Blaschke Addition and Convex Polyhedra
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Abstract. This is an extended version of a talk on October 4, 2004 at the research seminar "Differential geometry and applications" (headed by Academician A. T. Fomenko) at Moscow State University. The paper contains an overview of available (but far from well-known) results about the Blaschke addition of convex bodies, some new theorems on the monotonicity of the volume of convex bodies (in particular, convex polyhedra with parallel faces) as well as description of a software for visualization of polyhedra with prescribed outward normals and face areas.

1. The vector area of the surface of a polyhedron. In this article we take the following fact as well-known: *Let P be a compact convex polyhedron in $\mathbb{R}^m$ ($m \geq 2$), $n_1, \ldots, n_k$ be unit vectors of outward normals to its $(m-1)$-dimensional faces and let $F_1, \ldots, F_k$ be $(m-1)$-dimensional volumes of its $(m-1)$-dimensional faces. Then*

$$\sum_{j=1}^{k} F_j n_j = 0. \tag{1}$$

Sometimes this fact is stated as follows: "The vector area of the surface of a convex polyhedron equals zero" (see, for example, [2]). From a hydrodynamical point of view this means that a polyhedron submersed in a fluid under only the action of the forces of the fluid pressure remains in equilibrium. Convexity of the polyhedron is of course a redundant requirement. Item 1 is a direct consequence of the Stokes theorem. Its "smooth" analog says that the integral of the unit outward normal over a closed surface equals zero. This t is given, for example, in a celebrated calculus problem book by B. P. Demidovich (problem 4381). We do not prove Item 1 since it is obvious, but we give an analog of it for a domain on a sphere [3]:

Let $\mathbb{S}^2$ be a unit sphere in $\mathbb{R}^3$ and let D be a domain in $\mathbb{S}^2$ with piecewise smooth boundary. Let $\hat{N}$ stand for a vector field on the sphere $\mathbb{S}^2$ that maps each point of the sphere to a unit outward normal with base this point of the sphere . Let $\hat{n}$ denote a vector field on the smooth part of the boundary ∂D of D that maps each point to a unit outward (with respect to D) vector which is normal to ∂D and parallel to the plane tangent to the sphere. Let σ denote the standard measure on $\mathbb{S}^2$ and s denote the arc length on ∂D. Then

$$2 \iint_D \hat{N} \, d\sigma + \int_{\partial D} \hat{n} \, ds = 0. \tag{2}$$

It is natural to think that the first integral in (2) gives the barycenter of the domain D on the sphere. It would be interesting to find, by using (2), the "geographical centers" of domains on the Earth whose curvature cannot be neglected (for example, Russia).

Another basic point of this survey is the following celebrated theorem of H. Minkowski [2].

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2. Theorem (Minkowski). *Let $m \geq 2$, $F_1, \ldots, F_k$ be positive real, $n_1, \ldots, n_k$ be unit vectors in $\mathbb{R}^m$ not lying in a single hyperplane such that $\sum_{j=1}^{k} F_j n_j = 0$. Then there exists a compact convex polyhedron P in $\mathbb{R}^m$ such that the vectors $n_1, \ldots, n_k$ (and only they) are the outward normals to $(m-1)$-dimensional faces of P and $F_1, \ldots, F_k$ are the $(m-1)$-dimensional volumes of the $(m-1)$-dimensional faces of P. Moreover, such a polyhedron P is unique up to translation.*

In [4], this theorem is abstracted to the class of nonconvex polyhedra in 3-dimensional Euclidean space which are called herissons. The definition of herisson in [4] is rather complicated and requires injectivity of the spherical map of a nonconvex polyhedron. Referring to a recent talk of N. P. Dolbilin, A. T. Fomenko suggested to significantly simplify the definition of herisson as follows. A herisson is defined to be a set of vectors not lying in a single hyperplane and equipped with some numbers meeting (1).

We can define addition on the set of such herissons in a natural way. The sum of two herissons $\{n_1, \ldots, n_k; F_1, \ldots, F_k\}$ and $\{n'_1, \ldots, n'_l; F'_1, \ldots, F'_l\}$ is defined to be the set of vectors $\mathcal{N} = \{n_1, \ldots, n_k\} \cup \{n'_1, \ldots, n'_l\}$ together with the positive numbers determined as follows:

• if a vector $n \in \mathcal{N}$ belongs to only one of the sets $\{n_1, \ldots, n_k\}$ or $\{n'_1, \ldots, n'_l\}$ and equals, say, n_j, then assign the number F_j to the vector n;

• if $n \in \mathcal{N}$ belongs to both $\{n_1, \ldots, n_k\}$ and $\{n'_1, \ldots, n'_l\}$ and equals, say, n_j and n'_p, then assign the number $F_j + F'_p$ to the vector n.

It is clear that the set of vectors $\mathcal{N}$ does not lie in a hyperplane and the condition (1) holds, that is the sum of herissons is a herisson.

By the fact 1, for each compact convex polyhedron, there exists a herisson whose vectors are the unit outward normals to the faces of the highest dimension equipped with "areas" of the faces. Two polyhedra give rise to two herissons. The sum of these herissons is a herisson, which corresponds to a compact convex polyhedron by the Minkowski theorem. Such a polyhedron is determined up to translation. So we can define a new addition on the set of classes of translates of convex polyhedra. Examples below show that this operation is differed from the Minkowski sum of polyhedra.

Considering the importance of the Minkowski sum of convex bodies, it is natural to study the properties of this new operation. In order to complete the picture, let us first recall some basic facts about the Minkowski sum.

3. Minkowski sum of convex bodies [2, 6, 23, 24, 31]. The set $K + L = \{z \in \mathbb{R}^m | z = x + y, \ x \in K, \ y \in L\}$ is called the Minkowski (or vector) sum of nonempty convex compact sets $K, L \subset \mathbb{R}^m$.

Another definition of Minkowski sum uses the support function of a convex set K which is the function h_K defined on $\mathbb{R}^m$ by the formula

$$h_K(x) = \sup_{y \in K}(x, y),$$

where (x, y) stands for the usual inner product of vectors x and y in $\mathbb{R}^m$.

The support function of a convex compact set is a positive homogeneous convex function. Moreover, there is a one-to-one correspondence between positive homogeneous convex functions and convex compact sets in $\mathbb{R}^m$. Let us give another definition of the Minkowski sum (clearly equivalent to the previous) which is based on the bijectivity of this correspondence: the Minkowski (or vector) sum of nonempty convex compact sets K and L from $\mathbb{R}^m$ is the convex set $K + L$

whose support function h_{K+L} is the sum of the support functions of K and L; i.e., $h_{K+L} = h_K + h_L$.

It is not difficult to show that the sum of some translates (i.e., images of a convex body under some translation of the space) K and L is a translate of $K + L$. Therefore, we can speak correctly about the Minkowski sum of translates of convex bodies. Allowing some ambiguity, we will sometimes identify the class of translates of a convex body with the body itself.

The notion of the vector sum of convex bodies was introduced by Minkowski in study of some questions about the isoperimetric inequality [26]. Since then, this notion was well studied and found a wide application to various areas of mathematics; for example, see [15] and [31]. The most known result using the Minkowski sum is the Brunn–Minkowski inequality

$$(\text{Vol } (K + L))^{1/m} \geq (\text{Vol } K)^{1/m} + (\text{Vol } L)^{1/m}, \tag{3}$$

which holds for all convex compact sets $K, L \subset \mathbb{R}^m$.

4. Blaschke sum of convex bodies. Let us come back to the addition of convex polyhedra which was described in Section 2. The terminology is questionable to some degree in fact. For example, B. Grünbaum says [18, p. 339]: "the process we have called the Blaschke addition was first described by Blaschke [5] for smooth convex sets, although, even earlier, the corresponding addition of polytopes occurs implicitly in the work of Minkowski [25]."

The Blaschke sum is defined for arbitrary convex bodies in $\mathbb{R}^m$; i.e., compact convex sets with nonempty interior. (In fact, we may omit the claim of nonempty interior, but this leads to some technical difficulties that we are reluctant to discuss in this paper.) To this end ,we need the following definition: the surface area measure $S(K, \cdot)$ of a convex body K in $\mathbb{R}^m$ is the additive set function on the unit sphere $\mathbb{S}^{m-1}$ which associates to each subset $U \subset \mathbb{S}^{m-1}$ the (surface) measure of the set of the points $x \in \partial K$ admitting a unit outward normal to ∂K such that if we shift this vector to make its base coincide with the center of $\mathbb{S}^{m-1}$ then its endpoint will belong to U.

Let us give some standard examples of surface area measures.

(4.1) The surface area measure of a sphere of radius r in $\mathbb{R}^m$ is proportional to the surface area measure of the unit sphere $\mathbb{S}^{m-1}$ with rate r^{m-1}.

(4.2) The surface area measure of a polyhedron $\mathbb{R}^m$ is concentrated in the normals to the faces. Its value at the "endpoint" $x \in \mathbb{S}^{m-1}$ of such a normal equals the $(m-1)$-dimensional volume of the face.

(4.3) The surface area measure of a C^2-smooth convex body is given by the identity

$$S(K, U) = \int_U \frac{d\sigma}{\varkappa},$$

where $\varkappa$ is the Gaussian curvature of the surface ∂K and σ is the standard measure on the sphere $\mathbb{S}^{m-1}$.

The notion of surface area measure of a convex body was introduced in 1938 by A. D. Alexandrov [1] as well as by W. Fenchel and B. Jessen [11] independently. This notion is a typical ingredient of convex analysis and allows one to work uniformly with smooth and nonsmooth convex bodies.

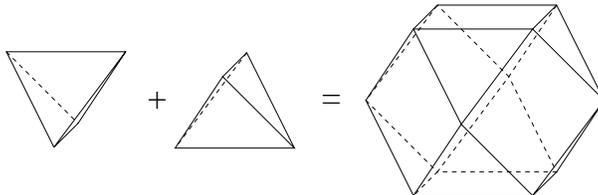We we are ready now to give the central definition of this article.

FIGURE 1. The Minkowski sum of two tetrahedra

The *Blaschke sum* (of the classes of translates) of two convex bodies K and L in $\mathbb{R}^m$ is the (class of translates of the) convex body $K\#L$ whose surface area measure is the sum of surface area measures of the summands: $S(K\#L, \cdot) = S(K, \cdot) + S(L, \cdot)$.

Correctness of the definition of Blaschke sum rests on the following form of the Minkowski theorem on unique existence of a convex body with a given surface area measure which was first proved by A. D. Alexandrov [1]:

4.4. Theorem. *Let μ be an additive set function on the unit sphere $\mathbb{S}^{m-1} \subset \mathbb{R}^m$ such that*

$$\int_{\mathbb{S}^{m-1}} x \, d\mu = 0$$

(here x denotes the radius-vector of a point on $\mathbb{S}^{m-1}$) and there is no hyperplane Π such that μ is concentrated in $\Pi \cap \mathbb{S}^{m-1}$. Then there exists a convex body $K \subset \mathbb{R}^m$ whose surface area measure coincides with μ; i.e,, $S(K, \cdot) = \mu(\cdot)$. Moreover, this body K is unique up to translation.

Theorem 2 is a particular case of Theorem 4.4.

The Blaschke sum of polyhedra coincides with the addition of herissons as described in Section 2.

5. Properties of the Blaschke sum. Although an analogy between the definitions of Minkowski and Blaschke sums is obvious, the Blaschke sum has been studied to a much lesser degree. Let us discuss the properties of the Blaschke sum which we managed to find in the literature.

(5.1) *On the Euclidean plane $\mathbb{R}^2$, the Blaschke sum of two convex polygons coincides with their Minkowski sum: $K\#L = K + L$.*

This can be proved for instance by inspecting a detailed construction of the Minkowski sum of convex polygons in [24].

(5.2) *In dimensions greater than or equal to 3, the Blaschke sum of convex polyhedra does not coincide necessarily with their Minkowski sum.*

Skipping details, we give the following picture from [2] as an example exhibiting the Minkowski sum of two regular tetrahedra one of which is obtained from the other by rotation through $90°$ around the vertical axis. From Figure 1 it is clear that the Minkowski sum of these tetrahedra is a polyhedron with 14 faces, while their Blaschke sum is a polyhedron with 8 faces by definition. The difference transpires.

(5.3) *For all convex bodies K, L in $\mathbb{R}^m$, the Kneser–Süss inequality holds:*

$$(\mathit{Vol}\ (K\#L))^{1-1/m} \geq (\mathit{Vol}\ K)^{1-1/m} + (\mathit{Vol}\ L)^{1-1/m},$$

moreover, equality takes place if and only if K and L are homothetic.

A proof can be found in [31, Theorem 7.1.3] and [19].

(5.4) *Every compact convex polyhedron with nonempty interior in $\mathbb{R}^m$ can be expressed as the Blaschke sum of finitely many simplices in $\mathbb{R}^m$. Furthermore, these*

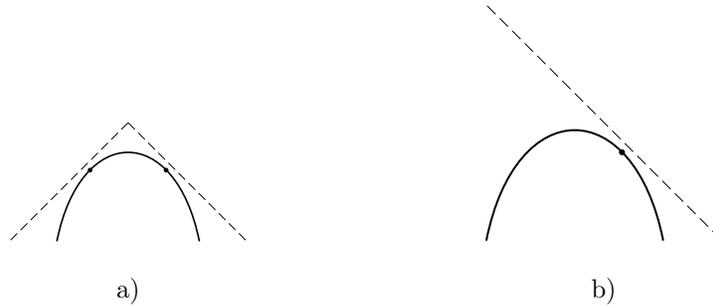a)   b)

FIGURE 2. Maximization of a concave functional

simplices can be chosen so that the sum contains at most $f - m$ summands, where f is the number of faces of the polyhedron.

A proof can be found in [18, pp. 334–335] and [13].

The Blaschke addition is not adequately explored. This is confirmed for example by the fact that it is still unknown whether we can generate functionals analogous to the mixed volumes generated by the Minkowski sum by using the Blaschke sum and prove inequalities for these functionals by analogy to the Minkowski and Alexandrov–Fenchel inequalities for mixed volumes.

6. The Blaschke sum and extremal problems. It is known that the classical isoperimetric problem consists in to finding a body of maximal volume among all convex sets in $\mathbb{R}^m$ of a given surface area. In order to apply the tools of analysis, the problem must be parametrized for example by identifying convex sets with their support functions. This leads to the problem of finding a maximum of a concave functional, namely, the volume of the set, in a vector space of continuous functions on the sphere, namely, support functions, subject to a constraint, namely, a fixed surface area. Difficulties in the proof of existence and uniqueness of a solution of the classical isoperimetric problem appear because both the constraint (surface area) and the objective functional (volume) are concave (see Figure 2 a)). Concavity of both functionals is guaranteed exactly by the Brunn–Minkowski inequality (3) (there is a corresponding analog for the surface area).

These difficulties naturally disappear if the problem is parametrized by identifying convex bodies with their surface area measures. Then the constraint (that the surface area be constant) turns into a hyperplane in the vector space of all surface area measures and the objective functional (volume) remains concave which is guaranteed by the Kneser–Süss inequality (5.3). In this situation, we can use the conventional methods of the theory of convex programming so that both existence and uniqueness of a solution is obtained "for free" (see Figure 2 b)) and a solution can be found from the Euler equation which says that in the point of extremum the tangent plane to the objective functional is parallel to the tangent plane to the constraint.

In [20], it was studied why the established modern theory of extremal problems helps little in solving isoperimetric problems. In that paper, a functional-analytical view of isoperimetric problems was developed, in particular, it was suggested to subdivide the problems into those that are "strategically right" to be parametrized by support functions (then the Minkowski sum plays a key rôle), by surface area measures (then the Blaschke sum is of fundamental importance), and those that

do not admit any parametrization that leads to convex extremal problems. Let us explain this by examples of some problems.

(6.1) *The Urysohn problem*: Find a body of maximal volume among convex bodies in $\mathbb{R}^m$ of diameter at most 1.

From a functional-analytical point of view, it is a question of maximization of a concave functional, namely, the volume, on a convex set in the vector space of support functions (indeed, the fact that the width of a convex body K in direction $n \in \mathbb{R}^m$ is at most 1 amounts to the linear inequality $h_K(n) - h_K(-n) \leq 1$). By a general argument, the problem has a unique solution that can be found from the Euler equation. A solution to the problem was found by P. S. Urysohn in 1924 [33]. He proved that the sought body is exactly a ball.

(6.2) *Inner isoperimetric problem*: Find a convex body of maximal volume among convex bodies in $\mathbb{R}^m$ of a prescribed surface area which lie in a given convex body T.

From a functional-analytical point of view, complexity of this problem consists in the fact that the set of convex surfaces lying a given convex body is convex with respect to the Minkowski addition (but not with respect to the Blaschke addition), while the surface area is linear with respect to the Blaschke addition (and with respect to the Minkowski addition, it is just concave in much the same way as volume). Such a "mixture of styles" when one part of conditions behaves well with respect to the Minkowski sum and another part of conditions behaves well with respect to the Blaschke sum essentially complicates the problem and results in the fact that the inner isoperimetric problem is less studied even in the case when T is a tetrahedron. From the necessary extremum condition (the Euler equation), it is clear that the boundary of the extremal body consists of flat pieces (where the body contacts the boundary of the tetrahedron) and pieces of surfaces of constant mean curvature (where its surface is inside the tetrahedron). In 1994, A. V. Pogorelov showed that for every tetrahedron T, not necessarily regular, and every positive number $H > 1/R$, where R is the radius of the ball inscribed in the tetrahedron T, there exists a smooth closed convex surface inside T that contacts with all of its faces such that the part of the surface contained strictly within T has a constant mean curvature H [30]. This surface can be described in the following way: take the Blaschke sum of the tetrahedron T with a ball of a special radius and then take the Minkowski sum of the Blaschke addition result with another ball [30]. However, a surface constructed in this way satisfies only the necessary extremum condition. It is still unclear whether it is indeed a solution to the inner isoperimetric problem and if the problem has a unique solution. There are no other significant advances in the solving of the inner isoperimetric problem.

The Blaschke sum was systematically used by S. S. Kutateladze [20, 21, 22] for solving isoperimetric problems with many subsidiary constraints.

We mention also that various generalizations of the Blaschke sum of convex compacts were considered by W. Firey [12, 14] and other authors [9, 10, 32].

7. New theorems on monotonicity of the volume of convex bodies.

7.1. Theorem. *Let K and L be convex bodies in $\mathbb{R}^m$ ($m \geq 2$) and let the surface area measure of K do not exceed the surface area measure of L, that is $S(K, \cdot) \leq S(L, \cdot)$. Then the volume of K does not exceed the volume of L, that is Vol $K \leq$ Vol L.*

Proof. Take $t \in (0,1)$. Consider an additive set function $\mu(\cdot)$ of the unit sphere $\mathbb{S}^{m-1}$ such that $\mu(\cdot) = S(L, \cdot) - tS(K, \cdot)$. Since $t < 1$, for every subset $U \subset \mathbb{S}^{m-1}$ such that $S(K, U) > 0$, we have $\mu(U) > 0$. Since $S(K, \cdot)$ is not concentrated in a hyperplane (more precisely, its support is not contained in $\Pi \cap \mathbb{S}^{m-1}$ for all hyperplanes Π), we conclude that μ is not concentrated in a hyperplane. Finally, the condition

$$\int_{\mathbb{S}^{m-1}} x \, d\mu = 0$$

holds, because analogous integrals with respect to additive set functions $S(K, \cdot)$ and $S(L, \cdot)$ vanish.

By the Minkowski theorem 4.4, there exists a convex body M (unique up to translation) in $\mathbb{R}^m$ whose surface area function coincides with μ, that is $\mu(\cdot) = S(M, \cdot)$. But then $S(L, \cdot) = S(M, \cdot) + tS(K, \cdot)$, and so $L = M\#(t^{1/(m-1)}K)$. By the Kneser–Süss inequality, we then have

$$(\text{Vol } L)^{1-1/m} = (\text{Vol } (M\#(t^{1/(m-1)}K)))^{1-1/m} \geq$$

$$\geq (\text{Vol } M)^{1-1/m} + (\text{Vol } (t^{1/(m-1)}K))^{1-1/m} =$$

$$= (\text{Vol } M)^{1/(m-1)} + t\,(\text{Vol } K)^{1-1/m} \geq t\,(\text{Vol } K)^{1-1/m}.$$

Tending t to 1, we get $(\text{Vol } L)^{1-1/m} \geq (\text{Vol } K)^{1-1/m}$, which completes the proof.

Let us give some obvious corollaries from Theorem 7.1.

7.2. Corollary. *Let $\{n_1, \ldots, n_k\}$ be the unit outward normals to $(m-1)$-dimensional faces of a compact convex polyhedron $P \subset \mathbb{R}^m$ with nonempty interior and let $\{n'_1, \ldots, n'_l\}$ be the unit outward normals to $(m-1)$-dimensional faces of a compact convex polyhedron $Q \subset \mathbb{R}^m$ with non-empty interior. Put $\mathcal{N} = \{n_1, \ldots, n_k\} \cup \{n'_1, \ldots, n'_l\}$. Given $n \in \mathcal{N}$, let $\Gamma_P(n)$ be the face of P with outward normal n and let $\Gamma_Q(n)$ be the face of Q with outward normal n. At least one of these faces is of dimension $m-1$. Suppose that for every vector n in $\mathcal{N}$, the $(m-1)$-dimensional volume of $\Gamma_P(n)$ is not less than the $(m-1)$-dimensional volume of $\Gamma_Q(n)$; that is $\text{Vol}_{m-1} \Gamma_P(n) \geq \text{Vol}_{m-1} \Gamma_Q(n)$. Then m-dimensional volume of P is not less than m-dimensional volume of Q; that is $\text{Vol}_m P \geq \text{Vol}_m Q$.*

Corollary 7.2 can be shortened as follows:.

7.3. Corollary. *Let P and Q be compact convex polyhedra in $\mathbb{R}^m$ such that the volume of every $(m-1)$-dimensional face of P is not less than the $(m-1)$-dimensional volume of the face of Q parallel to it, and the volume of every $(m-1)$-dimensional face of Q is not greater than the $(m-1)$-dimensional volume of the face of P parallel to it. Then m-dimensional volume of P is not less than the m-dimensional volume of Q; that is $\text{Vol}_m P \geq \text{Vol}_m Q$.*

7.4. Remark. To emphasize nontriviality of the claims of 7.1–7.3, we note that under the hypotheses of Corollary 7.3, we cannot say that the polyhedron Q can be placed inside the polyhedron P by an appropriate translation. To demonstrate, take in 3-dimensional space a cube with edge 10 as P and as Q a rectangular parallelepiped whose faces are parallel to the faces of P and the edges equal 1, 1 and 50. Then the areas of 2-dimensional faces of Q are less than the areas of corresponding parallel faces of P, but Q is "two long" to be put inside the cube P. However, according to Corollary 7.3, the volume of the cube P is indeed greater than the volume of the parallelepiped Q.

We cannot give explicitly an analogous example of surfaces with continuous curvature. Moreover, Blaschke proved that on the plane such an example is impossible

in principle [5]. However, we will show that such an example exists in $\mathbb{R}^3$; namely, there exist infinitely differentiable convex surfaces $F, G \subset \mathbb{R}^3$ such that at the points with parallel outward normals the Gaussian curvature of F is always less than or equal to the Gaussian curvature of G and, nevertheless, G cannot be placed in the convex body bounded by F by any translation.

Suppose that the faces of the cube P above are parallel to the coordinate planes in $\mathbb{R}^3$. It is clear that there exists α $(0 < \alpha < \pi/2)$ such that for each unit vector $a \in \mathbb{R}^3$ there exists a face p of P such that the outward normal to p makes with a an angle of at most α. Therefore, the area of the projection of p to the plane orthogonal to a is at least $100 \cos \alpha > 0$. Hence, the area of the projection of the cube onto any plane in $\mathbb{R}^3$ is uniformly separated from 0.

Complicate the situation. The surface area function $S(P, \cdot)$ of P is concentrated in the six points $\pm n_1, \pm n_2, \pm n_3$ on the unit sphere and takes the value 100 at each of these points. Given a natural s and an arbitrary unit vector n, denote by $U_s(n)$ the set of points $x \in \mathbb{S}^2$ such that the distance between x and the endpoint of n is less than $1/s$, put $N = \{\pm n_1, \pm n_2, \pm n_3\}$ and construct an infinitely differentiable function ν_s on $\mathbb{S}^2$ such that (a) ν_s equals 0 outside the set $\cup_{n \in N} U_s(n)$; (b) ν_s is positive in each of $U_s(n)$, $n \in N$; (c) ν_s is centrally symmetric, i.e., $\nu_s(x) = \nu_s(-x)$ for all $x \in \mathbb{S}^2$; (d) the integral of ν_s with respect to the standard measure $d\sigma$ on $\mathbb{S}^2$ over each of $U_s(n)$, $n \in N$, equals 100.

By the Minkowski theorem 4.4, there exists a unique convex body P_s in $\mathbb{R}^3$ whose surface area function has density $\nu_s + 1/s$:

$$S(P_s, V) = \int_V (\nu_s + 1/s) \, d\sigma, \qquad V \subset \mathbb{S}^2.$$

We will check later that each of the bodies P_s is infinitely differentiable and the sequence $P_1, \ldots, P_s, \ldots$ is uniformly bounded (i.e., lies in a ball of finite radius). By the Blaschke choice theorem [5], we can extract a convergent subsequence $P_{s_1}, \ldots, P_{s_j}, \ldots$ (convergence is meant with respect to the Hausdorff metric) from the sequence $P_1, \ldots, P_s, \ldots$. Let $P_{s_j} \to P_0$ as $j \to \infty$. Then, on the one hand, $S(P_{s_j}, \cdot) \to S(P_0, \cdot)$, and on the other hand, by construction, $S(P_{s_j}, \cdot) \to S(P, \cdot)$. Hence, $S(P_0, \cdot) = S(P, \cdot)$ and, by uniqueness of a convex body with a given surface area measure, (see the Minkowski theorem 4.4), $P_0 = P$. Therefore, the sequence of infinitely differentiable surfaces $P_{s_1}, \ldots, P_{s_j}, \ldots$ converges to the cube P.

Analogously, we can construct a sequence of infinitely differentiable surfaces $Q_{s_1}, \ldots, Q_{s_j}, \ldots$ that converges to the parallelepiped Q. Since the inequality $S(Q, \{n\}) < S(P, \{n\})$ holds for any $n \in N$, we can smoothen the surface area measure of the parallelepiped Q so that the density of the surface area measure of Q_{s_j} is not greater than the density of the surface area measure of P_{s_j} (that is than $\nu_{s_j} + 1/s_j$). With this construction, for each $j \in \mathbb{N}$, the hypotheses of Theorem 7.1 hold for P_{s_j} and Q_{s_j}, however, for all sufficiently large j, there is no translation that shifts Q_{s_j} inside the body bounded by P_{s_j}, because these surfaces are close to the boundaries of the polyhedra Q and P, respectively, and the (Euclidean) diameter of the parallelepiped Q is greater than the (Euclidean) diameter of the cube P.

Let us show now that each of the bodies P_s is infinitely differentiable. This follows directly from the following theorem of A. V. Pogorelov [27]: *Let $\varkappa$ be a regular k times differentiable $(k \geq 3)$ positive function on the unit sphere $\mathbb{S}^2$. Let*

it satisfy the condition

$$\int_{\mathbb{S}^2} \frac{x\,d\sigma}{\varkappa(x)} = 0,$$

where $d\sigma$ is the element of area on $\mathbb{S}^2$ and the integral is taken over the whole sphere. Then there exists a regular (at least $k+1$ times differentiable) surface F with Gaussian curvature $\varkappa(x)$ at a point with the outward normal x.

Multidimensional variants of this theorem took significant efforts and generated an extensive literature; for example, see [7]. These theorems were first proved by A. V. Pogorelov [28, 29]. Later they were re-proved by S. Y. Cheng and Sh. T. Yau [8]; moreover, the latter author was awarded the Fields medal for this series of articles in 1982. Therefore, sometimes these multidimensional theorems are referred as "Pogorelov's theorems for which Yau has got the Fields medal".

Finally, let us show that the sequence $P_1, \ldots, P_s, \ldots$ is uniformly bounded (i.e., lies in a Euclidean ball of finite radius). We will essentially follow the arguments of A. V. Pogorelov [27]. Suppose that the sequence of convex bodies under consideration is not bounded. Dropping to a subsequence, we can assume that the (Euclidean) diameter of P_s is greater than s. Then there exists a pair of points A_s and B_s inside the body P_s such that the distance between them is greater than s. Project the body P_s onto the plane Π orthogonal to the line $A_s B_s$. So we obtain a convex figure $\overline{P}_s$. We have denoted by α ($0 < \alpha < \pi/2$) the number such that for every unit vector $a \in \mathbb{R}^3$ there exists a face p of P such that the outward normal to p makes with a an angle of at most α. It follows that for every vector a there exists an above domain $U_s(n) \subset \mathbb{S}^2$, $n \in N = \{\pm n_1, \pm n_2, \pm n_3\}$ such that an arbitrary vector $x \in U_s(n)$ makes with a an angle of at most $\alpha - 1/s$. Therefore, the projection of $\widetilde{U}_s(n) \subset \partial P_s$, which corresponds to the domain $U_s(n) \subset \mathbb{S}^2$, onto the plane orthogonal to a has area separated from 0 uniformly (in a and in s):

$$\text{area of the domain projection } \widetilde{U}_s(n) = \int\limits_{U_s(n)} \frac{(\nu_s(x) + 1/s)\,d\sigma}{\cos\angle(x,a)} \geq \frac{100}{\cos(\alpha - 1/s)}.$$

Therefore, for all sufficiently large s, the area of the convex figure $\overline{P}_s$ is not less than some positive β. Since every convex figure of diameter γ is contained in some disc of radius γ, and so it has area at most $\pi\gamma^2/4$, the diameter of $\overline{P}_s$ is not less than $2\sqrt{\beta/\pi}$, that is not less than some (the same for all sufficiently large s) number $2\sqrt{\beta/\pi}$. Hence, there exist points $\overline{C}_s$ and $\overline{D}_s$ inside the convex figure $\overline{P}_s$ such that the Euclidean distance between them is at least $\sqrt{\beta/\pi}$. Let C_s and D_s be points of a convex body bounded by P_s which are projected to the points $\overline{C}_s$ and $\overline{D}_s$, respectively, on the plane Π. Construct a plane $\widetilde{\Pi}$ parallel to the lines $A_s B_s$ and $C_s D_s$. Project the surface P_s to the plane $\widetilde{\Pi}$. This projection contains a quadrilateral whose vertices are the projections of A_s, B_s, C_s and D_s. The area of this quadrilateral is obviously not less than the product of the lengths of the segments $A_s B_s$ and $C_s D_s$, that is not less than $s\sqrt{\beta/\pi}$. Therefore, taking s to be sufficiently large, we can make the area of the projection of P_s onto $\widetilde{\Pi}$ however large. Hence, taking s to be sufficiently large, we can make the area of the surface P_s however larger. The latter case is impossible since the convex surfaces P_s converge to the surface P of finite area. This contradiction completes the proof of the fact that there exist smooth surfaces in $\mathbb{R}^3$ satisfying the hypotheses of Theorem 7.1

which cannot be shifted inside each other by any translation. A multidimensional example can be constructed analogously.

7.5. Theorem. *For every two convex bodies K and L in $\mathbb{R}^m$ ($m \geq 2$), the volume of their Blaschke sum does not exceed the volume of their Minkowski sum: Vol $(K \# L) \leq$ Vol $(K + L)$.*

Proof. First of all, note that it is sufficient to prove the theorem for polyhedra. The general case is then obtained by passage to a limit.

For polyhedra, the proof is by induction by the dimension of the space $\mathbb{R}^m$. As it was noticed in Section 5.1, the Blaschke sum of convex polygons coincides with their Minkowski sum on the plane: $K \# L = K + L$. Therefore, for $m = 2$ the claim of Theorem 7.5 is trivial.

If $m = 3$ then it is known [24] that the Minkowski sum of two convex polyhedra is a convex polyhedron whose faces are obtained as the Minkowski sum of (a)the faces of the polyhedra under addition; (b) a face of one of these polyhedra and an edge or a vertex of the other; (c) nonparallel edges of the polyhedra. Here, all mentioned faces, edges and vertices lie in the supporting planes with parallel outward normals. The cases (a) and (b) are especially important for us.

Indeed, if the summands of the Minkowski sum have faces with the same normal, then the sum has a face with the same normal and area greater than or equal to the sum of areas of the summands; since, by (a), the parallel faces Γ_1 and Γ_2 must be added in the sense of Minkowski and, by the Brunn–Minkowski inequality (3),

$$\text{Vol } (\Gamma_1 + \Gamma_2)^{1/2} \geq \text{Vol } \Gamma_1^{1/2} + \text{Vol } \Gamma_2^{1/2}.$$

It follows that

$$\left[\frac{\text{Vol } (\Gamma_1 + \Gamma_2)}{\text{Vol } \Gamma_1} \right]^{1/2} \geq 1 + \left[\frac{\text{Vol } \Gamma_2}{\text{Vol } \Gamma_1} \right]^{1/2},$$

and so

$$\frac{\text{Vol } (\Gamma_1 + \Gamma_2)}{\text{Vol } \Gamma_1} \geq \left\{ 1 + \left[\frac{\text{Vol } \Gamma_2}{\text{Vol } \Gamma_1} \right]^{1/2} \right\}^2 \geq 1 + \frac{\text{Vol } \Gamma_2}{\text{Vol } \Gamma_1}$$

or

$$\text{Vol } (\Gamma_1 + \Gamma_2) \geq \text{Vol } \Gamma_1 + \text{Vol } \Gamma_2.$$

Therefore, if we add, in the sense of Minkowski, the faces with the same outward normals, then we obtain a face with the same normal and, moreover, the area of this face is not less than the sum of the areas of the summands. That is in this case the face of $K + L$ has area not less than the corresponding face of $K \# L$ parallel to it.

Considering the cases (b) and (c) analogously, we see that the area of each face of the Minkowski sum is not less than the area of a face of the Blaschke sum $K \# L$ parallel to it. Hence, Theorem 7.5 for polyhedra follows from Theorem 7.1 (and its Corollaries 7.2 or 7.3).

In the general case, the induction step is analogous to the above step from $m = 2$ to $m = 3$.

8. On exponents in the Brunn–Minkowski and Kneser–Süss inequalities. In the proof of Theorem 7.5 we have given, in fact, an "arithmetical" reason for the fact that the concavity of the square root of the Minkowski sum area for plane figures implies the concavity of the Minkowski sum area itself. This suggests that, firstly, exponents in the Brunn–Minkowski and Kneser–Süss inequalities can be changed in some range so that the inequalities keep, and, secondly, that there

must be an "unimprovable" exponent. As far as we know, the question was not posed in this form before and we think that it is worth to give here some (although very simple) arguments for that.

8.1. Theorem. *For every $a \geq 1$ and all convex bodies K and L in $\mathbb{R}^m$ $(m \geq 2)$ the following inequalities hold:*

$$(Vol \ (K + L))^{a/m} \geq (Vol \ K)^{a/m} + (Vol \ L)^{a/m} \tag{4}$$

$$(Vol \ (K \# L))^{a-a/m} \geq (Vol \ K)^{a-a/m} + (Vol \ L)^{a-a/m}. \tag{5}$$

Moreover, for $0 < a < 1$ there exist convex bodies K and L in $\mathbb{R}^m$ $(m \geq 2)$ such that the inequalities (4) and (5) fail.

In other words, wee can say that, increasing exponent in the Brunn–Minkowski or Kneser–Süss inequalities, we coarsen the inequalities but they still hold. The original Brunn–Minkowski and Kneser–Süss inequalities are optimal in the sense that the exponents cannot be decreased.

For the proof of Theorem 8.1 we need the following absolutely elementary lemma whose proof we leave to the reader.

8.2. Lemma. *For all $a \geq 1$ and $x > 0$ the following inequality holds:*

$$(1 + x)^a \geq 1 + x^a. \tag{6}$$

Moreover, for all $0 < a < 1$ there exists $x > 0$ such that (6) fails.

Proof of Theorem 8.1 is given only for the Brunn–Minkowski inequality, since for the Kneser–Süss inequality the same arguments can be applied.

For $a \geq 1$, by the Brunn–Minkowski inequality and the inequality (6), we have

$$\left[\frac{Vol \ (K + L)}{Vol \ K} \right]^{a/m} \geq \left\{ 1 + \left[\frac{Vol \ L}{Vol \ K} \right]^{1/m} \right\}^a \geq 1 + \left[\frac{Vol \ L}{Vol \ K} \right]^{a/m},$$

that proves (4) for $a \geq 1$.

Now, let $0 < a < 1$. Take a ball of radius r as a body K and a ball of radius R as L. Then $K + L$ is a ball of radius $r + R$ and we have Vol $K = \omega_m r^m$, Vol $L = \omega_m R^m$ and Vol $(K + L) = \omega_m (r + R)^m$, where ω_m is the volume of the unit ball in $\mathbb{R}^m$. Then (4) is equivalent to

$$\left[\omega_m (r + R)^m \right]^{a/m} \geq \left[\omega_m r^m \right]^{a/m} + \left[\omega_m R^m \right]^{a/m},$$

or

$$(r + R)^a \geq r^a + R^a, \qquad \left(1 + \frac{r}{R} \right)^a \geq 1 + \left[\frac{r}{R} \right]^a.$$

However, by Lemma 8.2 for $0 < a < 1$, the latter inequality certainly fails for some $x = r/R$.

9. Visualization of the Blaschke sum of convex polyhedra. On the base of the package OpenGeometry [16], the authors developed a computer program for visualization of the Blaschke sum of polyhedra in 3-dimensional Euclidean space. The main difficulty is to construct a polyhedron given by a set of outward normals and areas of faces. Let us describe an algorithm for that.

Suppose that we plan to construct a polyhedron P such that $n_1, \ldots, n_k$ are unit vectors of outward normals to its faces and $F_1, \ldots, F_k$ are areas of faces.

We start with a polyhedron P_0 that circumscribes the unit sphere with the center the origin and has outward normals $n_1, \ldots, n_k$, then we deform it by modifying the support numbers so that the vector $n_1, \ldots, n_k$ remain unchanged.

Step 1. Let P_0 be a polyhedron with outward normals $n_1, \ldots, n_k$ and support numbers $h_j = 1$ for all $j = 1, \ldots, k$. It is clear that such a polyhedron always exists. Denote by $F_1^0, \ldots, F_k^0$ areas of faces of P_0. Then $\sum\limits_{j=1}^{k} F_j^0 n_j = 0$.

Step 2. For every $t \in [0,1]$, consider a polyhedron P_t which is the Blaschke sum $(1-t)P_0 \# tP$, i.e., a polyhedron with outward normals $n_1, \ldots, n_k$ and the area of the j-th face is given by the formula $F_j^t = (1-t)F_j^0 + tF_j$. Clearly, for every $t \in [0,1]$, we have $\sum\limits_{j=1}^{k} F_j^t n_j = 0$ and so P_t exists.

Step 3: Computing the support numbers $h_1^t, \ldots, h_k^t$ of the planes $\alpha_1^t, \ldots, \alpha_k^t$ that bound P_t. Partition the interval $[0,1]$ into equal intervals of length Δt. Suppose that the support numbers $h_1^t, \ldots, h_k^t$ of P_t have been computed for the left endpoint of an interval of length Δt. The support numbers $h_1^{t+\Delta t}, \ldots, h_k^{t+\Delta t}$ of the polyhedron $P_{t+\Delta t}$ that corresponds to the right endpoint of this interval will be found from a linear system of algebraic equations. Since for $t = 0$ all support numbers are known (they are equal to 1), the problem will be solved at $t = 1$.

Let us describe the construction for a required system of equations.

Each polyhedron P_t is determined by its support numbers $h_1^t, \ldots, h_k^t$. Therefore, the set of all polyhedra P_t can be presented as a subset of k-dimensional space $\mathbb{R}^k$ with coordinates $h_1^t, \ldots, h_k^t$. This subset is open, because the faces do not disappear under small displacement.

Combine in a single class all polyhedra P_t that are equal and parallel to each other. Since each translation is determined by three components, such a class is determined by $k - 3$ variables. The set of these classes forms a $(k-3)$-dimensional manifold $\mathbf{A}_t$.

Consider the set $\mathbf{B}_t$ of all k-tuples of numbers $F_1^t, \ldots, F_k^t$ such that $F_j^t > 0$ for all $j = 1, \ldots, k$ and $\sum_{j=1}^{k} F_j^t n_j = 0$. The vectors $n_1, \ldots, n_k$ are assumed to be fixed. The vector identity $\sum_{j=1}^{k} F_j^t n_j = 0$ is equivalent to three scalar identities and so it determines a $(k-3)$-dimensional plane in $\mathbb{R}^k$. The inequalities $F_j^t > 0$ determine in this plane an open convex set which is exactly $\mathbf{B}_t$.

Thus, we have a natural one to one map $F : \mathbf{A}_t \to \mathbf{B}_t$ (see [2]).

Let $\Delta F_j = F_j^{t+\Delta t} - F_j^t$ and $\Delta h_j = h_j^{t+\Delta t} - h_j^t$. Then

$$\Delta F_j = \frac{\partial F_j}{\partial h_1} \Delta h_1 + \cdots + \frac{\partial F_j}{\partial h_k} \Delta h_k, \quad j = 1, \ldots, k. \tag{7}$$

Note that $\Delta F_j = \Delta t(-F_j^0 + F_j)$, i.e., ΔF_j is a constant for a given Δt.

We are left with computing $\frac{\partial F_j}{\partial h_i}$. Let j-th face of P_t is cut by planes $\alpha_{q_1}, \ldots, \alpha_{q_{m_j}}$. Denote by ℓ_{jp} the length of an edge lying on the line $\alpha_j \cap \alpha_p$. It is not difficult to show that the modification of the support number h_j implies the modification of the area of the j-th face to the value

$$\frac{\partial F_j}{\partial h_j} = \sum_{p \in \{q_1, \ldots, q_{m_j}\}} \ell_{jp} \mathrm{ctg} \angle (n_j, n_p),$$

and the modification of the area of the i-th face $(i \neq j)$ to the value

$$\frac{\partial F_i}{\partial h_j} = \frac{\ell_{ji}}{\sin \angle (n_j, n_i)}.$$

```
10
1 1 0 2.0412414523
0 1 1 2.0412414523
1 0 1 2.0412414523
-1 1 1 5
1 -1 1 5
1 1 -1 5
-1 -1 1 5
-1 1 -1 5
1 -1 -1 5
-1 -1 -1 5
```

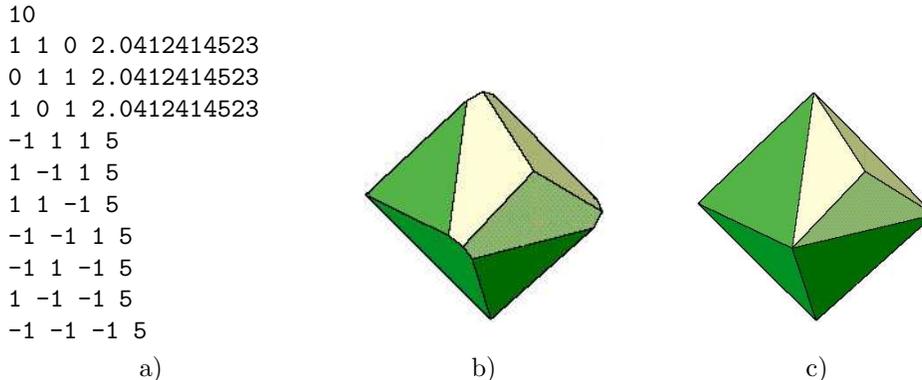a)   b)   c)

FIGURE 3. Construction of the Grünbaum polyhedron

Hence, the system (7) is a system of linear equations of rank $k - 3$ in k unknowns $\Delta h_1, \ldots, \Delta h_k$. We take three of the unknowns arbitrarily (for example, equal to 0) and find the other $k - 3$ unknowns from (7). Thus, we obtain $\Delta h_1, \ldots, \Delta h_k$, and so $h_j^{t+\Delta t} = \Delta h_j + h_j^t$.

Step 4. We now construct P_t with the support numbers $h_1^t, \ldots, h_k^t$. In order to do this, given every pair of planes α_i^t and α_j^t, we find a set of points $\{\alpha_i^t \cap \alpha_j^t \cap \alpha_p^t, p = 1, \ldots, k\}$ and then we pick up from this set those points that are vertices of the polyhedron. Since the polyhedron is convex, there are at most two of these points. We add these points to the list of vertices $\{V\}$ and, if the points were two, put the elements ℓ_{ij} and ℓ_{ji} to be equal to the distance between these points. In all other cases we put $\ell_{ij} = \ell_{ji} = 0$.

When the list $\{V\}$ is composed, we, for each plane α_i^t, pick up from $\{V\}$ those points that lie in α_i^t. So we determine the i-th face of P_t.

9.1. Example. In the process of deformation, P_t can change its combinatorial structure. In order to be convinced that the proposed algorithm works correctly with disappearing and appearing edges, consider a polyhedron P obtained from an octahedron by attaching a tetrahedron with a face congruent to a face of the octahedron (see Figure 3 c)).

The polyhedron P is input to the programme as shown in Figure 3 a). Here, the first line gives the number of faces and each subsequent line shows coordinates of an outward (not necessarily unit) normal to a face and the area of this face.

In [17] and [18, p. 286], it was shown that a polyhedron of such combinatorial type cannot circumscribe a sphere. A polyhedron P_0 that circumscribes a sphere with the same outward normals as P is shown in Figure 3 b). The difference in the combinatorial structures of P and P_0 is obvious.

9.2. Remark. Example 9.1 shows, in particular, that it is impossible to determine a combinatorial structure of a polyhedron if only outward normals are known. (That is it is impossible to determine which faces of the polyhedron have a common edge.)

9.3. Example: The Blaschke sum of a cube and icosahedron. In Figure 5 c), the Blaschke sum of a cube whose faces are parallel o the coordinate planes and face areas equal 2 and an icosahedron given as shown in Figure 4 is shown.

Icosahedron:

```
20
1 1 1 5
-1 1 1 5
1 -1 1 5
1 1 -1 5
-1 -1 1 5
-1 1 -1 5
1 -1 -1 5
-1 -1 -1 5
0 0.6180339887 1.618033989 5
0 -0.6180339887 1.618033989 5
0 0.6180339887 -1.618033989 5
0 -0.6180339887 -1.618033989 5
0.6180339887 1.618033989 0 5
-0.6180339887 1.618033989 0 5
0.6180339887 -1.618033989 0 5
-0.6180339887 -1.618033989 0 5
1.618033989 0 0.6180339887 5
-1.618033989 0 0.6180339887 5
1.618033989 0 -0.6180339887 5
-1.618033989 0 -0.6180339887 5
```

Dodecahedron:

```
12
0 1.618033989 1 3
0 1.618033989 -1 3
0 -1.618033989 1 3
0 -1.618033989 -1 3
1 0 1.618033989 3
-1 0 1.618033989 3
1 0 -1.618033989 3
-1 0 -1.618033989 3
1.618033989 1 0 3
-1.618033989 1 0 3
1.618033989 -1 0 3
-1.618033989 -1 0 3
```

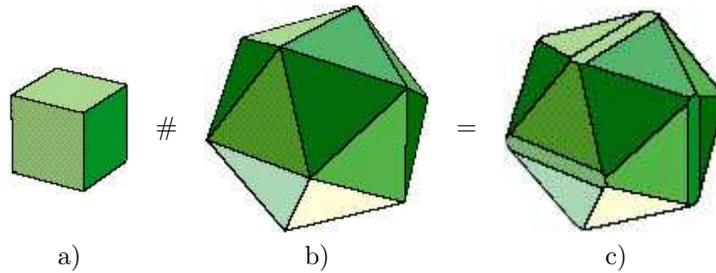FIGURE 4. Input data for an icosahedron and dodecahedron

a) # b) = c)

FIGURE 5. The Blaschke sum of a cube and icosahedron

The result enables us to conjecture that an icosahedron has parallel edges, which is, of course, true, and that there is a simple way to construct an icosahedron.

Take a cube with an edge 2 and construct segments with the endpoints at the edges as shown in Figure 6 a). On each segment, mark two points at the distance $t = (3 - \sqrt{5})/2$ from those edges of the cube to which this segment is orthogonal (Figure 6 b)). Cutting parts off the cube by planes that pass through triples of points according to the combinatorial structure of an icosahedron (for example, by planes through A, B and C, and through A, C and D), we get an icosahedron (Figure 6 c)).

9.4. Example: The Blaschke sum of a dodecahedron and icosahedron. In Figure 7 c), the Blaschke sum of a dodecahedron and icosahedron given as in Figure 4 is shown. Shortly speaking, the Blaschke sum of a dodecahedron and
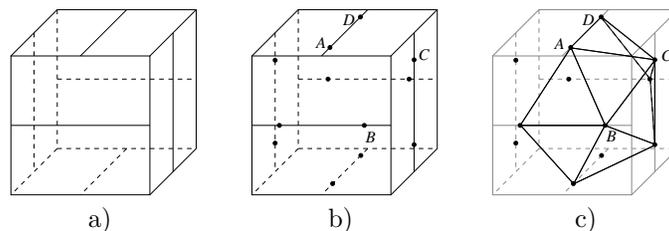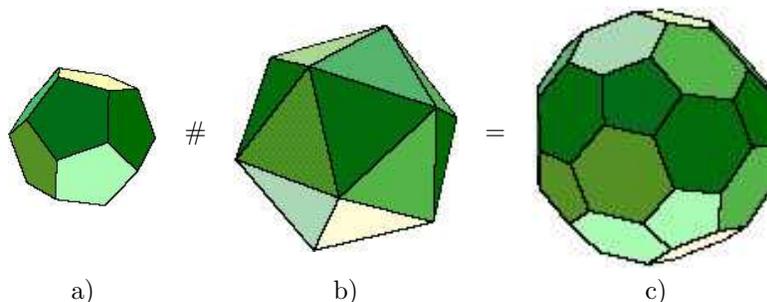
FIGURE 6. Construction of an icosahedron

a)   b)   c)

FIGURE 7. The Blaschke sum of a dodecahedron and icosahedron

icosahedron is a football (which is well known to be sewed from flat leather penta-
and hexagons so that each pentagon is sewed only with hexagons and pentagons
and hexagons are attached to a hexagon consequently). Although, the polyhedron
shown in Figure 7 c) has a quite scientific name: "truncated icosahedron" (see [34]).

9.5. Remark. Our program for visualization of a polyhedron with given areas
of faces and outward normals is not unique. Something like that was done in [35, 36]
for the purposes of computer graphics. Unfortunately, the papers [35, 36] are not
available for us.

9.6. Remark. Our program for visualization of the Blaschke sum enables us to
find, as a byproduct, the numerical values of various geometrical parameters of the
constructed polyhedra, for example, the volume, areas of faces, the integral mean
curvature etc. This enables us to master some geometric intuition and reject some
conjectures that arose during the work.

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