simplex by a regular spherical triangle δ , whose angles are equal to φ radians and whose sides are equal to α radians. Applying the spherical law of sines to δ in a way similar to the above described we get

$$\frac{\sin \alpha}{\sin(\pi/2)} = \frac{\sin(\alpha/2)}{\sin(\varphi/2)} \quad \text{or} \quad \cos \frac{\alpha}{2} = \frac{1}{2 \sin(\varphi/2)}. \quad (3)$$

From (3), it follows that $\varphi \rightarrow \arctan 2\sqrt{2}$ as $\alpha \rightarrow \pi/3$ and $\varphi \rightarrow \pi$ as $\alpha \rightarrow 2\pi/3$. Observe that $\arctan 2\sqrt{2}$ is, of course, the dihedral angle of a regular tetrahedron in the Euclidean 3-space.

Combining (2) and (3), we get

$$x = 2 \arccos \frac{\sin(\varphi/2)}{\sqrt{4 \sin^2(\varphi/2) - 1}}, \quad (4)$$

where $29\pi/74 \approx \arctan 2\sqrt{2} < \varphi < 2\pi/3$. Observe that, in (4), $x \rightarrow 0$ as $\varphi \rightarrow \arctan 2\sqrt{2}$ and $x \rightarrow x_* = \arccos(-1/3)$ as $\varphi \rightarrow \pi$. Below we will use (4) for the values of φ that lie in the interval $(\pi/2 - \pi/10, \pi/2 + \pi/10)$; this is possible according to the previous estimates.

Now recall the classical Schläfli differential formula [15]:

$$\frac{d}{dr} \text{vol } P_r = \frac{1}{2} \sum_j |\lambda_r^j| \frac{d}{dr} \beta_r^j. \quad (5)$$

Here $\{P_r\}_{r \in I}$ is a smooth family of polyhedra in \mathbb{S}^3 , $\text{vol } P_r$ stands for the volume of P_r , λ_r^j , $j = 1, \dots, J$, denote the edges of P_r , $|\lambda_r^j|$ is the length of the edge λ_r^j , and β_r^j is the dihedral angle of P_r attached to λ_r^j .

It follows immediately from (4) and (5) that

$$\frac{d}{d\varphi} \text{vol } \sigma(\varphi) = 6x = 12 \arccos \frac{\sin(\varphi/2)}{\sqrt{4 \sin^2(\varphi/2) - 1}}. \quad (6)$$

If we divide $\mathbb{S}^3 \subset \mathbb{R}^4$ into 16 pieces by 4 mutually orthogonal hyperplanes in \mathbb{R}^3 , each piece will be congruent to $\sigma(\pi/2)$. Hence, $\text{vol } \sigma(\pi/2) = \pi^2/8$. Taking this fact and (6) into account, we get

$$\text{vol } \sigma(\varphi) = \frac{\pi^2}{8} + 12 \int_{\pi/2}^{\varphi} \arccos \frac{\sin(\psi/2)}{\sqrt{4 \sin^2(\psi/2) - 1}} d\psi \quad (7)$$

for all $\pi/2 - \pi/10 < \varphi < \pi/2 + \pi/10$. In order to simplify the integral (7), we put $\varphi = \pi/2 + \pi t$, where $|t| < 1/10$, and make the following substitution $\psi = \pi/2 + s$. Then we get

$$\text{vol } \sigma\left(\frac{\pi}{2} + \pi t\right) = \frac{\pi^2}{8} + 12 \int_0^{\pi t} \arccos \frac{\sin(s/2) + \cos(s/2)}{\sqrt{2 + 4 \sin s}} ds. \quad (8)$$

Recall that [8, formula 1.626.2]:

$$2 \arccos x = \begin{cases} \arccos(2x^2 - 1), & 0 \leq x \leq 1; \\ 2\pi - \arccos(2x^2 - 1), & -1 \leq x \leq 0. \end{cases} \quad (9)$$

Taking into account that

$$0 < 0.636 < \frac{\sin(s/2) + \cos(s/2)}{\sqrt{2 + 4 \sin s}} < 0.952 < 1$$

for $-\pi/10 < s < \pi/10$ and using the first line in (9), we get from (8) after simplifications

$$\text{vol } \sigma\left(\frac{\pi}{2} + \pi t\right) = \frac{\pi^2}{8} + 6 \int_0^{\pi t} \arccos \frac{-\sin s}{1 + 2 \sin s} ds.$$

At last, using the formula $\arccos(-u) = \pi - \arccos u$ that is valid for all $-1 < u < 1$, we get

$$\text{vol } \sigma\left(\frac{\pi}{2} + \pi t\right) = \frac{\pi^2}{8} + 6\pi^2 t - 6 \int_0^{\pi t} \arccos \frac{\sin s}{1 + 2 \sin s} ds. \quad (10)$$

It follows from (10) that if, for some rational $t \in (-1/10, 1/10)$, the number $f(t)$, defined by the formula (1), is irrational, then the volume of the simplex $\sigma(\pi/2 + \pi t)$ is not a rational multiple of π^2 . This completes the proof of the theorem.

3 Remarks

(A) The problem to decide whether the number $f(t)$, defined by the formula (1), is rational or not is a difficult problem itself. For some results on similar problems, see, e.g. [7, 10, 17]. The classical methods are not applicable to $f(t)$ at least because no representation of $f(t)$ is known in the form of a rapidly convergent series. By the classical methods we mean those used by Charles Hermite and Ferdinand Lindemann on the exponential function, which allow them to prove the transcendence of e and π . A generalization of that classical method is known as the Siegel–Shidlovskij method for proving transcendence and algebraic independence of values of E-functions [14].

(B) Note that even if, on the contrary to the conditions of our theorem, the number $f(t)$ is rational for all rational t satisfying the inequality $|t| < 1/10$, we will get a new contribution to the following classical problem: *Given an interval $I \subset \mathbb{R}$, does there exist a transcendental entire function $g : \mathbb{C} \rightarrow \mathbb{C}$ such that $g(x) \in \mathbb{Q}$ for all $x \in I \cap \mathbb{Q}$?* For the first time this problem (even in a more general setting) was solved positively by Karl Weierstrass in 1886. The reader may find the construction of Weierstrass and related historical notes in [11, pages 254–255]. The Weierstrass’s solution represents the desired transcendental entire function as a convergent series, while the formula (1), of course if it is proved that the number $f(t)$ is rational for all rational t satisfying the inequality $|t| < 1/10$, provides us with a closed form solution which is of independent interest [3].

(C) Note also that, generally speaking, we may expect that what follows will lead to a geometric proof of the rationality of the number $f(t)$, defined by the formula (1), for all rational t satisfying the inequality $|t| < 1/10$.

By definition, let T_0 be the set composed of a single simplex in the spherical 3-space so that all its dihedral angles are rational multiples of π . Suppose that, for some $n \geq 0$, the set T_n is already constructed. Then we say that a spherical simplex τ belongs to the set T_{n+1} if and only if there is a spherical simplex τ' in $\cup_{k=0}^n T_k$ such that τ and τ' share a common face and are mutually symmetric with respect to the 2-dimensional plane containing this face. By definition, let V_n be the set of the vertices of the simplices $\tau \in T_n$ and let W be the set of points $x \in \mathbb{S}^3$ such that for every $\tau \in \cup_{n=0}^{\infty} T_n$ either $x \notin \tau$ or x is an interior point of τ .

Observe that the following two statements (if proved) will provide us (among other things) with a geometric proof of the rationality of the number $f(t)$, defined by the formula (1), for all rational t satisfying the inequality $|t| < 1/10$:

(h₁) the set $\cup_{n=0}^{\infty} V_n$ is finite;

(h₂) for every $x, y \in W$, the cardinality of the set of all simplices $\tau \in \cup_{n=0}^{\infty} T_n$ so that $x \in \tau$ is equal to the cardinality of the set of all simplices $\tau \in \cup_{n=0}^{\infty} T_n$ so that $y \in \tau$.

Of course, the hypothetical properties (h₁) and (h₂) resemble the discreteness of the Coxeter groups [4], [6] and the definition of the brunched covering.

(D) After reading the first version of this paper, Professor Johan Dupont shared with me the following information that he got from Professor Ruth Kellerhals: it turns out that the Rational Simplex Problem actually goes back to a paper by Ludwig Schläfli [13, pp. 267–269], where he has calculated the volume of several spherical orthoschemes whose dihedral angles are rational multiples of π . Schläfli proved that the volume of each of those orthoschemes is a rational multiple of π^2 and conjectured that, for all other orthoschemes, whose dihedral angles are rational multiples of π , their volume is not a rational multiple of π^2 .

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