The Reshetnyak formula and Natterer stability estimates in tensor tomography

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The Reshetnyak formula and Natterer stability estimates in tensor tomography

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Abstract

The Reshetnyak formula (also known as the Plancherel formula for the Radon transform) states that the Radon transform \( R \) is an isometry between \( L^2(\mathbb{R}^n) \) and \( H^{n-1/2, \alpha}(S^n \times \mathbb{R}) \), the latter being the Hilbert space of even functions on \( S^n \times \mathbb{R} \) furnished by some special norm. We generalize this statement to Sobolev spaces: \( R \) is an isometry between \( H^s(\mathbb{R}^n) \) and \( H^{n-1/2, \alpha}(S^n \times \mathbb{R}) \) for every real \( s \). Moreover, with the help of Riesz potentials, we define some new Hilbert spaces \( H^t(\mathbb{R}^n) \) and prove that \( R \) is an isometry between \( H^t(\mathbb{R}^n) \) and \( H^{n-1/2, \alpha}(S^n \times \mathbb{R}) \). The generalized Reshetnyak formula closely relates to the Natterer stability estimates:

\[
|f|_{H^t(\mathbb{R}^n)} \lesssim \|Rf\|_{H^{n-1/2, \alpha}(S^n \times \mathbb{R})} \lesssim |f|_{H^t(\mathbb{R}^n)}\text{ for functions } f \text{ supported in a fixed ball.}
\]

Then we obtain analogs of these statements for the x-ray transform of symmetric tensor fields.

Keywords: Reshetnyak formula, stability estimates, tensor tomography

1. Introduction

The Radon transform \( R \) and x-ray transform \( I \) are main mathematical tools of tomography. These operators have also many pure mathematical applications. As an example, let us recall that Radon [11] introduced the operator \( R \) while searching for a decomposition of an arbitrary solution to the wave equation to a sum of plane waves. As the most recent example, Nadirashvili and Vladuţ [7] apply the x-ray transform of tensor fields for studying the Euler equations for ideal fluids.

In tomography, the data \( Rf \) (or \( If \)) for recovering an unknown \( f \) always contain some errors since they are obtained by measurements. Therefore stability estimates for \( R \) and \( I \) are of great importance, first of all for designing tomographic hardware. The situation is similar in theoretical applications of the operators where the linear data \( Rf \) (or \( If \)) are mostly obtained...
by some asymptotic procedure from non-linear data. See for example [13, section 5.1] where linear data for the polarization tomography problem are obtained by using the Born approximation and shortwave asymptotics for the Maxwell equations.

The first result on stability for the Radon transform was obtained by Yu Reshetnyak (unpublished work, approximately 1960). For a function \( f \in L^2(\mathbb{R}^n) \), he proved the equality

\[
\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} = \|Rf\|_{\mathcal{H}^{n+1/2}(\mathbb{R}^{n+1} \times \mathbb{R})},
\]

The norm on the right-hand side is defined in section 2 below. A proof of (1.1) is presented (with a reference to Reshetnyak) in the book [3, section 1.1.5] by Gelfand et al. Gelfand calls (1.1) the Plancherel formula for the Radon transform. We will call (1.1) the Reshetnyak formula. In our opinion, the proof in [3] is too complicated; at least the cases of even and odd \( n \) should be considered separately. An easier proof of (1.1) is presented by Helgason [4, chapter 2, theorem 2.17]. We actually reproduce the latter proof with some modifications in our proof of theorem 2.1 below.

The Reshetnyak formula can be easily generalized to Sobolev norms:

\[
\|f\|_{\mathcal{H}^2(\mathbb{R}^n)} = \|Rf\|_{\mathcal{H}^{n+3/2}(\mathbb{R}^{n+1} \times \mathbb{R})},
\]

Surprisingly, but this result was not known before.

The standard Sobolev norm of the function \( f \) participates on (1.2) while some modified Sobolev norm of \( Rf \) should be used. Actually, the functions \( f \) and \( Rf \) are somewhat equal in their rights in theory of the Radon transform. This means in particular that some version of formula (1.2) should exist which involves the standard Sobolev norm of \( Rf \). Indeed, we prove

\[
\|f\|_{\mathcal{H}^1_{(1-n/2)}(\mathbb{R}^n)} = \|Rf\|_{\mathcal{H}^{n+1/2}(\mathbb{R}^{n+1} \times \mathbb{R})}.
\]

Formulas (1.2) and (1.3) can be obtained simultaneously in the scope of some universal approach. In section 2, we introduce Hilbert spaces \( \mathcal{H}^t(\mathbb{R}^n) \) \((t > -1/2)\) and \( \mathcal{H}^t(\mathbb{S}^{n-1} \times \mathbb{R}) \) \((t > -1/2)\) and prove the (generalized) Reshetnyak formula

\[
\|f\|_{\mathcal{H}^t(\mathbb{R}^n)} = \|Rf\|_{\mathcal{H}^{t+1/2}(\mathbb{R}^{n+1} \times \mathbb{R})}.
\]

Introducing the spaces, the author was sure that he was reinventing the wheel. Nevertheless, so far we have not found any paper where the spaces are studied explicitly. The norm \( \|\cdot\|_{\mathcal{H}^t} \) is episodically used by Fedotov and Volevich [2] without discussing the spaces \( \mathcal{H}^t \).

The Reshetnyak formula gives the best stability estimate for the problem of recovering \( f \) from \( Rf \). Nevertheless, the following stability estimates are mostly used in tomography:

\[
c\|f\|_{\mathcal{H}^t(\mathbb{R}^n)} \leq \|Rf\|_{\mathcal{H}^{t+1/2}(\mathbb{R}^{n+1} \times \mathbb{R})} \leq C\|f\|_{\mathcal{H}^t(\mathbb{R}^n)}.
\]

The estimates hold for functions \( f \) supported in a fixed ball. The constants \( c \) and \( C \) depend on \((n,s)\) and on the radius of the ball but are independent of \( f \). We call (1.4) the Natterer stability estimates although several other authors have done their contributions to the subject [5, 6, 8, 9, 14]. No one of these authors has mentioned the relation of the subject to the Reshetnyak formula.

All of our arguments in section 2 constitute a slight modification of Natterer’s arguments from the proof of [10, chapter 2, theorem 5.1]. Natterer’s proof consists of two parts. The result of the first part is equivalent to Reshetnyak formula (1.3) and the second part actually proves the equivalence of norms \( \|f\|_{\mathcal{H}^t(\mathbb{R}^n)} \) and \( \|f\|_{\mathcal{H}^t_{(1-n/2)}(\mathbb{R}^n)} \) for functions \( f \) supported in a fixed ball.

Then we proceed to studying the x-ray transform \( I \) on symmetric tensor fields of an arbitrary rank \( m \). Unlike the Radon transform, the operator \( I \) has a large null-space in the case
of $m > 0$. Given $If$, we can hope to recover the solenoidal part $f'$ of a tensor field $f$ only. The definition of the solenoidal part slightly varies depending on the space of tensor fields under consideration.

Section 3 consists of two parts. The first part contains preliminaries on symmetric tensors and tensor fields together with a survey of results known before. Starting from lemma 3.3, we present new results related to spaces $H^s_t$. Statements known before are called propositions while new statements are called theorems.

In section 4, we derive an analogous of the Reshetnyak formula for the operator $I$. As compared with the Radon transform, the main specifics here is that we have to compare the norms $\|f\|$ and $\|If\|$ since $If$ depends on the solenoidal part $f'$ only. Besides this, the Reshetnyak formula for tensor fields has a more complicated form involving some pure algebraic operators. Just because of this, we do not present the formula in the introduction. But in principle, the structure of the Reshetnyak formula for $I$ as well as its proof are very similar to (1.4).

At the end of section 4, we discuss an analogous of the Natterer stability estimates (1.5) for the x-ray transform. Here, the situation is much more interesting and difficult than for the Radon transform. The difficulty is caused by the following: given a compactly supported tensor field $f$, the solenoidal part $f'$ is not compactly supported in the general case. We obtain some partial results but actually the question remains open, see problem 4.6 below.

Finally, we shortly discuss a question aroused by the anonymous referee. The Natterer stability estimates in Sobolev norms are still valid for the attenuated Radon transform, see for example [12, theorems 4.1 and 4.2], as well as for the attenuated x-ray transform on scalar functions [13, theorem 8.1.1 and corollary 8.1.3] (the term ‘exponential ray transform’ is used instead of attenuated ray transform in the latter book). It would be interesting to obtain the corresponding estimates for the attenuated Radon transform in our $H^s_t$-norms. The author did not think on this yet. The attenuated x-ray transform on tensor fields can be easily defined but the following question remains open even in the case of a constant attenuation: Does there exist an appropriate analogous of the solenoidal part of a tensor field in theory of the attenuated x-ray transform?

This paper has several origins. As already mentioned, formula (1.1) was obtained by Yu. Reshetnyak in early 60’s. Stability estimates (1.5) for functions supported in a given ball were proved by Natterer and others in 80’s. Natterer’s book [10] contains also stability estimates for the x-ray transform in the scalar case of $m = 0$ ($f = f$ in this case). An analogous of formula (1.1) for $I$ was proved by the author [13, theorem 2.15.1] (and was called the Plancherel formula for the ray transform). In the case of a vector field $f$ on $\mathbb{R}^3$, a formula equating $\|If\|_{L^2}$ to some norm of $f'$ was obtained by Boman [1]. Boman’s formula is the partial case of our formula (4.6) for $(s,t,m,n) = (0, -1/2, 1, 3)$. In the present work, we just looked for a universal approach uniting these results.

2. The Radon transform

The Radon transform $R$ maps a function defined on $\mathbb{R}^n$ to the family of its integrals over hyperplanes

$$Rf(\xi, p) = \int_{\{x, \xi \cdot p = 0\}} f(x) \, dx \quad (\xi, p) \in \mathbb{S}^{n-1} \times \mathbb{R}. \quad (2.1)$$

The standard dot-product on $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$, and $|\cdot|$ is the corresponding norm; $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$. In (2.1), $dx$ means the $(n - 1)$-dimensional Lebesgue measure on the hyperplane $\{x \in \mathbb{R}^n | \langle x, \xi \rangle = 0\}$. Some decay condition should be imposed on the function $f$ in order integral (2.1) to converge in the classical or some generalized sense.
Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of smooth rapidly decaying together with all derivatives functions on $\mathbb{R}^n$ furnished by the standard topology (we use the term ’smooth’ as the synonym of ’$C^\infty$-smooth’). Initially, the Radon transform can be considered as the linear continuous operator

$$R : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{S}^{n-1} \times \mathbb{R}).$$

(2.2)

Here $\mathcal{S}(\mathbb{S}^{n-1} \times \mathbb{R})$ is the Schwartz space of functions on $\mathbb{S}^{n-1} \times \mathbb{R}$. More generally, the Schwartz space $\mathcal{S}(E)$ is well defined on the total space of a smooth vector bundle $E \to M$ over a compact manifold $M$.

Roughly speaking, the Radon transform increases the smoothness of a function by $(n - 1)/2$. The precise sense of this statement is expressed by theorem 2.1 below.

Recall that the Fourier transform $\hat{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, $f \mapsto \hat{f}$ is defined by

$$\hat{f}(y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\langle x, y \rangle} f(x) \, dx.$$  

For a real $s$ and $t > -n/2$, the Hilbert space $H^s_t(\mathbb{R}^n)$ is defined as the completion of $\mathcal{S}(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{H^s_t(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |y|^{2s} (1 + |y|^2)^{t+s} |\hat{f}(y)|^2 \, dy \right)^{1/2}.$$  

(2.3)

Let us compare (2.3) with the Sobolev norm

$$\|\varphi\|_{H^s_t(\mathbb{S}^{n-1})} = \left( \int_{\mathbb{S}^{n-1}} |\xi|^2 (1 + |\xi|^2)^{t+s} |\hat{\varphi}(\xi,q)|^2 \, d\xi d\xi \right)^{1/2}.$$  

(2.4)

The weights $|y|^{2s} (1 + |y|^2)^{t+s}$ and $(1 + |y|^2)^t$ have the same asymptotics as $|y| \to \infty$ but are very different near $y = 0$ if $t \neq 0$. Therefore $\|\cdot\|_{H^s_t(\mathbb{R}^n)}$ can be called ’the Sobolev norm with attenuated low frequencies’ in the case of $t > 0$ and ’the Sobolev norm with amplified low frequencies’ in the case of $t < 0$.

The Fourier transform $F : \mathcal{S}(\mathbb{S}^{n-1} \times \mathbb{R}) \to \mathcal{S}(\mathbb{S}^{n-1} \times \mathbb{R})$, $\varphi(\xi,p) \mapsto \hat{\varphi}(\xi,q)$ is just the one-dimensional Fourier transform with respect to the variable $p$ while $\xi$ is considered as a parameter

$$\hat{\varphi}(\xi,q) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-iwp} \varphi(\xi,p) \, dp.$$  

For a real $s$ and $t > -1/2$, the Hilbert space $H^s_t(\mathbb{S}^{n-1} \times \mathbb{R})$ is defined as the completion of $\mathcal{S}(\mathbb{S}^{n-1} \times \mathbb{R})$ with respect to the norm

$$\|\varphi\|_{H^s_t(\mathbb{S}^{n-1} \times \mathbb{R})} = \left( \frac{1}{2(2\pi)^n} \int_{\mathbb{S}^{n-1}} \int_{-\infty}^{\infty} |q|^{2s} (1 + q^2)^{t+s} |\hat{\varphi}(\xi,q)|^2 \, dq d\xi \right)^{1/2},$$  

(2.5)

where $d\xi$ is the volume form on $\mathbb{S}^{n-1}$ induced by the Euclidean metric of $\mathbb{R}^n$. Again $H^s_t(\mathbb{S}^{n-1} \times \mathbb{R}) = H^s_t(\mathbb{S}^{n-1} \times \mathbb{R})$ is the Sobolev space. The factor $1/(2(2\pi)^{n-1})$ is written in (2.5) to adopt the norm to theorem 2.1 below.

For general $(s,t)$, $H^s_t$ is not always a space of distributions. Therefore elements of the space should be treated with some care.

Let $\mathcal{S}_e(\mathbb{S}^{n-1} \times \mathbb{R}) \subset \mathcal{S}(\mathbb{S}^{n-1} \times \mathbb{R})$ be the subspace of even functions satisfying $\varphi(-\xi,-p) = \varphi(\xi,p)$ and let $H^s_{te}(\mathbb{S}^{n-1} \times \mathbb{R}) \subset H^s_t(\mathbb{S}^{n-1} \times \mathbb{R})$ be the completion of $\mathcal{S}_e(\mathbb{S}^{n-1} \times \mathbb{R})$ with respect to norm (2.5).
Theorem 2.1. For a real s and t > −n/2, the equality
\[ \|f\|_{H^s_t(\mathbb{R}^n)} = \|Rf\|_{H^{s+\frac{(n-1)}{2},t\frac{1}{2}}(S^{n-1} \times \mathbb{R})} \]  
holds for every function \( f \in \mathcal{S}(\mathbb{R}^d) \). Operator (2.2) uniquely extends to the bijective isometry of Hilbert spaces
\[ R : H^s_t(\mathbb{R}^d) \rightarrow H^{s+\frac{(n-1)}{2},t\frac{1}{2}}(S^{n-1} \times \mathbb{R}). \]  

Proof. Given \( f \in \mathcal{S}(\mathbb{R}^d) \), let \( \varphi = Rf \). By definition (2.5),
\[ \|\varphi\|_{H^{s+\frac{(n-1)}{2},t\frac{1}{2}}(S^{n-1} \times \mathbb{R})} = \frac{1}{2(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} |q|^{2s+\frac{n}{2}-1}(1 + q^2)^{-\frac{1}{2}} |\hat{\varphi}(\xi, q)|^2 \, dq \, d\xi. \]
The function \( \hat{\varphi} \) is even together with \( \varphi \). Therefore the latter integral can be rewritten as
\[ \|\varphi\|_{H^{s+\frac{(n-1)}{2},t\frac{1}{2}}(S^{n-1} \times \mathbb{R})} = (2\pi)^{(n-1)/2} \int_{\mathbb{R}^{n-1}} \int_{0}^{\infty} |q|^{2s+\frac{n}{2}-1}(1 + q^2)^{-\frac{1}{2}} |\hat{\varphi}(\xi, q)|^2 \, dq \, d\xi. \]
Changing integration variables as \( y = q\xi \), we transform this formula to the form
\[ \|\varphi\|_{H^{s+\frac{(n-1)}{2},t\frac{1}{2}}(S^{n-1} \times \mathbb{R})} = (2\pi)^{(n-1)/2} \int_{\mathbb{R}^d} |y|^{2s+1} |\hat{\varphi}(y)|^2 \, dy. \]
By the slice theorem [10, chapter 2, theorem 1.1], \( \hat{\varphi}(y) = (2\pi)^{(n-1)/2} f(y) \). Substituting this expression into the previous formula, we arrive to (2.6).

Now, we prove the second statement of the theorem. Obviously, \( Rf \in \mathcal{S}(S^{n-1} \times \mathbb{R}) \) for \( f \in \mathcal{S}(\mathbb{R}^d) \), i.e. instead of (2.2), we actually have
\[ R : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(S^{n-1} \times \mathbb{R}). \]
The operator satisfies (2.6) for every \( f \in \mathcal{S}(\mathbb{R}^d) \). Hence, it uniquely extends to completions, i.e. to operator (2.7). Equality (2.6) holds now for every \( f \in H^s_t(\mathbb{R}^d) \). Therefore (2.7) is an injective operator with a closed range. We have to prove the surjectivity of operator (2.7).

Let \( \Phi \in H^{s+\frac{(n-1)}{2},t\frac{1}{2}}(S^{n-1} \times \mathbb{R}) \) be orthogonal to the range of operator (2.7). In particular,
\[ (Rf, \Phi)_{H^{s+\frac{(n-1)}{2},t\frac{1}{2}}(S^{n-1} \times \mathbb{R})} = 0 \quad \text{for every} \quad f \in \mathcal{S}(\mathbb{R}^d). \]
Choose a sequence \( \varphi_k \in \mathcal{S}(S^{n-1} \times \mathbb{R}) \) (\( k = 1, 2, \ldots \)) converging to \( \Phi \) in \( H^{s+\frac{(n-1)}{2},t\frac{1}{2}}(S^{n-1} \times \mathbb{R}) \). Such a sequence exists by the very definition of the latter space. Then the sequence of norms \( \|\varphi_k\|_{H^{s+\frac{(n-1)}{2},t\frac{1}{2}}(S^{n-1} \times \mathbb{R})} \) is bounded
\[ (Rf, \varphi_k)_{H^{s+\frac{(n-1)}{2},t\frac{1}{2}}(S^{n-1} \times \mathbb{R})} \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty \quad \text{for every} \quad f \in \mathcal{S}(\mathbb{R}^d). \]
This can be written as
\[ \int_{S^{n-1}} \int_{0}^{\infty} |q|^{2s+\frac{n}{2}-1}(1 + q^2)^{-\frac{1}{2}} |\hat{\varphi}_k(\xi, q)|^2 \, dq \, d\xi \rightarrow 0. \]
The integration versus $q$ is restricted to $q \geq 0$ because the integrand is an even function in $q$. Changing integration variables as above, we obtain
\[
\int_{\mathbb{R}^n} y^2 (1 + |y|^2)^{r-1} \phi_k(\|y\|, |y|) f(y) \, dy \to 0. \tag{2.8}
\]
Functions $g(y) = |y|^r (1 + |y|^2)^{r-1/2} \tilde{f}(y)$ constitute a dense subset of $L^2(\mathbb{R}^n)$ when $f$ varies in $\mathcal{S}(\mathbb{R}^n)$. Introducing the functions $h_k \in L^2(\mathbb{R}^n)$ $(k = 1, 2, \ldots)$ by
\[
h_k(y) = |y|^r (1 + |y|^2)^{r-1/2} \phi_k(\|y\|, |y|) \quad (y \neq 0),
\]
we write (2.8) in the form
\[
(h_k, g)_{L^2(\mathbb{R}^n)} \to 0.
\]
Since this relation holds for a dense in $L^2(\mathbb{R}^n)$ set of functions $g$ and the sequence of norms
\[
\|h_k\|_{L^2(\mathbb{R}^n)} = \|\phi_k\|_{H^{(n-1)/2}(\mathbb{R}^n)}
\]
is bounded, the sequence $h_k$ weakly converges to zero in $L^2(\mathbb{R}^n)$.

Now, for an arbitrary even function $\psi \in \mathcal{S}(\mathbb{S}^{n-1} \times \mathbb{R})$,
\[
(\phi_k, \psi)_{H^{(n-1)/2}(\mathbb{S}^{n-1} \times \mathbb{R})} = (h_k, |y|^r (1 + |y|^2)^{r-1/2} \tilde{\psi}(\|y\|, |y|))_{L^2(\mathbb{R}^n)} \to 0.
\]
Since this relation holds for the dense in $H^{s+(n-1)/2}(\mathbb{S}^{n-1} \times \mathbb{R})$ set of functions $\psi$ and the sequence of norms $\|\phi_k\|_{H^{s+(n-1)/2}(\mathbb{S}^{n-1} \times \mathbb{R})}$ is bounded, the sequence $\phi_k$ weakly converges to zero in $H^{s+(n-1)/2}(\mathbb{S}^{n-1} \times \mathbb{R})$. Initially, the sequence was chosen so that $\phi_k \to \Phi$ strongly in $H^{s+(n-1)/2}(\mathbb{S}^{n-1} \times \mathbb{R})$. Therefore $\Phi = 0$. \hfill \Box

Let us write down the Reshetnyak formula in the cases of $t = 0$ and of $t = -(n-1)/2$
\[
\|f\|_{H^t(\mathbb{R}^n)} = \|Rf\|_{H^{(n-1)/2}(\mathbb{S}^{n-1} \times \mathbb{R})},
\]
\[
\|f\|_{H^{-t}(\mathbb{R}^n)} = \|Rf\|_{H^{(n-1)/2}(\mathbb{S}^{n-1} \times \mathbb{R})}.
\]
Thus, to compare Sobolev norms of $f$ and $Rf$, we have either to attenuate low frequencies of $Rf$ or to amplify low frequencies of $f$.

The following stability estimates are mostly used in tomography.

**Proposition 2.2.**  Given $A > 0$ and $s \in \mathbb{R}$, the estimates
\[
c \|f\|_{H^s(\mathbb{R}^n)} \leq \|Rf\|_{H^{s+(n-1)/2}(\mathbb{S}^{n-1} \times \mathbb{R})} \leq C \|f\|_{H^s(\mathbb{R}^n)} \tag{2.9}
\]
hold for every function $f \in H^s(\mathbb{R}^n)$ supported in the ball $\{x \in \mathbb{R}^n : |x| \leq A\}$, where $c$ and $C$ are some constants dependent on $(s, n, A)$ but independent of $f$.

If a function $f$ is supported in the ball $\{x \in \mathbb{R}^n : |x| \leq A\}$, then $Rf$ is supported in the set $\{(x, p) \in \mathbb{S}^{n-1} \times \mathbb{R} : |p| \leq A\}$. Therefore proposition 2.2 immediately follows from theorem 2.1 with the help of any of the following two statements.

**Lemma 2.3.**  For any real $s$ and for $t \in (-n/2, n/2)$, the norms $\|f\|_{H^s(\mathbb{R}^n)}$ and $\|f\|_{H^t(\mathbb{R}^n)}$ are equivalent on the space of functions $f \in \mathcal{S}(\mathbb{R}^n)$ supported in a fixed ball $\{x \in \mathbb{R}^n : |x| \leq A\}$. 

Lemma 2.4. For any real $s$ and for $t \in (-1/2, 1/2)$, the norms $\|\psi\|_{H^s(\mathbb{S}^{n-1} \times \mathbb{R})}$ and $\|\phi\|_{H^s(\mathbb{S}^{n-1} \times \mathbb{R})}$ are equivalent on the space of functions $\varphi \in \mathcal{S}(\mathbb{S}^{n-1} \times \mathbb{R})$ supported in the set \{$(\xi, p) \in \mathbb{S}^{n-1} \times \mathbb{R} \mid |p| \leq A$\} with a fixed $A$.

Formally speaking, these statements do not relate to the Radon transform.

We present the proof of lemma 2.3. The proof of lemma 2.4 is very similar.

Proof of lemma 2.3. We consider separately the cases of positive and negative $t$.

Let $0 \leq t < n/2$. On using the inequality $|y|^{2t} < (1 + |y|^2)^t$, we obtain
\[
\|f\|_{L^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |y|^{2t} (1 + |y|^2)^{-t} |\hat{f}(y)|^2 \, dy \leq \int_{\mathbb{R}^n} (1 + |y|^2)^t |\hat{f}(y)|^2 \, dy = \|f\|_{L^2(\mathbb{R}^n)}^2.
\] (2.10)

Now, we are going to estimate each of two integrals on the right-hand side of the equality
\[
\|f\|_{L^2(\mathbb{R}^n)}^2 = \int_{|y| \leq 1} (1 + |y|^2)^t |\hat{f}(y)|^2 \, dy + \int_{|y| > 1} (1 + |y|^2)^t |\hat{f}(y)|^2 \, dy.
\] (2.11)

The second of integrals (2.11) is estimated with the help of inequality $(1 + |y|^2)^t \leq 2^t |y|^{2t}$ as follows:
\[
\int_{|y| > 1} (1 + |y|^2)^t |\hat{f}(y)|^2 \, dy \leq 2^t \int_{|y| > 1} |y|^{2t} (1 + |y|^2)^t |\hat{f}(y)|^2 \, dy = 2^t \|f\|_{L^2(\mathbb{R}^n)}.
\] (2.12)

The first of integrals (2.11) admits the preliminary estimate
\[
\int_{|y| \leq 1} (1 + |y|^2)^t |\hat{f}(y)|^2 \, dy \leq \int_{|y| \leq 1} (1 + |y|^2)^t \sup_{|y| \leq 1} |\hat{f}(y)|^2.
\] (2.13)

To estimate the latter supremum, we introduce a function $\chi \in C^\infty_0(\mathbb{R}^n)$ satisfying $\chi(x) = 1$ for $|x| \leq A$ and denote $\chi_y(x) = e^{-i(y \cdot x)} \chi(x)$. Let $\hat{\chi}_y$ be the inverse Fourier transform of $\chi_y$. By the Plancherel formula,
\[
\hat{f}(y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \chi_y(x) f(x) \, dx = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{\chi}_y(z) \hat{f}(z) \, dz.
\]

We write this in the form
\[
\hat{f}(y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} |z|^{-t} (1 + |z|^2)^{-t/2} \hat{\chi}_y(z) z^t (1 + |z|^2)^{t/2} \hat{f}(z) \, dz
\]
and use the Schwartz inequality to obtain the estimate
\[
|\hat{f}(y)|^2 \leq (2\pi)^{-n} \int_{\mathbb{R}^n} |z|^{-2t} (1 + |z|^2)^{-(t-t)} |\hat{\chi}_y(z)|^2 \, dz \int_{\mathbb{R}^n} |z|^{2t} (1 + |z|^2)^{t-t} |\hat{f}(z)|^2 \, dz,
\]
i.e.
\[
|\hat{f}(y)|^2 \leq (2\pi)^{-n} \|\chi_y\|_{L^2(\mathbb{R}^n)}^2 \|\hat{f}\|_{L^2(\mathbb{R}^n)}^2.
\] (2.14)

The norm $\|\chi_y\|_{L^2(\mathbb{R}^n)}$ depends continuously on $y$, hence
\[ \sup_{|y| \leq 1} \| \chi_y \|_{L^2_t(\mathbb{R}^n)}^2 \leq c_2(s, t, n). \]

This implies together with (2.14)
\[ |\hat{f}(y)|^2 \leq c_2(s, t, n) \| \mathcal{F} \|_{L^2_t(\mathbb{R}^n)}^2 \quad \text{for} \quad |y| \leq 1. \] (2.15)

From (2.13) and (2.15),
\[ \int_{|y| \leq 1} (1 + |y|^2)^{y' - t} |\hat{f}(y)|^2 \, dy \leq c(s, t, n) \| \mathcal{F} \|_{L^2_t(\mathbb{R}^n)}^2. \]

Combining this with (2.11) and (2.12), we obtain
\[ \| \mathcal{F} \|_{L^2_t(\mathbb{R}^n)} \leq c(s, t, n) \| \mathcal{F} \|_{L^2_t(\mathbb{R}^n)}^2. \]

Together with (2.10), this gives the desired statement.

Let now \( t \in (-n/2, 0) \). Then
\[ \| \mathcal{F} \|_{L^2_t(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} (1 + |y|^2)^{y' - t} |\hat{f}(y)|^2 \, dy \leq \int_{\mathbb{R}^n} |y|^2 (1 + |y|^2)^{y' - t} |\hat{f}(y)|^2 \, dy \]
\[ = \| \mathcal{F} \|_{L^2_t(\mathbb{R}^n)}^2. \] (2.16)

Now, we are going to estimate each of two integrals on the right-hand side of the equality
\[ \| \mathcal{F} \|_{L^2_t(\mathbb{R}^n)}^2 = \int_{|y| \leq 1} |y|^2 (1 + |y|^2)^{y' - t} |\hat{f}(y)|^2 \, dy + \int_{|y| > 1} |y|^2 (1 + |y|^2)^{y' - t} |\hat{f}(y)|^2 \, dy. \] (2.17)

On using the inequality \( |y|^2 \leq 2^{-2}(1 + |y|^2)^t \), the second of integrals (2.17) is estimated as follows:
\[ \int_{|y| > 1} |y|^2 (1 + |y|^2)^{y' - t} |\hat{f}(y)|^2 \, dy \leq 2^{-t} \int_{|y| > 1} (1 + |y|^2)^t |\hat{f}(y)|^2 \, dy = 2^{-t} \| \mathcal{F} \|_{L^2_t(\mathbb{R}^n)}^2. \] (2.18)

The first of integrals (2.17) admits the estimate
\[ \int_{|y| \leq 1} |y|^2 (1 + |y|^2)^{y' - t} |\hat{f}(y)|^2 \, dy \leq \int_{|y| \leq 1} |y|^2 (1 + |y|^2)^{y' - t} \sup_{|y| \leq 1} |\hat{f}(y)|^2 \, dy \]
\[ = c_5(s, t, n) \sup_{|y| \leq 1} |\hat{f}(y)|^2. \] (2.19)

Quite similarly to (2.14)–(2.15), we obtain
\[ |\hat{f}(y)|^2 \leq (2\pi)^{-n} \| \chi_y \|_{L^2_t(\mathbb{R}^n)}^2 \| \mathcal{F} \|_{L^2_t(\mathbb{R}^n)}^2 \]
and
\[ |\hat{f}(y)|^2 \leq c_5(s, t, n) \| \mathcal{F} \|_{L^2_t(\mathbb{R}^n)}^2 \quad \text{for} \quad |y| \leq 1. \]

This gives together with (2.19)
\[
\int_{|x| \leq 1} |y|^p (1 + |y|^2)^{r-1} \left| \tilde{f}(y) \right|^2 \, dy \leq C_r(s,t,n) \|f\|_{L^p(\mathbb{R}^n)}^2.
\]

Combining this with (2.17)–(2.18), we obtain
\[
\|f\|_{L^p(\mathbb{R}^n)}^2 \leq C_2(s,t,n) \|f\|_{L^p(\mathbb{R}^n)}^2.
\]

Together with (2.16), this gives the desired statement.

In author’s opinion, lemma 2.3 is not true for \( t \neq n/2 \). But the latter fact is not proved yet.

3. The solenoidal part of a tensor field

For an integer \( m \geq 0 \), let \( S^m \mathbb{R}^n \) be the complex vector space of all symmetric \( \mathbb{R} \)-multilinear maps
\[
f : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \longrightarrow \mathbb{C}.
\]

The dimension of \( S^m \mathbb{R}^n \) is \( \frac{n + m}{m} \). Its elements are called (complex) rank \( m \) symmetric tensors on \( \mathbb{R}^n \). For such a tensor \( f \in S^m \mathbb{R}^n \), the complex-valued form \( f(\xi_1, \ldots, \xi_m) \) is well defined for vectors \( \xi_i \in \mathbb{R}^n \) (1 \( \leq i \leq m \)); the form depends linearly on each argument and is symmetric with respect to any permutation of arguments. The tensor \( f \) is said to be real if \( f(\xi_1, \ldots, \xi_m) \) is a real-valued form. Obviously \( S^m \mathbb{R}^n = \mathbb{R} \). It is convenient to agree that \( S^{-m} \mathbb{R}^n = 0 \) for \( m < 0 \).

Now, we briefly discuss the coordinate representation of symmetric tensors. Given a basis \((e_1, \ldots, e_n)\) of \( \mathbb{R}^n \), every vector \( \xi \in \mathbb{R}^n \) is uniquely written as \( \xi = \xi_i e_i \). In this formula and further formulas, we use the Einstein summation rule: the summation from 1 to \( n \) is assumed over an index repeated in a monomial in low and upper positions. Then, for \( f \in S^m \mathbb{R}^n \),
\[
f(\xi_1, \ldots, \xi_m) = f_{h_{1} \ldots i_{m}} \xi_{1}^{h_{1}} \cdots \xi_{m}^{i_{m}}.
\]

The coefficients \( f_{h_{1} \ldots i_{m}} \) of the sum are called the covariant coordinates (or components) of the tensor \( f \) with respect to the given basis. The coefficients are symmetric in all indices \( (i_{1}, \ldots, i_{m}) \).

The dot-product on \( S^m \mathbb{R}^n \) is defined as follows. Given a basis \((e_1, \ldots, e_n)\) of \( \mathbb{R}^n \), we set \( g_{ij} = \langle e_i, e_j \rangle \) and let \((g^{ij})\) be the inverse matrix of \((g_{ij})\). Being defined by
\[
\langle f, h \rangle = g^{i_{1} \ldots i_{m}} g^{h_{1} \ldots h_{m}} f_{i_{1} \ldots i_{m}} h_{1} \cdots h_{m} \quad (f, h \in S^m \mathbb{R}^n),
\]

the dot-product is independent of the choice of a basis. The corresponding norm will be denoted by \( |f|^2 \). In the case of an orthonormal basis,
\[
|f|^2 = \sum_{i_{1} \ldots i_{m}} |f_{i_{1} \ldots i_{m}}|^2.
\]

Orthonormal bases are sufficient for the most of our considerations.

The metric tensor \((g_{ij})\) allows us to introduce the contravariant coordinates of a tensor \( f \in S^m \mathbb{R}^n \) by
\[
f^{h_{1} \ldots i_{m}} = g^{h_{1} i_{1}} \cdots g^{h_{m} i_{m}} f_{i_{1} \ldots i_{m}}.
\]

Let us emphasize that we do not discuss covariant and contravariant tensors but speak on covariant and contravariant coordinates of the same tensor \( f \in S^m \mathbb{R}^n \). Formula (3.2) is now written in the shorter form.
\[ \langle f, h \rangle = \sum_{i_1, \ldots, i_m} f_{i_1 \ldots i_m} h_{i_1 \ldots i_m}. \]

In particular, \( S^m \mathbb{R}^n \) is now identified with \( \mathbb{R}^n \) by \( f(\xi) = \langle f, \xi \rangle \) for \( f, \xi \in \mathbb{R}^n \).

We will need some algebraic operations on symmetric tensors. Let \( \otimes^m \mathbb{R}^n \) be the space of all rank \( m \) tensors on \( \mathbb{R}^n \), i.e., the space of \( \mathbb{R} \)-multilinear maps
\[ f : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{C}. \]

Unlike (3.1), symmetry is not required here. There is the canonical projection (symmetrization) \( \sigma : \otimes^m \mathbb{R}^n \to S^m \mathbb{R}^n \) defined by
\[ (\sigma f)(\xi_1, \ldots, \xi_m) = \frac{1}{m!} \sum_{\pi \in \Pi_m} f(\xi_{\pi(1)}, \ldots, \xi_{\pi(m)}), \]
where the summation is performed over the set \( \Pi_m \) of all permutations of the set \( \{1, \ldots, m\} \).

Let us recall that, for \( f \in \otimes^k \mathbb{R}^n \) and \( h \in \otimes^m \mathbb{R}^n \), the tensor product \( f \otimes h \in \otimes^{k+m} \mathbb{R}^n \) is defined by
\[ (f \otimes h)(\xi_1, \ldots, \xi_{k+m}) = f(\xi_1, \ldots, \xi_k) h(\xi_{k+1}, \ldots, \xi_{k+m}). \]

Now, for \( f \in S^k \mathbb{R}^n \) and \( h \in S^m \mathbb{R}^n \), the symmetric tensor product \( f h \in S^{k+m} \mathbb{R}^n \) is defined by \( f h = \sigma(f \otimes h) \). Being furnished with this product, \( S \mathbb{R}^n = \bigoplus_{m=0}^\infty S^m \mathbb{R}^n \) becomes a commutative graded algebra, the algebra of symmetric tensors, which is actually isomorphic to the algebra of polynomials in \( n \) variables.

For a fixed tensor \( f \in S^k \mathbb{R}^n \), we denote by \( f^i \) the operator of symmetric multiplication by \( f \), i.e.
\[ (f^i)(\xi) = \sum_{i_1, \ldots, i_k} f_{i_1 \ldots i_k} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_k}. \]

The adjoint of \( f^i \) is the operator \( f^j \) of contraction with \( f \) which is written in coordinates as
\[ (f^j h)(i_1, \ldots, i_m) = f_{i_1 \ldots i_m} h(i_1, \ldots, i_m). \]

For \( y \in \mathbb{R}^n \), the following commutator formula for operators \( i_j \) and \( j_y \) is valid [13, lemma 3.3.3]:
\[ j_y i_j f = \frac{|y|^2}{m+1} f + \frac{m}{m+1} i_j j_y f \quad \text{for } f \in S^m \mathbb{R}^n. \]

Take the scalar product of this equality with \( f \) to obtain
\[ |i_j f|^2 = \frac{|y|^2}{m+1} |f|^2 + \frac{m}{m+1} |j_y f|^2. \]

In particular,
\[ |f|^2 \leq (m+1) |y|^2 |i_j f|^2 \quad (f \in S^m \mathbb{R}^n, \ y \neq 0). \]  
(3.3)

Since \( |i_j f|^2 = (i_j f, i_j f) = (j_y i_j f, f) \leq |j_y i_j f||f| \) we obtain one more useful inequality
\[ |f| \leq (m+1) |y| |j_y i_j f| \quad (f \in S^m \mathbb{R}^n, \ y \neq 0). \]  
(3.4)

Now, we briefly discuss symmetric tensor fields. Roughly speaking, a rank \( m \) symmetric tensor field \( f \) on \( \mathbb{R}^n \) is a function \( f : \mathbb{R}^n \to S^m \mathbb{R}^n, \ x \mapsto f(x) \in S^m \mathbb{R}^n \). More precisely, let \( \mathcal{F}(\mathbb{R}^n) \) be a functional space of functions on \( \mathbb{R}^n \). By \( \mathcal{F}(\mathbb{R}^n, S^m \mathbb{R}^n) \), we denote the space of all maps \( f : \mathbb{R}^n \to S^m \mathbb{R}^n \) such that all coordinate functions \( f_{i_1 \ldots i_m}(x) \) belong to \( \mathcal{F}(\mathbb{R}^n) \). For the most of
spaces $\mathcal{F}(\mathbb{R}^n)$, the resulting space $\mathcal{F}(\mathbb{R}^n; S^m\mathbb{R}^n)$ is independent of the choice of a basis of $\mathbb{R}^n$ participating in the definition of coordinates of a tensor. In particular, the following spaces are defined by this scheme.

The space $C^k(\mathbb{R}^n; S^m\mathbb{R}^n)$ of rank $m$ symmetric tensor fields whose partial derivatives of order $\leq k$ are continuous in $\mathbb{R}^n$.

The Sobolev space $H^s(\mathbb{R}^n; S^m\mathbb{R}^n)$ ($s \in \mathbb{R}$).

The space $S(\mathbb{R}^n; S^m\mathbb{R}^n)$ of smooth rapidly decaying symmetric tensor fields.

The space $S'(\mathbb{R}^n; S^m\mathbb{R}^n)$ of tempered tensor fields-distributions.

The space $\mathcal{E}(\mathbb{R}^n; S^m\mathbb{R}^n)$ of compactly supported tensor fields-distributions.

Each of these spaces is furnished with the corresponding topology. There is no problem with the definition of the topology since $S^m\mathbb{R}^n$ is a finite-dimensional vector space.

The Fourier transform $\hat{F} : S(\mathbb{R}^n; S^m\mathbb{R}^n) \rightarrow S(\mathbb{R}^n; S^m\mathbb{R}^n)$, $f \mapsto \hat{f}$ is defined componentwise, i.e. $\hat{f}_{\lambda_1...\lambda_n} = (f_{\lambda_1...\lambda_n})^\lambda$; the result is independent of the choice of a basis. Now, formula (2.4) makes sense for a tensor field $f$ and defines the Hilbert space structure on $H^s(\mathbb{R}^n; S^m\mathbb{R}^n)$ ($s \in \mathbb{R}$). The Hilbert space $H^t(\mathbb{R}^n; S^m\mathbb{R}^n)$ ($s \in \mathbb{R}$, $t > -n/2$) is defined similarly to (2.3).

We use affine coordinates on $\mathbb{R}^n$ only and, in particular, Cartesian coordinates. Given an affine coordinate system $(x^1, \ldots, x^n)$ with the coordinate basis $(e_1, \ldots, e_n)$, a tensor field $f \in C^\infty(\mathbb{R}^n; S^m\mathbb{R}^n)$ can be considered as a family of smooth functions $f_{\lambda_1...\lambda_n}(x^1, \ldots, x^n)$ that are symmetric in all indices. Partial derivatives $\partial f_{\lambda_1...\lambda_n}/\partial x^j$ are well defined. In particular, $f \in C^\infty(\mathbb{R}^n; S^0\mathbb{R}^n)$ is just a smooth function on $\mathbb{R}^n$. Since $S^0\mathbb{R}^n$ is identified with $\mathbb{R}^n$, a first rank tensor field $f \in C^\infty(\mathbb{R}^n; S^1\mathbb{R}^n)$ can be identified with the vector field $f = f^i e_i$ as well as with the one-form $f = f^i dx^i$ (recall that $f^i = g^{ij} f_j$).

Now, we introduce two important first order differential operators. The inner derivative $d : C^\infty(\mathbb{R}^n; S^m\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n; S^{m+1}\mathbb{R}^n)$ is defined in affine coordinates by

$$(df)_{\lambda_1...\lambda_{m+1}} = \sigma\left(\frac{\partial f_{\lambda_1...\lambda_n}}{\partial x^{\lambda_{m+1}}}\right),$$

where $\sigma$ is the symmetrization in all indices. In particular, in the case of $m = 0$, $df = \frac{df}{\partial x^i} dx^i$ is the differential of the function $f$. In the case of $m = 1$,

$$(df)_{ij} = \frac{1}{2}\left(\frac{\partial f_i}{\partial x^j} + \frac{\partial f_j}{\partial x^i}\right);$$

in the case of $m = 2$,

$$(df)_{ijk} = \frac{1}{3}\left(\frac{\partial f_i}{\partial x^j} + \frac{\partial f_j}{\partial x^k} + \frac{\partial f_k}{\partial x^i}\right);$$

and so on.

The divergence $\delta : C^\infty(\mathbb{R}^n, S^{m+1}\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n, S^m\mathbb{R}^n)$ is defined in affine coordinates by
(\delta f)_{i_1 \ldots i_n} = g^{pq} \frac{\partial f_{i_1 \ldots i_n}}{\partial x^q} \tag{3.7}

(recall that \( g_{ij} = \langle e_i, e_j \rangle \) is the metric tensor and \((g^{ij})^{-1}\)).

One easily checks the correctness of these definitions, i.e. that formulas (3.5)–(3.7) are invariant under a change of affine coordinates. Formulas (3.5)–(3.7) are actually valid in curvilinear coordinates too, but partial derivatives must be replaced with covariant derivatives in the latter case.

The operators \( d \) and \( -\delta \) are formally adjoint to each other with respect to the above-introduced \( L^2 \)-product, i.e.

\[ \int \langle du, v \rangle \, dx = -\int \langle u, \delta v \rangle \, dx \]

for \( u \in C^1(\mathbb{R}^n; S' \mathbb{R}^n) \) and \( v \in C(\mathbb{R}^n; S^{m+1}\mathbb{R}^n) \) if either \( u \) or \( v \) is compactly supported.

The classical Helmholtz decomposition of a vector field can be generalized to arbitrary rank symmetric tensor fields. In different situations, the decomposition of a tensor field to potential and solenoidal parts should be defined in slightly different ways. We start with tensor fields from the Schwartz space.

**Proposition 3.1.** For every \( f \in S(\mathbb{R}^n; S' \mathbb{R}^n) \) \((n \geq 2)\), there exist uniquely determined fields \( \varepsilon f \in C^\infty(\mathbb{R}^n; S^{m-1}\mathbb{R}^n) \) and \( \nu \in C^\infty(\mathbb{R}^n; S^{m-1}\mathbb{R}^n) \) such that

\[ f = \varepsilon f + \nu, \quad \delta \varepsilon f = 0, \]

\( \varepsilon f(x) \to 0, \quad \nu(x) \to 0 \) as \( |x| \to \infty \).

These fields satisfy the estimates

\[ |\varepsilon f(x)| \leq C(1+|x|)^{1-n}, \quad |\nu(x)| \leq C(1+|x|)^{2-n}, \quad |\nu(x)| \leq C(1+|x|)^{1-n} \tag{3.8} \]

with a constant \( C \) independent of \( f \). The fields \( \varepsilon f \) and \( \nu \) belong to \( L^2(\mathbb{R}^n; S' \mathbb{R}^n) \). The Fourier transform \( \widehat{\varepsilon f}(y) \) of the field \( \varepsilon f \) belongs to \( C^\infty(\mathbb{R}^n \setminus \{0\}; S' \mathbb{R}^n) \) is bounded on \( \mathbb{R}^n \) and decays rapidly as \(|y| \to \infty| \).

The proof is presented in [13, theorem 2.6.2]. We call \( \varepsilon f \) and \( \nu \) the solenoidal and potential parts of the field \( f \) respectively.

Next, we present a version of proposition 3.1 for compactly supported tensor fields. We say that a field \( u \in S'(\mathbb{R}^n; S' \mathbb{R}^n) \) decays at infinity if \( u \) is continuous outside some compact set and \( u(x) \to 0 \) as \(|x| \to \infty| \).

**Proposition 3.2.** For every \( f \in \mathcal{E}(\mathbb{R}^n; S' \mathbb{R}^n) \) \((n \geq 2)\), there exist uniquely determined fields \( \varepsilon f \in S(\mathbb{R}^n; S^{m-1}\mathbb{R}^n) \) and \( \nu \in S(\mathbb{R}^n; S^{m-1}\mathbb{R}^n) \) decaying at infinity and such that

\[ f = \varepsilon f + \nu, \quad \delta \varepsilon f = 0, \]

where the operators \( d \) and \( \delta \) are understood in the distribution sense. The fields \( \varepsilon f \) and \( \nu \) are smooth outside \( \text{supp} f \) and satisfy estimates (3.8) outside some compact set.

The proof is presented in [13, theorem 2.14.1].

Finally, we present an analogous of proposition 3.1 for Sobolev spaces \( H^s(\mathbb{R}^n; S' \mathbb{R}^n) \). Before doing this, let us discuss partial derivatives on \( H^s(\mathbb{R}^n; S' \mathbb{R}^n) \).

**Lemma 3.3.** Every partial derivative \( D_j = -i \frac{\partial}{\partial e^j} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \) uniquely extends to the bounded operator
\[ D_f : H^{r+1}_{r+1}(\mathbb{R}^n) \to H^r(\mathbb{R}^n) \quad (s \in \mathbb{R}, t > -n/2). \]  

**Proof.** On using \( \bar{D} \mu = \xi \hat{u} \), we easily obtain for \( u \in \mathcal{S}(\mathbb{R}^n) \)

\[ \|D_f u\|_{H^r(\mathbb{R}^n)} \leq \|u\|_{H^{r+1}_{r+1}(\mathbb{R}^n)}. \]  

This estimate allows us to define operator \( (3.9) \) as follows. Given \( f \in H^{r+1}_{r+1}(\mathbb{R}^n) \), choose a sequence \( u_k \in \mathcal{S}(\mathbb{R}^n) \) converging to \( f \) in \( H^{r+1}_{r+1}(\mathbb{R}^n) \). By (3.10), \( D \mu_k \) is a Cauchy sequence in \( H^r(\mathbb{R}^n) \) and hence determines some \( g \in H^r(\mathbb{R}^n) \). We set \( D_f g = g \). One easily checks the correctness of the definition as well as the boundedness of operator \( (3.9) \) \( \square \).

**Proof.**

**Theorem 3.5.** Every tensor field \( f \in H^r(\mathbb{R}^n; S^m\mathbb{R}^n) \) \( (s \in \mathbb{R}, t > -n/2, m \geq 0) \) can be uniquely represented in the form

\[ f = f + \delta v, \quad \delta f = 0, \]  

where \( f \in H^r(\mathbb{R}^n; S^m\mathbb{R}^n) \), \( v \in H^{r+1}_{r+1}(\mathbb{R}^n; S^{m-1}\mathbb{R}^n) \) and \( \delta v \in H^r(\mathbb{R}^n; S^m\mathbb{R}^n) \). The differential operators \( d \) and \( \delta \) are understood in the sense of lemma 3.3 here. The estimates

\[ \|f\|_{H^r(\mathbb{R}^n)} \leq C \|f\|_{H^r(\mathbb{R}^n)}, \quad \|v\|_{H^{r+1}_{r+1}(\mathbb{R}^n)} \leq C \|f\|_{H^r(\mathbb{R}^n)}, \quad \|\delta v\|_{H^r(\mathbb{R}^n)} \leq C \|f\|_{H^r(\mathbb{R}^n)} \]  

(3.12)
are valid with a constant $C$ independent of $f$. In particular, $\mathbf{f}$ and $\mathbf{v}$ are smooth if $f$ is smooth. (We hope the reader is not confused by the double sense of the index $s$: $\mathbf{f}$ is the solenoidal part and $H^s$ is the Sobolev space. Both notations are standard.)

**Proof.** The uniqueness statement follows from lemma 3.4. Indeed, let $\mathbf{f}, \mathbf{v} \in H^s_0(\mathbb{R}^n; S^n \mathbb{R}^n)$ satisfy

$$\mathbf{f} + \nabla \mathbf{v} = 0, \quad \delta \mathbf{f} = 0.$$  

Then $\delta \nabla \mathbf{v} = 0$. Applying lemma 3.4, we see that $\mathbf{v} = 0$. Now, the previous equation implies $\mathbf{f} = 0$.

It suffices to prove the existence statement and estimates (3.12) for $f \in \mathcal{S}(\mathbb{R}^n; S^{m-1} \mathbb{R}^n)$. Then the statement for $f \in H^s_0(\mathbb{R}^n; S^m \mathbb{R}^n)$ will follow with the help of the same completeness arguments as used in the proof of lemma 3.3.

Let $\hat{f}(\mathbf{y})$ be the Fourier transform of a field $f \in \mathcal{S}(\mathbb{R}^n; S^m \mathbb{R}^n)$. Assume for a moment the existence of decomposition (3.11). Applying the Fourier transform to (3.11), we obtain

$$\hat{\mathbf{f}}(\mathbf{y}) = \hat{\mathbf{g}}(\mathbf{y}) + i \mathbf{i} \hat{\mathbf{v}}(\mathbf{y}), \quad j_y \hat{\mathbf{g}}(\mathbf{y}) = 0.$$  

(3.13)

Since $j_y$ is the adjoint of $i_y$, representation (3.13) exists and is unique for every tensor $\hat{\mathbf{f}}(\mathbf{y})$. Conversely, starting with decomposition (3.13) of $\hat{\mathbf{f}}$, we apply the inverse Fourier transform to obtain (3.11).

In terms of [13, lemma 2.6.1], the tensor $\hat{\mathbf{g}}(\mathbf{y})$ is called the tangential part of $\hat{\mathbf{f}}(\mathbf{y})$ and is expressed through the letter tensor by the formula

$$\hat{\mathbf{g}}_{i_1 \ldots i_n}(\mathbf{y}) = \lambda_{i_1 \ldots i_n}^h \hat{\mathbf{f}}_{i_1 \ldots i_n}(\mathbf{y}).$$  

(3.14)

where

$$\lambda_{i_1 \ldots i_n}^h(y) = \left( \delta_{i_1}^h - \frac{y_{i_1} y^h}{|y|^2} \right) \ldots \left( \delta_{i_n}^h - \frac{y_{i_n} y^h}{|y|^2} \right)$$  

(3.15)

and $\delta_i^j$ is the Kronecker tensor. Since the coefficients $\lambda_{i_1 \ldots i_n}^h(y)$ are bounded, we have the estimate

$$|\hat{\mathbf{g}}(\mathbf{y})|^2 \leq C |\hat{\mathbf{f}}(\mathbf{y})|^2$$  

(3.16)

with some constant $C$ independent of $f$. This implies the validity of the first of estimates (3.12).

On using (3.13) and (3.16), we estimate

$$|i_y \hat{\mathbf{v}}(\mathbf{y})|^2 = |\hat{\mathbf{f}}(\mathbf{y}) - \hat{\mathbf{g}}(\mathbf{y})|^2 \leq 2 |\hat{\mathbf{f}}(\mathbf{y})|^2 + 2 |\hat{\mathbf{g}}(\mathbf{y})|^2 \leq 2(C + 1) |\hat{\mathbf{f}}(\mathbf{y})|^2.$$  

This implies the last of estimates (3.12) because $\hat{\mathbf{g}}(\mathbf{y}) = i_y \hat{\mathbf{v}}(\mathbf{y})$.

Applying the operator $j_y$ to the first of equalities (3.13), we see that

$$j_y i_y \hat{\mathbf{e}}(\mathbf{y}) = j_y \hat{\mathbf{f}}(\mathbf{y}).$$

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Taking the scalar product of this equality with \( \hat{v}(y) \), we obtain

\[
|\hat{i}_\nu \hat{v}(y)|^2 = \langle j, \hat{f}(y), \hat{v}(y) \rangle.
\]

This implies the estimate

\[
|\hat{i}_\nu \hat{v}(y)|^2 \leq C_1 |y| \| \hat{f}(y) \| \| \hat{v}(y) \|
\]

with some constant \( C_1 \) dependent on \((m, n)\) only. On using (3.3), this implies

\[
|\hat{v}(y)|^2 \leq (m + 1) |y|^{-2} |\hat{i}_\nu \hat{v}(y)|^2 \leq C_2 |y|^{-1} \| \hat{f}(y) \| \| \hat{v}(y) \|
\]

i.e.

\[
|\hat{v}(y)|^2 \leq C_3 |y|^{-2} \| \hat{f}(y) \|^2.
\]

Integrating this inequality, we obtain

\[
\|v\|_{H^{2,1}} = \int_{\mathbb{R}^n} |y|^{2(r+1)} (1 + |y|^2)^{-r} |\hat{v}(y)|^2 \, dy \leq C_3 \int_{\mathbb{R}^n} |y|^{2r} (1 + |y|^2)^{-r} \| \hat{f}(y) \|^2 \, dy = C_4 \|f\|_{H^r}^2.
\]

This proves the second of estimates (3.12).

Observe also that formulas (3.14) and (3.15) imply

\[
\langle \hat{f}(y), \xi \rangle = \langle \hat{f}(y), \xi \rangle \quad \text{for} \quad y \in \mathbb{R}^n, \xi \in \mathbb{S}^{n-1}, \langle y, \xi \rangle = 0. \tag{3.17}
\]

This important relation will be used in the next section.

4. The x-ray transform

The family of oriented straight lines in \( \mathbb{R}^n \) is parameterized by points of the manifold

\[
T \mathbb{S}^{n-1} = \{ (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n | \langle \xi, \xi \rangle = 1, \langle x, \xi \rangle = 0 \}
\]

that is the tangent bundle of the unit sphere \( \mathbb{S}^{n-1} \). Namely, a point \((x, \xi) \in T \mathbb{S}^{n-1}\) determines the line \( \{ x + t\xi | t \in \mathbb{R} \} \).

The x-ray transform is initially defined as the linear continuous operator

\[
I : \mathcal{S}(\mathbb{R}^n; \mathbb{S}^{m-1}) \to \mathcal{S}(T \mathbb{S}^{n-1})
\]

by

\[
(I f)(x, \xi) = \int_{-\infty}^{\infty} f_{t, \xi}(x + t\xi) \xi_1 \ldots \xi_m \, dt = \int_{-\infty}^{\infty} \langle f(x + t\xi), \xi_m \rangle \, dt \quad ((x, \xi) \in T \mathbb{S}^{n-1}). \tag{4.2}
\]

Being initially defined on the Schwartz space, the x-ray transform then extends to wider spaces of symmetric tensor fields. First of all we observe that integral (4.2) converges in the classical sense if a field \( f \in C(\mathbb{R}^n; \mathbb{S}^{m-1}) \) decays at infinity so that \( |f(x)| \leq C(1 + |x|)^{-1-\varepsilon} \) with some \( \varepsilon > 0 \). The most important feature of the x-ray transform is the presence of a big null-space in the case of \( m > 0 \). If a tensor field \( v \in C(\mathbb{R}^n; \mathbb{S}^{m-1}) \) decays at infinity so that \( v(x) \to 0 \) as \(|x| \to \infty\), then \( I(dv) = 0 \). Indeed, by the definition of \( d, \)
\((dv)(x + t\xi), \xi^m) = (dv)_{h_1,…,h_n}(x + t\xi) \xi^h \cdots \xi^m
\quad = \frac{1}{m} \left( \frac{\partial v_{h_1,…,h_n}}{\partial x^h}(x + t\xi) + \cdots + \frac{\partial v_{h_1,…,h_n}}{\partial x^m}(x + t\xi) \right) \xi^h \cdots \xi^m
\quad = \xi^j \frac{\partial v_{h_1,…,h_n}}{\partial x^j}(x + t\xi) \xi^h \cdots \xi^{k-1} = \frac{d}{dt}(v(x + t\xi), \xi^{m-1}).

On using this identity, we derive
\[
(I dv)(x, \xi) = \int_{-\infty}^{\infty} (dv)(x + t\xi), \xi^m) \, dt = \int_{-\infty}^{\infty} \frac{d}{dt}(v(x + t\xi), \xi^{m-1}) \, dt
\quad = \lim_{T \to \infty} \{ (v(x + T\xi), \xi^{m-1}) - (v(x - T\xi), \xi^{m-1}) \} = 0.
\]

In other words, the x-ray transform vanishes on potential tensor fields. Therefore, given \(I f\), we can hope to recover the solenoidal part of the field \(f\) only. These easy observations have a specification in each of situations discussed in propositions 3.1–3.2 and theorem 3.5.

First of all, the situation is well studied for symmetric tensor fields belonging to the Schwartz space. Let \(f = f^s + dv\) be the decomposition of a tensor field \(f \in S(\mathbb{R}^n, S^m \mathbb{R}^e)\) into solenoidal and potential parts in the sense of proposition 3.1. Then \(I dv = 0\) and \(f^s\) can be recovered from \(If\) by an explicit inversion formula of Radon type [13, theorem 2.12.2]. Some stability estimate can be obtained from the formula.

Next, we consider compactly supported tensor fields-distributions. The x-ray transform extends to the continuous operator
\[
I : \mathcal{E}(\mathbb{R}^n; S^m \mathbb{R}^e) \to \mathcal{E}(T S^{n-1},)
\]
see details in [13, section 2.5]. The following statement is proved in [13, theorems 2.2.1 and 2.5.1]:

**Proposition 4.1.** For a compactly supported tensor field-distribution \(f \in \mathcal{E}(\mathbb{R}^n; S^m \mathbb{R}^e)\), \(I f = 0\) if and only if there exists \(v \in \mathcal{E}(\mathbb{R}^n, S^{m-1} \mathbb{R}^e)\) such that \(f = dv\) and the support of \(v\) is contained in the convex hull of \(\text{supp} f\). Moreover, if \(f \in C^k(\mathbb{R}^n, S^m \mathbb{R}^e)\) with \(k \geq \max\{m, 2\}\), then \(v \in C^{k+1}(\mathbb{R}^n, S^{m-1} \mathbb{R}^e)\).

The Fourier transform \(F : S(T S^{n-1}) \to S(T S^{n-1}), \ u \mapsto \hat{u}\) is defined by
\[
\hat{u}(y, \xi) = (2\pi)^{\frac{1}{2}} \int_{\xi^1} e^{-i(y \cdot x) \xi^1} u(x, \xi) \, dx,
\]
where \(dx\) is the \((n-1)\)-dimensional Lebesgue measure on \(\xi^1 = \{ x \in \mathbb{R}^n \mid \{ x, \xi \} = 0 \}\). This is the standard Fourier transform in the \((n-1)\)-dimensional variable \(x\), where \(\xi\) stands as a parameter.

There is the following important relation between the x-ray transform and Fourier transform. If \(\varphi = If \in S(T S^{n-1})\) for \(f \in S(\mathbb{R}^n, S^m \mathbb{R}^e)\), then
\[
\hat{If}(y, \xi) = (2\pi)^{\frac{1}{2}} \hat{f}(y), \xi^m \quad \text{for} \quad (y, \xi) \in T S^{n-1}.
\]
The proof is presented in [13, section 2.1]. This statement can be considered as an analogous of the well known slice theorem for the Radon transform. Formulas (3.17) and (4.3) imply
\[
\hat{If}(y, \xi) = (2\pi)^{\frac{1}{2}} \hat{f}(y), \xi^m \quad \text{for} \quad (y, \xi) \in T S^{n-1}.
\]

For \(s \in \mathbb{R}\) and \(t > -(n-1)/2\), the Sobolev space \(H^s(T S^{n-1})\) is defined as the completion of \(S(T S^{n-1})\) with respect to the norm
\[
\| f \|_{H^s(T S^{n-1})} = \left( \int_{T S^{n-1}} |(dv)(x, \xi)|^2 \, d\xi \right)^{\frac{1}{2}}.
\]
Theorem 4.2. For every $s \in \mathbb{R}, t > -n/2, m \geq 0, n \geq 2$, the x-ray transform uniquely extends to the bounded operator

$$I : H^s_0(\mathbb{R}^n; S^n) \to H^{s+1/2}(T^{n-1}).$$

(4.5)

Given $f \in H^s_0(\mathbb{R}^n; S^n)$, let $f = f_\text{sol} + f_\text{pot}$ be the decomposition into solenoidal and potential parts in the sense of theorem 3.5. Then $I(f) = 0$ and the following Reshetnyak formula is valid for the solenoidal part:

$$\|f\|_{H^{s+1/2}(T^{n-1})} \leq \sum_{k=0}^{\lfloor m/2 \rfloor} a_k \|j^k(f)\|_{L^2(\mathbb{R}^n)},$$

(4.6)

where $\lfloor m/2 \rfloor$ is the integer part of $m/2$, $j : S^n \to S^{n-2}$ is the operator of contraction with the metric tensor $g$ (recall that $g_{ij} = \delta_{ij}$ in Cartesian coordinates), and coefficients $a_k$ are given by

$$a_k = a_0(m, n) = \frac{2^{m+1}n(n-1)/2}{(2m)!} \frac{1}{\Gamma(m + \frac{n-1}{2})} \frac{2^{2k}k!}{(2m)!} \frac{1}{(m - 2k)!}.$$

Corollary 4.3. For $f \in H^s_0(\mathbb{R}^n; S^n)$ ($t > -n/2$), the estimates

$$c \|f\|^2_{H^s_0(\mathbb{R}^n)} \leq \|f\|^2_{H^{s+1/2}(T^{n-1})} \leq C \|f\|^2_{H^s_0(\mathbb{R}^n)},$$

(4.7)

hold with some constants $c$ and $C$ dependent on $(s, t, m, n)$ but independent of $f$.

Proof. The norm of the operator $j$ depends on $(m, n)$ only. Therefore the right-hand side of (4.6) is bounded from above by $C \|f\|^2_{H^s_0(\mathbb{R}^n)}$ with a constants $C$ dependent on $(s, t, m, n)$ but independent of $f$.

All summands on the right-hand side of (4.6) are non-negative. The sum is bounded from below by the summand corresponding to $k = 0$, i.e. by $a_0 \|j^0(f)\|_{L^2(\mathbb{R}^n)}$. By the way, this gives the precise value for the constant $c$ on the left-hand side of (4.7):

$$c = a_0 = \frac{2^{m+1}n(n-1)/2}{m!(2m)!} \frac{1}{\Gamma(m + \frac{n-1}{2})} \frac{2^{2k}k!}{(m - 2k)!}.$$

Most probably, the value is sharp.

In the case of $s = t = 0$, theorem 4.2 is proved in [13, theorem 2.15.1]. The proof is actually the same in the general case. Here, we reproduce the scheme of the proof referring to [13] for technical details.

Proof of theorem 4.2. The second statement of the theorem, $I(f) = 0$, is not quite obvious in our case since $\psi(x)$ does not need to tend to zero as $|x| \to \infty$. Nevertheless, the statement follows from the Reshetnyak formula (4.6). Indeed, $df$ belongs to $H^s_0(\mathbb{R}^n; S^n)$ and the solenoidal part of $df$ is equal to zero. We can write the Reshetnyak formula with $f$ replaced by $df$. 
The right-hand side of the latter formula is equal to zero and we obtain the desired statement 
\( I(\nu) = 0 \). The boundedness of operator (4.5) also follows from (4.6). So, it suffices to prove

the Reshetnyak formula (4.6) for \( f \in \mathcal{S}(\mathbb{R}^n; \mathbb{S}^m \mathbb{R}^n) \).

For \( f \in \mathcal{S}(\mathbb{R}^n; \mathbb{S}^m \mathbb{R}^n) \), by the definition of the norm \( \| \cdot \|_{H^{m/2} \mathbb{R}^n} \),

\[
\| f \|_{H^{m/2} \mathbb{R}^n}^2 = \frac{1}{2\pi} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |y^{2r+1} (1+|y|^2)^{r-t} | \mathcal{F}(y, \xi) |^2 \; dy \; d\xi. \tag{4.8}
\]

Substitute value (4.4) for \( \mathcal{F}(y, \xi) \),

\[
\| f \|_{H^{m/2} \mathbb{R}^n}^2 = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |y^{2r+1} (1+|y|^2)^{r-t} \langle \mathcal{F}(y), \xi \rangle^2 | \; dy \; d\xi.
\]

Changing the order of integrations according to [13, lemma 2.15.3], we obtain

\[
\| f \|_{H^{m/2} \mathbb{R}^n}^2 = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |y^{2r+1} (1+|y|^2)^{r-t} \langle \mathcal{F}(y), \xi \rangle^2 | \; dy \; d\xi.
\]

This can be written in the form

\[
\| f \|_{H^{m/2} \mathbb{R}^n}^2 = \int_{\mathbb{R}^n} |y^{2r+1} (1+|y|^2)^{r-t} \langle \mathcal{F}(y), \xi \rangle^2 | \; dy \; d\xi.
\]

The inner integral can be easily computed [13, lemma 2.15.4]

\[
\int_{\mathbb{R}^n} \xi_{i_1} \ldots \xi_{i_m} \xi_{j_1} \ldots \xi_{j_m} \, d\xi = \frac{2\Gamma(m+n/2)}{\Gamma(m+n/2)} \varepsilon_{i_1 \ldots i_m j_1 \ldots j_m}^m.
\tag{4.10}
\]

where the tensor field \( \varepsilon \in C^\infty(\mathbb{R}^n \{0\}; \mathbb{S}^2 \mathbb{R}^n) \) is defined in Cartesian coordinates by

\[ \varepsilon_{ij}(y) = \delta_{ij} - y_i y_j / |y|^2 \] and \( \varepsilon^m \) is the \( m \)th symmetric power of \( \varepsilon \).

The following important simplification is possible while substituting (4.10) into (4.9). The
tensor \( \mathcal{F} \) satisfies \( \langle \mathcal{F}(y), \varepsilon \rangle = 0 \), i.e. \( y_i \mathcal{F}_{ij} \ldots \langle \varepsilon \rangle = 0 \), see (3.13). Due to this fact, we can replace \( \varepsilon^m \) with \( \delta^m \) on the right-hand side of (4.10) before substituting this expression into (4.9),

where \( \delta \) is the Kronecker tensor. An easy algebraic calculation with the help of induction in

\( m \) gives

\[
(\delta^m)_{i_1 \ldots i_m} = \frac{2^m (m!)^3}{(2m)!} \sigma(i_1, \ldots, i_m) \sigma(i_1, \ldots, i_m) \sum_{k=0}^{[m/2]} \frac{1}{2^k (m-k)! (m-2k)!} \varepsilon_{i_1 \ldots i_m}^m \varepsilon_{i_1 \ldots i_m}^m \varepsilon_{i_1 \ldots i_m}^m \varepsilon_{i_1 \ldots i_m}^m \varepsilon_{i_1 \ldots i_m}^m \varepsilon_{i_1 \ldots i_m}^m \varepsilon_{i_1 \ldots i_m}^m \varepsilon_{i_1 \ldots i_m}^m \varepsilon_{i_1 \ldots i_m}^m.
\tag{4.11}
\]

where \( \sigma(i_1 \ldots i_m) \) stands for the symmetrization in indices \( (i_1, \ldots, i_m) \). The operators \( \sigma(i_1, \ldots, i_m) \) and

\( \sigma(j_1, \ldots, j_m) \) can be omitted while substituting (4.11) into (4.9) since the product

\( \langle \mathcal{F}(y), \varepsilon \rangle \langle \mathcal{F}(y), \varepsilon \rangle \) is symmetric in indices \( (i_1, \ldots, i_m) \) as well as in indices \( (j_1, \ldots, j_m) \). After

substituting (4.10) into (4.9), each summand of (4.11) gives the integral
\[
\int_{\mathbb{R}^d} |y|^{2r} (1 + |y|^2)^{r+1} \left[ \delta_{j_0} \cdots \delta_{j_{2k-2} + j_{2k-1}} y^{j_{2k-1} - j_0}(y) \right] \delta_{\delta_{j_0} \cdots \delta_{j_{2k-2} + j_{2k-1}}} y^{j_{2k-1} - j_0}(y) \right] dy
\]

In this way we arrive to (4.6).

\[\square\]

**Corollary 4.4.** For every \( s \in \mathbb{R} \), the stability estimate

\[\|f\|_{H^s} \leq C\|f\|_{H^{s+1/2}(\mathbb{R}^{2n-1})}\]  

(4.12)

holds for \( f \in H^s(\mathbb{R}^n; S^m(\mathbb{R}^n)) \) with a constant \( C \) independent of \( f \).

This follows immediately from (4.7) and (4.8):

\[
c\|f\|_{H^s(\mathbb{R}^n)} \leq \|f\|_{H^{s+1/2}(\mathbb{R}^{2n-1})} = \frac{1}{2\pi} \int_{\mathbb{R}^{2n-1}} \int_{\xi} (1 + |y|^2)^{s+1/2} |\tilde{f}(y, \xi)|^2 \, dy \, d\xi
\]

This implies the inequality

\[
\|f\|_{H^{s+1/2}(\mathbb{R}^{2n-1})} \leq \frac{1}{2\pi} \int_{\mathbb{R}^{2n-1}} \int_{\xi} (1 + |y|^2)^{s+1/2} |\tilde{f}(y, \xi)|^2 \, dy \, d\xi
\]

Changing the order of integrations in the last integral, we obtain

\[
\|f\|_{H^{s+1/2}(\mathbb{R}^{2n-1})} \leq \frac{1}{2\pi} \int_{\mathbb{R}^{2n-1}} \int_{\xi} (1 + |y|^2)^{s+1/2} |\tilde{f}(y, \xi)|^2 \, dy \, d\xi
\]
where $\omega_{n-2}$ is the volume of the unit sphere $S^{n-2} = S^{n-1} \cap y^\perp$. The rest of the proof follows the proof of [10, chapter 2, theorem 5.1] word by word. □

Corollary 4.3 and theorem 4.5 give the analogous of the Natterer stability estimates for the x-ray transform:

$$c\|f\|_{H^p(\mathbb{R}^p)} \leq \|f\|_{H^{\frac{1}{2}+1/2(\mathbb{T}^{n-1})}} \leq C\|f\|_{H^p(\mathbb{R}^p)}$$

which are valid for tensor fields $f \in H^p(\mathbb{R}^p; S^m \mathbb{R}^p)$ satisfying $\text{supp} f \subset \{x \in \mathbb{R}^p | |x| \leq A\}$. However, the estimates

$$c\|f\|_{H^p(\mathbb{R}^p)} \leq \|f\|_{H^{\frac{1}{2}+1/2(\mathbb{T}^{n-1})}} \leq C\|f\|_{H^p(\mathbb{R}^p)}$$

are more interesting because $f$ depends on the solenoidal part of $f$ only. The first of these estimates holds by corollary 4.3. But the second estimate is very problematic. Let us present the question precisely.

**Problem 4.6.** Does the estimate

$$\|f\|_{H^{\frac{1}{2}+1/2(\mathbb{T}^{n-1})}} \leq C\|f\|_{H^p(\mathbb{R}^p)}$$

(4.14)

hold for all $f \in H^p(\mathbb{R}^p; S^m \mathbb{R}^p)$ satisfying $\text{supp} f \subset \{x \in \mathbb{R}^p | |x| \leq A\}$ with a constant $C$ dependent on $(A, s, m, n)$ but independent of $f$?

So far, the author can neither prove this statement nor find a counter-example. The difficulty is caused by the fact: the solenoidal part of $f$ does not need to be compactly supported if $f$ is a compactly supported tensor field. Therefore arguments from the proof of [10, chapter 2, theorem 5.1] do not work here.

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References