Linearized inverse problem for the Dirichlet-to-Neumann map on differential forms

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Abstract

For a compact n-dimensional Riemannian manifold \((M, g)\) with boundary \(i : \partial M \subset M\), the Dirichlet-to-Neumann (DN) map \(\Lambda_g : \Omega^k(\partial M) \to \Omega^{n-k-1}(\partial M)\) is defined on exterior differential forms by \(\Lambda_g \varphi = i^*(\star d\omega)\), where \(\omega\) solves the boundary value problem \(\Delta \omega = 0, i^* \omega = \varphi, i^* \delta \omega = 0\). For a symmetric second rank tensor field \(h\) on \(M\), let \(\dot{\Lambda}_h = d\Lambda_g + h/dt|_{t=0}\) be the Gateaux derivative of the DN map in the direction \(h\). We study the question: for a given \((M, g)\), how large is the subspace of tensor fields \(h\) satisfying \(\dot{\Lambda}_h = 0\)? Potential tensor fields belong to the subspace since the DN map is invariant under isomeries fixing the boundary. For a manifold of an even dimension \(n\), the DN map on \((n/2 − 1)\)-forms is conformally invariant, therefore spherical tensor fields belong to the subspace in the case of \(k = n/2 - 1\). The manifold is said to be \(\Omega^k\)-rigid if there is no other \(h\) satisfying \(\dot{\Lambda}_h = 0\). We prove that the \(\Omega^k\)-rigidity is equivalent to the density of the range of some bilinear form on the space \(\mathcal{H}^{k+1}(M)\) of exact harmonic fields.

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1. Posing the problem and formulating main statements

Throughout the paper, \((M, g)\) is a smooth compact connected oriented Riemannian manifold of dimension \(n \geq 2\) with a nonempty boundary. The term “smooth” is used as the synonym of
"$C^\infty$-smooth". By $i: \partial M \subset M$ we denote the identical embedding. Let $\Omega(M) = \bigoplus_{k=0}^{n} \Omega^k(M)$ be the graded algebra of smooth exterior differential forms. To simplify notations, we will consider only real forms although all the results can be obviously generalized to the complex case. The only exception is Section 9 where we need to use complex valued forms. We use the standard operators $d, \delta, \Delta$, and $\star$ on $\Omega(M)$.

Recall the definition of the Dirichlet-to-Neumann (DN) map introduced in [2]

$$\Lambda = \Lambda_g : \Omega(\partial M) \to \Omega(\partial M).$$

For a form $\varphi \in \Omega(\partial M)$, one has to solve the boundary value problem

$$\begin{cases}
\Delta \omega = 0, \\
i^{\ast}\omega = \varphi, \quad i^{\ast}(\delta \omega) = 0,
\end{cases} \quad (1.1)$$

and then to set

$$\Lambda \varphi = i^{\ast}(\star d \omega). \quad (1.2)$$

Problem (1.1) is solvable for any $\varphi$ and the solution is unique up to a Dirichlet harmonic field. The solution satisfies

$$\delta \omega = 0 \quad (1.3)$$

as is shown in Lemma 3.1 of [2]. If the metric $g$ is fixed, we use the short notation $\Lambda$ for the DN map. Otherwise, if different metrics are considered simultaneously, we use the notation $\Lambda_g$. Let

$$\Lambda^k = \Lambda^k_g : \Omega^k(\partial M) \to \Omega^{n-k-1}(\partial M) \quad (0 \leq k \leq n - 1),$$

be the restriction of $\Lambda$ to $k$-forms.

The inverse problem is posed as follows: to what extent is a Riemannian metric $g$ on a compact manifold determined either by the data $(\partial M, \Lambda_g)$ for all forms or by the data $(\partial M, \Lambda_k^g)$ for $k$-forms with some fixed $k$? The case of $k = 0$ is of a particular interest since this is the electro-impedance tomography problem for anisotropic media. In the case of $k = n - 1$, the full answer is given by the following

**Proposition 1.1.** For two Riemannian metrics $g$ and $g'$ on an $n$-dimensional manifold, $\Lambda_{g'}^{n-1} = \Lambda_g^{n-1}$ if and only if $\text{Vol}(M, g) = \text{Vol}(M, g')$.

This follows from Theorem 6.1 of [2].

Isometric metrics have obviously the same DN map. More precisely, the following statement holds.

**Proposition 1.2.** Let a diffeomorphism $\varphi : M \to M$ satisfy $\varphi|_{\partial M} = 1$ and, for a metric $g$, let $\varphi^* g$ be the pull back of $g$. Then $\Lambda_{\varphi^* g} = \Lambda_g$.

For a manifold of an even dimension $n$, the case of $k = n/2 - 1$ is the exceptional case since the DN map possesses the following conformal invariance.

**Proposition 1.3.** Let $n = \dim M$ be even. Then $\Lambda^{n/2-1}_{\lambda g} = \Lambda^{n/2-1}_g$ for any positive function $\lambda \in C^\infty(M)$. 
In the case of $n = 2$, the converse statement to Propositions 1.2 and 1.3 is proved in [1,6]. Besides the latter statement and Proposition 1.1, almost nothing is known on the inverse problem. The only exceptions are the case of a real analytic metric [6] and the boundary determination of the metric $g$ through $A^0_g$ [7] and through $A^k_g$ for $0 < k \neq n/2 - 1$ [5].

In this paper, we mostly study the linearized version of the inverse problem. Let $C^\infty(S^2\tau'_M)$ stand for the space of smooth symmetric covariant tensor fields of second rank on $M$. Given a Riemannian manifold $(M, g)$ and tensor field $h \in C^\infty(S^2\tau'_M)$, let us consider the one-parameter variation $g' = g + th$ of the metric $g$. For a sufficiently small $|t|$, $g'$ is a Riemannian metric. The DN map of the metric $g'$ admits the representation

$$
A_{g'} = A_g + t\dot{A}_{g,h} + o(t).
$$

We have thus defined the linear operator

$$
\dot{A}_h = \dot{A}_{g,h} : \Omega(\partial M) \to \Omega(\partial M)
$$

which is just the Gateaux derivative of the map $g' \mapsto A_{g'}$ at the point $g$ in the direction $h$. Like for the DN map, we use the short notation $\dot{A}_h$ if the metric $g$ is assumed to be fixed, and the notation $\dot{A}_{g,h}$ is used if different metrics are considered simultaneously. Let

$$
\dot{A}^k_h = \dot{A}^k_{g,h} : \Omega^k(\partial M) \to \Omega^{n-k-1}(\partial M) \quad (0 \leq k \leq n - 1),
$$

be the restriction of the operator to $k$-forms. The linearized inverse problem is posed as follows: given a Riemannian manifold $(M, g)$, to what extent is a tensor field $h$ determined either by the operator $\dot{A}_h$ known on all forms or by the data $\dot{A}^k_h$ known for some fixed $k$? Since $\dot{A}_h$ depends linearly on $h$, the question is equivalent to the problem of finding all tensor fields $h$ satisfying $\dot{A}_h = 0$ (or $\dot{A}^k_h = 0$ for a fixed $k$).

Linearized versions of Propositions 1.1 and 1.3 sound as follows.

**Proposition 1.4.** For a tensor field $h \in C^\infty(S^2\tau'_M)$ on an $n$-dimensional manifold, $\dot{A}^{n-1}_h = 0$ if and only if

$$
\int_M \text{tr} h \mu = 0.
$$

Hereafter $\mu$ is the Riemannian volume form and the trace of a symmetric tensor is defined with making use of local coordinates by $\text{tr} h = h_i^j = g^{ij}h_{ij}$, where $(g^{ij})$ is the inverse matrix of $(g_{ij})$.

**Proposition 1.5.** In the case of an even $n = \dim M$, $\dot{A}^{n/2-1}_{g,h} = 0$ for every function $\lambda \in C^\infty(M)$.

For the linearized version of Proposition 1.2, we need a couple of definitions. Given a Riemannian manifold $(M, g)$, the differential operators

$$
\Omega^1(M) \xrightarrow{\delta_s} \frac{d_s}{\delta_s} C^\infty(S^2\tau'_M)
$$
are defined in local coordinates by the formulas

\[(dsv)_{ij} = \frac{1}{2}(\nabla_i v_j + \nabla_j v_i), \quad (\delta s f)_i = -\nabla_j f_{ij},\]

where \(\nabla_i\) is the covariant derivative in the metric \(g\) and \(\nabla^i = g^{ij}\nabla_j\). The operator \(d_s\) is named the \textit{symmetrized covariant derivative} and \(\delta_s\), the \textit{divergence}. They are dual to each other with respect to the natural \(L^2\)-products on the spaces \(\Omega^1 M\) and \(C^\infty(S^2\tau'_M)\). Warning: the operators \(d_s\) and \(\delta_s\) are denoted by \(d\) and \(-\delta\) respectively in [10].

A tensor field \(f \in C^\infty(S^2\tau'_M)\) is said to be \textit{potential} if it can be represented as \(f = dsv\) with some \(v \in \Omega^1 M\) satisfying \(v|_{\partial M} = 0\). The linearized version of Proposition 1.2 sounds as follows.

**Proposition 1.6.** \(\dot{\Lambda}_h = 0\) for a potential tensor field \(h \in C^\infty(S^2\tau'_M)\).

Propositions 1.4–1.6 should be true as far as Propositions 1.1–1.3 are true. Nevertheless, we will give explicit proofs of Propositions 1.3–1.6 based on Theorem 1.7 below.

Let us recall the definition of the space of \textit{exact harmonic fields}, compare with [9],

\[\mathcal{H}^k_{ex}(M) = \{\omega \in \Omega^k(M) \mid \delta \omega = 0 \text{ and } \omega = d\alpha \text{ for some } \alpha \in \Omega^{k-1}(M)\}.\]

For a tensor field \(h \in C^\infty(S^2\tau'_M)\), we introduce the symmetric bilinear form

\[Q^k_h : \mathcal{H}^k_{ex}(M) \times \mathcal{H}^k_{ex}(M) \to \mathbb{R}\]

by

\[Q^k_h(\omega, \varepsilon) = \int_M \left(k h^p_q - \frac{1}{2} \text{tr } h \cdot \delta^p_q\right) \omega^{q_{12\ldots ik}} \varepsilon_{p_{12\ldots ik}} \mu, \quad (1.5)\]

where \((\delta^p_q)\) is the Kronecker tensor. The integrand on (1.5) is written with making use of local coordinates. Nevertheless, the integrand is obviously independent of the choice of coordinates. See Section 2 below for the definition of contravariant coordinates of a form.

Our main result is the following

**Theorem 1.7.** Let \((M, g)\) be a Riemannian manifold of dimension \(n\). For \(h \in C^\infty(S^2\tau'_M)\) and \(0 \leq k \leq n - 1\), \(\dot{\Lambda}^k_h = 0\) if and only if the bilinear form \(Q^{k+1}_h\) is identically equal to zero on \(\mathcal{H}^{k+1}_{ex}(M)\).

Let us discuss this statement in the particular case of \(k = 0\) and of a tensor field \(h\) being a scalar multiple of the metric tensor, i.e., \(h = \lambda g\) with \(\lambda \in C^\infty(M)\). The space \(\mathcal{H}^1_{ex}(M)\) consists of differentials of harmonic functions and formula (1.5) becomes

\[Q^1_h(\omega, \varepsilon) = \frac{2 - n}{2} \int_M \lambda \langle \omega, \varepsilon \rangle \mu,\]

where \(\langle \cdot, \cdot \rangle\) stands for the scalar product. In the case of \(n \geq 3\), Theorem 1.7 states that \(\dot{\Lambda}^0_h = 0\) for \(h = \lambda g\) if and only if the function \(\lambda\) is \(L^2\)-orthogonal to the scalar product of differentials of any two harmonic functions. This coincides with one of results of paper [3] by Calderon.
The space $H^k_{\text{ex}}(M) = \{ c\mu \mid c = \text{const} \}$ is one-dimensional and definition (1.5) becomes in the case of $k = n$

$$Q^n_h(c_1\mu, c_2\mu) = \frac{c_1c_2}{2(n-1)!} \int_M \text{tr } h \mu.$$

This implies Proposition 1.4.

Proposition 1.5 follows also immediately from Theorem 1.7. Indeed, $Q^{n/2}_h = 0$ for $h = \lambda g$ with $\lambda \in C^\infty(M)$.

Let us demonstrate that Proposition 1.3 follows from Proposition 1.5. For a positive function $\lambda \in C^\infty(M)$, we set $h = \lambda g - g = (\lambda - 1)g$ and define the one-parameter family of metrics $g^t = g + th = (t\lambda - t + 1)g \quad (0 \leq t \leq 1)$.

Let $\Lambda^{n/2-1}_{g^t} : \Omega^{n/2-1}(\partial M) \rightarrow \Omega^{n/2}(\partial M)$ be the DN map of the metric $g^t$ on $(n/2-1)$-forms. Then

$$\frac{d}{dt} \Lambda^{n/2-1}_{g^t} = \Lambda^{n/2-1}_{g^t, h} \quad . \quad (1.6)$$

Since $h$ is a scalar multiple of $g^t$, the right-hand side of (1.6) is equal to zero for every $t$ by Proposition 1.5. Therefore $\Lambda^{n/2-1}_{g^t} = \Lambda^{n/2-1}_{g^t} = \Lambda^{n/2-1}_{g^t}$.

Definition (1.5) can be written in the form

$$Q^n_h(\omega, \varepsilon) = \left( kh - \frac{1}{2} \text{tr } h \cdot g, F(\omega \otimes \varepsilon) \right)_{L^2}, \quad (1.7)$$

where the tensor field $F(\omega \otimes \varepsilon) \in C^\infty(S^2\tau'_M)$ is defined by

$$F(\omega \otimes \varepsilon) = f, \quad f_{ij} = \frac{1}{2}(\omega^p_{i_1p_{j_2}...p_{k_2}}\varepsilon_{j_2p_{j_3}...p_{k_3}} + \omega^p_{j_1p_{i_2}...p_{k_2}}\varepsilon_{i_2p_{i_3}...p_{k_3}}). \quad (1.8)$$

How large is the subspace of $C^\infty(S^2\tau'_M)$ consisting of tensor fields $f$ that can be represented in form (1.8) with some $\omega, \varepsilon \in H^k_{\text{ex}}(M)$? The following statement gives a partial answer to the question.

**Proposition 1.8.** If a tensor field $f \in C^\infty(S^2\tau'_M)$ can be represented in form (1.8) with some $\omega, \varepsilon \in H^k_{\text{ex}}(M)$, then it satisfies the equation

$$kd^s f + \frac{1}{2} d(\text{tr } f) = 0. \quad (1.9)$$

This statement is quite expectable. Indeed, by Proposition 1.6, $\dot{\Lambda}_{d^s v} = 0$ for every $v \in \Omega^1(M)$ satisfying the boundary condition $v|_{\partial M} = 0$. In view of Theorem 1.7, this means that

$$Q^k_{d^s v}(\omega, \varepsilon) = \left( kd^s v - \frac{1}{2} \text{tr } (d^s v \cdot g, F(\omega \otimes \varepsilon)) \right)_{L^2} = 0.$$

for any such \( v \). In other words, the tensor field \( f = F(\omega \otimes \varepsilon) \) is orthogonal to the range of the operator \( v \mapsto kd_v - \frac{1}{2} \text{tr}(d_v \cdot g) \). Therefore \( f \) belongs to the null-space of the dual operator that is just the operator on the left-hand side of (1.9). We will present a more explicit proof of Proposition 1.8 in Section 6.

For an integer \( k \) satisfying \( 0 \leq k \leq n - 1 \) and \( k \neq n/2 \), let us denote by

\[
C^\infty_k(S^2\tau'_M) = \left\{ f \in C^\infty(S^2\tau'_M) \mid k\delta_s f + \frac{1}{2} d(\text{tr } f) = 0 \right\}
\]  

(1.10)

the subspace of tensor fields satisfying equation (1.9). In virtue of Proposition 1.8, the linear operator

\[
F : \mathcal{H}^k_{ex}(M) \otimes \mathcal{H}^k_{ex}(M) \to C^\infty_k(S^2\tau'_M)
\]  

(1.11)

is well defined by formula (1.8). This operator plays the crucial role in our study of the linear inverse problem.

Our main definition is as follows.

**Definition 1.9.** Let \( 0 \leq k \leq n - 2 \) and \( k + 1 \neq n/2 \). An \( n \)-dimensional manifold \( (M, g) \) is said to be \( \Omega^k \)-rigid if \( \Lambda^k_{\delta_s} = 0 \) only for potential tensor fields \( h \).

We have not included the case of \( k = n - 1 \) to the definition because an \( n \)-dimensional manifold is never \( \Omega^{n-1} \)-rigid. Indeed, there are many tensor fields satisfying (1.4) which are not potential. The case of \( k = n/2 - 1 \) is not included in view of Proposition 1.5, this exceptional case will be discussed later.

**Theorem 1.10.** Let \( 0 \leq k \leq n - 2 \) and \( k + 1 \neq n/2 \). An \( n \)-dimensional manifold \( M \) is \( \Omega^k \)-rigid if the range of the operator

\[
F : \mathcal{H}^{k+1}_{ex}(M) \otimes \mathcal{H}^{k+1}_{ex}(M) \to C^\infty_{k+1}(S^2\tau'_M)
\]  

(1.12)

is dense in the space \( C^\infty_{k+1}(S^2\tau'_M) \) endowed with the \( L^2 \)-topology. Conversely, if there exists a nonzero tensor field in \( C^\infty_{k+1}(S^2\tau'_M) \) which is \( L^2 \)-orthogonal to the range of operator (1.12), then \( M \) is not \( \Omega^k \)-rigid.

We emphasize that two statements of Theorem 1.10 are not strongly converse to each other. Let \( L^2_{k+1}(S^2\tau'_M) \) be the closure of \( C^\infty_{k+1}(S^2\tau'_M) \) in \( L^2(S^2\tau'_M) \) and \( (\text{Ran } F)^\perp \) be the orthogonal complement of \( \text{Ran } F \) in \( L^2_{k+1}(S^2\tau'_M) \). The theorem states that \( M \) is not \( \Omega^k \)-rigid if \( (\text{Ran } F)^\perp \cap C^\infty_{k+1}(S^2\tau'_M) \neq 0 \). But in principle there can be no smooth field in \( (\text{Ran } F)^\perp \neq 0 \). The author has no idea whether such a situation really happens.

Let us now address to the exceptional case of \( k = n/2 - 1 \) for a manifold of an even dimension \( n \). In this case, we replace definition (1.10) with the following one:

\[
C^\infty_{n/2}(S^2\tau'_M) = \left\{ f \in C^\infty(S^2\tau'_M) \mid \delta_s f = 0 \text{ and } \text{tr } f = 0 \right\}.
\]  

(1.13)

For \( \omega, \varepsilon \in \mathcal{H}^{n/2}_{ex}(M) \), define the tensor field \( f = F(\omega \otimes \varepsilon) \) by formula (1.8). By Proposition 1.8, \( f \) satisfies

\[
n\delta_s f + d(\text{tr } f) = 0.
\]  

(1.14)
The trace-free part $\tilde{f}$ of the field $f$ belongs to the space $C^\infty_{n/2}(S^2\tau'_M)$. Indeed, $f$ and $\tilde{f}$ are related by the equation $f = \tilde{f} + \frac{1}{n} \text{tr} f \cdot g$. Substituting this expression into (1.14), we obtain $\delta s \tilde{f} = 0$. Therefore the operator
\[ G : \mathcal{H}^{n/2}_{ex}(M) \otimes \mathcal{H}^{n/2}_{ex}(M) \to C^\infty_{n/2}(S^2\tau'_M), \]
(1.15)
is well defined.

In view of Proposition 1.5, Definition 1.9 is now replaced with the following one:

**Definition 1.11.** A manifold $(M, g)$ of an even dimension $n$ is said to be $\Omega^{n/2-1}$-rigid if $\tilde{\Lambda}^{n/2-1} = 0$ only for tensor fields $h \in C^\infty(S^2\tau'_M)$ which can be represented as the sum of a potential and spherical tensor fields, i.e.,
\[ h = d_s v + \lambda g, \quad v|_{\partial M} = 0 \]
for some $v \in \Omega^1(M)$ and $\lambda \in C^\infty(M)$.

**Theorem 1.12.** A manifold $M$ of an even dimension $n$ is $\Omega^{n/2-1}$-rigid if the range of operator (1.15) is dense in the space $C^\infty_{n/2}(S^2\tau'_M)$ endowed with the $L^2$-topology. Conversely, if there exists a nonzero tensor field in $C^\infty_{n/2}(S^2\tau'_M)$ which is $L^2$-orthogonal to the range of operator (1.15), then $M$ is not $\Omega^{n/2-1}$-rigid.

In [3], Calderon used exponential harmonic functions to prove the injectivity of the Gateaux derivative of the DN map in the scalar case. We use the same method for proving the following

**Theorem 1.13.** Let $M$ be a bounded domain in $\mathbb{R}^n$ with a smooth boundary and $g$ be the Euclidean metric on $M$. The Riemannian manifold $(M, g)$ is $\Omega^k$-rigid for every $0 \leq k \leq n - 2$.

**2. Preliminaries**

A $k$-form $\omega \in \Omega^k(M)$ can be considered as a skew-symmetric tensor field of rank $k$. In local coordinates it can be written as
\[ \omega = \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \]
where the summation from 1 to $n$ is performed over all repeated indices, and $\omega_{i_1 \ldots i_k}$ are smooth functions in the domain of the coordinate system which are assumed to be skew-symmetric, i.e.,
\[ \omega_{\pi(i_1) \ldots \pi(i_k)} = \sigma(\pi) \omega_{i_1 \ldots i_k} \]
for any permutation $\pi$ of the set $\{1, \ldots, k\}$, where $\sigma(\pi)$ is the sign of the permutation. In particular, $\omega_{i_1 \ldots i_k} = 0$ if two indices coincide. There is another version of the representation
\[ \omega = k! \sum_{1 \leq i_1 < \cdots < i_k \leq n} \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \]
which is obviously equivalent to the previous one. We will use both the representations.

The point-wise scalar product of two forms $\omega, \varepsilon \in \Omega^k(M)$ is defined in coordinates by
\[ \langle \omega, \varepsilon \rangle = k! g^{i_1 j_1} \cdots g^{i_k j_k} \omega_{i_1 \ldots i_k} \varepsilon_{j_1 \ldots j_k} = k! \omega^{i_1 \ldots i_k} \varepsilon_{i_1 \ldots i_k}. \]
(2.1)
For forms (as for general tensor fields), we use both covariant and contravariant coordinates
\[ \omega_{i_1 \ldots i_k} = g_{i_1 j_1} \ldots g_{i_k j_k} \omega_{j_1 \ldots j_k}, \quad \omega_{i_1 \ldots i_k} = g_{i_1 j_1} \ldots g_{i_k j_k} \omega_{j_1 \ldots j_k}. \]

Quite similarly, mixed coordinates can be defined where some indices are in upper position and other indices are in low position. For example, formula (1.8) involves
\[ \omega_{i p_2 \ldots p_k} = g_{p_2 q_2} \ldots g_{p_k q_k} \omega_{iq_2 \ldots q_k}. \]

All tensor operations; like raising and lowering indices, covariant differentiation, and so on; are assumed to be done with respect to the fixed metric \( g \).

Recall that the Hodge star \( \star : \Omega^k(M) \to \Omega^{n-k}(M) \) is defined by the formula
\[ \omega \wedge \star \epsilon = \langle \omega, \epsilon \rangle \mu. \]

The coefficient \( k! \) on (2.1) is needed to make \( \star \) an isometry. Recall that on \( k \)-forms
\[ \star \star = (-1)^{k(n-k)} . \] (2.2)

The \( L^2 \)-product is defined on forms by
\[ (\omega, \epsilon) = \int_M \langle \omega, \epsilon \rangle \mu = \int_M \omega \wedge \star \epsilon = \int_M \epsilon \wedge \star \omega. \]

The operators \( d \) and \( \delta \) are dual to each other with respect to the \( L^2 \)-product. Moreover, Green’s formula holds
\[ (d \omega, \epsilon) - (\omega, d \epsilon) = \int_{\partial M} i^* (\omega \wedge \star \epsilon). \] (2.3)

Recall also the relations
\[ \star \delta = (-1)^k d \star, \quad \star d = (-1)^{k+1} \delta \star \quad \text{on } \Omega^k(M). \] (2.4)

3. The operator \( \dot{\Lambda}_h \)

So, we consider a Riemannian manifold \((M, g)\) and the variation \( g^t = g + th \) of the metric \( g \) with a smooth symmetric tensor field \( h = (h_{ij}) \). Let \( \star_t, \delta_t, \) and \( \Delta_t \) be the Hodge star, codifferential, and Laplacian with respect to the metric \( g^t \). The operators depend smoothly on \( t \) and admit the representations
\[ \star_t = \star + t \dot{\star} + o(t), \quad \delta_t = \delta + t \dot{\delta} + o(t), \quad \Delta_t = \Delta + t \dot{\Delta} + o(t), \]

where \( \star, \delta, \) and \( \Delta \) are the corresponding operators for the metric \( g \). The formula \( \Delta_t = \delta_t d + d \delta_t \) implies that
\[ \dot{\Delta} = \dot{\delta} d + d \dot{\delta}, \] (3.1)

and relation (2.2) implies
\[ \star \star + \star \dot{\star} = 0. \]

Now, we calculate the operator \( \dot{\Lambda}_h = d \Lambda_{g^t}/dt |_{t=0} \). For a form \( \varphi \in \Omega(\partial M) \), we have
\[ \dot{\Lambda}_{g^t} \varphi = i^* (\star_t d \omega_t), \] (3.2)
where $\omega_t \in \Omega(M)$ is a solution to the B.V.P.
\[
\begin{aligned}
\Delta_t \omega_t &= 0, \\
i^* \omega_t &= \varphi, \quad i^*(\delta_t \omega_t) = 0. 
\end{aligned}
\tag{3.3}
\]
Representing the form $\omega_t$ as
\[
\omega_t = \omega + t\dot{\omega} + o(t)
\]
and substituting the representation into (3.2) and (3.3), we easily obtain
\[
\begin{aligned}
\Delta \omega &= 0, \\
i^* \omega &= \varphi, \quad i^*(\delta \omega) = 0, \\
\Delta \dot{\omega} &= -\dot{\Delta} \omega, \\
i^* \dot{\omega} &= 0, \quad i^* \delta \dot{\omega} = -i^* \dot{\delta} \omega, \\
\hat{A}_h \varphi &= i^*(\star d \dot{\epsilon} + \dot{\star} d \epsilon) \tag{3.6}
\end{aligned}
\]
Formulas (3.4)–(3.6) give us the following description of the operator $\hat{A}_h$. Given a form $\varphi \in \Omega^k(\partial M)$, one has to solve the B.V.P. (3.4), then to solve the B.V.P. (3.5), and finally to define $\hat{A}_h \varphi$ by formula (3.6). Observe that the B.V.P. (3.4) is independent of $h$, while the right-hand side and boundary condition on (3.5) depend linearly on $h$ through the operators $\dot{\Delta}$ and $\dot{\delta}$.

The operator $\hat{A}_h$ depends linearly on $h$.

4. When $\hat{A}_h = 0$?

Let us introduce the space
\[
A^k(M) = \{ \omega \in \Omega^k(M) \mid \Delta \omega = 0 \text{ and } \delta \omega = 0 \}.
\]
We first prove the following

**Lemma 4.1.** For a smooth symmetric tensor field $h$, $\hat{A}_h^k = 0$ if and only if
\[
(\omega, \dot{\Delta} \epsilon) + \int_{\partial M} i^* (\omega \wedge \star d \epsilon - \dot{\delta} \epsilon \wedge \star \omega) = 0 \tag{4.1}
\]
for any $\omega, \epsilon \in A^k(M)$.

**Proof.** Green’s formula for $\Delta$ looks as follows:
\[
(\Delta \alpha, \beta) - (\alpha, \Delta \beta) = \int_{\partial M} i^* (\alpha \wedge \star d \beta - \beta \wedge \star d \alpha + \delta \alpha \wedge \star \beta - \delta \beta \wedge \star \alpha). \tag{4.2}
\]
Let $\varphi, \psi \in \Omega^k(\partial M)$ be two arbitrary boundary forms. Let $\omega, \dot{\omega} \in \Omega^k(M)$ solve the B.V.P.’s (3.4) and (3.5). Similarly, let $\epsilon, \dot{\epsilon} \in \Omega^k(M)$ solve the B.V.P.’s
\[
\begin{aligned}
\Delta \epsilon &= 0, \\
i^* \epsilon &= \psi, \quad i^*(\delta \epsilon) = 0 \quad \text{and} \quad \Delta \dot{\epsilon} &= -\dot{\Delta} \epsilon, \\
i^* \dot{\epsilon} &= 0, \quad i^* \dot{\delta} \epsilon = -i^* \dot{\delta} \epsilon, 
\end{aligned}
\]
then
\[
\hat{A}_h \psi = i^*(\star d \dot{\epsilon} + \dot{\star} d \epsilon).
\]
The forms $\omega$ and $\epsilon$ belong to $A^k(M)$ by (1.3).
Set $\alpha = \omega$ and $\beta = \dot{\varepsilon}$ in Green's formula (4.2)

$$(\Delta \omega, \dot{\varepsilon}) - (\omega, \Delta \dot{\varepsilon}) = \int_{\partial M} i^* (\omega \wedge \star d \dot{\varepsilon} - \dot{\varepsilon} \wedge \star \omega + \delta \omega \wedge \star \dot{\delta} \varepsilon - \delta \dot{\varepsilon} \wedge \star \omega).$$

The first term on the left-hand side is zero as well as the second and third terms on the right-hand side. The formula is thus simplified to the following one:

$$-(\omega, \Delta \dot{\varepsilon}) = \int_{\partial M} i^* (\omega \wedge \star d \dot{\varepsilon} - \delta \dot{\varepsilon} \wedge \star \omega).$$

Substitute the values $\Delta \dot{\varepsilon} = -\dot{\Delta} \varepsilon$ and $i^* \delta \dot{\varepsilon} = -i^* \dot{\delta} \varepsilon$ to obtain

$$(\omega, \dot{\Delta} \varepsilon) = \int_{\partial M} i^* (\omega \wedge \star d \dot{\varepsilon} + \dot{\delta} \varepsilon \wedge \star \omega).$$

Finally, substitute $i^* \star d \dot{\varepsilon} = \dot{\Lambda} h \psi - i^* \dot{\star} d \varepsilon$ to obtain

$$(\omega, \dot{\Delta} \varepsilon) + \int_{\partial M} i^* (\omega \wedge \dot{\star} d \varepsilon - \dot{\delta} \varepsilon \wedge \star) = \int_{\partial M} \varphi \wedge \dot{\Lambda} h \psi.$$

Since $\varphi$ and $\psi$ are arbitrary forms, this proves the statement of the lemma. $\square$

Next, we transform the first term on the left-hand side of (4.1). We start with Green's formula (2.3) for $d$ and $\delta$. It can be written in the form

$$\int_M d \alpha \wedge \star \beta - \int_M \alpha \wedge \star \delta \beta = \int_{\partial M} i^* (\alpha \wedge \star \beta). \quad (4.3)$$

Formula (4.3) holds for any two forms $\alpha, \beta$ and for any Riemannian metric on $M$. Let us consider the formula for fixed $\alpha, \beta$ and for the metric $g^t = g + th$. Both parts of the formula are smooth functions of $t$. Differentiate the formula with respect to $t$ at $t = 0$

$$\int_M d \alpha \wedge \dot{\star} \beta - \int_M \alpha \wedge \dot{\star} \delta \beta - \int_M \alpha \wedge \dot{\star} \delta \beta = \int_{\partial M} i^* (\alpha \wedge \dot{\star} \beta).$$

If the form $\beta$ is co-closed

$$\delta \beta = 0, \quad (4.4)$$

then the second integral on the left-hand side is zero, and we obtain

$$\int_M \alpha \wedge \dot{\star} \delta \beta = \int_M d \alpha \wedge \dot{\star} \beta - \int_{\partial M} i^* (\alpha \wedge \dot{\star} \beta). \quad (4.5)$$

Let now $\omega, \varepsilon \in \mathcal{A}^k(M)$. Set $\alpha = \omega$ and $\beta = d \varepsilon$ in (4.5). Condition (4.4) is satisfied because $\delta d \varepsilon = 0$. We thus obtain

$$\int_M \omega \wedge \dot{\star} \delta d \varepsilon = \int_M d \omega \wedge \dot{\star} d \varepsilon - \int_{\partial M} i^* (\omega \wedge \dot{\star} d \varepsilon)$$

or

$$(\omega, \dot{\delta} d \varepsilon) = \int_M d \omega \wedge \dot{\star} d \varepsilon - \int_{\partial M} i^* (\omega \wedge \dot{\star} d \varepsilon). \quad (4.6)$$
Since $\dot{\Lambda} = \dot{\delta}d + d\dot{\delta}$, we can write

$$(\omega, \dot{\Lambda}\varepsilon) = (\omega, \delta d\varepsilon) + (\omega, d\dot{\delta}\varepsilon) = (\omega, \delta d\varepsilon) + (\delta\omega, \dot{\delta}\varepsilon) + \int_{\partial M} i^*(\dot{\delta}\varepsilon \wedge \ast \omega).$$

The second term on the right-hand side is equal to zero since $\delta\omega = 0$, and we have

$$(\omega, \dot{\Lambda}\varepsilon) = (\omega, \delta d\varepsilon) + \int_{\partial M} i^*(\dot{\delta}\varepsilon \wedge \ast \omega).$$

With the help of the last formula, (4.6) gives

$$(\omega, \dot{\Lambda}\varepsilon) = \int_M d\omega \wedge \dot{\varepsilon} + \int_{\partial M} i^*(\dot{\delta}\varepsilon \wedge \ast \omega - \omega \wedge \dot{\varepsilon}).$$

Substituting the latter value for $(\omega, \dot{\Lambda}\varepsilon)$ into (4.1), we arrive at

**Lemma 4.2.** For a tensor field $h \in C^\infty(S^2\tau'_M)$, $\dot{\Lambda}^k_h = 0$ if and only if

$$R^{k+1}_h(d\omega, d\varepsilon) = 0$$

for any $\omega, \varepsilon \in A^k(M)$, where the bilinear form $R^k_h$ is defined by

$$R^k_h(\omega, \varepsilon) = \int_M \omega \wedge \dot{\varepsilon}.$$  

(4.7)

Observe that

$$dA^k(M) = H^{k+1}_{ex}(M).$$

(4.9)

Indeed, if $\omega \in A^k(M)$, then $\lambda = d\omega$ is an exact form. Moreover, $\lambda$ is a harmonic field since $d\lambda = dd\omega = 0$ and $\delta\lambda = \delta d\omega = \Delta\omega = 0$. Conversely, let $\lambda = d\alpha$ be an exact harmonic $(k+1)$-field. Expand $\alpha$ by Hodge–Morrey (see [9] for the latter decomposition),

$$\alpha = d\beta + \delta\gamma + \lambda', \quad d\lambda' = \delta\lambda' = 0,$$

and substitute the expansion into the equality $\lambda = d\alpha$ to obtain $\lambda = d\omega$ with $\omega = \delta\gamma$. The form $\omega$ belongs to $A^k(M)$ since $\delta\omega = \delta\delta\gamma = 0$ and $\Delta\omega = \delta d\omega = \delta\lambda = 0$. Therefore $\lambda = d\omega \in dA^k(M)$.

In view of (4.9), Lemma 4.2 is equivalent to the following

**Lemma 4.3.** $\dot{\Lambda}^k_h = 0$ if and only if $R^{k+1}_h(\omega, \varepsilon) = 0$ for any two forms $\omega, \varepsilon \in H^{k+1}_{ex}(M)$.

5. Proof of Theorem 1.7

What we need is the coincidence of the forms $-k!Q^k_h$ and $R^k_h$ on the space $\Omega^k(M)$. Indeed, if the coincidence is proved, Theorem 1.7 is the same as Lemma 4.3. To prove the equality $R^k_h = -k!Q^k_h$, we need the following

**Lemma 5.1.** Given a form $\omega \in \Omega^k(M)$, the following coordinate representation holds for $\ast \omega$:

$$\ast \omega = (-1)^{(k+1)/2}! \sum_{1 \leq i_1 < \cdots < i_k \leq n} (-1)^{i_1+\cdots+i_k} \omega^{i_1\cdots i_k} \sqrt{g} dx^1 \wedge \cdots \wedge \hat{dx}^{i_1} \wedge \cdots \wedge \hat{dx}^n.$$  

Hereafter the symbol $\wedge$ above a factor means that the factor is omitted, and $\sqrt{g} = \sqrt{\det(g_{ij})}$.  

(5.1)
Proof. A form \( \lambda \in \Omega^k(M) \) can be written as
\[
\lambda = k! \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1 \ldots i_k} \, dx^{i_1} \wedge \cdots \wedge dx^{i_k}.
\]
From two last formulas
\[
\lambda \wedge \star \omega = (-1)^{k(k+1)/2} (k!)^2 \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}
\]
\[
\wedge \sum_{1 \leq j_1 < \cdots < j_k \leq n} (-1)^{j_1 + \cdots + j_k} \omega_{j_1 \ldots j_k}
\]
\[
\times \sqrt{g} \, dx^{i_1} \wedge \cdots \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} \wedge \cdots \wedge dx^n.
\]
The product of the summands is nonzero only if \((i_1, \ldots, i_k) = (j_1, \ldots, j_k)\). So,
\[
\lambda \wedge \star \omega = (-1)^{k(k+1)/2} (k!)^2 \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1 \ldots i_k} \omega^{i_1 \ldots i_k} (-1)^{i_1 + \cdots + i_k}
\]
\[
\times (\sqrt{g} \, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{1} \wedge \cdots \wedge \hat{dx}^{i_1} \wedge \cdots \wedge \hat{dx}^{i_k} \wedge \cdots \wedge dx^n). \tag{5.2}
\]
The form in the parentheses coincides with the volume form \( \mu \) up to a sign. Let us calculate the sign. We have to count the number of transpositions that are needed to transform the product
\[
dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{1} \wedge \cdots \wedge \hat{dx}^{i_1} \wedge \cdots \wedge \hat{dx}^{i_k} \wedge \cdots \wedge dx^n
\]
to \( dx^{1} \wedge \cdots \wedge dx^n \).

The number of transpositions can be calculated by the following sequence:

1. move \( dx^{i_k} \) to the \( i_k \)th position by \( i_k - 1 - (k - 1) = i_k - k \) transpositions;
2. move \( dx^{j_k - 1} \) to the \( i_k \)th position by \( i_k - 1 - (k - 2) = i_k - 1 - (k - 1) \) transpositions;

\ldots \ldots 

(k) move \( dx^{i_1} \) to the \( i_1 \)th position by \( i_1 - 1 = i_1 - 1 \) transpositions.

So, the number of transpositions is
\[
i_1 + \cdots + i_k - \sum_{j=1}^{k} j = i_1 + \cdots + i_k - k(k + 1)/2.
\]

Thus, formula (5.2) takes the form
\[
\lambda \wedge \star \omega = (k!)^2 \left( \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1 \ldots i_k} \omega^{i_1 \ldots i_k} \right) \mu
\]
which is equivalent to
\[
\lambda \wedge \star \omega = k! \lambda_{i_1 \ldots i_k} \omega^{i_1 \ldots i_k} \mu.
\]

By (2.1), the latter formula can be also written as
\[
\lambda \wedge \star \omega = \langle \lambda, \omega \rangle \mu.
\]
The last formula is just the definition of \( \star \). This proves (5.1). \( \square \)
Lemma 5.2. For any $k$-form $\lambda$ ($k \geq 1$),
\[
(\ast \ast \lambda)_{i_1 \ldots i_k} = (-1)^{k(n-k)+1} \alpha(i_1 \ldots i_k) \left[ (k h_{i_1}^p - \frac{1}{2} \text{tr} h \cdot \delta_i^p) \lambda_{pi_2 \ldots i_k} \right],
\]
where $\alpha(i_1 \ldots i_k)$ is the alternation in the indices $i_1, \ldots, i_k$.

Proof. Apply formula (5.1) to the form $\ast \ast \lambda$
\[
(-1)^{k(n-k)+k(k+1)/2} \ast \lambda = \ast \ast \lambda = \ast (\ast \ast \lambda)
\]
\[
= k! \sum_{1 \leq i_1 < \ldots < i_k \leq n} (-1)^{i_1 + \ldots + i_k} (\ast \ast \lambda)_{i_1 \ldots i_k}
\times \sqrt{g} dx^1 \wedge \ldots \wedge dx^{i_1} \wedge \ldots \wedge dx^k \wedge \ldots \wedge dx^n.
\]
So
\[
\frac{(-1)^{k(n-k)+k(k+1)/2}}{k!} \dot{\lambda} = \sum_{1 \leq i_1 < \ldots < i_k \leq n} (-1)^{i_1 + \ldots + i_k} (\ast \ast \lambda)_{i_1 \ldots i_k}
\times \sqrt{g} dx^1 \wedge \ldots \wedge dx^{i_1} \wedge \ldots \wedge dx^k \wedge \ldots \wedge dx^n.
\]
(5.4)

For two $k$-forms $\lambda$ and $\omega$, we have by the definition of the Hodge star
\[
\ast \lambda \wedge \omega = (-1)^{k(n-k)} \omega \wedge \ast \lambda = (-1)^{k(n-k)}(\omega, \lambda) \mu.
\]
Write down the right-hand side in coordinates
\[
\ast \lambda \wedge \omega = (-1)^{k(n-k)} k! \lambda_{i_1 \ldots i_k} \omega_{i_1 \ldots i_k} \sqrt{g} dx^1 \wedge \ldots \wedge dx^n.
\]
(5.5)

Formula (5.5) holds for any forms $\lambda$ and $\omega$ and for any Riemannian metric. Let us fix the forms $\lambda$ and $\omega$ and consider (5.5) for the metric $g^t = g + th$. Both sides of (5.5) depend smoothly on $t$. Differentiate (5.5) with respect to $t$ at $t = 0$. The covariant coordinates $\lambda_{i_1 \ldots i_k}$ and $\omega_{i_1 \ldots i_k}$ are independent of $t$. So, after differentiation we obtain
\[
\dot{\lambda} \wedge \omega = (-1)^{k(n-k)} k! \lambda_{i_1 \ldots i_k} \left( \omega_{i_1 \ldots i_k} + \sqrt{g} (\sqrt{g})^{-1} \omega_{i_1 \ldots i_k} \right) \mu,
\]
(5.6)
where
\[
\omega_{i_1 \ldots i_k} = \partial \omega_{i_1 \ldots i_k} / \partial t |_{t=0} \text{ and } \dot{\sqrt{g}} = \partial \sqrt{g} / \partial t |_{t=0} = \frac{1}{2} \sqrt{g} \text{tr} \, h.
\]

From the expression of the contravariant coordinates through covariant ones
\[
\omega^i_{j_1 \ldots j_k} = g^{i_1 j_1} \ldots g^{i_k j_k} \omega_{j_1 \ldots j_k}
\]
with the help of the relation $\partial g^{ij} / \partial t |_{t=0} = - h^{ij}$, we obtain
\[
\omega^i_{j_1 \ldots j_k} = - \sum_{a=1}^k g^{i_j a} g^{i_a+1 j_a+1} h_{a+1}^{j_a} g^a j_a+1 \ldots g^{i_k j_k} \omega_{j_1 \ldots j_k}
\]
\[
\quad = - \sum_{a=1}^k h^{i_j a} \omega_{i_1 \ldots i_a+1 \ldots i_k} = - k \alpha(i_1 \ldots i_k) (h^{i_j a} \omega_{i_1 \ldots i_k}).
\]

Substituting this value into (5.6), we obtain
\[
\dot{\lambda} \wedge \omega = (-1)^{k(n-k)+1} k! \lambda_{i_1 \ldots i_k} \left( kh^{i_j a} - \frac{1}{2} \text{tr} h \cdot \delta_i^a \right) \omega_{i_1 \ldots i_k} \mu.
\]
(5.7)
For a $k$-form $\omega$, let us denote by $\tilde{\omega}$ the $k$-form defined by

$$\tilde{\omega}_{i_1 \ldots i_k} = \alpha(i_1 \ldots i_k) \left[(kh^p_i - \text{tr} h \cdot \delta^p_i)\omega_{p i_2 \ldots i_k}\right]. \quad (5.8)$$

Then formula (5.7) can be written as

$$\ast \lambda \wedge \omega = (-1)^{k(n-k)+1} \langle \lambda, \tilde{\omega} \rangle \mu. \quad (5.9)$$

Substitute expression (5.4) for $\ast \lambda$ and the expression

$$\omega = k! \sum_{1 \leq j_1 < \ldots < j_k \leq n} \omega_{j_1 \ldots j_k} dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$

into the left-hand side of (5.9)

$$(k!)^2 \sum_{1 \leq i_1 < \ldots < i_k \leq n} \sum_{1 \leq j_1 < \ldots < j_k \leq n} (-1)^{i_1 + \cdots + i_k} (\ast \ast \lambda)_{i_1 \ldots i_k} \omega_{j_1 \ldots j_k}
\times \left(\sqrt{g} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge \cdots \wedge dx^n \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}\right)
= (-1)^{k(n-k)/2+1} \langle \lambda, \tilde{\omega} \rangle \mu.$$

The form in the parentheses is not zero only if $(i_1, \ldots, i_k) = (j_1, \ldots, j_k)$. In the latter case the form coincides with $\mu$ up to a sign. The sign can be calculated as before in the proof of Lemma 5.1, and it turns out to be

$$(-1)^{i_1 + \cdots + i_k} \lambda_{i_1 \ldots i_k} \tilde{\omega}_{i_1 \ldots i_k}.$$

Then formula (5.9) gives

$$(k!)^2 \sum_{1 \leq i_1 < \ldots < i_k \leq n} (\ast \ast \lambda)_{i_1 \ldots i_k} \omega_{i_1 \ldots i_k} = (-1)^{k(n-k)+1} \langle \lambda, \tilde{\omega} \rangle \mu$$

or

$$\langle \ast \ast \lambda, \omega \rangle = (-1)^{k(n-k)+1} \langle \lambda, \tilde{\omega} \rangle. \quad (5.10)$$

Write down (5.10) again in the coordinate form

$$(\ast \ast \lambda)_{i_1 \ldots i_k} \omega^{i_1 \ldots i_k} = (-1)^{k(n-k)+1} \lambda_{i_1 \ldots i_k} \tilde{\omega}^{i_1 \ldots i_k}. \quad (5.11)$$

Substitute the expression

$$\tilde{\omega}^{i_1 \ldots i_k} = \alpha(i_1 \ldots i_k) \left(kh^p_i \omega^{pi_2 \ldots i_k}\right), \quad \text{where} \quad kh^p_i = kh^i_p - \frac{1}{2} \text{tr} h \cdot \delta^i_p,$$

into (5.11)

$$(\ast \ast \lambda)_{i_1 \ldots i_k} \omega^{i_1 \ldots i_k} = (-1)^{k(n-k)+1} \lambda_{i_1 \ldots i_k} kh^i_p \omega^{pi_2 \ldots i_k}.$$

After changing summation indices, this can be written as

$$(\ast \ast \lambda)_{i_1 \ldots i_k} \omega^{i_1 \ldots i_k} = (-1)^{k(n-k)+1} k \cdot kh^p_i \lambda_{pi_2 \ldots i_k} \omega^{i_1 \ldots i_k}. \quad (5.12)$$

Formula (5.12) holds for an arbitrary $k$-form $\omega$, i.e., for an arbitrary skew-symmetric family of numbers $(\omega^{i_1 \ldots i_k})$. This can be valid iff

$$(\ast \ast \lambda)_{i_1 \ldots i_k} = (-1)^{k(n-k)+1} \alpha(i_1 \ldots i_k) \left(kh^p_i \lambda_{pi_2 \ldots i_k}\right).$$

This coincides with the statement of Lemma 5.2. $\Box$

Lemma 5.2 implies the coincidence of $R^k_h$ and $-k!Q^k_h$. Indeed, for $\omega, \varepsilon \in \Omega^k(M)$,
\[ R^k_h(\omega, \varepsilon) = \int_M \omega \wedge \dot{*}\varepsilon = (-1)^{(n-k)} \int_M \omega \wedge \star \star\varepsilon = (-1)^{(n-k)} (\omega, \star\varepsilon) \]
\[ = (-1)^{(n-k)} \int_M \langle \omega, \star\varepsilon \rangle \mu = (-1)^{(n-k)} k! \int_M \omega^{i_1...i_k} (\star\varepsilon)_{i_1...i_k} \mu. \]

Substituting expression (5.3) for \((\star\varepsilon)_{i_1...i_k}\), we obtain
\[ R^k_h(\omega, \varepsilon) = -k! \int_M \omega^{i_1...i_k} \left( k h^p_{i_1} - \frac{1}{2} \text{tr} h \cdot \delta^p_{i_1} \right) \varepsilon_{p i_2...i_k} \mu \]
\[ = -k! \int_M \left( k h^q_{i} - \frac{1}{2} \text{tr} h \cdot \delta^q_{i} \right) \omega^{q i_2...i_k} \varepsilon_{p i_2...i_k} \mu = -k! Q^k_h(\omega, \varepsilon). \]

This finishes the proof of Theorem 1.7.

6. \( \dot{A}_h = 0 \) for a potential field \( h \)

Here we prove Propositions 1.6 and 1.8. We will need the coordinate representation of the codifferential:
\[ (\delta\omega)_{i_2...i_k} = -k \nabla^p \omega_{pi_2...i_k} \] (6.1)
for \( \omega \in \Omega^k(M) \). This easily follows from the definition of \( \delta \).

Following [9], we introduce the space of harmonic \( k \)-fields
\[ \mathcal{H}^k(M) = \left\{ \omega \in \Omega(M) \mid d\omega = 0, \delta\omega = 0 \right\}. \]

The following statement is stronger than Proposition 1.8.

**Proposition 6.1.** For any two forms \( \omega, \varepsilon \in \mathcal{H}^k(M) \) the tensor field \( f \in C^\infty(S^2 \tau'_M) \) defined by
\[ f_{ij} = \frac{1}{2} \left( \omega^{p_2...p_k} \varepsilon_{j p_2...p_k} + \omega^{p_2...p_k} \varepsilon_{i p_2...p_k} \right) \]
(6.2)
satisfies the equation
\[ k \delta_s f + \frac{1}{2} d(\text{tr} f) = 0. \]

**Proof.** From (6.2)
\[ \text{tr} f = \frac{1}{k!} (\omega, \varepsilon) = \omega^{p_1...p_k} \varepsilon_{p_1...p_k}. \]

Differentiating this equality, we obtain
\[ \left( d(\text{tr} f) \right)_i = \nabla_i \omega^{p_1...p_k} \varepsilon_{p_1...p_k} + \nabla_i \varepsilon^{p_1...p_k} \omega_{p_1...p_k}. \]
(6.3)

By differentiation of (6.2), we obtain
\[ (\delta_s f)_i = -\nabla^q f_{q i} = -\frac{1}{2} \left( \nabla^q \omega^p_{q} \varepsilon^{p_2...p_k}_{i p_2...p_k} + \omega^p_{q} \varepsilon^{p_2...p_k} \nabla^q \varepsilon_{i p_2...p_k} \right. \]
\[ + \nabla^q \omega^p_{i} \varepsilon^{p_2...p_k} \varepsilon_{q p_2...p_k} + \omega^p_{i} \varepsilon^{p_2...p_k} \nabla^q \varepsilon_{q p_2...p_k} \).
Since \( \omega \) and \( \varepsilon \) are co-closed forms, the first and last terms on the right-hand side of this formula are equal to zero as is seen by comparing with (6.1). Changing notation for summation indices in two other terms, we transform the formula to the following one:

\[
(\delta_s f)_i = -\frac{1}{2} \left( \nabla_{p_1} \omega_{p_2 \ldots p_k} \varepsilon^{p_1 \ldots p_k} + \nabla_{p_1} \varepsilon_{i p_2 \ldots p_k} \omega^{p_1 \ldots p_k} \right).
\]  

(6.4)

From (6.3) and (6.4)

\[
\left( k \delta_s f + \frac{1}{2} d(\text{tr} f) \right)_i = \frac{1}{2} \left( \nabla_i \omega_{p_1 \ldots p_k} - k \nabla_{p_1} \omega_{i p_2 \ldots p_k} \right) \varepsilon^{p_1 \ldots p_k}
\]

\[
+ \frac{1}{2} \left( \nabla_i \varepsilon_{p_1 \ldots p_k} - k \nabla_{p_1} \varepsilon_{i p_2 \ldots p_k} \right) \omega^{p_1 \ldots p_k}.
\]

Let us show that both terms on the right-hand side of the last formula are equal to zero. Indeed, since the factor \( \varepsilon^{p_1 \ldots p_k} \) is skew-symmetric in all indices, we can write

\[
(\nabla_i \omega_{p_1 \ldots p_k} - k \nabla_{p_1} \omega_{i p_2 \ldots p_k}) \varepsilon^{p_1 \ldots p_k} = \left[ \alpha(i p_1 \ldots p_k) (\nabla_i \omega_{p_1 \ldots p_k} - k \nabla_{p_1} \omega_{i p_2 \ldots p_k}) \right] \varepsilon^{p_1 \ldots p_k},
\]

where \( \alpha(i p_1 \ldots p_k) \) is the alternation in the indices \( (i, p_1, \ldots, p_k) \). The expression in brackets is just \( \frac{1}{k+1} (d \omega)_{i p_1 \ldots p_k} \). It is equal to zero since \( \omega \) is closed.

\[\Box\]

**Proof of Proposition 1.6.** It follows easily from Theorem 1.7 and Proposition 1.8. Let \( h \) be a potential tensor field, i.e., \( h = d s v \) for some \( v \in \Omega^1(M) \) satisfying \( v|_{\partial M} = 0 \). We have to prove that \( \dot{A}_h^{k-1} = 0 \) for every \( k \). By Theorem 1.7, this is equivalent to the statement: for any \( \omega, \varepsilon \in \mathcal{H}_{ex}^k \),

\[
Q_h^k(\omega, \varepsilon) = \left( kd_s v - \frac{1}{2} \text{tr}(d_s v) g, f \right)_{L^2} = 0,
\]

where \( f = F(\omega \otimes \varepsilon) \) is defined by (1.8). Obviously \( \text{tr}(d_s v) = -\delta_s v \). With the help of Green’s formula for operators \( d_s \) and \( \delta_s \) (see Theorem 3.3.1 of [10]), we obtain

\[
\left( kd_s v - \frac{1}{2} \text{tr}(d_s v) g, f \right)_{L^2} = \left( v, k \delta_s f + \frac{1}{2} d(\text{tr} f) \right)_{L^2}.
\]

The right-hand side of this formula is equal to zero by Proposition 1.8. \[\Box\]

7. **The operator** \( D_c = d_s + c g \delta \)

This section contains some preliminaries needed for proving Theorems 1.10 and 1.12. For a real constant \( c \), let us consider the differential operator

\[ D_c = d_s + c g \delta : \Omega^1(M) \to C^\infty(S^2 \tau' M) \]

on an \( n \)-dimensional Riemannian manifold \( (M, g) \). The dual operator is expressed by

\[ D_c^* = \delta_s + c d \text{tr} : C^\infty(S^2 \tau' M) \to \Omega^1(M). \]

Let us calculate the product

\[ D_c^* D_c v = \delta_s d_s v + c \delta_s (g \delta v) + c d(\text{tr}(d_s v)) + c^2 d(\text{tr}(g \delta v)). \]

Substituting the values

\[ \delta_s (g \delta v) = -d \delta v, \quad \text{tr}(d_s v) = -\delta v, \quad \text{tr}(g \delta v) = n \delta v, \]

we get the desired result.
we obtain
\[ D_c^* D_c = \delta_s d_s + c(nc - 2)d\delta. \] (7.1)

Let the Hilbert spaces \( H^2(\tau'_M) \) and \( H^1(S^2\tau'_M) \) be defined as completions of the spaces \( \Omega^1(M) \) and \( C^\infty(S^2\tau'_M) \) with respect to the norms
\[
\|v\|^2_{H^2} = \|v\|^2_{L^2} + \|\nabla v\|^2_{L^2} + \|\nabla\nabla v\|^2_{L^2}, \quad \|f\|^2_{H^1} = \|f\|^2_{L^2} + \|\nabla f\|^2_{L^2},
\]
respectively. Let \( H^2_0(\tau'_M) \) be the subspace of \( H^2(\tau'_M) \) consisting of covector fields \( v \) vanishing on the boundary, \( v|_{\partial M} = 0 \). Let us consider the sequence of Hilbert spaces and operators
\[
H^2_0(\tau'_M) \xrightarrow{D_c} H^1(S^2\tau'_M) \xrightarrow{D^*_c} L^2(\tau'_M).
\] (7.2)

**Lemma 7.1.** For \( c(nc - 2) > -1 \), the operator \( D^*_c D_c \) is elliptic with a positive definite principal symbol. The range of \( D_c \) in sequence (7.2) is closed and the \( L^2 \)-orthogonal decomposition holds
\[
H^1(S^2\tau'_M) = \text{Ran} \, D_c \oplus \text{Ker} \, D^*_c.
\] (7.3)

**Proof.** For a covector \( \xi \), principal symbols of the operators \( d_s \) and \( \delta_s \) are
\[
\sigma_1(d_s) = \sqrt{-1}i\xi, \quad \sigma_1(\delta_s) = -\sqrt{-1}j\xi,
\]
where \( i\xi \) is the operator of symmetric multiplication by \( \xi \) and \( j\xi \) is the contraction with \( \xi \). Using last formulas, we obtain from (7.1)
\[
\sigma_2(D^*_c D_c) = j\xi i\xi + c(nc - 2)i\xi j\xi.
\] (7.4)
The operators \( i\xi \) and \( j\xi \) satisfy the commutator formula (see Lemma 3.3.3 of [10])
\[
j\xi i\xi = \frac{1}{2}(|\xi|^2 I + i\xi j\xi),
\]
where \( I \) is the identity operator. With the help of the last formula, (7.4) becomes
\[
\sigma_2(D^*_c D_c) = \frac{|\xi|^2}{2} I + \left(\frac{1}{2} + c(nc - 2)\right)i\xi j\xi.
\]
If \( c(nc - 2) \geq -1 + \varepsilon \) with some \( \varepsilon > 0 \), then
\[
\langle \sigma_2(D^*_c D_c) v, v \rangle = \frac{1}{2} |\xi|^2 |v|^2 + \left(\frac{1}{2} + c(nc - 2)\right) \langle i\xi j\xi v, v \rangle
\]
\[
= \frac{1}{2} |\xi|^2 |v|^2 + \left(\frac{1}{2} + c(nc - 2)\right) (\xi, v)^2 \geq \varepsilon |\xi|^2 |v|^2.
\]
This proves the first statement of the lemma.

As is known, a boundary value problem with the Dirichlet boundary condition is elliptic if the principal symbol of the differential operator is positive definite, see Proposition 11.10 of [12]. Now, the second statement of the lemma is proved by standard arguments of theory of elliptic boundary value problems. \( \square \)

**Lemma 7.2.** For \( \text{const} = c \leq 1/n \), the homogeneous boundary value problem
\[
(\delta_s d_s - cd\delta)v = 0, \quad v|_{\partial M} = 0
\] (7.5)
on an \( n \)-dimensional manifold has only the zero solution \( v \in H^2(\tau'_M) \).
**Proof.** Let \( v \) be a solution to the problem. Then
\[
(d_s v, d_s v) = c(\delta v, \delta v).
\] (7.6)
If \( c \leq 0 \), this implies \( d_s v = 0 \) and \( v = 0 \).

Let now \( c > 0 \). Represent \( d_s v \) in the form
\[
d_s v = \lambda g + f, \quad \text{tr} f = 0
\] (7.7)
with some scalar function \( \lambda \). Applying the operator \( \text{tr} \) to this equation, we obtain
\[
\delta v = -n\lambda.
\]
The terms on the right-hand side of (7.7) are orthogonal to each other, therefore
\[
\langle d_s v, d_s v \rangle = |\lambda|^2 |g|^2 + |f|^2 = n|\lambda|^2 + |f|^2.
\]
Substituting two last values into (7.6), we obtain
\[
\|f\|^2 = n(cn^2 - 1)\|\lambda\|^2.
\] (7.8)
If \( cn - 1 < 0 \), (7.8) implies that \( f = 0 \) and \( \lambda = 0 \), therefore \( d_s v = 0 \) and \( v = 0 \). For \( cn - 1 = 0 \), (7.8) gives \( f = 0 \), i.e., \( d_s v = \lambda g \). This means that \( v \) is a conformal Killing covector field. By Theorem 1.3 of [4], a conformal Killing field is identically zero if it vanishes on the boundary. The latter fact also follows from the main result of [8]. \( \square \)

8. Rigidity and density

Here we prove Theorems 1.10 and 1.12.

For \( 1 \leq k \leq n \), we introduce the differential operator
\[
\partial_k = k d_s + \frac{1}{2} g \delta : \Omega^1(M) \to C^\infty(S^2 \tau'_M).
\]
In notation of Section 7, \( \partial_k = k D_{1/2k} \). Write down sequence (7.3) for \( \partial_k \)
\[
H^2_0(\tau'_M) \xrightarrow{\partial_k} H^1(S^2 \tau'_M) \xrightarrow{\partial^*_k} L^2(\tau'_M).
\] (8.1)
By Lemma 7.1, the range of \( \partial_k \) is closed and the \( L^2 \)-orthogonal decomposition holds
\[
H^1(S^2 \tau'_M) = \text{Ran} \partial_k \oplus \text{Ker} \partial^*_k.
\] (8.2)

**Proof of Theorem 1.10.** Assume the range of operator (1.12) to be dense in \( C^\infty_{k+1}(M) \). Let a field \( h \in C^\infty(S^2 \tau'_M) \) be such that \( \hat{A}^k_h = 0 \). By Theorem 1.7, the bilinear form
\[
Q^k_{\epsilon}(\omega, \epsilon) = \left( (k+1)h - \frac{1}{2} \text{tr} h \cdot g, F(\omega \otimes \epsilon) \right)_{L^2}
\]
is equal to zero for all \( \omega, \epsilon \in H^k_{ex}(M) \). Since \( \text{Ran} F \) is dense in \( C^\infty_{k+1}(M) \), this implies that
\[
\left( (k+1)h - \frac{1}{2} \text{tr} h \cdot g, f \right)_{L^2} = 0 \quad \text{for all} \quad f \in C^\infty_{k+1}(M).
\]
By the definition of the space \( C^\infty_{k+1}(M) \), its closure in \( H^1(S^2 \tau'_M) \) coincides with \( \text{Ker} \partial^*_k \). Therefore the last equation means that
\[
(k+1)h - \frac{1}{2} \text{tr} h \cdot g \in \left( \text{Ker} \partial^*_k \right)^\perp.
\]
In virtue of (8.2), this is equivalent to the statement 
\[(k + 1)h - \frac{1}{2} \text{tr} h \cdot g \in \text{Ran} \partial_{k+1},\]
i.e., there exists a covector field \(v \in H^2_0(\tau'_M)\) such that
\[(k + 1)d_v + \frac{1}{2} g \delta v = (k + 1)h - \frac{1}{2} \text{tr} h \cdot g.\]  
(8.3)
Applying the operator \(\partial^*_{k+1}\) to this equation, we obtain
\[\partial^*_{k+1} d_{k+1} v = \partial^*_{k+1} \left( (k + 1)h - \frac{1}{2} \text{tr} h \cdot g \right).\]
Since \(\partial^*_{k+1} \partial_{k+1}\) is an elliptic operator and the right-hand side of the equation is smooth, the field \(v\) is smooth too.

We have thus proved the existence of \(v \in \Omega^1(M)\) satisfying equation (8.3) and boundary condition
\[v|_{\partial M} = 0.\]  
(8.4)
Applying the operator \(\text{tr}\) to equation (8.3), we obtain
\[-(k + 1 - n/2) \delta v = (k + 1 - n/2) \text{tr} h.\]
If \(k + 1 - n/2 \neq 0\), this implies
\[\delta v = -\text{tr} h\]
and Eq. (8.3) becomes \(d_v v = h\). Together with boundary condition (8.4), this means that \(h\) is a potential field.

Conversely, assume the existence of a nonzero field \(\tilde{h} \in C^\infty(S^2\tau'_M)\) satisfying
\[\partial^*_{k+1} \tilde{h} = (k + 1) \delta_{\tilde{h}} + \frac{1}{2} d(\text{tr} \tilde{h}) = 0\]  
(8.5)
and
\[\left( \tilde{h}, F(\omega \otimes \varepsilon) \right)_{L^2} = 0 \quad \text{for all } \omega, \varepsilon \in \mathcal{H}^{k+1}_{ex}(M).\]  
(8.6)
Let us show that \(\tilde{h}\) can be represented in the form
\[\tilde{h} = (k + 1)h - \frac{1}{2} \text{tr} h \cdot g\]  
(8.7)
with some \(h \in C^\infty(S^2\tau'_M)\). Indeed, applying the operator \(\text{tr}\) to Eq. (8.7), we obtain
\[\text{tr} h = \frac{1}{k + 1 - n/2} \text{tr} \tilde{h}\]
and therefore
\[h = \frac{1}{k + 1} \left( \tilde{h} + \frac{1}{2(k + 1 - n/2)} \text{tr} \tilde{h} \cdot g \right).\]  
(8.8)
Conversely, if \(h\) is defined by (8.8), then (8.7) holds. We note that \(h\) is a nonzero field as is seen from the same formula (8.7).
Substituting expression (8.7) into (8.6), we obtain
\[ Q_{h}^{k+1}(\omega, \varepsilon) = \left( (k+1)h - \frac{1}{2} \text{tr} h \cdot g, F(\omega \otimes \varepsilon) \right)_{L^2} = 0 \quad \text{for all } \omega, \varepsilon \in H_{\text{ex}}^{k+1}(M). \]

This means, by Theorem 1.7, that \( \dot{\Lambda}_{h}^{k} = 0 \).

Finally, let us show that \( h \) is not a potential field. To this end we substitute expression (8.7) into (8.5) and obtain
\[ (k+1)^2 \delta_{h} + (k+1 - n/4) d(\text{tr} h) = 0. \]  
(8.9)

Assume \( h \) to be potential, i.e., \( h = d_{s} v, \ v|_{\partial M} = 0 \). Substituting \( h = dv \) into (8.9), we arrive to the boundary value problem (7.5) with the constant \( c = \frac{k+1-n/4}{(k+1)^2} \leq 1/n \). By Lemma 7.2, the boundary value problem has only trivial solution, i.e., \( v = 0 \). Therefore \( h = d_{s} v = 0 \). This contradicts to our above conclusion that \( h \neq 0 \).

Before proving Theorem 1.12, let us give the following remark. Theorem 3.3 of [11] states that every second rank symmetric tensor field can be represented as the sum of three fields such that the first field is potential, the second field is spherical, and the last field is trace-free and solenoidal. The theorem is proved in [11] under the assumption that the manifold is conformally rigid. But later in [4] it was proved that every connected Riemannian manifold with nonempty boundary is conformally rigid, see also [8]. So, we can apply the theorem.

**Proof of Theorem 1.12.** For a manifold of an even dimension \( n \), assume the range of operator (1.15) to be dense in \( C_{n/2}(M) \). Let a tensor field \( h \in C_{\infty}(S^2_{\tau'_{M}}) \) be such that \( \dot{\Lambda}_{h}^{n/2-1} = 0 \). By Theorem 3.3 of [11], \( h \) can be uniquely represented in the form
\[ h = d_{s} v + \lambda g + \tilde{h}, \ v|_{\partial M} = 0, \]  
(8.10)

where \( v \in \Omega^1(M), \ \lambda \in C_{\infty}(M) \), and the field \( \tilde{h} \in C_{\infty}(S^2_{\tau'_{M}}) \) satisfies
\[ \delta_{s} \tilde{h} = 0, \quad \text{tr} \tilde{h} = 0. \]  
(8.11)

We have to prove that \( \tilde{h} \) is identically equal to zero.

From (8.10)
\[ 0 = \dot{\Lambda}_{h}^{n/2-1} = \dot{\Lambda}_{d_{s} v}^{n/2-1} + \dot{\Lambda}_{\lambda g}^{n/2-1} + \dot{\Lambda}_{\tilde{h}}^{n/2-1}. \]

The first two terms on the right-hand side are equal to zero by Propositions 1.5 and 1.6. Therefore \( \dot{\Lambda}_{\tilde{h}}^{n/2-1} = 0 \). By Theorem 1.7, the bilinear form
\[ Q_{h}^{n/2}(\omega, \varepsilon) = \frac{n}{2} (\tilde{h}, F(\omega \otimes \varepsilon))_{L^2} = \frac{n}{2} (\tilde{h}, G(\omega \otimes \varepsilon))_{L^2} \]
is equal to zero for all \( \omega, \varepsilon \in H_{\text{ex}}^{n/2}(M) \). Since \( \text{Ran} G \) is dense in \( C_{n/2}(S^2_{\tau'_{M}}) \), this means that \( \tilde{h} \) is \( L^2 \)-orthogonal to \( C_{n/2}(S^2_{\tau'_{M}}) \). On the other hand, \( \tilde{h} \) belongs to \( C_{n/2}(S^2_{\tau'_{M}}) \) as is seen from (1.13) and (8.11). Thus, \( \tilde{h} \) must be identically equal to zero.

Conversely, assume that the range of operator (1.15) is not dense. This means the existence of a nonzero field \( h \in C_{\infty}(S^2_{\tau'_{M}}) \) such that
\[ \text{tr} h = 0, \quad \delta_{s} h = 0 \]  
(8.12)
and
\[(h, G(\omega \otimes \varepsilon))_{L^2} = (h, F(\omega \otimes \varepsilon))_{L^2} = 0 \text{ for all } \omega, \varepsilon \in \mathcal{H}^{n/2}_{ex}(M) .\]

This can be written in the form
\[\left( \frac{n}{2} h - \frac{1}{2} \text{tr} h \cdot g, F(\omega \otimes \varepsilon) \right)_{L^2} = 0 \text{ for all } \omega, \varepsilon \in \mathcal{H}^{n/2}_{ex}(M) .\]

By Theorem 1.7, this means that \(\Lambda^{n/2-1}_h = 0\). It remains to note that \(h\) cannot be represented as a sum of potential and spherical fields since it is orthogonal to all such sums in virtue of (8.12).

9. Euclidean case

We prove Theorem 1.13 here. Define the closed subspace \(H^1_k(S^2 \tau'_M)\) of \(H^1(S^2 \tau'_M)\) by
\[H^1_k(S^2 \tau'_M) = \{ u \in H^1(S^2 \tau'_M) \mid k \delta_s u + \frac{1}{2} d(\text{tr} u) = 0 \} \text{ for } k \neq n/2, \]
\[H^1_{n/2}(S^2 \tau'_M) = \{ u \in H^1(S^2 \tau'_M) \mid \delta_s u = 0, \text{ tr} u = 0 \} .\]

Theorem 1.13 follows from Theorems 1.10 and 1.12 with the help of the following

Lemma 9.1. Let \(M \subset \mathbb{R}^n (n \geq 2)\) be a bounded domain and \(g\) be the Euclidean metric. In the case of \(0 \leq k \leq n - 1, k + 1 \neq n/2\), if a tensor field \(u \in H^1_{k+1}(S^2 \tau'_M)\) is \(L^2\)-orthogonal to the range of operator (1.12), then \(u = 0\). In the case of an even \(n\), if a tensor field \(u \in H^1_{n/2}(S^2 \tau'_M)\) is \(L^2\)-orthogonal to the range of operator (1.15), then \(u = 0\).

Proof. We present the proof of the first statement only, the second statement is proved in the same way with some simplifications. Assume a tensor field \(u \in H^1(S^2 \tau'_M)\) to be \(L^2\)-orthogonal to the range of operator (1.12) and to satisfy the equation
\[(k + 1)\delta_s u + \frac{1}{2} d(\text{tr} u) = 0 .\]

We have to prove that \(u\) is identically equal to zero. The last equation is written in Cartesian coordinates as
\[2(k + 1) \sum_{j=1}^{n} \frac{\partial u_{ij}(x)}{\partial x_j} - \frac{\partial(\text{tr} u(x))}{\partial x_i} = 0 \text{ for } 1 \leq i \leq n .\]

Applying the Fourier transform, we obtain
\[2(k + 1) \sum_{j=1}^{n} \zeta_j \hat{u}_{ij}(\zeta) - \zeta_i \text{tr} \hat{u}(\zeta) = 0 \text{ for } 1 \leq i \leq n .\]

Let two vectors \(\eta, \zeta \in \mathbb{R}^n\) be such that
\[|\eta| = |\zeta|, \quad \langle \eta, \zeta \rangle = 0 .\]

and let \(c\) be a \(k\)-form on \(\mathbb{R}^n\)
\[c = \sum_{i_1, \ldots, i_k=1}^{n} c_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} .\]
with real constant coordinates $c_{i_1...i_k}$ that are skew-symmetric in $(i_1, \ldots, i_k)$ and satisfy

$$|c|^2 = k! \sum_{i_1, \ldots, i_k=1}^n c_{i_1...i_k}^2 = 1. \tag{9.4}$$

Assume $\eta$, $\zeta$ and $c$ to be related by the equations

$$\sum_{p=1}^n \eta_p c_{pj_2\ldots i_k} = \sum_{p=1}^n \zeta_p c_{pj_2\ldots i_k} = 0. \tag{9.5}$$

The complex vector $\xi = \frac{1}{2}(\eta + i\zeta) \in \mathbb{C}^n$ satisfies

$$\xi^2 = \sum_{j=1}^n \xi_j^2 = 0, \quad (\xi c)_{j_2\ldots j_k} = \sum_{p=1}^n \xi_p c_{pj_2\ldots j_k} = 0. \tag{9.6}$$

The $(k + 1)$-forms

$$\omega = d(e^{\xi x} c) = e^{\xi x} \xi \wedge c \quad \text{and} \quad \epsilon = -d(e^{-\bar{\xi} x} c) = e^{-\bar{\xi} x} \bar{\xi} \wedge c \tag{9.7}$$

belong to the space $H_{\alpha x}^{k+1}(\mathbb{R}^n)$ as is easily seen from (9.6). Let us calculate coordinates of the tensor field $f = F(\omega \otimes \epsilon)$. Substituting expressions (9.7) into (1.8), we obtain

$$f_{ij}(x) = e^{i\xi x} \sigma(ij) \left[ \sum_{p_1, \ldots, p_k=1}^n (\xi \wedge c)_{ip_1 \ldots p_k} (\bar{\xi} \wedge c)_{jp_1 \ldots p_k} \right]. \tag{9.8}$$

where $\sigma(ij)$ stands for the symmetrization with respect to the indices $i$ and $j$. Using the formula

$$(\xi \wedge c)_{ip_1 \ldots p_k} = \xi_i c_{p_1 \ldots p_k} - k\alpha(p_1 \ldots p_k)(\bar{\xi}_{p_1} c_{ip_2 \ldots p_k}),$$

we transform the expression in the brackets on (9.8) as follows

$$\sum_{p_1, \ldots, p_k=1}^n (\xi \wedge c)_{ip_1 \ldots p_k} (\bar{\xi} \wedge c)_{jp_1 \ldots p_k}$$

$$= \xi_i \bar{\xi}_j \sum_{p_1, \ldots, p_k=1}^n c_{p_1 \ldots p_k}^2 - k\xi_i \sum_{p_1, \ldots, p_k=1}^n c_{p_1 \ldots p_k} \alpha(p_1 \ldots p_k)(\bar{\xi}_{p_1} c_{jp_2 \ldots p_k})$$

$$- k\bar{\xi}_j \sum_{p_1, \ldots, p_k=1}^n c_{p_1 \ldots p_k} \alpha(p_1 \ldots p_k)(\xi_{p_1} c_{ip_2 \ldots p_k})$$

$$+ k^2 \sum_{p_1, \ldots, p_k=1}^n \alpha(p_1 \ldots p_k)(\xi_{p_1} c_{ip_2 \ldots p_k}) \alpha(p_1 \ldots p_k)(\bar{\xi}_{p_1} c_{jp_2 \ldots p_k}).$$

The first sum on the right-hand side is equal to $1/k!$ by (9.4), while the second and third sums are equal to zero by (9.6). Substituting the latter expression into (9.8), we obtain

$$f_{ij}(x) = e^{i\xi x} \sigma(ij) \left[ \frac{1}{k!} \xi_i \bar{\xi}_j ight.$$

$$+ k^2 \sum_{p_1, \ldots, p_k=1}^n \alpha(p_1 \ldots p_k)(\xi_{p_1} c_{ip_2 \ldots p_k}) \alpha(p_1 \ldots p_k)(\bar{\xi}_{p_1} c_{jp_2 \ldots p_k}) \right]. \tag{9.9}$$
In order to simplify formula (9.9), let us consider the last sum on the right-hand side of the formula in more details. By the definition of the alternation,

\[
\sum_{p_1, \ldots, p_k = 1}^{n} \alpha(p_1 \ldots p_k)(\xi_{p_1 c_{ip_2 \ldots p_k}})\alpha(p_1 \ldots p_k)(\bar{\xi}_{p_1 c_{jp_2 \ldots p_k}})
\]

\[\frac{1}{(k!)^2} \sum_{\rho, \pi \in \Pi_k} \sum_{p_1, \ldots, p_k = 1}^{n} \sigma(\rho)\sigma(\pi)\xi_{p_{\rho(1)} c_{ip_{\rho(2)} \ldots p_{\rho(k)}}} \bar{\xi}_{p_{\pi(1)} c_{jp_{\pi(2)} \ldots p_{\pi(k)}}},
\]

where \( \Pi_k \) is the group of all permutations of the set \( \{1, \ldots, k\} \) and \( \sigma(\rho) \) is the sign of the permutation \( \rho \). By (9.6), the inner sum is equal to zero if \( \rho(1) \neq \pi(1) \). There are \( k((k - 1)!)^2 \) pairs \( (\rho, \pi) \) satisfying \( \rho(1) = \pi(1) \). For every such pair, the inner sum is equal to

\[
\sum_{p = 1}^{n} \xi_{p c_{ip}} \bar{\xi}_{p c_{jp}} \sum_{q_2, \ldots, q_k = 1}^{n} c_{iq_2 \ldots q_k} c_{jq_2 \ldots q_k}.
\]

Therefore

\[
\sum_{p_1, \ldots, p_k = 1}^{n} \alpha(p_1 \ldots p_k)(\xi_{p_1 c_{ip_2 \ldots p_k}})\alpha(p_1 \ldots p_k)(\bar{\xi}_{p_1 c_{jp_2 \ldots p_k}})
\]

\[\frac{1}{k} \sum_{p = 1}^{n} \xi_{p c_{ip}} \bar{\xi}_{p c_{jp}} \sum_{q_2, \ldots, q_k = 1}^{n} c_{iq_2 \ldots q_k} c_{jq_2 \ldots q_k}.
\]

Inserting this expression into (9.9), we obtain

\[
f_{ij}(x) = e^{i\zeta x} \sigma(ij) \left( \frac{1}{k!} \xi_{i c_{ip}} \xi_{j c_{jp}} + k \sum_{p = 1}^{n} \xi_{p c_{ip}} \bar{\xi}_{p c_{jp}} \sum_{q_2, \ldots, q_k = 1}^{n} c_{iq_2 \ldots q_k} c_{jq_2 \ldots q_k} \right).
\]

With the help of the relations

\[
\sigma(ij)(\xi_{i c_{ip}} \xi_{j c_{ip}}) = \frac{1}{4}(\eta_i \eta_j + \zeta_i \zeta_j), \quad \sum_{p = 1}^{n} \xi_{p c_{ip}} \bar{\xi}_{p c_{jp}} = \frac{1}{2}|\eta|^2
\]

which follows from \( \xi = \frac{1}{2}(\eta + i\zeta) \) and (9.3), formula (9.10) takes the final form

\[
f_{ij}(x) = \frac{1}{4} e^{i\zeta x} \left( \frac{1}{k!} (\eta_i \eta_j + \zeta_i \zeta_j) + 2k|\eta|^2 \sigma(ij) \sum_{p_2, \ldots, p_k = 1}^{n} c_{ip_2 \ldots p_k} c_{jp_2 \ldots p_k} \right).
\]

The tensor field \( u \) is \( L^2 \)-orthogonal to the field \( f = F(\omega \otimes \varepsilon) \), i.e.,

\[
\sum_{i, j = 1}^{n} \int_{M} u_{ij}(x) f_{ij}(x) \, dx = 0.
\]

Substituting expression (9.11) into (9.12), we obtain

\[
\sum_{i, j = 1}^{n} \left( \frac{1}{k!} (\eta_i \eta_j + \zeta_i \zeta_j) + 2k|\eta|^2 \sum_{p_2, \ldots, p_k = 1}^{n} c_{ip_2 \ldots p_k} c_{jp_2 \ldots p_k} \right) \hat{u}_{ij}(\zeta) = 0.
\]
Remark. The reader may be confused with the real form of the $L^2$-product used on (9.12): the integrand on (9.12) should be probably replaced with $u_{ij} f_{ij}$ since $f$ is now a complex valued tensor. Besides this, the bilinear form $F(\omega \otimes \varepsilon)$ should be probably replaced with the corresponding sesquilinear form for complex valued $\omega$ and $\varepsilon$. Any such replacement does not change our arguments since the complex conjugate $\bar{\omega}$ belongs to $H^k_{ex}(M)$ for every $\omega \in H^k_{ex}(M)$.

Now, we consider the system obtained by uniting Eqs. (9.2) and (9.13). We are going to demonstrate that $\hat{u}(\zeta) = 0$ if Eqs. (9.2) and (9.13) hold for a fixed $\zeta \in \mathbb{R}^n$ and for all $\eta, c$ satisfying (9.3)–(9.5). For a fixed $0 \neq \zeta \in \mathbb{R}^n$, we choose Cartesian coordinates in $\mathbb{R}^n$ such that $\zeta = (0, \ldots, 0, |\zeta|)$. Conditions (9.3) hold if and only if $\eta = |\zeta| \eta'$ with $\eta' = (\eta'_1, \ldots, \eta'_{n-1}, 0) \in \mathbb{R}^{n-1}$ satisfying $|\eta'| = 1$. Conditions (9.5) are now written as

$$c_{ni_2 \ldots i_k} = 0 \quad (9.14)$$

and

$$\sum_{p=1}^{n-1} \eta'_p c_{pi_2 \ldots i_k} = 0. \quad (9.15)$$

Equations (9.2) and (9.13) take the form

$$\hat{u}_{in}(\zeta) = 0 \quad \text{for } 1 \leq i \leq n - 1,$$

$$2(k + 1)\hat{u}_{nn}(\zeta) - \text{tr} \hat{u}(\zeta) = 0,$$

$$\sum_{i,j=1}^{n-1} \left( \frac{1}{k!} \eta'_i \eta'_j + 2k \sum_{p_2, \ldots, p_k=1}^{n-1} c_{ip_2 \ldots p_k} c_{jp_2 \ldots p_k} \right) \hat{u}_{ij}(\zeta) + \frac{1}{k!} \hat{u}_{nn}(\zeta) = 0. \quad (9.16)$$

Expressing

$$\hat{u}_{nn}(\zeta) = \frac{1}{2k + 1} \sum_{i=1}^{n-1} \hat{u}_{ii}(\zeta)$$

from (9.16) and substituting the expression into (9.17), we obtain

$$\sum_{i,j=1}^{n-1} \left( \hat{u}_{ij}(\zeta) + \frac{1}{2k + 1} \delta_{ij} \sum_{p=1}^{n-1} \hat{u}_{pp}(\zeta) \right) \eta'_i \eta'_j = -2kk! \sum_{i,j=1}^{n-1} \hat{u}_{ij}(\zeta) \sum_{p_2, \ldots, p_k=1}^{n-1} c_{ip_2 \ldots p_k} c_{jp_2 \ldots p_k} \quad (9.17)$$

This equation holds for every unit vector $\eta' \in \mathbb{R}^{n-1}$ and for every $k$-vector $c \in \Lambda^k \mathbb{R}^n$ satisfying (9.4) and (9.14)–(9.15).

Further arguments are slightly different in the cases of $n \leq 3$ and of $n \geq 4$. Let us first consider the case of $n \geq 4$. We are going to demonstrate that the left-hand side of (9.18) is independent of the unit vector $\eta' \in \mathbb{R}^{n-1}$. Let $\eta'$ and $\tilde{\eta}'$ be two such vectors. We choose a unit vector $e_1 \in \mathbb{R}^{n-1}$ orthogonal to both $\eta'$ and $\tilde{\eta}'$. Then we complete the vector $e_1$ to an orthonormal system $(e_1, \ldots, e_k)$ of vectors belonging to $\mathbb{R}^{n-1}$ and set $c = e_1 \wedge \cdots \wedge e_k$. The $k$-vector $c$ satisfies (9.4), (9.14)–(9.15) and the equation obtained from (9.18) by replacing $\eta'$ with $\tilde{\eta}'$. Therefore Eq. (9.18) holds for the chosen $c$, as well as the equation obtained from (9.18) by replacing $\eta'$
with $\tilde{\eta}'$. Right-hand sides of two latter equations are the same, so the left-hand sides must be equal.

Thus, the left-hand side of (9.18) is the same for any unit vector $\eta' \in \mathbb{R}^{n-1}$. This implies immediately

$$\hat{u}_{ij}(\zeta) = \lambda(\zeta)\delta_{ij} \quad \text{for } 1 \leq i, j \leq n - 1,$$

(9.19)

with some scalar $\lambda(\zeta)$. Now, with the help of (9.4), the right-hand side of (9.18) is simplified as follows

$$k! \sum_{i,j=1}^{n-1} \hat{u}_{ij}(\zeta) \sum_{p_2,\ldots,p_k=1}^{n-1} c_{i_2\ldots p_k} c_{j_2\ldots p_k} = \lambda(\zeta)k! \sum_{p_1,\ldots,p_k=1}^{n-1} c_{p_1\ldots p_k}^2 = \lambda(\zeta),$$

and Eq. (9.18) is reduced to the following one:

$$\lambda(\zeta) + \frac{n-1}{2k+1} \lambda(\zeta) = -2k\lambda(\zeta).$$

This gives $\lambda(\zeta) = 0$ and therefore $\hat{u}(\zeta) = 0$.

In the two-dimensional case $n = 2$, (9.19) holds trivially and implies $\hat{u}(\zeta) = 0$ as before.

It remains to consider the three-dimensional case. In the case of $n = 3$ and $k = 0$, Eq. (9.18) is as follows

$$\sum_{i,j=1}^{2} \left( \hat{u}_{ij}(\zeta) + \delta_{ij} \sum_{p=1}^{2} \hat{u}_{pp}(\zeta) \right) \eta'_i\eta'_j = 0.$$

This equation holds for every $\eta' \in \mathbb{R}^2$. Therefore

$$\hat{u}_{ij}(\zeta) + \delta_{ij} \sum_{p=1}^{2} \hat{u}_{pp}(\zeta) = 0 \quad (1 \leq i, j \leq 2).$$

This implies immediately that $\hat{u}(\zeta) = 0$.

Finally, in the case of $n = 3$ and $k = 1$, let $(e_1, e_2, e_3)$ be an orthonormal basis of $\mathbb{R}^3$ such that $\zeta = |\zeta|e_3$. We choose $\eta' = \cos \theta e_1 + \sin \theta e_2$ and $c = -\sin \theta e_1 + \cos \theta e_2$ with an arbitrary $\theta$. For the chosen $\eta'$ and $c$, Eq. (9.18) is as follows

$$\left( \hat{u}_{11} + \frac{1}{3}(\hat{u}_{11} + \hat{u}_{22}) \right) \cos^2 \theta + 2\hat{u}_{12} \cos \theta \sin \theta + \left( \hat{u}_{22} + \frac{1}{3}(\hat{u}_{11} + \hat{u}_{22}) \right) \sin^2 \theta$$

$$= -2(\hat{u}_{11} \sin^2 \theta - 2\hat{u}_{12} \cos \theta \sin \theta + \hat{u}_{22} \sin^2 \theta),$$

where $\hat{u}_{ij} = \hat{u}_{ij}(\zeta)$. We rewrite the equation in the form

$$(4\hat{u}_{11}(\zeta) + 7\hat{u}_{22}(\zeta)) \cos^2 \theta - 6\hat{u}_{12}(\zeta) \cos \theta \sin \theta + (7\hat{u}_{11}(\zeta) + 4\hat{u}_{22}(\zeta)) \sin^2 \theta = 0.$$

Since $\theta$ is arbitrary, this implies $\hat{u}_{ij}(\zeta) = 0 \ (1 \leq i, j \leq 2)$. \qed

**References**


