DERIVATIVE STRUCTURES
IN MODEL THEORY
AND GROUP THEORY

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\[
\int f(x) \, dx = \text{ cows}
\]
\[
f(x) = \text{ milk}
\]
\[
f'(x) = \text{ dairy products}
\]
We consider a series of derivative structures used for a classification of structures and their elementary theories:

- Rudin–Keisler preorders and distribution functions for limit models of a given theory, producing spectra of countable models (with B.Sh. Kulpeshov);

- Algebras for distributions of formulas over families of types (with B.Sh. Kulpeshov and D.Yu. Emel’yanov);

- Hypergraphs of models of a theory (with B.Sh. Kulpeshov);

- Generic classes and their limits (with P. Stefaneas and Y. Kiouvrekis);

- Topologies, closures, generating sets and e-spectra for families of theories with respect to $P$-operators and $E$-operators.
Weakly o-minimal structures\(^1\)

**Definition**

A *weakly o-minimal structure* is a linearly ordered structure \( \mathcal{M} = \langle M, =, <, \ldots \rangle \) such that any definable (with parameters) subset of the structure \( M \) is a finite union of convex sets in \( \mathcal{M} \).

We recall that such a structure \( \mathcal{M} \) is said to be *o-minimal* if any definable (with parameters) subset of \( \mathcal{M} \) is the union of finitely many intervals and points in \( \mathcal{M} \).

Weakly orthogonal types\textsuperscript{2}

**Definition**

Assuming that $\mathcal{M}$ is an $|A|^+$-saturated weakly o-minimal structure, $A, B \subseteq M$, and $p, q \in S^1(A)$ are non-algebraic types, we say that $p$ is not weakly orthogonal to $q$ ($p \not\perp w q$) if there are an $A$-definable formula $H(x, y)$, $a \in p(\mathcal{M})$, and $b_1, b_2 \in q(\mathcal{M})$ such that $b_1 \in H(\mathcal{M}, a)$ and $b_2 \notin H(\mathcal{M}, a)$.

**Lemma (B.S. Baizhanov)**

The relation $\not\perp w$ of the weak non-orthogonality is an equivalence relation on $S^1(A)$.

Quite o-minimal theories

**Definition**

We say that $p$ is not *quite orthogonal* to $q$ ($p \not\perp^q q$) if there is an $A$-definable bijection $f : p(M) \to q(M)$. We say that a weakly o-minimal theory is *quite o-minimal* if the relations of weak and quite orthogonality coincide for 1-types over arbitrary sets of models of the given theory.

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Definition

A type $p \in S_1(\emptyset)$ is called simple if for any $n \in \omega$, any non-trivial $\emptyset$-definable $n$-ary function $f(x_1, \ldots, x_n)$, and any realizations $a_1, \ldots, a_n$ of $p$, $f(a_1, \ldots, a_n)$ does not realize the type $p$. 

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Definition

Let $T$ be a weakly o-minimal theory, $\mathcal{M}$ be a sufficiently saturated model of $T$, $\phi(x)$ be a $\mathcal{M}$-definable formula with one free variable. The convexity rank of the formula $\phi(x)$ ($RC(\phi(x))$) is defined as follows:

1) $RC(\phi(x)) \geq 1$ if $\phi(\mathcal{M})$ is infinite;
2) $RC(\phi(x)) \geq \alpha + 1$ if there are a parametrically definable equivalence relation $E(x,y)$ and an infinite sequence $b_i, i \in \omega$ of elements such that:
   - for any $i, j \in \omega$, with $i \neq j$ we have $\mathcal{M} \models \neg E(b_i, b_j)$;
   - for any $i \in \omega$, $RC(E(x, b_i)) \geq \alpha$ and $E(\mathcal{M}, b_i)$ is a convex subset of $\phi(\mathcal{M})$;

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3) $RC(\phi(x)) \geq \delta$ if $RC(\phi(x)) \geq \alpha$ for all $\alpha < \delta$, where $\delta$ is a limit ordinal.

If $RC(\phi(x)) = \alpha$ for some $\alpha$ then we say that $RC(\phi(x))$ is defined. Otherwise (i.e., if $RC(\phi(x)) \geq \alpha$ for all $\alpha$) we set $RC(\phi(x)) = \infty$.

The \textit{convexity rank of 1-type $p$} ($RC(p)$) is the infimum of the set $\{RC(\phi(x)) \mid \phi(x) \in p\}$, i.e., $RC(p) := \inf\{RC(\phi(x)) \mid \phi(x) \in p\}$. 
The convexity rank of an arbitrary unary formula $\phi(x)$ is called *binary* and it is denoted by $RC_{bin}(\phi(x))$ if in Definition above the parametrically definable equivalence relations are replaced by $\emptyset$-definable (i.e., binary) equivalence relations.
We say that a quite o-minimal theory $T$ has few countable models if $T$ has fewer than $2^\omega$ countable models up to isomorphisms.

**Theorem**

Let $T$ be a quite o-minimal theory in a countable language. Then either $T$ has $2^\omega$ countable models or $T$ has exactly $3^k \cdot 6^s$ countable models, where $k$ and $s$ are natural numbers. Moreover, for any $k, s \in \omega$ there is a quite o-minimal theory $T$ with exactly $3^k \cdot 6^s$ countable models.


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Quite o-minimal theories with few countable models

Theorem
Let $T$ be a quite o-minimal theory with few countable models. Then
(1) $T$ is binary;
(2) each non-algebraic type $p \in S_1(\emptyset)$ is simple and has a finite convexity rank.

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Let $T$ be a small theory, i.e., a theory with countably many types ($|S(T)| = \omega$).

Then any countable model $M$ of $T$ is either prime over a tuple ($M = M(\bar{a})$), or limit, i.e., it is not prime over tuples and

$$M = \bigcup_{n \in \omega} M(\bar{a}_n),$$

where $(M(\bar{a}_n))_{n \in \omega}$ is an elementary chain of prime models over tuples $\bar{a}_n$. 
Number of pairwise non-isomorphic models

- \( I(T, \omega) \) is the number of countable models of theory \( T \),
- \( P(T) \) is the number of models of \( T \), being prime over tuples,
- \( L(T) \) is the number of limit models of \( T \).

\[
I(T, \omega) = P(T) + L(T)
\]
A theory $T$ is *countably categorical* or $\omega$-*categorical* if $I(T, \omega) = 1$.

A theory $T$ is *Ehrenfeucht* if $1 < I(T, \omega) < \omega$. 
A theory $T$ is $p$-categorical ($l$-categorical) if $P(T) = 1$ (respectively, $L(T) = 1$).

A theory $T$ is $p$-Ehrenfeucht ($l$-Ehrenfeucht) if $1 < P(T) < \omega$ (respectively, $1 < L(T) < \omega$).
A theory $T$ is $p$-categorical $\iff T$ countably categorical.

A theory $T$ is $p$-Ehrenfeucht $\iff$ its structure $\text{RK}(T) = \langle M, \leq_{\text{RK}} \rangle$ is finite and has at least two elements, where $M$ is the set of isomorphism types of prime models $M(\bar{a})$ over tuples $\bar{a}$ and

$$M(\bar{a}) \leq_{\text{RK}} M(\bar{b}) \iff M(\bar{b}) \models \text{tp}(\bar{a}).$$

A small theory $T$ is $p$-Ehrenfeucht and $1 \leq L(T) < \omega$ $\iff T$ is Ehrenfeucht.
Notations

- $\sim_{RK} \triangleq \leq_{RK} \cap \leq_{RK}^{-1}$;

- $\tilde{M}$ is the $\sim_{RK}$-class containing the isomorphism type $M$ for a prime model over a finite set;

- $IL(\tilde{M})$ is the number of limit models being unions of elementary chains of models with isomorphism types in $\tilde{M}$. 
Characterization of \( I(T, \omega) < \omega \) with respect to limit models

**Theorem**

For any countable complete theory \( T \), the following conditions are equivalent:

1. \( I(T, \omega) < \omega \);
2. \( T \) is small, \( |RK(T)| < \omega \) and \( IL(\tilde{M}) < \omega \) for any \( \tilde{M} \in RK(T)/\sim_{RK} \).

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If (1) or (2) holds then $T$ possesses the following properties:

(a) $\text{RK}(T)$ has a least element $\mathcal{M}_0$ (an isomorphism type of a prime model) and $\text{IL}(\mathcal{M}_0) = 0$;

(b) $\text{RK}(T)$ has a greatest $\sim_{\text{RK}}$-class $\mathcal{M}_1$ (a class of isomorphism types of all prime models over realizations of powerful types) and $|\text{RK}(T)| > 1$ implies $\text{IL}(\mathcal{M}_1) \geq 1$;

(c) if $|\mathcal{M}| > 1$ then $\text{IL}(\mathcal{M}) \geq 1$.

Moreover, the following decomposition formula holds:

$$I(T, \omega) = |\text{RK}(T)| + \sum_{i=0}^{\text{|RK}(T)/\sim_{\text{RK}}|-1} \text{IL}(\mathcal{M}_i),$$

where $\mathcal{M}_0, \ldots, \mathcal{M}_{|\text{RK}(T)/\sim_{\text{RK}}|-1}$ are all elements of the partially ordered set $\text{RK}(T)/\sim_{\text{RK}}$. 


Examples

\[ I(T, \omega) = 3 \]

\[ I(T, \omega) = 4 \]
Examples

\[ I(T, \omega) = 5 \]
Quite $\omega$-minimal theories with $I(T, \omega) = 3$ and $I(T, \omega) = 6$
Quite $o$-minimal theories with $I(T, \omega) = 3^2$ and $I(T, \omega) = 3^3$
Quite o-minimal theories with $I(T,\omega) = 6^2$ and $I(T,\omega) = 6^3$
Quite $\sigma$-minimal theories with $I(T, \omega) = 3 \cdot 6$ and $I(T, \omega) = 3^2 \cdot 6$
Quite \( \sigma \)-minimal theories with \( I(T, \omega) = 3 \cdot 6^2 \)
Applying Theorem on numbers of countable models for quite $o$-minimal theories we have the following representation of Decomposition formula (1):

$$3^k \cdot 6^s = 2^k \cdot 3^s + \sum_{t=0}^{k} \sum_{m=0}^{s} 2^{s-m} \cdot (2^t \cdot 4^m - 1) \cdot C_k^t \cdot C_s^m.$$
Almost $\omega$-categorical theories\textsuperscript{10} \textsuperscript{11}

Let $p_1(x_1), \ldots, p_n(x_n)$ be 1-types in $S(T)$ with disjoint sets of free variables. A type $q(x_1, \ldots, x_n) \in S(T)$ is a $(p_1, \ldots, p_n)$-type if $q(x_1, \ldots, x_n) \supseteq \bigcup_{i=1}^n p_i(x_i)$. The set of all $(p_1, \ldots, p_n)$-types of theory $T$ is denoted by $S_{p_1,\ldots,p_n}(T)$.

A countable theory $T$ is almost $\omega$-categorical if for any types $p_1(x_1), \ldots, p_n(x_n) \in S(T)$ there are only finitely many types

$$q(x_1, \ldots, x_n) \in S_{p_1,\ldots,p_n}(T).$$

A countable theory $T$ in a predicate language is called 1-locally countably categorical (or 1-locally $\omega$-categorical), or briefly LCC1-theory, if $T$ has only finitely many non-principal 1-types $p_1, \ldots, p_n$ and for any formulas $\varphi_i(x) \in p_i(x)$, $i = 1, \ldots, n$, and any tuple $\bar{a}$, whose each coordinate realizes some type $p_i$, the structure, which is defined by formulas with parameters in $\bar{a}$, on the set, defined by the formula $\neg \varphi_1(x) \land \ldots \land \neg \varphi_n(x)$, is $\omega$-categorical.

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Almost $\omega$-categorical and 1-locally countably categorical theories$^{13}$

**Theorem**

Let $T$ be a countable theory in a predicate language and with finitely many nonisolated types $p_1, \ldots, p_n \in S_1(\emptyset)$ such that the set $S_{p_1,\ldots,p_n}(T)$ is finite. Then the following conditions are equivalent:

1. $T$ is almost $\omega$-categorical;
2. $T$ is 1-locally $\omega$-categorical.

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Theorem

Let $T$ be a LCC1-theory with a strict dense linear order $<$ and with finitely many non-principal 1-types $p_0(x), \ldots, p_m(x)$ such that sets of realizations for each $p_i$ are convex, each predicate, which differs from $<$ and some finite set of unary predicates $P_1, \ldots, P_k$, does not have tuples with at least one coordinate realizing some $p_i(x)$, and each $(p_1, \ldots, p_m)$-type is isolated by the types $p_1, \ldots, p_m$ and some set of formulas $(x \approx y)\delta, x < y, P_j^{\delta}(x), \delta \in \{0, 1\}, j \in \{1, \ldots, k\}$. Then $T$ is an Ehrenfeucht theory.
Ehrenfeucht almost \( \omega \)-categorical theories\(^{15}\)

**Theorem**
Any Ehrenfeucht quite o-minimal theory is almost \( \omega \)-categorical.

**Corollary**
Any Ehrenfeucht quite o-minimal theory in a predicate language and with finitely many non-isolated 1-types in \( S_1(\emptyset) \) is a LCC1-theory.

We say that a set $\Gamma \subseteq S_1(\emptyset)$ is *independent* if for any set $\Gamma'$, consisting of exactly one realization of each type in $\Gamma$, for any $c' \in \Gamma'$ we have $c' \not\in \text{dcl}(\Gamma' \setminus \{c'\})$. We say that $p \in S_1(\emptyset)$ *depends on* $\Gamma$ (or $p$ and $\Gamma$ are *dependent*) if $\Gamma \cup \{p\}$ is not independent. The *dimension* of set $\Gamma \subseteq S_1(\emptyset)$ (denoted by $\dim(\Gamma)$) is the cardinality of a maximal independent subset of $\Gamma$.

**Corollary**

Let $T$ be a binary quite $\sigma$-minimal theory, $\Gamma$ be the set of all non-isolated types in $S_1(\emptyset)$, $1 \leq \dim(\Gamma) < \omega$. Then $T$ is Ehrenfeucht $\iff$ $T$ is almost $\omega$-categorical.

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**Definition.** Let $T$ be a complete theory, $\mathcal{M} \models T$. Consider types $p(x), q(y) \in S(\emptyset)$, realized in $\mathcal{M}$, and all $(p, q)$-semi-isolating, or $(p, q)$-preserving formulas $\varphi(x, y)$ of $T$, i.e., formulas for which there is $a \in M$ such that $\models p(a)$ and $\varphi(a, y) \vdash q(y)$. Now, for each such a formula $\varphi(x, y)$, we define a binary relation $R_{p,\varphi,q} \coloneqq \{(a, b) \mid \mathcal{M} \models p(a) \land \varphi(a, b)\}$. If $(a, b) \in R_{p,\varphi,q}$, then $(a, b)$ is called a $(p, \varphi, q)$-arc. If $\varphi(a, y)$ is principal (over $a$), the $(p, \varphi, q)$-arc $(a, b)$ is also principal. If besides $\varphi(x, b)$ is a principal formula (over $b$) then the set $[a, b] \coloneqq \{(a, b), (b, a)\}$ is a principal $(p, \varphi, q)$-edge.
For types $p(x), q(y) \in S(\emptyset)$, we denote by $\text{PF}(p, q)$ the set

$$\{ \varphi(x, y) \mid \varphi(a, y) \text{ is a principal formula, } \varphi(a, y) \vdash q(y), \text{ where } \models p(a) \}.$$ 

Let $\text{PE}(p, q)$ be the set of all pairs $(\varphi(x, y), \psi(x, y))$ of formulas in $\text{PF}(p, q)$ such that for any (some) realization $a$ of $p$ the sets of solutions for $\varphi(a, y)$ and $\psi(a, y)$ coincide.

Clearly, $\text{PE}(p, q)$ is an equivalence relation on the set $\text{PF}(p, q)$. 
Thus the quotient $\text{PF}(p, q)/\text{PE}(p, q)$ is represented as a disjoint union of sets $\text{PFS}(p, q)$ and $\text{PFN}(p, q)$, where $\text{PFS}(p, q)$ consists of $\text{PE}(p, q)$-classes corresponding to principal edges and $\text{PFN}(p, q)$ consists of $\text{PE}(p, q)$-classes corresponding to irreversible principal arcs.
Let $T$ be a complete theory, $U = U^- \cup \{0\} \cup U^+$ be an alphabet of cardinality $\geq |S(T)|$, consisting of negative elements $u^- \in U^-$, positive elements $u^+ \in U^+$, and zero $0$. As usual, we write $u < 0$ for any $u \in U^-$ and $u > 0$ for any $u \in U^+$. The set $U^- \cup \{0\}$ is denoted by $U^{\leq 0}$ and $U^+ \cup \{0\}$ is denoted by $U^{\geq 0}$. Elements of $U$ are called labels.
Let $\nu(p, q): PF(p, q)/PE(p, q) \to U$ be an injective labelling functions, $p(x), q(y) \in S(\emptyset)$, for which negative elements correspond to classes in $PFN(p, q)/PE(p, q)$ and non-negative elements correspond to classes in $PFS(p, q)/PE(p, q)$ such that 0 is defined only for $p = q$ and is represented by the formula $(x \approx y)$, $\nu(p) \equiv \nu(p, p)$. We additionally suppose that $\rho_{\nu(p)} \cap \rho_{\nu(q)} = \{0\}$ for $p \neq q$ (where, as usual, we denote by $\rho_f$ the image of the function $f$) and $\rho_{\nu(p, q)} \cap \rho_{\nu(p', q')} = \emptyset$ if $p \neq q$ and $(p, q) \neq (p', q')$. Labelling functions with the properties above as well families of these functions are said to be regular. Below we shall consider only regular labelling functions and their regular families.
We denote by $\theta_{p,u,q}(x, y)$ a formula in $PF(p, q)$ with the label $u \in \rho_{\nu(p,q)}$. If a type $p$ is fixed and $p = q$ then a formula $\theta_{p,u,q}(x, y)$ is denoted by $\theta_u(x, y)$.

For types $p_1, p_2, \ldots, p_{k+1} \in S^1(\emptyset)$ and sets $X_1, X_2, \ldots, X_k \subseteq U$ of labels we denote by

$$P(p_1, X_1, p_2, X_2, \ldots, p_k, X_k, p_{k+1})$$

the set of all labels $u \in U$ corresponding to formulas $\theta_{p_1,u,p_{k+1}}(x, y)$ satisfying, for realizations $a$ of $p_1$ and some $u_1 \in X_1, \ldots, u_k \in X_k$, the following condition:

$$\theta_{p_1,u,p_{k+1}}(a, y) \vdash \theta_{p_1,u_1,p_2,u_2,\ldots,p_k,u_k,p_{k+1}}(a, y),$$

where

$$\theta_{p_1,u_1,p_2,u_2,\ldots,p_k,u_k,p_{k+1}}(x, y) \Rightarrow$$

$$\Rightarrow \exists x_2, x_3, \ldots x_k (\theta_{p_1,u_1,p_2}(x, x_2) \land \theta_{p_2,u_2,p_3}(x_2, x_3) \land \ldots \land \theta_{p_{k-1},u_{k-1},p_k}(x_{k-1}, x_k) \land \theta_{p_k,u_k,p_{k+1}}(x_k, y)).$$
Thus the Boolean $\mathcal{P}(U)$ of $U$ is the universe of an algebra of distributions of binary isolating formulas with $k$-ary operations

$$P(p_1, \cdot, p_2, \cdot, \ldots, p_k, \cdot, p_{k+1}),$$

where $p_1, \ldots, p_{k+1} \in S^1(\emptyset)$. 
Definitions

If all types $p_i$ equal to a type $p$ then we write $P_p(X_1, X_2, \ldots, X_k)$ and $P_p(u_1, u_2, \ldots, u_k)$ as well as $[X_1, X_2, \ldots, X_k]_p$ and $[u_1, u_2, \ldots, u_k]_p$ instead of

$$P(p_1, X_1, p_2, X_2, \ldots, p_k, X_k, p_{k+1})$$

and

$$P(p_1, u_1, p_2, u_2, \ldots, p_k, u_k, p_{k+1})$$

respectively.

We set $\mathcal{P}_{\nu(p)} = \langle \mathcal{P}(\rho_{\nu(p)}) \setminus \{\emptyset\}; [\cdot, \cdot]_p \rangle$. The groupoid $\mathcal{P}_{\nu(p)}$ is called the groupoid of binary isolating formulas over the labelling function $\nu(p)$ or the $I_{\nu(p)}$-groupoid. Below the operation $[\cdot, \cdot]$ will be also denoted by $\cdot$ and we write $uv$ instead of $u \cdot v$. 
Since by the choice of the label 0 for the formula \((x \approx y)\) the equalities \(X \cdot \{0\} = X\) and \(\{0\} \cdot X = X\) are true for any \(X \subseteq \rho_{\nu(p)}\), the groupoid \(\mathcal{P}_{\nu(p)}\) has the unit \(\{0\}\), and it is a monoid if the algebra is right semi-associative. We have

\[ Y \cdot Z = \bigcup \{yz \mid y \in Y, z \in Z\} \]

for any sets \(Y, Z \in \mathcal{P}(\rho_{\nu(p)}) \setminus \{\emptyset\}\) in this structure.

Groupoids \(\mathcal{P}_{\nu(p)}\) are naturally extensible to groupoids \(\mathcal{P}_{\nu(R)}\) for binary isolating formulas on nonempty families \(R \subseteq S^1(\emptyset)\) of types.
Quasirational and irrational types

Definition

Let $M$ be a weakly o-minimal structure, $A \subseteq M$, $p \in S_1(A)$ be a non-algebraic type.

We say that $p$ is *quasirational to the right (left)* if there is an $A$-definable convex formula $U_p(x) \in p$ such that $U_p(N)^+ = p(N)^+$ ($U_p(N)^- = p(N)^-$) for a sufficiently saturated model $N \succ M$.

A non-isolated 1-type is called *quasirational* if it is either quasirational to the right or quasirational to the left.

A non-quasirational non-isolated 1-type is called *irrational*.

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We construct by induction a sequence of embedded monoids $\mathcal{A}_n$, $n \in \omega$.

For $\mathcal{A}_0$, we take the algebra of distributions of isolating formulas with unique label 0 and the operation defined by equation $0 \cdot 0 = \{0\}$ (the algebra for an algebraic type $r$).

The algebra $\mathcal{A}_1$ (for $RC(r) = 1$) is defined by the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>${0}$</td>
<td>${1}$</td>
<td>${2}$</td>
</tr>
<tr>
<td>1</td>
<td>${1}$</td>
<td>${1}$</td>
<td>${0, 1, 2}$</td>
</tr>
<tr>
<td>2</td>
<td>${2}$</td>
<td>${0, 1, 2}$</td>
<td>${2}$</td>
</tr>
</tbody>
</table>
For $A_2$ (with $RC(r) = 2$), we take the monoid defined by the following table:

$$
\begin{array}{|c|c|c|c|c|}
\hline
\cdot & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & \{0\} & \{1\} & \{2\} & \{3\} & \{4\} \\
1 & \{1\} & \{1\} & \{0, 1, 2\} & \{3\} & \{4\} \\
2 & \{2\} & \{0, 1, 2\} & \{2\} & \{3\} & \{4\} \\
3 & \{3\} & \{3\} & \{3\} & \{3\} & \{0, 1, 2, 3, 4\} \\
4 & \{4\} & \{4\} & \{4\} & \{0, 1, 2, 3, 4\} & \{4\} \\
\hline
\end{array}
$$

The monoids $A_0$, $A_1$, $A_2$ form the chain $A_0 \subset A_1 \subset A_2$. 
If the monoid $A_n$ (for $RC(r) = n$) is already constructed, we define its extension $A_{n+1}$ (for $RC(r) = n + 1$) by adding of the labels $2n + 1$ and $2n + 2$, with the operation $\cdot$ on Boolean of the set $\{0, 1, \ldots, 2n+2\}$ defined by the following table:

<table>
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<tr>
<th></th>
<th>0</th>
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<th>2</th>
<th>$\ldots$</th>
<th>$2n + 1$</th>
<th>$2n + 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>${0}$</td>
<td>${1}$</td>
<td>${2}$</td>
<td>$\ldots$</td>
<td>${2n + 1}$</td>
<td>${2n + 2}$</td>
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<tr>
<td>1</td>
<td>${1}$</td>
<td>${1}$</td>
<td>${0, 1, 2}$</td>
<td>$\ldots$</td>
<td>${2n + 1}$</td>
<td>${2n + 2}$</td>
</tr>
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<td>2</td>
<td>${2}$</td>
<td>${0, 1, 2}$</td>
<td>${2}$</td>
<td>$\ldots$</td>
<td>${2n + 1}$</td>
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<td>${2n + 1}$</td>
<td>${0, 1, \ldots, 2n + 2}$</td>
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<tr>
<td>$2n + 2$</td>
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<td>${2n + 2}$</td>
<td>${2n + 2}$</td>
<td>$\ldots$</td>
<td>${0, 1, \ldots, 2n + 2}$</td>
<td>${2n + 2}$</td>
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</tbody>
</table>
Any monoid, isomorphic to a monoid \( \mathfrak{A}_n \), is called the monoid of isolating formulas for a 1-type of a countably categorical weakly o-minimal theory, having the binary convexity rank \( n \), or briefly the \((P, \aleph_0, n)\)-wom-monoid.

**Theorem**

Let \( T \) be a countably categorical weakly o-minimal theory. Then for any type \( r \in S^1(\emptyset) \) and a natural \( n \) the following conditions are equivalent:

1. the algebra \( \mathfrak{P}_{\nu}(r) \) is a \((P, \aleph_0, n)\)-wom-monoid;
2. \( RC_{\text{bin}}(r) = n \).

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The monoid of binary isolating formulas for a quasirational to the right 1-type $p$, with the convexity rank $n$, in an arbitrary weakly o-minimal theory, is called the $(P, QR, n)$-wom-monoid and denoted by $A_n^{QR}$.

Similarly, the monoid of binary isolating formulas for a quasirational to the left 1-type, with the convexity rank $n$, is called the $(P, QL, n)$-wom-monoid and denoted by $A_n^{QL}$. 
We assert that the type $p$ has the algebra $A^{QR}_n$ of isolating formulas, which consists of $2n$ labels and defined by the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>$2n - 3$</th>
<th>$2n - 2$</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{1}</td>
<td>{2}</td>
<td>...</td>
<td>{2n - 3}</td>
<td>{2n - 2}</td>
<td>{−1}</td>
</tr>
<tr>
<td>1</td>
<td>{1}</td>
<td>{1}</td>
<td>{0, 1, 2}</td>
<td>...</td>
<td>{2n - 3}</td>
<td>{2n - 2}</td>
<td>{−1}</td>
</tr>
<tr>
<td>2</td>
<td>{2}</td>
<td>{0, 1, 2}</td>
<td>{2}</td>
<td>...</td>
<td>{2n - 3}</td>
<td>{2n - 2}</td>
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<tr>
<td>...</td>
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</tr>
<tr>
<td>$2n - 3$</td>
<td>{2n - 3}</td>
<td>{2n - 3}</td>
<td>{2n - 3}</td>
<td>...</td>
<td>{2n - 3}</td>
<td>{0, 1, ..., 2n - 2}</td>
<td>{−1}</td>
</tr>
<tr>
<td>$2n - 2$</td>
<td>{2n - 2}</td>
<td>{2n - 2}</td>
<td>{2n - 2}</td>
<td>...</td>
<td>{0, 1, ..., 2n - 2}</td>
<td>{2n - 2}</td>
<td>{−1}</td>
</tr>
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<td>−1</td>
<td>{−1}</td>
<td>{−1}</td>
<td>{−1}</td>
<td>...</td>
<td>{−1}</td>
<td>{−1}</td>
<td>{−1}</td>
</tr>
</tbody>
</table>
Replacing the sign $<$, in the structure $\mathcal{M}$ and in the formulas $\theta$, by $>$, we obtain the algebra $\mathcal{A}_n^{QL}$ with the same table. Notice that the algebras $\mathcal{A}_n^{QR}$ and $\mathcal{A}_n^{QL}$ are really the monoids. Here the monoids $\mathcal{A}_n^{QR}$ and $\mathcal{A}_n^{QL}$ have submonoids generated by non-negative labels $0, 1, \ldots, 2n - 2$ and by non-positive labels $0, -1$. 
We assert that an irrational type \( p \) of the convexity rank \( n \) has the algebra \( \mathcal{A}_n^I \) of isolating formulas, which consists of \( 2n - 1 \) labels and defined by the following table:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>1</th>
<th>2</th>
<th>( 2n - 3 )</th>
<th>( 2n - 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{1}</td>
<td>{2}</td>
<td>( \ldots )</td>
<td>{2n - 3}</td>
</tr>
<tr>
<td>1</td>
<td>{1}</td>
<td>{1}</td>
<td>{0, 1, 2}</td>
<td>( \ldots )</td>
<td>{2n - 3}</td>
</tr>
<tr>
<td>2</td>
<td>{2}</td>
<td>{0, 1, 2}</td>
<td>{2}</td>
<td>( \ldots )</td>
<td>{2n - 3}</td>
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<tr>
<td></td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>2n - 3</td>
<td>{2n - 3}</td>
<td>{2n - 3}</td>
<td>{2n - 3}</td>
<td>( \ldots )</td>
<td>{2n - 3}</td>
</tr>
<tr>
<td>2n - 2</td>
<td>{2n - 2}</td>
<td>{2n - 2}</td>
<td>{2n - 2}</td>
<td>{0, 1, \ldots, 2n - 2}</td>
<td>{2n - 2}</td>
</tr>
</tbody>
</table>

Clearly, the algebra \( \mathcal{A}_n^I \) is isomorphic to the \((P, \aleph_0, n)\)-wom-monoid \( \mathcal{A}_{n-1} \).
Theorem (D.Yu. Emel’yanov, B.Sh. Kulpeshov, S.V. Sudoplatov)

Let $T$ be a quite o-minimal theory with few countable models, $p \in S_1(\emptyset)$. Then there is $n < \omega$ such that:

1. if $p$ is isolated then the algebra $\mathcal{V}_\nu(p)$ is the $(P, \aleph_0, n)$-wom-monoid consisting of $2n + 1$ labels;
2. if $p$ is quasirational to the right (left) then the algebra $\mathcal{V}_\nu(p)$ is the $(P, QR, n)$-wom-monoid ($(P, QL, n)$-wom-monoid) consisting of $2n$ labels;
3. if $p$ is irrational then the algebra $\mathcal{V}_\nu(p)$ is the $(P, I, n)$-wom-monoid consisting of $2n - 1$ labels.
Corollary (D.Yu. Emel’yanov, B.Sh. Kulpeshov, S.V. Sudoplatov)

Let $T$ be a quite o-minimal theory with few countable models, $p, q \in S_1(\emptyset)$. Then the algebras $\mathcal{P}_\nu(p)$ and $\mathcal{P}_\nu(q)$ are isomorphic if and only if $RC(p) = RC(q)$ and the types $p$ and $q$ are simultaneously either isolated, or quasirational, or irrational.
Definition

We say that the algebra $\mathfrak{A}_\nu(\{p, q\})$ is generalized commutative if there is a bijection $\pi : \rho_\nu(p) \to \rho_\nu(q)$ witnessing that the algebras $\mathfrak{A}_\nu(p)$ and $\mathfrak{A}_\nu(q)$ are isomorphic (i.e., the defining tables coincide up to $\pi$) and for any $l \in \rho_\nu(p, q)$, $m \in \rho_\nu(q, p)$ we have $\pi(l \cdot m) = m \cdot l$. 

---

Theorem

Let $T$ be a countably categorical weakly o-minimal theory, $p, q \in S^1(\emptyset)$. Then the following conditions are equivalent:

1. the algebra $\mathfrak{P}_\nu(\{p, q\})$ is a generalized commutative monoid;
2. $RC_{bin}(p) = RC_{bin}(q)$.

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Algebras of formulas for families of 1-types of quite o-minimal theories with few countable models

Theorem (D.Yu. Emel’yanov, B.Sh. Kulpeshov, S.V. Sudoplatov)

Let $T$ be a quite o-minimal theory with few countable models, $p, q \in S^1(\emptyset)$, $p \not\perp^w q$. Then the algebra $\mathfrak{P}_\nu(\{p, q\})$ is a generalized commutative monoid.
Almost deterministic algebras for polygonometrical theories

An algebra $\mathcal{A}$ of binary formulas of a theory is called \textit{(almost) deterministic} if for any labels $u$ and $v$ the set $u \cdot v$ is a singleton (finite). For a deterministic algebra $\mathcal{A}$ with a set $U$ of labels the algebra $\langle U; \ast \rangle$ with the operation $u \ast v = w$, where $u \cdot v = \{w\}$, is denoted by $\mathcal{A}'$.

**Theorem**

An algebra $\mathfrak{X}_{\nu(p)}$ of binary isolating formulas of a polygonometrical theory $T(pm)$, where $pm = pm(G_1, G_2, \mathcal{P})$, is deterministic if and only if some of the following conditions is satisfied:

1. $|G_1| = 1$ and $c(pm) \leq 2$;
2. $1 < |G_1| < \omega$, $|G_2| = 1$ and $c(pm) = 1$;
3. $|G_1| \geq \omega$ and $|G_2| = 1$.

Here, in the case (1), the algebra $\mathfrak{X}'_{\nu(p)}$ is isomorphic to the unit group or the group $\mathbb{Z}_2$, and for the cases (2) and (3) this algebra is isomorphic to the group $G_1$.

---

Theorem

An algebra $\mathfrak{P}_{\nu(p)}$ of binary isolating formulas of a polygonometrical theory $T(pm)$ of an everywhere finitely defined polygonometry $pm = pm(G_1, G_2, \mathcal{P})$ is almost deterministic if and only if the group $G_1$ is a singleton or the group $G_2$ is finite.

---

An algebra $\mathfrak{P}_{\nu(p)}$ for a theory $T(\text{spm})$ is called pseudo-euclidian if for every labels $u$ and $v$, correspondent to predicates $Q_{g_1}$ and $Q_{g'_1}$, $g_1, g'_1 \in \text{Pos}(G_1)$, the set $u \cdot v$ consists of all labels $w$ correspondent to predicates $Q_g$ with $|g_1 - g'_1| \leq g \leq g_1 + g'_1$.

Basic examples of pseudo-euclidian algebras are represented by the theories of classical and spherical trigonometries.

**Theorem (D.Yu. Emel’yanov, S.V. Sudoplatov)**

For any commutative linearly ordered group $G_1$ there is a $s$-trigonometry $\text{strm} = \text{strm}(G_1, G_2, \mathcal{P})$ such that the theory $T(\text{strm})$ has a pseudo-euclidian algebra of binary isolating formulas.
An algebra $P_{\nu}(p)$ for a theory $T(spm)$ is called interval if for every labels $u$ and $v$, correspondent to predicates $Q_{g_1}$ and $Q_{g_1'}$, $g_1, g_1' \in \text{Pos}(G_1)$, the set $u \cdot v$ consists of all labels $w$, correspondent to predicates $Q_g$ such that the elements $g$ form an interval $I_{g_1, g_1'}$ in $\text{Pos}(G_1)$. Here each set $u \cdot v$ is denoted by $I_{u, v}$ and forms an interval with respect to the linear order of the group $G_1$.

Thus each pair $(u, v)$ defines both an interval $I_{g_1, g_1'}$ and an interval $I_{u, v}$ obtained by $I_{g_1, g_1'}$.

Note that since $|g_1 - g_1'|, g_1 + g_1' \in I_{g_1, g_1'}$ then

$$[|g_1 - g_1'|, g_1 + g_1'] \subseteq I_{g_1, g_1'}.$$ (2)

The system $I$ of intervals $I_{g_1, g_1'}$, satisfying (2), is called coordinated with multiplications of labels.

Clearly, equalities in (2) means that the algebra is pseudo-euclidian.
Theorem (D.Yu. Emel’yanov, S.V. Sudoplatov)

For any commutative linearly ordered group $G_1$ and a coordinated system $\mathcal{I}$ of intervals there is a $s$-trigonometry $\text{strm} = \text{strm}(G_1, G_2, \mathcal{P})$ such that the theory $T(\text{strm})$ has an interval algebra of binary isolating formulas defining the system $\mathcal{I}$. 
Hypergraphs of models: references

Recall that a *hypergraph* is a pair \((X, Y)\) of sets, where \(Y\) is a subset of the set \(\mathcal{P}(X)\) being the set of all subsets of \(X\).

Let \(\mathcal{M}\) be a model of a theory \(T\). Denote by \(Y(\mathcal{M})\) the set of all subsets in the universe \(M\) of \(\mathcal{M}\) such that these subsets are universes of elementary submodels of \(\mathcal{M}\). Thus we have the *hypergraph* \((\mathcal{M}, Y(\mathcal{M}))\) of elementary submodels of \(\mathcal{M}\).

Restricting \(Y(\mathcal{M})\) to the families of prime, minimal, prime minimal submodels, respectively, we get hypergraphs \(H_{pr}(\mathcal{M}), H_{min}(\mathcal{M}), H(\mathcal{M})\), with the universe \(\mathcal{M}\).
Definition. Let $\mathcal{M}$ be a model of a theory $T$, with a hypergraph $\mathcal{H} = (\mathcal{M}, H(\mathcal{M}))$ of elementary submodels, $A$ be an infinite definable set in $\mathcal{M}$, of arity $n$: $A \subseteq M^n$. The set $A$ is called $\mathcal{H}$-free if for any infinite set $A' \subseteq A$ we have $A' = A \cap Z^n$ for some $Z \in H(\mathcal{M})$ containing parameters for $A$. Two $\mathcal{H}$-free sets $A$ and $B$ of arities $m$ and $n$, respectively, are called $\mathcal{H}$-independent if for all infinite sets $A' \subseteq A$ and $B' \subseteq B$ there is $Z \in H(\mathcal{M})$ containing parameters for $A$ and $B$ and such that $A' = A \cap Z^m$ and $B' = B \cap Z^n$. 

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Properties

1. Any two tuples of \( \mathcal{H} \)-free set \( A \), whose tuples do not have common coordinates, have same type.

2. If \( A \subseteq M \) is a \( \mathcal{H} \)-free set then \( A \) does not have non-trivial definable sets, with parameters in \( A \), i.e., subsets which differ from subsets defined by (in)equalities with elements in \( A \).

3. If \( A \) and \( B \) are \( \mathcal{H} \)-independent, where \( A \cup B \) does not have distinct tuples with common coordinates, then \( A \cap B = \emptyset \).
**Definition.** A complete union of hypergraphs \((X_i, Y_i), i \in I\), is a hypergraph \(\left( \bigcup_{i \in I} X_i, Y \right)\), where \(Y = \left\{ \bigcup_{i \in I} Z_i \mid Z_i \in Y_i \right\}\). If the sets \(X_i\) are disjoint then the complete union is called *disjoint* too. If the sets \(X_i\) form a \(\subseteq\)-chain then the complete union is called *chain*.

**Theorem (with B.Sh. Kulpeshov)**

Any restriction of a hypergraph \(\mathcal{H} = (M, H(M))\) to a union of family of \(\mathcal{H}\)-free \(\mathcal{H}\)-independent sets \(A_i \subseteq M\) can be represented as a disjoint complete union of restrictions \(\mathcal{H}_i\) of \(\mathcal{H}\) to the sets \(A_i\).

Applying to theories of unars, if sets \(A_i = f^{-k_i}(a_i), k_i > 0\), are \(\mathcal{H}\)-independent then the restriction \(\mathcal{H}\) to \(\bigcup_{i} A_i\) is representable as a disjoint complete union of restrictions \(\mathcal{H}_i = \mathcal{H}\big|_{A_i}\).
Denote by \((M, H_{\text{dlo}}(M))\) the hypergraph of (prime) elementary submodels of a countable model \(M\) of the theory of dense linear order without endpoints.

Notice that the class of hypergraphs \((M, H_{\text{dlo}}(M))\) is closed under countable chain complete unions having encompassed by a dense linear order. Thus any hypergraph \((M, H_{\text{dlo}}(M))\) is representable as a countable chain complete union of some their proper subhypergraphs.

**Theorem (with B.Sh. Kulpeshov)**

A hypergraph of prime models of any countable model of an Ehrenfeucht-type theory is representable as a disjoint complete union of some hypergraphs \((M, H_{\text{dlo}}(M))\) as well as some singletons \((\{c\}, \{\{c\}\})\).
Theorem

For any Abelian group \( A \) exactly one of the following conditions is satisfied:

1) \( \text{Hp}(A) = \emptyset \);
2) \( |\text{Hp}(A)| = 1 \) (with finite Szmielew invariants \( \alpha_{p,n}, \beta_p, \gamma_p \) besides that ones for which \( |\{ n \mid \alpha_{p,n} \neq 0 \}| = \omega \), as well as with \( |\text{Hp}(A/T(A))| = 1 \) for the periodic part \( T(A) \) of \( A \), and with at most one group \( Q \) obtained from \( A \) as direct summand forming a prime model \( A' \preceq A \));
3) \( |\text{Hp}(A)| \geq \omega \); here the presence of infinite invariant \( \alpha_{p,n} \), or infinite \( \beta_p \), or infinite \( \gamma_p \) besides that ones for which \( |\{ n \mid \alpha_{p,n} \neq 0 \}| = \omega \), or the group \( Q^{(\omega)} \), obtained from \( A \) as direct summand, with \( Q \) in a prime model \( A' \preceq A \), implies \( |\text{Hp}(A)| \geq 2^\omega \); if \( A \) has \( \lambda \geq \omega \) copies of the group \( Q \), forming the direct sum, then \( |\text{Hp}(A)| = 2^{\lambda} \).
Theorem

For any Abelian group $\mathcal{A}$ or the form
\[
\bigoplus_{p, n} \mathbb{Z}^{(\alpha_p, n)}(p, n) \bigoplus \bigoplus \mathbb{Z}^{(\beta_p)}(p, \infty) \bigoplus \bigoplus \mathbb{R}^{(\gamma_p)}(p) \bigoplus \mathbb{Q}(\varepsilon), \quad \varepsilon \in \{0, 1\}, \quad (3)
\]

exactly one of the following conditions is satisfied:
1) $\text{Hm}(\mathcal{A}) = \emptyset$;
2) $|\text{Hm}(\mathcal{A})| = 1$ (with finite Szmielew invariants $\alpha_p, n$, $\beta_p$, $\gamma_p$ besides that ones for which $|\{ n \mid \alpha_p, n \neq 0 \}| = \omega$, as well as with $|\text{Hm}(\mathcal{A}/ T(\mathcal{A}))| = 1$).

Allowing to vary $\varepsilon$ in the Abelian group $\mathcal{A}$ of the form (3) we have $|\text{Hm}(\mathcal{A})| \geq \omega$ for $\varepsilon \geq 2$, $\beta_p = \gamma_p = 0$ with any $p$ and finite $|\{ \langle p, n \rangle \mid \alpha_p, n \neq 0 \}| < \omega$. If additionally $\varepsilon \geq \omega$ then $|\text{Hm}(\mathcal{A})| \geq 2\omega$. 
Hypergraphs for Abelian groups

Theorem
For any Abelian group $\mathcal{A}$ the following conditions are equivalent:
1) $H(\mathcal{A}) \neq \emptyset$;
2) all Szmielew invariants for $\mathcal{A}$ are finite and some of the following conditions is satisfied: a) $\mathcal{A} \equiv T(\mathcal{A})$; b) the reduced part of $T(\mathcal{A})$ is bounded and $H(\mathcal{A}/T(\mathcal{A})) \neq \emptyset$, i.e., there is a finite set $P_0$ of prime numbers and $\lambda \in \omega$ such that $\gamma_p(\mathcal{A}/T(\mathcal{A})) = \lambda$ for $p \in P_0$ and $\gamma_p(\mathcal{A}/T(\mathcal{A})) = 0$ for $p \notin P_0$. 

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Almost disjoint hypergraphs for Abelian groups

For theories in a class $\mathcal{T}_H^a$, with acyclic hypergraphs of minimal prime models, there are three possibilities for these hypergraphs: disjoint (i.e., with disjoint universes), almost disjoint (with finite pairwise intersections) with diameter 2, almost disjoint with infinite diameter.

Theorem (with K.A. Baikalova)

A theory $\text{Th}(\mathcal{A})$ of an infinite Abelian group $\mathcal{A}$ has an almost disjoint hypergraph of minimal prime models, for an $\omega$-saturated model, if and only if there are only finitely many non-zero Szmielew invariants $\alpha_{p,n}$, and all invariants $\beta_p$, $\gamma_p$ equal zero.
Generic classes and their limits: references

- S.V. Sudoplatov, Y. Kiouvrekis, P. Stefaneas, Definable sets in generic structures and their cardinalities // Siberian Advances in Mathematics. (accepted)


Let $P = (P_i)_{i \in I}$, be a family of nonempty unary predicates, $(\mathcal{A}_i)_{i \in I}$ be a family of structures such that $P_i$ is the universe of $\mathcal{A}_i$, $i \in I$, and the symbols $P_i$ are disjoint with languages for the structures $\mathcal{A}_j$, $j \in I$. The structure $\mathcal{A}_P \models \bigcup_{i \in I} \mathcal{A}_i$ expanded by the predicates $P_i$ is the $P$-union of the structures $\mathcal{A}_i$, and the operator mapping $(\mathcal{A}_i)_{i \in I}$ to $\mathcal{A}_P$ is the $P$-operator. The structure $\mathcal{A}_P$ is called the $P$-combination of the structures $\mathcal{A}_i$ and denoted by $\text{Comb}_P(\mathcal{A}_i)_{i \in I}$ if $\mathcal{A}_i = (\mathcal{A}_P|_{\mathcal{A}_i})|_{\Sigma(\mathcal{A}_i)}$, $i \in I$. Structures $\mathcal{A}'$, which are elementary equivalent to $\text{Comb}_P(\mathcal{A}_i)_{i \in I}$, will be also considered as $P$-combinations.
Clearly, all structures $A' \equiv \text{Comb}_P(A_i)_{i \in I}$ are represented as unions of their restrictions $A'_i = (A'|_{P_i})|_{\Sigma(A_i)}$ if and only if the set $p_\infty(x) = \{-P_i(x) | \ i \in I\}$ is inconsistent. If $A' \neq \text{Comb}_P(A'_i)_{i \in I}$, we write $A' = \text{Comb}_P(A'_i)_{i \in I \cup \{\infty\}}$, where $A'_\infty = A'|_{\bigcap_{i \in I} \overline{P_i}}$, maybe applying Morleyzation. Moreover, we write $\text{Comb}_P(A_i)_{i \in I \cup \{\infty\}}$ for $\text{Comb}_P(A_i)_{i \in I}$ with the empty structure $A_\infty$. 
Note that if all predicates $P_i$ are disjoint, a structure $A_P$ is a $P$-combination and a disjoint union of structures $A_i$. In this case the $P$-combination $A_P$ is called *disjoint*. Clearly, for any disjoint $P$-combination $A_P$, $\text{Th}(A_P) = \text{Th}(A'_P)$, where $A'_P$ is obtained from $A_P$ replacing $A_i$ by pairwise disjoint $A'_i \equiv A_i$, $i \in I$. Thus, in this case, similar to structures the $P$-operator works for the theories $T_i = \text{Th}(A_i)$ producing the theory $T_P = \text{Th}(A_P)$, being $P$-combination of $T_i$, which is denoted by $\text{Comb}_P(T_i)_{i \in I}$. In general, for non-disjoint case, the theory $T_P$ will be also called a $P$-combination of the theories $T_i$, but in such a case we will keep in mind that this $P$-combination is constructed with respect (and depending) to the structure $A_P$, or, equivalently, with respect to any/some $A' \equiv A_P$. 
For an equivalence relation $E$ replacing disjoint predicates $P_i$ by $E$-classes we get the structure $\mathcal{A}_E$ being the $E$-union of the structures $\mathcal{A}_i$. In this case the operator mapping $(\mathcal{A}_i)_{i \in I}$ to $\mathcal{A}_E$ is the $E$-operator. The structure $\mathcal{A}_E$ is also called the $E$-combination of the structures $\mathcal{A}_i$ and denoted by $\text{Comb}_E(\mathcal{A}_i)_{i \in I}$; here $\mathcal{A}_i = (\mathcal{A}_E |_{\mathcal{A}_i}) |_{\Sigma(\mathcal{A}_i)}$, $i \in I$. Similar above, structures $\mathcal{A}'$, which are elementary equivalent to $\mathcal{A}_E$, are denoted by $\text{Comb}_E(\mathcal{A}'_j)_{j \in J}$, where $\mathcal{A}'_j$ are restrictions of $\mathcal{A}'$ to its $E$-classes. The $E$-operator works for the theories $T_i = \text{Th}(\mathcal{A}_i)$ producing the theory $T_E = \text{Th}(\mathcal{A}_E)$, being $E$-combination of $T_i$, which is denoted by $\text{Comb}_E(T_i)_{i \in I}$ or by $\text{Comb}_E(T)$, where $T = \{T_i \mid i \in I\}$. 
Clearly, $\mathcal{A}' \equiv \mathcal{A}_P$ realizing $p_\infty(x)$ is not elementary embeddable into $\mathcal{A}_P$ and can not be represented as a disjoint $P$-combination of $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. At the same time, there are $E$-combinations such that all $\mathcal{A}' \equiv \mathcal{A}_E$ can be represented as $E$-combinations of some $\mathcal{A}'_j \equiv \mathcal{A}_i$. We call this representability of $\mathcal{A}'$ to be the $E$-representability.
If there is $\mathcal{A}' \equiv \mathcal{A}_E$ which is not $E$-representable, we have the $E'$-representability replacing $E$ by $E'$ such that $E'$ is obtained from $E$ adding equivalence classes with models for all theories $T$, where $T$ is a theory of a restriction $\mathcal{B}$ of a structure $\mathcal{A}' \equiv \mathcal{A}_E$ to some $E$-class and $\mathcal{B}$ is not elementary equivalent to the structures $\mathcal{A}_i$. The resulting structure $\mathcal{A}_{E'}$ (with the $E'$-representability) is a e-completion, or a e-saturation, of $\mathcal{A}_E$. The structure $\mathcal{A}_{E'}$ itself is called e-complete, or e-saturated, or e-universal, or e-largest.
For a structure $\mathcal{A}_E$ the number of new structures with respect to the structures $\mathcal{A}_i$, i.e., of the structures $\mathcal{B}$ which are pairwise elementary non-equivalent and elementary non-equivalent to the structures $\mathcal{A}_i$, is called the \textit{e-spectrum} of $\mathcal{A}_E$ and denoted by $e\text{-Sp}(\mathcal{A}_E)$. The value $\sup\{e\text{-Sp}(\mathcal{A}') \mid \mathcal{A}' \equiv \mathcal{A}_E\}$ is called the \textit{e-spectrum} of the theory $\text{Th}(\mathcal{A}_E)$ and denoted by $e\text{-Sp}(\text{Th}(\mathcal{A}_E))$. 
If $A_E$ does not have $E$-classes $A_i$, which can be removed, with all $E$-classes $A_j \equiv A_i$, preserving the theory $\text{Th}(A_E)$, then $A_E$ is called $e$-prime, or $e$-minimal.

For a structure $A' \equiv A_E$ we denote by $\text{TH}(A')$ the set of all theories $\text{Th}(A_i)$ of $E$-classes $A_i$ in $A'$.

By the definition, an $e$-minimal structure $A'$ consists of $E$-classes with a minimal set $\text{TH}(A')$. If $\text{TH}(A')$ is the least for models of $\text{Th}(A')$ then $A'$ is called $e$-least.
**Definition** [?]. Let $\mathcal{T}$ be the class of all complete elementary theories of relational languages. For a set $\mathcal{T} \subset \mathcal{T}$ we denote by $\text{Cl}_E(\mathcal{T})$ the set of all theories $\text{Th}(\mathcal{A})$, where $\mathcal{A}$ is a structure of some $E$-class in $\mathcal{A}' \equiv \mathcal{A}_E$, $\mathcal{A}_E = \text{Comb}_E(\mathcal{A}_i)_{i \in I}$, $\text{Th}(\mathcal{A}_i) \in \mathcal{T}$. As usual, if $\mathcal{T} = \text{Cl}_E(\mathcal{T})$ then $\mathcal{T}$ is said to be $E$-closed.

The operator $\text{Cl}_E$ of $E$-closure can be naturally extended to the classes $\mathcal{T} \subset \mathcal{T}$ as follows: $\text{Cl}_E(\mathcal{T})$ is the union of all $\text{Cl}_E(\mathcal{T}_0)$ for subsets $\mathcal{T}_0 \subset \mathcal{T}$.

For a set $\mathcal{T} \subset \mathcal{T}$ of theories in a language $\Sigma$ and for a sentence $\varphi$ with $\Sigma(\varphi) \subseteq \Sigma$ we denote by $\mathcal{T}_\varphi$ the set $\{ \mathcal{T} \in \mathcal{T} \mid \varphi \in \mathcal{T} \}$. 
**Proposition**

If $\mathcal{T} \subset \overline{\mathcal{T}}$ is an infinite set and $\mathcal{T} \in \overline{\mathcal{T}} \setminus \mathcal{T}$ then $\mathcal{T} \in \text{Cl}_E(\mathcal{T})$ (i.e., $\mathcal{T}$ is an accumulation point for $\mathcal{T}$ with respect to $E$-closure $\text{Cl}_E$) if and only if for any formula $\varphi \in \mathcal{T}$ the set $\mathcal{T} \varphi$ is infinite.
Theorem

If $\mathcal{T}_0'$ is a generating set for an $E$-closed set $\mathcal{T}_0$ then the following conditions are equivalent:

1. $\mathcal{T}_0'$ is the least generating set for $\mathcal{T}_0$;
2. $\mathcal{T}_0'$ is a minimal generating set for $\mathcal{T}_0$;
3. any theory in $\mathcal{T}_0'$ is isolated by some set $(\mathcal{T}_0')_{\varphi}$, i.e., for any $T \in \mathcal{T}_0'$ there is $\varphi \in T$ such that $(\mathcal{T}_0')_{\varphi} = \{ T \}$;
4. any theory in $\mathcal{T}_0'$ is isolated by some set $(\mathcal{T}_0)_{\varphi}$, i.e., for any $T \in \mathcal{T}_0'$ there is $\varphi \in T$ such that $(\mathcal{T}_0)_{\varphi} = \{ T \}$.
Lattices and semilattices for families of $E$-closed sets

Let $X$ be a nonempty family of $E$-closed sets $T \subset \overline{T}$. The operations $T_1 \land T_2 \equiv T_1 \cap T_2$ and $T_1 \lor T_2 \equiv \text{Cl}_E(T_1 \cup T_2)$, for $E$-closed sets $T_1, T_2 \subset \overline{T}$, generate a set $Y$ and form a structure $\langle Y; \land, \lor \rangle$ denoted by $L(X)$. Clearly, $L(X)$ is a lattice which can be naturally expended till a complete lattice. Now we consider the following restrictions for the lattices $L(X)$. For a nonempty set $X$ of $E$-closed families with least generating sets the operation $\lor$ generates a set $Z \subset Y$ and forms an upper semilattice $\text{SLLGS}(X) = \langle Z; \lor \rangle$ restricting the universe and the language for $L(X)$:
Semilattices for families of $E$-closed sets

Theorem

1. For any nonempty set $X$ of $E$-closed families with least generating sets the structure $\text{SLLGS}(X)$ is a upper semilattice.
2. There is a upper semilattice $\text{SLLGS}(X)$ with elements $x_1, x_2 \in X$ having least generating sets and such that $x_1 \cap x_2$ does not have the least generating set.
3. There is a upper semilattice $\text{SLLGS}(X)$ which can not be extended to a complete semilattice consisting of families with least generating sets.
Now we consider a nonempty set $X$ of $E$-closed families with least generating sets and $\mathcal{T}_1, \mathcal{T}_2 \in X$ with least generating sets $\mathcal{T}'_1$ and $\mathcal{T}'_2$, respectively. Denote by $\mathcal{T}_1 \wedge' \mathcal{T}_2$ the family $\mathcal{T}_0 \subseteq \mathcal{T}_1 \cap \mathcal{T}_2$ with the greatest generating set $\mathcal{T}'_0$ consisting of all isolated points for $\mathcal{T}_1 \cap \mathcal{T}_2$. For the set $X$ the operations $\wedge'$ and $\vee$ generate a set $U \supseteq X$ with the structure $\text{LLGS}(X) \equiv \langle U; \wedge', \vee \rangle$.

**Theorem**

Any structure $\text{LLGS}(X)$ is a distributive lattice.