REMARKS ON EFIMOV'S THEOREM ABOUT DIFFERENTIAL TESTS OF HOMEOMORPHISM

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In [1] N. V. Efimov has proved the following remarkable theorems

THEOREM 1. *Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1-mapping, moreover for all $x \in \mathbb{R}^2$ the Jacobian $\det f'(x)$ is negative. Let, in addition, be a positive function $a = a(x) > 0$ and non-negative constants C_1, C_2 so that for all $x, y \in \mathbb{R}^2$ the inequality*

$$(1) \qquad |1/a(x) - 1/a(y)| \leqslant C_1 \; |x - y| + C_2$$

holds. Then if for all $x \in \mathbb{R}^2$ the inequality

$$(2) \qquad |\det \; f'(x)| \geqslant a(x) \; |\mathrm{rot} \; f(x)| + a^2(x)$$

is valid, then $f(\mathbb{R}^2)$ is a convex domain and f maps $\mathbb{R}^2$ onto $f(\mathbb{R}^2)$ homeomorphically (Here rot *$f(x)$ denotes, as usual, the rotor of the mapping f on the point $x = (x_1, x_2)$, i.e. rot $f(x) = \partial f_2/\partial x_1\,(x) - \partial df_1/\partial x_2\,(x)$).*

THEOREM 2. *Let the conditions of Theorem 1 be satisfied. If inequality* (2) *holds with some constant $a = \mathrm{const} > 0$, in particular if*

$$|\det \; f'(x)| \geqslant \mathrm{const} > 0, \quad |\mathrm{rot} \cdot f(x)| \leqslant \mathrm{const},$$

then $f(\mathbb{R}^2)$ is either the whole plane, or a half-plane, or an infinite band between two parallel straight lines.

The proofs of these theorems are very difficult. They are based on a geometric technique developed by Efimov for a generalization of Hilbert's theorem about the impossibility of isometric immersion of the Lobachevsky plane into $\mathbb{R}^3$ [2,3]. Evidently Efimov's results were not noticed by specialists in the theory of univalent functions, theorems on global inverse function and quasiconformal mappings. In any case I do not know any paper which deals with many-dimensional analog of Theorems 1,2 or which studies its nature from the functional — theoretical point of view. The main aim of the present paper is to fill in these gaps if only partially.

Efimov's theorems were developed in the articles [4,5], investigating the question: which of the possibilities mentioned in Theorem 2 is really achieved? For the linear map $f_1(x_1, x_2) = x_2$, $f_2(x_1, x_2) = x_1$ the domain f $(\mathbb{R}^2)$ is the whole plane. B. E. Kantor in [4] prooved that if

$$\det \; f'(x) \equiv \mathrm{const} < 0, \; \mathrm{rot} \; f(x) \equiv 0,$$

then $f(\mathbb{R}^2)$ cannot be a band. He gives also an example of a mapping for which the last conditions are valid and $f(\mathbb{R}^2)$ is a half-plane. S. P. Geisberg in [5] gives some generalizations of Kantor's result showing that $f(\mathbb{R}^2)$ cannot be a band if either

(a) rot $f(x) \equiv 0$, det $f'(x) \equiv -g^2(x)$, where g is convex and non--vanishing; or

(b) rot $f(x) \equiv 0$ and det $f'(x)$ is a polynomial which takes only negative vaules.

The main result of the present paper is the following theorem, which asserts that $f(\mathbb{R}^2) = \mathbb{R}^2$ and f is a diffeomorphism under conditions similar to the ones of Efimov's, Kantor's and Geisberg's theorems.

THEOREM 3. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1-mapping, here for all $x \in \mathbb{R}^2$* det $f'(x) \neq 0$. *Let in addition be a monotone function $L: [0, +\infty) \to (0, +\infty)$ such that*

$$\int^{+\infty} L(t)\,dt = +\infty \tag{3}$$

and for all $x \in \mathbb{R}^2$ the inequalities

$$|\det f'(x)| \geqslant L(|x|)\ |\mathrm{rot}\, f(x)| + L^2(|x|), \tag{4}$$

$$|\mathrm{tr}\ f'(x)| \leqslant L(|x|). \tag{5}$$

hold. Then f is a diffeomorphism of $\mathbb{R}^2$ onto itself (Here tr $f'(x)$ *denotes as usual the trace of the mapping $f'(x)$,* i.e. tr $f'(x) = \partial f_1/\partial x_1(x) + \partial f_2/\partial x_2(x)$).

Remark 1. It is assumed det $f'(x) > 0$ in Theorem 3.

Remark 2. If a mapping f satisfies the conditions of Theorem 1 then there exists a function L for which relations (3) (4) hold. (One can take L of the form

$$L(t) = (C_1 t + C_2 + 1/a(0,0))^{-1}).$$

Remark 3. Inequality (5) is additional with respect to the conditions of Theorem 1. In that sense it is similar to the additional conditions of Kantor's and Geisberg's theorem mentioned above, the most important of which is rot $f \equiv 0$.

The following lemma will permit us to deduce Theorem 3 from the Hadamard-Levy-John's global inverse function theorem [6].

LEMMA. *Let the conditions of Theorem 3 be fulfilled. Then for all $x \in \mathbb{R}^2$ the norm of the linear map $(f\circ J)'(x)^{-1}$, which is inverse to $(f\circ J)'(x)$, satisfy the inequality*

$$\|(f\circ J)'(x)^{-1}\| = \sup_{|y|=1} |(f\circ J)'(x)^{-1} y| \leqslant \sqrt{6}/L(|x|).$$

(Here $J : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear map given by the matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

at the canonical basis in $\mathbb{R}^2$).

Proof of the Lemma. Let $F = f \circ J$, $L = L(x)$. Then the inequality (4) is equivalent to

$$|\det F'(x)| \geqslant L\,|\operatorname{tr} F'(x)| + L^2. \tag{6}$$

Let us denote the eigenvalues of the linear map $F'(x)$ by m_1, m_2 and assume that $|m_1| \leqslant |m_2|$. Then (6) is equivalent to

$$|m_1 m_2| \geqslant L\,|m_1 + m_2| + L^2.$$

Hence $|m_1| \geqslant L$, and the spectral radius of $F'(x)^{-1}$ is no greater than $1/|m_1| \leqslant 1/L$.

According to Schur's theorem [7] let us choose an orthonormal basis in C^2 so that matrix $F'(x)^{-1}$ is triangular. Then its diagonal elements coincide with eigenvalues of $F'(x)^{-1}$. But we have estimated its absolute values. Therefore we must estimate only the unique non-zero element n standing outside the diagonal. Direct calculation using inequality (5) but not (4) gives $|n| \leqslant 2/L$. Hence

$$\|F'(x)^{-1}\| \leqslant (1/|m_1|^2 + 1/|m_2|^2 + |n|^2)^{1/2} \leqslant \sqrt{6}/L, \text{ Q.E.D.}$$

Let us remind the Hadamard-Levy-John's global inverse function theorem [6].

THEOREM 4. *Let B, b be Banach spaces, $f : B \to b$ a continuously differentiable map and $M : [0, +\infty) \to (0, +\infty)$ a non-decreasing function such that the following conditions hold*

1) *for all $X \in B$ the linear map $f'(X)$ has continuous inverse one;*

2) *for all $X \in B$ the norm $\|f'(X)^{-1}\|$ of $f'(X)^{-1}$ is less or equal to $M(|X|)$;*

3) $\displaystyle\int^{+\infty} \frac{dt}{M(t)} = +\infty.$

Then f is a diffeomorphism onto b.

Proof of Theorem 3. For all $x \in \mathbb{R}^2$ we have by lemma

$$\|(f \circ J)'(x)^{-1}\| \leqslant \sqrt{6}/L(|x|).$$

Hence if we define a function M by equality

$$M(t) = \sqrt{6}/L(t), \quad 0 \leqslant t < +\infty,$$

then the conditions of Theorem 4 is fulfild for the map $f \circ J$ and function M. Therefore $f \circ J$ is a diffeomorphism. Q.E.D.

Now let us discuss some perspectives of further investigations connected with Efimov's theorems, which arise from our proof of Theorem 3. First of all we have seen that the enigmatic conditions (1), (2) of Theorem 1 imply a restriction on a growth of the spectral radius of the mapping $f'(x)^{-1}$ as $x \to \infty$. Is it true that the last restriction (and not only (1), (2)) already imply the injectivity of f? This question induces an analogy between the Efimov's theorems and the Hadamard-Levy-John's global inverse function theorem where the restriction is imposed on the growth of a norm of the mapping $f'(x)^{-1}$ as $x \to \infty$ instead of its spectral radius. In this connection one can hope for obtaining many-dimensional analogs of Efimov's theorems.

For a better understanding I am formulating a simpler theorem which must be proved in the way under discussion.

Hypothetical theorem. Let $n \geqslant 2$ and $f: \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable map. Denote by $m_1 = m_1(x)$, $m_2 = m_2(x), \ldots$, $m_n = m_n(x)$ eigenvalues of the linear map $f(x)$ ordered with respect to absolute values $|m_1| \leqslant |m_2| \leqslant \ldots \leqslant |m_n|$. If there exists a constant K such that for all $x \in \mathbb{R}^n$ the inequalities

$$1/K \leqslant |m_1(x)| \leqslant |m_n(x)| \leqslant K$$

holds, then $f(\mathbb{R}^n)$ is a convex domain and f is injective.

Remark 4. If $n = 2$ and $\det f'(x) < 0$, then the hypothetical theorem easily follows from Theorem 2. In fact under above assumptions

$$\det (f \circ J)'(x) < 0,$$
$$|\det (f \circ J)'(x)| = |m_1 \cdot m_2| \geqslant K^{-2} = \text{const.} > 0,$$
$$|\operatorname{rot} (f \circ J)(x)| = |\operatorname{tr} f'(Jx)| = |m_2 + m_1| \leqslant 2K = \text{const.} < \infty,$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Later on it is clear that one can develop investigations on Efimov's theorems by analogy with the theory of quasiconformal and quasi-isometric mappings. One can consider here a class A of the mappings, dealing with our hypothetical theorem, as an analogy of the K-quasi-isometric mappings class [8]. It is naturally to consider a class B of maps defined by the inequality $|m_n(x)/m_1(x)| \leqslant K$ as an analogy for the class of K-quasiconformal maps. Is there an analogy for Zoric's theorem on global homeomorphism of quasiconformal maps for one of the classes A and B [9, 10]? Similarly, is there an analogy for the theorem on radius of injectivity [10] and stability of quasi-isometric and quasiconformal maps [11, 12]? Let us point out two circumstances that complicate the proof of analogies under consideration. Firstly, neither class A nor B is closed with respect to composition. Secondly, maps from classes A and B have not any unique metric property because one can determine eigenvalues independently on a norm in $\mathbb{R}^n$. Similar difficulties are overcome in proofs of the Gale-Nikaido-Inada's and Parthasarathy's global univalence theorems [13]. However our statements of the question differ essentially from the last theorems because classes A and B were defined independently on the choice of a coordinate system in $\mathbb{R}^n$.

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REFERENCES

1. N. V. Efimov, *Differential homeomorphism tests of certain mappings with an application in surface theory*. Mat. Sb. (N.S) **76**(**118**) (1968), 499—512.
2. N. V. Efimov, *Generation of singularities on surface of negative curvature (Russian)*. Mat. Sb. (N.S) **64**(**106**) (1964), 286—320.
3. T. Milnor Klotz, *Efimov's theorem about complete immessed surfaces of negative curvature.* Advances in Math. **8** (1972), 474—543.
4. B. E. Kantor, *On the question of the normal form of a complete surface of negative curvature (Russian)*. Mat. Sb. (N.S) **82**(**124**) (1970), 220—223.
5. S. P. Geisberg, *The properties of a normal map that is generated by the equation* $rt - s^2 = -f^2(x, y)$ *(Russian)*. Ibid, 224—232.
6. B. H. Pourciau, *Global invertibility of nonsmooth mappings*. J. Math. Anal. Appl. **131** (1988,) 170—179.
7. R. Bellman, *Introduction to matrix analysis*. Mc Graw-Hill, New-York—Toronto—London, 1960.
8. F. John, *On quasi-isometric mappings I*. Comm. Pure and Appl. Math. 21(1968), 77—110; II, Ibid **22** (1969), 265—278.
9. V. A. Zoric, *M. A. Lavrent' ev' s theorem on quasiconformal space maps (Russian)*. Mat. Sb. (N.S.) **74**(**116**) (1967), 417—433.
10. Yu. G. Reshetnyak, *Spatial mappings with bounded distortion* (Russian). Nauka Sibirsk. Otdel., Novosibirsk, 1982.
11. F. John, *Rotation and strain*. Comm. Pure and Appl. Math. **14** (1961), 391—413.
12. Yu. G. Reshetnyak, *Stability theorems in geometry and analysis (Russian)*. Nauka Sibirsk. Otdel., 1982.
13. T. Parthasarathy, *On global univalence theorems*. Springer-Verlag, Berlin, 1983.