THE LACHLAN PROBLEM

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Let $T$ be a complete first order theory, $I(T,\lambda)$ be the number of pairwise nonisomorphic models of $T$ and of cardinality $\lambda$.

Remind the characterization for $I(T,\omega) = 1$ ($T$ is a countably categorical theory).

**THEOREM (C. Ryll-Nardzewski).** A theory $T$ is countably categorical iff for any $n \in \omega$ the set of types of $T$ and of $n$ fixed variables is finite ($|S_n(T)| < \omega$).

Ryll-Nardzewski function: a function $f \in \omega^\omega$ such that $f(n) = |S_n(T)|$.

If $1 < I(T,\omega) < \omega$ then the theory $T$ is called Ehrenfeucht.
Theories with finitely many countable models

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PROBLEM
ON CHARACTERIZATION OF EHRENFEUCHT THEORIES.
Problems on Ehrenfeucht theories

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LACHLAN PROBLEM
ON EXISTENCE OF STABLE EHRENFEUCHT THEORIES.
A type $p(\bar{x}) \in S(T)$ is called powerful type of theory $T$ if for any models $\mathcal{M}$ of $T$ realizing $p$ the model $\mathcal{M}$ realizes any type $q \in S(T): \mathcal{M} \models S(T)$.
If $I(T, \omega) < \omega$ then $T$ has a powerful type.
A type $p(\bar{x}) \in S(T)$ is called powerful type of theory $T$ if for any models $\mathcal{M}$ of $T$ realizing $p$ the model $\mathcal{M}$ realizes any type $q \in S(T) : \mathcal{M} \models S(T)$.

If $I(T, \omega) < \omega$ then $T$ has a powerful type. An existence of powerful type implies the smallness of theory $T$ i.e. the set $S(T)$ is countable. It also implies that there are prime models $\mathcal{M}_{\bar{a}}$ over tuples $\bar{a}$ for any type $p \in S(T)$ and any its realization $\bar{a}$. Since all prime models over realizations of $p$ are isomorphic, these models are denoted by $\mathcal{M}_p$. 
Basic characteristics of theories with finitely many countable models

For any types \( p, q \in S(T) \) we write \( p \leq_{RK} q \) and say that \( p \) is not more than \( q \) under the Rudin—Keisler preorder if \( M_q \) has a realization of type \( p \). At the same time we write \( M_p \leq_{RK} M_q \) if \( p \leq_{RK} q \). By \( RK(T) \) we denote the set of all isomorphism types of models \( M_p \) with the \( RK \)-relation induced by the relation \( \leq_{RK} \) for models \( M_p \).

We say that models \( M_p \) and \( M_q \) are \( RK \)-equivalent if

\[
M_p \leq_{RK} M_q \quad \text{and} \quad M_q \leq_{RK} M_p.
\]

Isomorphism types \( M_1 \) and \( M_2 \) from \( RK(T) \) are \( RK \)-equivalent:

\[
M_1 \sim_{RK} M_2,
\]

if their representatives are \( RK \)-equivalent.
A model $\mathcal{M}$ is (strongly) \textit{limit over a type $p$} if $\mathcal{M}$ is a union of an elementary chain $(\mathcal{M}_n)_{n \in \omega}$ such that $\mathcal{M}_n \cong \mathcal{M}_p$, $n \in \omega$, and $\mathcal{M} \not\cong \mathcal{M}_p$.

Let $\text{RK}(T)$ be a finite system. For any class $\tilde{\mathcal{M}} \in \text{RK}(T)/\sim_{\text{RK}}$ consisting of isomorphism types of $\text{RK}$-equivalent models $\mathcal{M}_{p_1}, \ldots, \mathcal{M}_{p_n}$ we denote by $\text{IL}(\tilde{\mathcal{M}})$ the number of pairwise non-isomorphic limit models each of which is limit over some type $p_i$. 
Syntactic characterization of theories with finitely many countable models

THEOREM 1.

For any countable complete theory $T$ the following conditions are equivalent:

1. $I(T, \omega) < \omega$;
2. $T$ is small, $|\text{RK}(T)| < \omega$ and $\text{IL}(\tilde{M}) < \omega$ for any $\tilde{M} \in \text{RK}(T)/\sim_{\text{RK}}$.

If the condition (1) (or (2)) is true, then $T$ satisfies the following conditions:

a. $\text{RK}(T)$ has the least element $M_0$ (the isomorphism type of a prime model) and $\text{IL}(\tilde{M}_0) = 0$;

b. $\text{RK}(T)$ has the greatest $\sim_{\text{RK}}$-class $\tilde{M}_1$ (the class of isomorphism types of all prime models over realizations of powerful types), and $|\text{RK}(T)| > 1$ implies $\text{IL}(\tilde{M}_1) \geq 1$;

c. If $|\tilde{M}| > 1$ then $\text{IL}(\tilde{M}) \geq 1$.

Moreover the following decomposition formula is true:

$$I(T, \omega) = |\text{RK}(T)| + \sum_{i=0}^{|\text{RK}(T)/\sim_{\text{RK}}| - 1} \text{IL}(\tilde{M}_i),$$

where $\tilde{M}_0, \ldots, \tilde{M}_{|\text{RK}(T)/\sim_{\text{RK}}| - 1}$ are all elements of the partially ordered set $\text{RK}(T)/\sim_{\text{RK}}$. 

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THE LACHLAN PROBLEM
Examples

\[ I(T, \omega) = 3 \]

\[ I(T, \omega) = 4 \]
Examples

\[ I(T, \omega) = 5 \]
Realization of basic characteristics of theories with finitely many countable models

**THEOREM 2.**

Let \( \langle X; \leq \rangle \) be a finite preordered set with the least element \( x_0 \) and the greatest class \( \tilde{x}_1 \) in the ordered factor-set \( \langle X; \leq \rangle / \sim \) by the relation \( \sim \) (where \( x \sim y \iff x \leq y \) and \( y \leq x \)), \( f : X / \sim \to \omega \) be a function (a distribution function) satisfying the following conditions:

(a) \( f(\tilde{x}_0) = 0 \);
(b) \( |X| > 1 \) implies \( f(\tilde{x}_1) \geq 1 \).
(c) \( |\tilde{y}| > 1 \) implies \( f(\tilde{y}) \geq 1 \).

Then there exists a stable (unstable) theory \( T \) and an isomorphism \( g : \langle X; \leq \rangle \xrightarrow{\sim} RK(T) \) such that \( IL(g(\tilde{y})) = f(\tilde{y}) \) for any \( \tilde{y} \in X / \sim \).
THEOREM 3.

For any $n \in \omega \setminus \{0, 2\}$ there exists a stable theory $T_n$ with $I(T_n, \omega) = n$. 

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Language
unary disjoint predicates $\text{Col}_m, m \in \omega$, and disjoint $P_1, \ldots, P_n$,
\[
\vdash \forall x \bigvee_{i=1}^{n} P_i(x), \text{ with given number } n \text{ of prime models over realizations of non-principal 1-types } p_1(x), \ldots, p_n(x);
\]
unary disjoint predicates $\text{Col}_m$, $m \in \omega$, and disjoint $P_1, \ldots, P_n$,
$\vdash \forall x \bigvee_{i=1}^{n} P_i(x)$, with given number $n$ of prime models over
realizations of non-principal 1-types $p_1(x), \ldots, p_n(x)$;

the countable set of pairwise disjoint antisymmetric irreflexive binary relations $Q_n$, $n \in \omega$ defining acyclic digraphs with unbounded lengths of shortest $Q^*$-routes ($Q^* \iff \bigcup_{n \in \omega} Q_n$) on the structures of $p_i(x)$ and on their neighbourhoods;
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realizations of non-principal 1-types $p_1(x), \ldots, p_n(x)$;
the countable set of pairwise disjoint antisymmetric irreflexive binary relations $Q_n$, $n \in \omega$ defining acyclic digraphs with unbounded lengths of shortest $Q^*$-routes ($Q^* \equiv \bigcup_{n \in \omega} Q_n$) on the structures of $p_i(x)$ and on their neighbourhoods;
the countable set of pairwise disjoint symmetric irreflexive binary relations $P_{i,k,l}$, $i, k \in \omega$, $l = 1, \ldots, n$, allowing to connect elements $a$ of infinite color \[ \left( \models \bigwedge_{m \in \omega} \neg \text{Col}_m(a) \right) \] with elements of finite colors $m$ by principal formulas over $a$;
the countable set of pairwise disjoint symmetric irreflexive binary relations $R_j, j \in \omega$, connecting only elements of the same color and the same $P_i$, guaranteeing the coincidence of prime models over realizations of $p_i(x)$ if these realizations are connected by $R_j$;
the countable set of pairwise disjoint symmetric irreflexive binary relations $R_j, j \in \omega$, connecting only elements of the same color and the same $P_i$, guaranteeing the coincidence of prime models over realizations of $p_i(x)$ if these realizations are connected by $R_j$;

predicates $R'_s, s \in \omega$, guaranteeing realization-equivalence of $\bigvee_{i=1}^{n} p_i$ with all nonprincipal types.
syntactic modifications of Hrushovski — Herwig generic construction;
Tools and objects

- syntactic modifications of Hrushovski — Herwig generic construction;
- syntactic modifications of Hrushovski fusion;
syntactic modifications of Hrushovski — Herwig generic construction;
syntactic modifications of Hrushovski fusion;
powerful directed graphs with almost inessential ordered colors;
syntactic modifications of Hrushovski — Herwig generic construction;
syntactic modifications of Hrushovski fusion;
powerful directed graphs with almost inessential ordered colors;
expressions for realizations of non-principal types in prime models over 1-types;
Tools and objects

- syntactic modifications of Hrushovski — Herwig generic construction;
- syntactic modifications of Hrushovski fusion;
- powerful directed graphs with almost inessential ordered colors;
- expansions for realizations of non-principal types in prime models over 1-types;
- corealization amalgams.
The results and generalizations for the class of all small theories are presented in:
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The book is available in: