REMARKS ON THE THEOREM OF M. AND S. RĂDULESCU ABOUT AN INITIAL VALUE PROBLEM FOR THE DIFFERENTIAL EQUATION $x^{(n)} = f(t, x)$

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We generalize the theorem of M. and S. Rădulescu about an initial value problem for the equation $x^{(n)} = f(t, x)$ in a Banach space. In contrast to M. and S. Rădulescu, we don't assume any differentiability of f, and we impose conditions for the growth of f instead of f'_x. Our proof is elementary and in the case $n = 1$ is based on A. Wintner's theorem about continuation of solutions of differential equations in finite dimensional Banach spaces.

In what follows $J = [a, b] \subset \mathbb{R}$ is a closed interval and E a Banach space with norm $|\cdot|$.

Using a global inverse function theorem [4], M. and S. Rădulescu proved the following theorem (for $n = 1$ in [7], for arbitrary n in [8]).

1. THEOREM. *Let $n \geqslant 1$ be a natural number, $f : J \times E \to E$ a continuous mapping, for which its partial derivative with respect to the second argument, denoted by f'_x, exists and is continuous. Suppose that there exist a constant $c > 0$ and a continuous increasing function $m : \mathbb{R}_+ \to (1, \infty)$, which satisfy the following conditions:*

(i)
$$\int_0^\infty \frac{dt}{m(t)} = \infty;$$

(ii)
$$\|f'_x(t, x)\| \leqslant c \ln^n m(|x|), \quad t \in J, \quad x \in E.$$

Then, for every $x_0, x_1, \ldots, x_{n-1} \in E$, the differential equation

$$x^{(n)}(t) = f(t, x(t)) \tag{1}$$

with initial conditions $x^{(i)}(a) = x_i$, $i = 0, 1, \ldots, n-1$, has a unique solution $x : J \to E$.

The main results of the present paper are the following two theorems:

2. THEOREM. *Let $n \geqslant 1$ be a natural number, $f : J \times E \to E$ a continuous mapping, satisfying the following conditions:*

(i) *for every bounded set $B \subset E$ there exists a constant $L(B)$ such that*

$$|f(t, x) - f(t, y)| \leqslant L(B)\, |x - y|, \quad t \in J, \quad x, y \in B;$$

(ii) *there exists a continuous non-decreasing function $M : [0, \infty) \to (0, \infty)$ for which*

$$\int_1^\infty \frac{dr}{\sqrt[n]{r^{n-1} M(r)}} = \infty \tag{2}$$

a

$$|f(t, x)| \leqslant M(|x|), \quad x \in E, \ t \in J.$$

Then, for every $x_0, x_1, \ldots, x_{n-1} \in E$, *the differential equation* (1) *with initial conditions* $x^{(i)}(a) = x_i$, $i = 0,1,\ldots, n-1$, *has a unique solution* $x : J \to E$.

3. THEOREM. *Let* E *be finite dimensional and* f *satisfies all the conditions of Theorem* 2 *except for* (i). *Then for every* $c \in J$, $c > a$, *every solution* $x : [a, c] \to E$ *of equation* (1) *is extendable to the solution* $X : J \to E$ *of equation* (1).

4. *Remark*. Theorem 1 follows from Theorem 2. In fact, condition (ii) of Theorem 1 implies that the derivative f'_x is bounded on $J \times B$ for every bounded set $B \subset E$, hence, the condition (i) of Theorem 2 is fulfilled. Let's verify condition (ii) of Theorem 2. From condition (ii) of Theorem 1 we have

$$|f(t, x)| \leqslant |f(t, 0)| + \sup_{|y| \leqslant |x|} \|f'_y(t, y)\| \, |x| \leqslant$$
$$\leqslant r_0 + c \ln^n m(|x|)|x|,$$

where $r_0 = \max\limits_{t \in J} |f(t, 0)|$. Therefore,

$$|f(t, x)| \leqslant M(|x|), \quad x \in E, \ t \in J,$$

where $M(r) = r_0 + cr \ln^n m(r)$. The validity of (2) for this function follows from the relations

$$\int_1^\infty \frac{dr}{\sqrt[n]{r^{n-1}(r_0 + cr \ln^n m(r))}} \geqslant \int_1^\infty \frac{dr}{cr \ln m(r)} = \infty.$$

which hold for large c. The last integral equals infinity because of the following assertion.

5. LEMMA. *Let* $m : \mathbb{R}_+ \to (1, \infty)$ *be a non-decreasing function and*

$$\int_0^\infty \frac{dr}{m(r)} = \infty.$$

Then

$$\int_1^\infty \frac{dr}{r \ln m(r)} = \infty.$$

Proof. It is sufficient to consider just the case where m is unbounded. There exists a sequence $0 < r_1 < r_2 < \ldots$ such that $m(r_1) > 1$, $m(r_j) \leqslant r_j^2$ and $\ln r_j > j \ln r_{j-1}$, $j \geqslant 2$.
Hence

$$\int_{r_1}^{\infty} \frac{\mathrm{d}r}{r \ln m(r)} = \sum_{j=2}^{\infty} \int_{r_{j-1}}^{r_j} \frac{\mathrm{d}r}{r \ln m(r)} \geqslant \sum_{j=2}^{\infty} \int_{r_{j-1}}^{r_j} \frac{\mathrm{d}r}{r \ln m(r_j)} \geqslant$$

$$\geqslant \sum_{j=2}^{\infty} \int_{r_{j-1}}^{r_j} \frac{\mathrm{d}r}{2r \ln r_j} = \frac{1}{2} \sum_{j=2}^{\infty} \left(1 - \frac{\ln r_{j-1}}{\ln r_j}\right) \geqslant \frac{1}{2} \sum_{j=2}^{\infty} \left(1 - \frac{1}{j}\right) = \infty.$$

The lemma is proved.

6. *Remark*. From the proof of Theorem 2 it will be seen that condition (i) is of technical nature and may be substituted by any con ition guaranteeing the existence of local solutions of equation (1). At the same time, we cannot exclude such condition completely because it is known that if for every continuous function $f : J \times E \to E$ and every $x_0 \in E$ the equation $x' = f(t, x)$ with an initial condition $x(a) = x_0$ has a solution, defined in some neighborhood of a, then E is finite dimensional [2].

Theorems 2 and 3 can be derived by standard methods from the following statements.

7. Lemma. *Let $M : [0, \infty) \to (0, \infty)$ be a continuous non-decreasing function for which equality* (2) *is fulfilled. Then, for every $J = [a, b$, there exists a constant $C = C(J, M)$, depending only on J and M, such that for every $c \in J$, $c > a$, and every C^n function $g : J' = [a, c) \to \mathbb{R}_+$ satisfying the inequalities*

$$g^{(j)}(t) \geqslant 0,\ j = 0,1, \ldots, n,\ t \in J';$$

$$g^{(j)}(a) = 0,\ j = 0,1, \ldots, n-1;$$

$$g^{(n)}(t) \leqslant M(g(t)),\ t \in J',$$

the estimate

$$|g^{(n-1)}(t)| \leqslant C,\ t \in J',$$

holds.

Proof. For $j = 1,2, \ldots, n-2$ and $t \in J'$ we have

$$[g^{(j)}(t)]^2 = 2 \int_a^t g^{(j)}(s) g^{(j+1)}(s) \mathrm{d}s \leqslant$$

(3)

$$\leqslant 2 g^{(j+1)}(t) \int_a^t g^{(j)}(s) \mathrm{d}s = 2 g^{(j+1)}(t) g^{(j-1)}(t).$$

Similarly,

$$[g^{(n-1)}(t)]^2 = 2\int_a^t g^{(n-1)}(s)g^{(n)}(s)\mathrm{d}s \leqslant$$

(4)

$$\leqslant 2\int_a^t g^{(n-1)}(s)M(g(s))\mathrm{d}s \leqslant 2M(g(t))g^{(n-2)}(t).$$

Let us show that (3) implies

(5) $$g^{(j-1)}(t) \leqslant 2^{a_j}[g(t)]^{1/j}[g^{(j)}(t)]^{1-1/j}$$

for $j = 2,3,\ldots, n-1$, $t \in J'$, with some constants a_j. Indeed, (5) is valid for $j = 2$ and by induction we have

$$[g^{(j)}(t)]^2 \leqslant 2g^{(j+1)}(t)g^{(j-1)}(t) \leqslant 2^{1+a_j}g^{(j+1)}(t)[g(t)]^{1/j}[g^{(j)}(t)]^{1-1/j}$$

or

$$g^{(j)}(t) \leqslant 2^{(1+a_j)/(1+1/j)}[g(t)]^{1/j+1}[g^{(j+1)}(t)]^{1-1/j+1}.$$

This proves (5).

Using (4) and (5) with $j = n-1$ we obtain

$$[g^{(n-1)}(t)]^2 \leqslant 2^{1+a_{n-1}}M(g(t))[g(t)]^{1/n-1}[g^{(n-1)}(t)]^{1-1/n-1}$$

or

$$g^{(n-1)}(t) \leqslant 2^{a_n}[g(t)]^{1/n}[M(g(t))]^{1-1/n}.$$

Let us introduce the function $M_0(r) = 2^{a_n} r^{1/n}[M(r)]^{1-1/n}$. Then

$$g^{(n-1)}(t) \leqslant M_0(g(t))$$

and

$$\int_1^\infty \frac{\mathrm{d}r}{\sqrt[n-1]{r^{n-2}M_0(r)}} \geqslant \int_1^\infty \frac{\mathrm{d}r}{2^{a_n/(n-1)}\sqrt[n]{r^{n-1}M(r)}} = \infty.$$

Finally, the conclusion of Lemma 7 is obtained by induction on n.

8. ***Remark.*** The following is known as A. Wintner's theorem.

9. THEOREM [3, 9]. *Let E be finite dimensional and let $f : J \times E \to E$ be a continuous mapping such that*

$$|f(t, x)| \leqslant M(|x|),\ t \in J,\ x \in E,$$

where $M : [0, \infty) \to (0, \infty)$ *is a continuous function for which*

$$\int_1^{\infty} \frac{\mathrm{d}r}{M(r)} = \infty .$$

Then for every $x_0 \in E$ *the initial value problem*

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t, x(t)), \quad x(a) = x_0,$$

has a solution $x : J \to E$.

Theorem 2 for $n = 1$ can be considered as an analog of Theorem 9 in infinite dimensional spaces. Consequently we may consider Theorems 2 and 3 for arbitrary n as an analog of Theorem 9 for equation (1).

10. LEMMA. *Let* $M : [0, \infty) \to (0, \infty)$ *be a continuous non-decreasing function and assume that* (2) *holds. Then, for every* $J = [a, b]$, *there exists a constant* $C = C(J, M)$, *depending only on* J *and* M, *so that for every* $c \in J$, $c > a$, *and every solution* $x : J' = [a, c) \to E$ *of the initial value problem*

$$x^{(n)}(t) = f(t, x(t)), \quad t \in J',$$

$$x^{(j)}(a) = 0, \; j = 0, 1, \ldots, n-1,$$

with the mapping $f : J \times E \to E$ *satisfying*

$$|f(t, x)| \leqslant M(|x|), \quad t \in J, \quad x \in E,$$

the estimate $|x^{(n-1)}(t)| \leqslant C$, $t \in J'$, *holds*.

Proof. Since

$$x(t) = \int_a^t \int_a^{t_1} \cdots \int_a^{t_{n-2}} x^{(n-1)}(t_{n-1})\, \mathrm{d}t_{n-1} \ldots \mathrm{d}t_2 \mathrm{d}t_1,$$

then

$$|x(t)| \leqslant \int_a^t \int_a^{t_1} \cdots \int_a^{t_{n-2}} |x^{(n-1)}(t_{n-1})|\, \mathrm{d}t_{n-1} \ldots \mathrm{d}t_2 \mathrm{d}t_1. \tag{6}$$

Put

$$D_R^+ z(t) = \limsup_{s \to t+0} \frac{z(s) - z(t)}{s - t} .$$

One can show that

$$D_R^+(|x^{(n-1)}(t)|^2) \leqslant 2\,|x^{(n-1)}(t)|\,|x^{(n)}(t)|.$$

Hence, from the comparison principle,

$$|x^{(n-1)}(t)|^2 \leqslant 2\int_a^t |x^{(n-1)}(s)|\,|x^{(n)}(s)|\mathrm{d}s \leqslant$$

(7)

$$\leqslant 2\int_a^t |x^{(n-1)}(s)|\,M(|x(s)|)\,\mathrm{d}s.$$

Denoting $y(t) = |x^{(n-1)}(t)|$, from (6) and (7) we obtain

$$y^2(t) \leqslant 2\int_a^t y(s)\,M\left(\int_a^s\int_a^{t_1}\cdots\int_a^{t_{n-2}} y(t_{n-1})\mathrm{d}t_{n-1}\,\ldots\,\mathrm{d}t_2\mathrm{d}t_1\right)\mathrm{d}s.$$

Let $V^2(t)$, $V(t) \geqslant 0$, denote the right hand side of the last inequality. Evidently $y(t) \leqslant V(t)$. By differentiation of the equation

$$V^2(t) = 2\int_a^t y(s)\,M\left(\int_a^s\int_a^{t_1}\cdots\int_a^{t_{n-2}} y(t_{n-1})\mathrm{d}t_{n-1}\,\ldots\,\mathrm{d}t_2\mathrm{d}t_1\right)\mathrm{d}s$$

we obtain

$$2\,V(t)\,V'(t) = 2\,y(t)\,M\left(\int_a^t\int_a^{t_1}\cdots\int_a^{t_{n-2}} y(t_{n-1})\,\mathrm{d}t_{n-1}\,\ldots\,\mathrm{d}t_2\mathrm{d}t_1\right) \leqslant$$

$$\leqslant 2\,V(t)\,M\left(\int_a^t\int_a^{t_1}\cdots\int_a^{t_{n-2}} V(t_{n-1})\mathrm{d}t_{n-1}\,\ldots\,\mathrm{d}t_2\mathrm{d}t_1\right).$$

It follows that

(8) $$V'(t) \leqslant M\left(\int_a^t\int_a^{t_1}\cdots\int_a^{t_{n-2}} V(t_{n-1})\mathrm{d}t_{n-1}\,\ldots\,\mathrm{d}t_2\mathrm{d}t_1\right).$$

Introducing the notation

$$W(t)=\int_a^t\int_a^{t_1}\cdots\int_a^{t_{n-2}} V(t_{n-1})\mathrm{d}t_{n-1}\ \ldots\ \mathrm{d}t_2\mathrm{d}t_1,$$

we may rewrite (8) in the form

$$W^{(n)}(t)\leqslant M(W(t)).$$

As far as

$$W^{(j)}(t)\geqslant 0,\ t\in J',\ j=0,1,\ldots,n,$$

$$W^{(j)}(a)=0,\ j=0,1,\ldots,n-1,$$

Lemma 7 yields the constant $C=C(J,M)$ for which

$$W^{(n-1)}(t)\leqslant C,\qquad t\in J'.$$

Taking into account that $|x^{(n-1)}(t)|=y(t)\leqslant V(t)=W^{(n-1)}(t)$ we conclude the proof of the lemma.

To prove Theorem 2, we need the following Picard theorem, that we formulate in a form suitable to our purposes.

11. THEOREM [1]. *Let* $g: J\times E\to E$ *be a continuous mapping satisfying condition* (i) *of Theorem* 2. *Let* $x_0\in E$, r, $K\in\mathbb{R}_+$ such that

$$|g(t,x)|\leqslant K,\ t\in J,\ |x-x_0|\leqslant r.$$

Then the equation $x'(t)=g(t,x(t))$ *with an initial condition* $x(t_0)=x_0$, $t_0\in J$, *has a unique solution defined on the interval* $J\cap[t_0-\alpha, t_0+\alpha]$, *for some* $\alpha>0$.

12. *Proof of Theorem* 2. Performing the change of function

$$y(t)=x(t)-\sum_{j=0}^{n-1}x_j(t-a)^j/_j!,$$

we get an analogous problem for y, but with vanishing initial conditions. Consequently, it is sufficient to prove Theorem 2 under the assumptions $x_j=0$, $j=0,1,\ldots,n-1$.

Equation (1) is equivalent to the system

$$(9)\qquad \begin{cases} z_1'(t)=z_2(t),\\ z_2'(t)=z_3(t),\\ \ldots\\ z_{n-1}'(t)=z_n(t),\\ z_n'(t)=f(t,z_1(t))\end{cases}\qquad \text{or}\quad \frac{\mathrm{d}}{\mathrm{d}t}Z(t)=F(t,Z(t)),$$

where $z_1(t) = x(t)$, $Z = (z_1, z_2, \ldots, z_n): J \to E^n$, and the mapping $F: J \times E^n \to E^n$ is defined as

$$F(t, Z) = (z_2, z_3, \ldots, z_{n-1}, z_n, f(t, z_1)).$$

It is obvious that F satisfies the conditions of Theorem 11. By this theorem, equation (9) with vanishing initial conditions has a solution $Z^*(t) = (z_1^*(t), z_2^*(t), \ldots, z_n^*(t))$ defined in some interval $J' = [a, c)$, $c > a$. Theorem 2 will be proved if for every $c > a$ we shall be able to extend this solution to some interval $[a, d)$ for $d > c$. Lemma 10 implies that the mapping $z_n^*(t) = x^{*(n-1)}(t)$ is bounded in J'. Consequently, the mapping $Z^*(t)$ is bounded in J' by some constant L. Therefore there exists the limit $Z^*(c) = \lim_{t \uparrow c} Z^*(t)$. According to Theorem 11, the initial value problem for system (9) with the condition $Z(t_0) = Z^*(t_0)$ has a solution $Z^{**}(t)$ in the interval $J \cap [c, d]$ for some $d > c$. Thus Theorem 2 is proved.

13. The proof of Theorem 3 is based on the fact that in a finite dimensional space, a solution of equation (1) must go to infinity on the border of the maximal existence interval [3]. But this contradicts Lemma 10.

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REFERENCES

1. H. Cartan, *Calcul différentiel. Formes différentielles.* Hermann, Paris, 1967.
2. A. N. Godunov, *On Peano's theorem in Banach spaces.* Funkcional. Anal. i Prilozen. 9(1975), vyp. 1, 59—60. (Russian; English translation: Functional Anal. Appl. 9(1975), no. 1, 53—55).
3. P. Hartman, *Ordinary Differential Equations.* Wiley, New York, 1964.
4. F. John, *On quasi-isometric mappings,* 1. Comm. Pure Appl. Math. 21(1968), 77—110.
5. M. A. Krasnosel'skii, *The Operator of Translation Along the Trajectories of Differential Equations.* Nauka, Moskow, 1966. (Russian; English translation: Translations of Mathematical Monographs, Vol. 19, Amer. Math. Soc., Providence, R.I., 1968).
6. M. A. Krasnosel'skii and S. G. Krein, *On the theory of ordinary differential equations in Banach spaces.* Voronez. Gos. Univ. Trudy Sem. Funkcional. Anal. 2 (1956), 3—23 (Russian).
7. M. Rădulescu and S. Rădulescu, *Global inversion theorems and applications to differential equations.* Nonlinear Anal. 4(1980), 951—965.
8. M. Rădulescu and S. Rădulescu, *An application of Hadamard-Levy's theorem to an initial value problem for the differential equation $x^{(n)} = f(t, x)$.* Rev. Roumaine Math. Pures Appl. 33(1988), 695—700.
89. A. Wntner, *The non-local existence problem of ordinary differential equations.* Amer. J. Math. 67(1945), 277—284.