MOMENTUM RAY TRANSFORMS

VENKATESWARAN P. KRISHNAN†, RAMESH MANNA†, SUMAN KUMAR SAHOO† AND VLADIMIR A. SHARAFUTDINOV♯

Abstract. The momentum ray transform \( I_k \) integrates a rank \( m \) symmetric tensor field \( f \) over lines in \( \mathbb{R}^n \) with the weight \( t^k \): \( (I_k f)(x,\xi) = \int_{-\infty}^{\infty} t^k (f(x + t\xi),\xi^m) \, dt. \) In particular, the ray transform \( I = I^0 \) was studied by several authors since it had many tomographic applications. We present an algorithm for recovering \( f \) from the data \((I^0 f, I^1 f, \ldots, I^m f)\). In the cases of \( m = 1 \) and \( m = 2 \), we derive the Reshetnyak formula that expresses \( \|f\|_{H^s(\mathbb{R}^n)} \) through some norm of \((I^0 f, I^1 f, \ldots, I^m f)\). The \( H^s \)-norm is a modification of the Sobolev norm weighted differently at high and low frequencies. Using the Reshetnyak formula, we obtain a stability estimate.

1. Introduction. The ray transform integrates symmetric tensor fields over straight lines. Let \( \langle \cdot, \cdot \rangle \) be the standard dot-product on \( \mathbb{R}^n \) and \( |\cdot| \), the corresponding norm. The family of oriented straight lines in \( \mathbb{R}^n \) is parameterized by points of the manifold

\[ T\mathbb{S}^{n-1} = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid |\xi| = 1, \langle x,\xi \rangle = 0 \} \subset \mathbb{R}^n \times \mathbb{R}^n \]

that is the tangent bundle of the unit sphere \( \mathbb{S}^{n-1} \). Namely, a point \((x,\xi) \in T\mathbb{S}^{n-1}\) determines the line \( \{x + t\xi \mid t \in \mathbb{R}\} \). (In topology, points of \( T\mathbb{S}^{n-1} \) are traditionally denoted by pairs \((\xi,x)\), where \( \xi \in \mathbb{S}^{n-1} \) and \( x \in T_\xi \mathbb{S}^{n-1} \). In integral geometry, the reverse notation \((x,\xi)\) is commonly used because the pair \((x,\xi)\) is thought as the “line through the point \( x \) in direction \( \xi \)”, see for example [4]. We use the latter notation since we refer to [4] many times.)

Along with the space \( C^\infty(T\mathbb{S}^{n-1}) \) of smooth functions, we use the Schwartz space \( \mathcal{S}(T\mathbb{S}^{n-1}) \) that is defined as follows. Given a function \( \varphi \in C^\infty(T\mathbb{S}^{n-1}) \), we extend it to some neighborhood of \( T\mathbb{S}^{n-1} \) in \( \mathbb{R}^n \times \mathbb{R}^n \) so that (the extension is again denoted by \( \varphi \))

\[ \varphi(x, r\xi) = \varphi(x,\xi) \ (r > 0), \quad \varphi(x + r\xi,\xi) = \varphi(x,\xi) \ (r \in \mathbb{R}). \]

2010 Mathematics Subject Classification. Primary: 44A12, 65R32; Secondary: 46F12.

Key words and phrases. Ray transform, Reshetnyak formula, inverse problems, tensor analysis, stability estimates.

The first author was supported by US NSF grant DMS 1616564 and a SERB Matrics Grant, MTR/2017/000837.

The second author was supported by SERB National Postdoctoral fellowship, PDF/2017/002780.

The first three authors were supported by Airbus Corporate Foundation Chair grant “Mathematics of Complex Systems” established at TIFR CAM and TIFR ICTS, Bangalore, India.

The work was started when the last author visited TIFR CAM January 2017. The author is grateful to the institute for the support and hospitality.

The last author was supported by RFBR, Grant 17-51-150001.
We say that a function \( \varphi \in C^\infty(TS^{n-1}) \) belongs to \( S(TS^{n-1}) \) if the seminorm
\[
\| \varphi \|_{k,\alpha,\beta} = \sup_{(x,\xi) \in TS^{n-1}} (1 + |x|)^k |\partial_x^\alpha \partial_\xi^\beta \varphi(x,\xi)|
\]
is finite for every \( k \in \mathbb{N} \) and for all multi-indices \( \alpha \) and \( \beta \). The family of these seminorms defines the topology on \( S(TS^{n-1}) \).

Let \( S^m\mathbb{R}^n \) be the complex vector space of rank \( m \) symmetric tensors on \( \mathbb{R}^n \). The dimension of \( S^m\mathbb{R}^n \) is \( \binom{n+m-1}{m} \). In particular, \( S^0\mathbb{R}^n = \mathbb{C} \) and \( S^1\mathbb{R}^n = \mathbb{C}^n \). Let \( S(\mathbb{R}^n; S^m\mathbb{R}^n) \) be the Schwartz space of \( S^m\mathbb{R}^n \)-valued functions that are called rank \( m \) smooth fast decaying symmetric tensor fields on \( \mathbb{R}^n \). The ray transform is the linear bounded operator
\[
I : S(\mathbb{R}^n; S^m\mathbb{R}^n) \to S(TS^{n-1})
\]
that is defined, for \( f = (f_{i_1...i_m}) \in S(\mathbb{R}^n; S^m\mathbb{R}^n) \), by
\[
I f(x,\xi) = \int_{-\infty}^{\infty} f_{i_1...i_m}(x+t\xi) \xi^{i_1}...\xi^{i_m} \, dt = \int_{-\infty}^{\infty} \langle f(x+t\xi), \xi^m \rangle \, dt \quad ((x,\xi) \in TS^{n-1}).
\]

We use the Einstein summation rule: the summation from 1 to \( n \) is assumed over every index repeated in lower and upper positions in a monomial. To adopt our formulas to the Einstein summation rule, we use either lower or upper indices for denoting coordinates of vectors and tensors. For instance, \( \xi^i = \xi_i \) in (1.2). There is no difference between covariant and contravariant tensors since we use Cartesian coordinates only. Being initially defined by (1.2) on smooth fast decaying tensor fields, the operator (1.1) then extends to some wider spaces of tensor fields.

The study of the ray transform is motivated by several applications. In the case of \( m = 0 \) (when \( f \) is a function), the ray transform is the main mathematical tool of Computer Tomography. In the case of \( m = 1 \) (when \( f \) is a vector field), the operator \( I \) is called the Doppler transform and serves as the main mathematical tool of Doppler Tomography. In the cases of \( m = 2 \) and of \( m = 4 \), the operator \( I \) and some its relatives are applied to various problems of tomography of anisotropic media, see [4, Chapters 6,7] and [3, 6]. We are not aware of any tomographic application where the operators \( I^k \) \( (k > 0) \) on tensor fields of rank \( m > 0 \) arise naturally. Nevertheless, the operators \( I^k \) \( (k > 0) \) on scalar functions (i.e., on tensor fields of rank \( m = 0 \)) are of some applied interest, see [2].

The operator \( I \) has a big null-space in the case of \( m > 0 \). A symmetric tensor field can be uniquely decomposed into its solenoidal and potential parts [4, Theorem 2.6.2], and the potential part lies in the null-space. Given \( I f \), one can recover only the solenoidal part of \( f \), and there is a reconstruction formula [4, Theorem 2.12.2]. This naturally leads to the question of what additional information should be added to the data \( I f \) for the unique recovery of the entire tensor field \( f \). One possibility is to consider the momentum ray transforms
\[
I^k : S(\mathbb{R}^n; S^m\mathbb{R}^n) \to S(TS^{n-1})
\]
that are defined for \( k = 0,1,\ldots \) as follows:
\[
( I^k f)(x,\xi) = \int_{-\infty}^{\infty} t^k \langle f(x+t\xi), \xi^m \rangle \, dt \quad ((x,\xi) \in TS^{n-1}).
\]


In particular, $I^0 = I$. A rank $m$ symmetric tensor field $f$ is uniquely determined by the functions $(I^0 f, I^1 f, \ldots, I^m f)$, see [4, Theorem 2.17.2] and [1].

The momentum ray transforms are primary objects of study of this paper, and we have three goals:

1. To obtain an algorithm for recovering a rank $m$ symmetric tensor field $f$ from the data $(I^0 f, I^1 f, \ldots, I^m f)$.
2. To derive a version of the Reshetnyak formula [5] that expresses the norm $\|f\|_{H_t}$ through some norms of the functions $(I^0 f, I^1 f, \ldots, I^m f)$. The $H_t^*$-norm is a modification of the Sobolev norm weighted differently at high and low frequencies, see Section 2 for the precise definition.
3. To obtain stability estimates in terms of $H_t^*$-norms.

We hope the reader is not confused by two different senses of the parameter $t$: as the integration parameter in (1.3) and as an index in the notation $H_t^*$. Both the notations are standard.

The first goal is achieved for arbitrary $m$ in any dimension $n \geq 2$. The Reshetnyak formula and stability estimate are obtained in the cases of $m = 1$ and $m = 2$ only. We believe our approach works for any $m$, but the bulkiness of the Reshetnyak formula grows very fast with $m$.

The paper is organized as follows. In Section 2, we discuss basic properties of momentum ray transforms and state a few preliminaries. Section 3 presents the inversion algorithm. Section 4 is devoted to the Reshetnyak formula and stability estimates. Finally, in Section 5, we restrict ourselves to the 2-dimensional case and propose an alternate approach based on the fact that there are natural coordinates on $T^*S^1 = S^1 \times \mathbb{R}$.

2. Preliminaries.

2.1. Basic properties of momentum ray transforms. First of all we observe that the right-hand side of (1.3) makes sense for all $(x, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. We define the continuous linear operators

$$J^k : S(\mathbb{R}^n; S^m \mathbb{R}^n) \to C^\infty(TS^{n-1}) \quad (k = 0, 1, 2, \ldots) \quad (2.1)$$

by

$$(J^k f)(x, \xi) = \int_{-\infty}^{\infty} t^k \langle f(x + t\xi), \xi^m \rangle \, dt \quad \text{for} \quad (x, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}). \quad (2.2)$$

The data $(I^0 f, I^1 f, \ldots, I^m f)$ and $(J^0 f, J^1 f, \ldots, J^m f)$ are equivalent as we will demonstrate right now. Therefore the operators (2.1) are also called momentum ray transforms. The function $J^k f$ is sometimes more convenient than $I^k f$ because the partial derivatives $\frac{\partial (J^k f)}{\partial x^i}$ and $\frac{\partial (J^k f)}{\partial \xi^i}$ are well defined. On the other hand, the function $(I^k f)(x, \xi)$ obeys good decay conditions in the first argument.

For a tensor field $f \in S(\mathbb{R}^n; S^m \mathbb{R}^n)$, the function $(J^k f)(x, \xi)$ possesses the following homogeneity in the second argument

$$(J^k f)(x, r\xi) = r^{m-k} \frac{1}{|r|} (J^k f)(x, \xi) \quad (0 \neq r \in \mathbb{R}) \quad (2.3)$$

and has the following property in the first argument

$$(J^k f)(x + r\xi, \xi) = \sum_{\ell=0}^{k} \binom{k}{\ell} (-r)^{k-\ell} (J^\ell f)(x, \xi) \quad (r \in \mathbb{R}). \quad (2.4)$$
Compare with [4, Formulas (2.1.11)–(2.1.12)]. These properties easily follow from the definition (2.2).

Comparing (1.3) and (2.2), we see that

\[ I^k f = J^k f |_{\mathbb{T}^{n-1}}. \tag{2.5} \]

On the other hand,

\[ (J^k f)(x, \xi) = |\xi|^{m-2k-1} \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} |\xi|^{\ell} \langle \xi, x \rangle^{k-\ell} (I^{\ell} f) \left( x - \frac{\langle \xi, x \rangle}{|\xi|^2} \xi, \frac{\xi}{|\xi|} \right) \tag{2.6} \]

as easily follows from (2.3)–(2.4). Compare with [4, Formula (2.1.13)].

Formulas (2.4) and (2.6) mean in particular that the operator \( I^k \) must always be considered together with lower order momenta \((I^0, \ldots, I^{k-1})\), i.e., the data \((I^0 f, \ldots, I^k f)\) must always be used instead of \(I^k f\).

There are two important first order differential operators on symmetric tensor fields: the inner derivative \( d : C^\infty(\mathbb{R}^n; S^m \mathbb{R}^n) \to C^\infty(\mathbb{R}^n; S^{m+1} \mathbb{R}^n) \) and the divergence \( \delta : C^\infty(\mathbb{R}^n; S^m \mathbb{R}^n) \to C^\infty(\mathbb{R}^n; S^{m-1} \mathbb{R}^n) \). These operators are defined as follows [4, 5]: Given a symmetric tensor field \( f = (f_{i_1 \ldots i_m}) \),

\[ (df)_{i_1 \ldots i_m} = \frac{1}{m+1} \sum_{k=1}^{m+1} \partial f_{i_1 \ldots i_k+1 \ldots i_{m+1}} \]

and

\[ (\delta f)_{i_1 \ldots i_{m-1}} = \sum_{i=1}^{n} \frac{\partial f_{i_1 \ldots i_{m-1} i}}{\partial x_i}. \]

Operators \( I^k \) are related to the inner derivative by the formula

\[ I^k(df) = -k I^{k-1} f \]

which is valid at least for \( f \in \mathcal{S}(\mathbb{R}^n; S^n \mathbb{R}^n) \). This easily follows from (2.2) with the help of integration by parts. In particular, \( I^k(df^\ell f) = 0 \) for \( k < \ell \).

Let us also mention the transformation law for operators \( I^k \) under a change of the origin in \( \mathbb{R}^n \). Given \( a \in \mathbb{R}^n \), set \( f_a(x) = f(x + a) \). As easily follows from (2.5) and (2.6),

\[ (I^k f_a)(x, \xi) = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \langle a, \xi \rangle^{k-\ell} (I^{\ell} f)(x + a - \langle a, \xi \rangle \xi, \xi) \quad \text{for} \quad (x, \xi) \in \mathbb{T}^{n-1}. \tag{2.7} \]

We are going to derive a formula that expresses \( \|f\|_{H^l} \) through \((I^0 f, I^1 f, \ldots, I^m f)\). Since \( \|f\|_{H^l} = \|f_a\|_{H^l} \), the expression must be invariant under the transformation (2.7).

2.2. Momentum ray transforms and the Fourier transform. We use the Fourier transform \( F : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n), f \mapsto \hat{f} \) in the following form (hereafter \( i \) is the imaginary unit and \( y \) is the Fourier dual variable of \( x \)):

\[ \hat{f}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-iy \cdot x} f(x) \, dx. \]

The Fourier transform \( F : \mathcal{S}(\mathbb{R}^n; S^m \mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n; S^m \mathbb{R}^n), f \mapsto \hat{f} \) on symmetric tensor fields is defined componentwise, i.e., \( \hat{f}_{i_1 \ldots i_m} = \hat{f}_{i_1 \ldots i_m} \) (we use Cartesian coordinates only). Introduce also the Fourier transform \( F : \mathcal{S}(\mathbb{T}^{n-1}) \to \mathcal{S}(\mathbb{T}^{n-1}), \varphi \mapsto \hat{\varphi} \).
\( \hat{\varphi} \) on \( TS^{n-1} \) by
\[
\hat{\varphi}(y, \xi) = \frac{1}{(2\pi)^{(n-1)/2}} \int_{\xi^\perp} e^{-i(y,x)}\varphi(x, \xi) \, dx, \tag{2.8}
\]
where \( dx \) is the \((n-1)\)-dimensional Lebesgue measure on the hyperplane \( \xi^\perp = \{ x \in \mathbb{R}^n \mid \langle \xi, x \rangle = 0 \} \). It is the standard Fourier transform in the \((n-1)\)-dimensional variable \( x \) while \( \xi \in S^{n-1} \) is considered as a parameter.

The Fourier transform of the momentum ray transform is given by the formula
\[
\hat{\mathcal{M}}f(y, \xi) = (2\pi)^{1/2}F_{x \to y} \left[ \langle \xi, x \rangle^k \langle f(x), \xi^m \rangle \right] = (2\pi)^{1/2} k \langle \hat{\varphi}(y), \xi^{m+k} \rangle \tag{2.9}
\]
for \((y, \xi) \in TS^{n-1}\). Indeed,
\[
\hat{\mathcal{M}}f(y, \xi) = (2\pi)^{(1-n)/2} \int_{\xi^\perp} e^{-i(y,x)} (\mathcal{M}f)(x, \xi) \, dx \]
\[
= (2\pi)^{(1-n)/2} \int_{\xi^\perp} \int_{-\infty}^{\infty} e^{-i(y,x)} t^k \langle f(x + t\xi), \xi^m \rangle \, dt \, dx. \tag{2.10}
\]
On assuming \((y, \xi) \in TS^{n-1}\), change the integration variables as \( z = x + t\xi \)
\[
\hat{\mathcal{M}}f(y, \xi) = (2\pi)^{1/2} \langle 2\pi \rangle^{n-2} \int_{\mathbb{R}^n} e^{-i(y,z)} \langle \xi, z \rangle^k f(z) \, dz, \xi^m).
\]
On using the equality
\[
(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i(y,z)} \langle \xi, z \rangle^k f(z) \, dz = i^k \langle \xi, \partial_y \rangle^k \hat{f}(y),
\]
we obtain
\[
\hat{\mathcal{M}}f(y, \xi) = (2\pi)^{1/2} i^k \langle \xi, \partial_y \rangle^k \hat{f}(y), \xi^m).
\]
This coincides with (2.9).

### 2.3. The Reshetnyak formula for scalar functions.

Recall [5] that the Hilbert space \( H_t^s(\mathbb{R}^n) \) is defined for \( s \in \mathbb{R} \) and \( t > -n/2 \) as the completion of \( \mathcal{S}(\mathbb{R}^n) \) with respect to the norm
\[
\| f \|_{H_t^s(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |y|^{2t} (1 + |y|^2)^{s-t} |\hat{f}(y)|^2 \, dy. \tag{2.11}
\]
Similarly, the Hilbert space \( H_t^s(TS^{n-1}) \) is defined for \( s \in \mathbb{R} \) and \( t > -(n-1)/2 \) as the completion of \( \mathcal{S}(TS^{n-1}) \) with respect to the norm
\[
\| \varphi \|_{H_t^s(TS^{n-1})}^2 = \frac{1}{2\pi} \int_{S^{n-1}} \int_{\xi^\perp} |y|^{2t} (1 + |y|^2)^{s-t} |\hat{\varphi}(y, \xi)|^2 \, d\xi \, dx \tag{2.12}
\]
where \( d\xi \) is the volume form on the sphere \( S^{n-1} \) induced by the Euclidean metric of \( \mathbb{R}^n \).

The following statement is a particular case of \( m = 0 \) in [5, Theorem 4.2]. Given a function \( f \in \mathcal{S}(\mathbb{R}^n) \), the Reshetnyak formula
\[
\| f \|_{H_t^s(\mathbb{R}^n)}^2 = a_n \| I f \|_{H_{t+1/2}^{s+1/2}(TS^{n-1})}^2 \tag{2.13}
\]
coefficients are independent of \((s, t)\) in Theorem 4.2 of [5]. The right formula is presented in [4, Formula (2.15.3)]. The coefficients are independent of \((s, t)\). Formula (2.12) will be used throughout the paper.

3. Inversion algorithm for momentum ray transforms. Let us recall the definition of the partial symmetrization

\[
\sigma(i_1 \ldots i_r)u_{i_1 \ldots i_r j_1 \ldots j_e} = \frac{1}{r!} \sum_{\pi \in \Pi_r} u_{i_{\pi(1)} \ldots i_{\pi(r)} j_1 \ldots j_e},
\]

where the summation is performed over the set \(\Pi_r\) of all permutations of the set \(\{1, \ldots r\}\).

The following statement is the main ingredient of our algorithm for recovering a rank \(m\) symmetric tensor field \(f\) from the data \((I^0f, \ldots, I^mf)\).

**Theorem 3.1.** Given a tensor field \(f = (f_{i_1 \ldots i_m}) \in \mathcal{S}(\mathbb{R}^n; S^m \mathbb{R}^n)\), equalities

\[
Jf_{i_1 \ldots i_m} = \frac{1}{m!} \sigma(i_1 \ldots i_m) \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{\partial^m (J^k f)}{\partial x^{i_1} \ldots \partial x^{i_k} \partial \xi^{i_{k+1}} \ldots \partial \xi^{i_m}}
\]

hold for all indices \((i_1, \ldots, i_m)\), where the left-hand side is the result of applying the ray transform \(J = J^0\) to the coordinate \(f_{i_1 \ldots i_m}\) considered as a scalar function on \(\mathbb{R}^n\) (we use Cartesian coordinates only).

**Proof.** Equality (3.1) trivially holds for \(m = 0\). We proceed by induction on \(m\).

Assume (3.1) to be valid for tensor fields \(f \in \mathcal{S}(\mathbb{R}^n; S^m \mathbb{R}^n)\) with some \(m\).

Now assume \(f \in \mathcal{S}(\mathbb{R}^n; S^{m+1} \mathbb{R}^n)\). Definition (2.2) can be written as

\[
(J^k f)(x, \xi) = \xi^{p_1} \ldots \xi^{p_{m+1}} \int_{-\infty}^{\infty} t^k f_{p_1 \ldots p_{m+1}}(x + t\xi) \, dt.
\]

The right-hand side of the equality depends on \(\xi^{i_{m+1}}\) for every \(1 \leq i_{m+1} \leq n\) since the summation is assumed over repeating indices. We differentiate this equality with respect to \(\xi^{i_{m+1}}\)

\[
\frac{\partial (J^k f)}{\partial \xi^{i_{m+1}}} = (m + 1)\xi^{p_1} \ldots \xi^{p_m} \int_{-\infty}^{\infty} t^k f_{p_1 \ldots p_m i_{m+1}}(x + t\xi) \, dt
\]

\[
+ \frac{\partial}{\partial x^{i_{m+1}}} \left( \xi^{p_1} \ldots \xi^{p_m+1} \int_{-\infty}^{\infty} t^{k+1} f_{p_1 \ldots p_{m+1}}(x + t\xi) \, dt \right). \tag{3.2}
\]

Let us fix a value of the index \(i_{m+1}\) and define the tensor field \(\tilde{f} \in \mathcal{S}(\mathbb{R}^n; S^m \mathbb{R}^n)\) by

\[
\tilde{f}_{i_1 \ldots i_m} = f_{i_1 \ldots i_m i_{m+1}}.
\]

The definition is correct assuming Cartesian coordinates to be fixed. Then (3.2) can be written as

\[
\frac{\partial (J^k f)}{\partial \xi^{i_{m+1}}} = (m + 1)J^k \tilde{f} + \frac{\partial (J^{k+1} f)}{\partial x^{i_{m+1}}},
\]
From this,
\[ J^k f = \frac{1}{m+1} \left( \frac{\partial (J^k f)}{\partial \xi_{m+1}} - \frac{\partial (J^{k+1} f)}{\partial x_{m+1}} \right). \]
Differentiate this equality to obtain
\[ \frac{\partial^m (J^k f)}{\partial x^i_1 \cdots \partial x^i_k \partial \xi_{i+1} \cdots \partial \xi_m} = \frac{1}{m+1} \sum_{k=0}^{m+1} \left( \frac{\partial^{m+1} (J^k f)}{\partial x^i_1 \cdots \partial x^i_k \partial \xi_{i+1} \cdots \partial \xi_m} - \frac{\partial^{m+1} (J^{k+1} f)}{\partial x^i_1 \cdots \partial x^i_k \partial \xi_{i+1} \cdots \partial x_{m+1} \partial \xi_{i+1} \cdots \partial \xi_m} \right). \]
By the induction hypothesis,
\[ J f_{1 \ldots i_{m+1}} = J f_{i_1 \ldots i_m} = \frac{1}{m!} \sigma(i_1 \ldots i_m) \sum_{k=0}^{m} (-1)^k \binom{m}{k} \partial^m (J^k f) \frac{\partial^{m+1} (J^k f)}{\partial x^i_1 \cdots \partial x^i_k \partial \xi_{i+1} \cdots \partial \xi_m}. \]
Substitute value (3.3) into the last formula
\[
J f_{i_1 \ldots i_{m+1}} = \frac{1}{(m+1)!} \sigma(i_1 \ldots i_{m+1}) \sum_{k=0}^{m} (-1)^k \binom{m}{k} \partial^{m+1} (J^k f) \frac{\partial^{m+1} (J^k f)}{\partial x^i_1 \cdots \partial x^i_k \partial x_{i+1} \partial \xi_{i+1} \cdots \partial \xi_m}.
\]
We have replaced the symmetrization \( \sigma(i_1 \ldots i_m) \) by the stronger operator \( \sigma(i_1 \ldots i_{m+1}) \) because the left-hand side is symmetric in the indices \( (i_1, \ldots, i_{m+1}) \). In the second sum on the right-hand side, we change the summation index as \( k = k'-1 \). After the change, we again use the notation \( k \) instead of \( k' \). In such the way, we transform the formula to the form
\[
J f_{i_1 \ldots i_{m+1}} = \frac{1}{(m+1)!} \sigma(i_1 \ldots i_{m+1}) \sum_{k=0}^{m+1} (-1)^k \binom{m}{k} \partial^{m+1} (J^k f) \frac{\partial^{m+1} (J^k f)}{\partial x^i_1 \cdots \partial x^i_k \partial x_{i+1} \partial \xi_{i+1} \cdots \partial x^m}.
\]
We assume binomial coefficients \( \binom{m}{k} = \frac{m!}{k!(m-k)!} \) to be defined for all integers \( m \) and \( k \) under the agreement: \( \binom{m}{0} = \binom{m}{m} = 1 \) for either \( m < 0 \) or \( k < 0 \) or \( k > m \). Therefore both summations can be extended to the limits \( 0 \leq k \leq m + 1 \). Besides this, we can write indices \( (i_1, \ldots, i_{m+1}) \) in an arbitrary order on the right-hand side because of the presence of the symmetrization \( \sigma(i_1 \ldots i_{m+1}) \). With the help of the Pascal relation \( \binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k} \), the formula takes the form
\[
J f_{i_1 \ldots i_{m+1}} = \frac{1}{(m+1)!} \sigma(i_1 \ldots i_{m+1}) \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} \partial^{m+1} (J^k f) \frac{\partial^{m+1} (J^k f)}{\partial x^i_1 \cdots \partial x^i_k \partial x_{i+1} \partial \xi_{i+1} \cdots \partial x^m}.
\]
This finishes the induction step.

Let us recall the inversion formula for recovering a scalar function \( f \in \mathcal{S}(\mathbb{R}^n) \) from \( If \):
\[ f(x) = \frac{\Gamma((n-1)/2)}{4\pi^{(n+1)/2}} \Delta^{1/2} \int_{\mathbb{R}^{n-1}} (If)(x - \langle \xi, x \rangle \xi, \xi) \, d\xi. \]  
This is a particular case of \( m = 0 \) of [4, Theorem 2.12.2].
Our algorithm for recovering a symmetric m-tensor field \( f \) from the collection of functions \( (I^f, \ldots, I^m f) \) is as follows. Given \( (I^f, \ldots, I^m f) \), we find the data \( (J^f, \ldots, J^m f) \) by formula (2.6). Then we find the functions \( Jf_{i_1\ldots i_m} \) for all values of indices by (3.1) and find \( Jf_{i_1\ldots i_m} = Jf_{i_1\ldots i_m}|_{TS^{n-1}} \). From the latter data we recover all components \( f_{i_1\ldots i_m} \) of the field \( f \) by formula (3.4).

The stability of the recovery procedure for a scalar function is completely described by the Reshetnyak formula (2.12). For higher rank tensor fields, the stability question is more delicate because of the presence of \( m \)th order derivatives in formula (3.1). We will investigate the stability question in the next section.

4. Reshetnyak formula for momentum ray transforms. In this section, we derive the Reshetnyak formula and stability estimate for \( m = 1 \) and for \( m = 2 \).

4.1. Operators \( X_i \) and \( \Xi_i \). In the cases of \( m = 1 \) and of \( m = 2 \), formula (3.1) takes the forms

\[
Jf_i = \frac{\partial (J^1 f)}{\partial \xi^i} - \frac{\partial (J^1 f)}{\partial x^i} \tag{4.1}
\]

and

\[
Jf_{ij} = \frac{1}{2} \left( \frac{\partial^2 (J^2 f)}{\partial \xi^i \partial \xi^j} - \frac{\partial^2 (J^1 f)}{\partial x^i \partial \xi^j} - \frac{\partial^2 (J^1 f)}{\partial x^i \partial x^j} + \frac{\partial^2 (J^2 f)}{\partial x^i \partial x^j} \right) \tag{4.2}
\]

respectively. We are going to rewrite these formulas in intrinsic terms of the manifold \( TS^{n-1} \).

Introduce the vector fields on \( \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) = \{ (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid \xi \neq 0 \} \)

\[
\hat{X}_i = \frac{\partial}{\partial x^i} - \xi_i \xi^p \frac{\partial}{\partial x^p} \quad (1 \leq i \leq n),
\]

\[
\hat{\Xi}_i = \frac{\partial}{\partial \xi^i} - x_i \xi^p \frac{\partial}{\partial x^p} - \xi_i \xi^p \frac{\partial}{\partial \xi^p} \quad (1 \leq i \leq n).
\tag{4.3}
\]

The notations \( \hat{X}_i \) and \( \hat{\Xi}_i \) are chosen because the derivatives \( \frac{\partial}{\partial x^i} \) and \( \frac{\partial}{\partial \xi^i} \) are in some sense leading terms on the right-hand sides of (4.3).

**Lemma 4.1.** At every point \( (x, \xi) \in TS^{n-1} \), vectors \( \hat{X}_i(x, \xi) \) and \( \hat{\Xi}_i(x, \xi) \) \( (1 \leq i \leq n) \) are tangent to \( TS^{n-1} \). Let \( X_i \) and \( \Xi_i \) be the restrictions of vector fields \( \hat{X}_i \) and \( \hat{\Xi}_i \) to the manifold \( TS^{n-1} \) respectively. Thus, \( X_i \) and \( \Xi_i \) are smooth vector fields on \( TS^{n-1} \) and can be considered as first order differential operators

\[
X_i, \Xi_i : C^\infty(TS^{n-1}) \rightarrow C^\infty(TS^{n-1}).
\]

The operators satisfy

\[
[X_i, X_j] = 0, \quad [\Xi_i, \Xi_j] = 0, \quad [X_i, \Xi_j] = x_i X_j - x_j X_i + \xi_i \Xi_j - \xi_j \Xi_i, \tag{4.4}
\]

At every point \( (x, \xi) \in TS^{n-1} \), vectors \( X_i(x, \xi), \Xi_i(x, \xi) \) \( (1 \leq i \leq n) \) generate the tangent space \( T_{(x,\xi)}(TS^{n-1}) \) and satisfy

\[
\xi^i X_i(x, \xi) = 0, \quad \xi^i \Xi_i(x, \xi) = 0. \tag{4.7}
\]

**Proof.** Definition (4.3) implies

\[
\hat{X}_i |\xi|^2 = 0, \quad \hat{\Xi}_i |\xi|^2 = 2\xi_i (1 - |\xi|^2), \quad \hat{X}_i(\xi, x) = \xi_i (1 - |\xi|^2), \quad \hat{\Xi}_i(\xi, x) = x_i (1 - |\xi|^2) - \xi_i (\xi, x).
\]

Right-hand sides of these equalities vanish on \( TS^{n-1} \). This proves the first statement.
From Definition (4.3),
\[ \xi^i \tilde{X}_i = (1 - |\xi|^2) \xi^p \frac{\partial}{\partial x^p} \quad \xi^i \tilde{\Xi}_i = (\xi, x) \xi^p \frac{\partial}{\partial x^p} - (1 - |\xi|^2) \xi^p \frac{\partial}{\partial \xi^p}. \]

Right-hand sides of these equalities vanish on $T\mathbb{S}^{n-1}$. This proves (4.7).

Recall the well known formula:
\[ \{fX, gY\} = fg[X, Y] + f(Xg)Y - g(Yf)X \]
for vector fields $X, Y$ and for functions $f, g$. On using this formula, one easily derives from Definition (4.3)
\[ [\tilde{X}_i, \tilde{\Xi}_j] = \xi_i \frac{\partial}{\partial x^j} - \xi_j \xi^p \frac{\partial}{\partial \xi^p}. \quad (4.8) \]

On the other hand,
\[ \xi_i \tilde{X}_j = \xi_i \left( \frac{\partial}{\partial x^j} - \xi_j \xi^p \frac{\partial}{\partial \xi^p} \right) = \xi_i \frac{\partial}{\partial x^j} - \xi_j \xi^p \frac{\partial}{\partial \xi^p}. \quad (4.9) \]

Comparing (4.8) and (4.9), we see that $[\tilde{X}_i, \tilde{\Xi}_j] = \xi_i \tilde{X}_j$. This proves (4.6). Formulas (4.4)–(4.5) are proved in a similar way.

Finally we prove that, at a point $(x, \xi) \in T\mathbb{S}^{n-1}$, the vectors $X_i(x, \xi), \Xi_i(x, \xi) (1 \leq i \leq n)$ generate the tangent space $T_{(x, \xi)}(T\mathbb{S}^{n-1})$. To this end we have to demonstrate that any linear dependence between these vectors is actually a corollary of (4.7). Assume that
\[ \alpha^i X_i + \beta^i \Xi_i = 0 \quad (4.10) \]
with some coefficients $\alpha^i, \beta^i$. Substitute values (4.3) into this equality
\[ \alpha^i \left( \frac{\partial}{\partial x^i} - \xi_i \xi^p \frac{\partial}{\partial x^p} \right) + \beta^i \left( \frac{\partial}{\partial x^i} - x_i \xi^p \frac{\partial}{\partial x^p} - \xi_i \xi^p \frac{\partial}{\partial \xi^p} \right) = 0. \]

This can be written in the form
\[ (\alpha^i (\delta^p_i - \xi_i \xi^p) - \beta^i x_i \xi^p) \frac{\partial}{\partial x^p} + \beta^i (\delta^p_i - \xi_i \xi^p) \frac{\partial}{\partial \xi^p} = 0, \]
where $\delta^p_i$ is the Kronecker tensor. Since vectors $\frac{\partial}{\partial x^p}, \frac{\partial}{\partial \xi^p} (p = 1, \ldots, n)$ are linearly independent, the last equation implies
\[ (\delta^p_i - \xi_i \xi^p) \alpha^i - x_i \xi^p \beta^i = 0 \quad (p = 1, \ldots, n), \quad (4.11) \]
\[ (\delta^p_i - \xi_i \xi^p) \beta^i = 0 \quad (p = 1, \ldots, n). \quad (4.12) \]

At a point $(x, \xi) \in T\mathbb{S}^{n-1}$, the rank of the matrix $(\delta^p_i - \xi_i \xi^p)$ of system (4.12) is equal to $n - 1$ and any solution to the system is of the form $\beta^i = \beta_0 \xi^i$. System (4.11) takes now the form
\[ (\delta^p_i - \xi_i \xi^p) \alpha^i = 0 \quad (p = 1, \ldots, n). \]
As before, this implies $\alpha^i = \alpha_0 \xi^i$. Thus, equation (4.10) is actually of the form
\[ \alpha_0 \xi^i X_i + \beta_0 \xi^i \Xi_i = 0. \]

This means that any linear dependence between vectors $X_i(x, \xi), \Xi_i(x, \xi) (1 \leq i \leq n)$ is actually a corollary of (4.7).

We can now present the intrinsic forms of equations (4.1) and (4.2).
Theorem 4.2. For a vector field \( f = (f_i) \in S(\mathbb{R}^n; \mathbb{C}^n) \), the equality
\[
I f_i = (\Xi_i + \xi_i)(I^0 f) - X_i(I^f)
\] (4.13)
holds on \( T\mathbb{S}^{n-1} \) for every \( i \), where the left-hand side is the result of applying the ray transform \( I = I^0 \) to the coordinates \( f_i \) considered as scalar functions on \( \mathbb{R}^n \).

Theorem 4.3. For a tensor field \( f = (f_{ij}) \in S(\mathbb{R}^n; S^2 \mathbb{R}^n) \), the equality
\[
I f_{ij} = \frac{1}{2} \sigma(ij) \left[ X_i X_j (I^2 f) - 2 \Xi_i X_j (I^f) - 4 \xi_i X_j (I^f) + 3 \xi_i \Xi_j (I^f) + 3 \delta_{ij} (I^0 f) - \xi_i \xi_j (I^f) \right]
\] (4.14)
holds on \( T\mathbb{S}^{n-1} \) for all indices \((i, j)\), where \( \delta_{ij} \) is the Kronecker tensor and the left-hand side is the result of applying the ray transform \( I = I^0 \) to the coordinates \( f_{ij} \) considered as scalar functions on \( \mathbb{R}^n \).

We present the proof of Theorem 4.2. Theorem 4.3 is proved in the same way although all involved calculations are more cumbersome.

Proof of Theorem 4.2. By the very definition of \( X_i \) and \( \Xi_i \), the equalities
\[
X_i \varphi = \tilde{X}_i \psi, \quad \Xi_i \varphi = \tilde{\Xi}_i \psi
\]
hold on \( T\mathbb{S}^{n-1} \) for an arbitrary function \( \varphi \in C^\infty(T\mathbb{S}^{n-1}) \), where \( \psi \) is an arbitrary smooth extension of \( \varphi \) to some neighborhood of \( T\mathbb{S}^{n-1} \) in \( \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \). For every \( k \), the function \( J^k f \) is an extension of \( I^k f \). Therefore
\[
X_i (I^k f) = \tilde{X}_i (J^k f), \quad \Xi_i (I^k f) = \tilde{\Xi}_i (J^k f) \quad \text{on} \quad T\mathbb{S}^{n-1}.
\]
Substitute values (4.3) to obtain
\[
\begin{align*}
X_i (I^k f) &= \left( \frac{\partial}{\partial x^i} - \xi_i \frac{\partial}{\partial x^p} \right) (J^k f), \\
\Xi_i (I^k f) &= \left( \frac{\partial}{\partial \xi^i} - x_i \xi^p \frac{\partial}{\partial \xi^p} - \xi_i \xi^p \frac{\partial}{\partial \xi^p} \right) (J^k f)
\end{align*}
\] (4.15)
on \( T\mathbb{S}^{n-1} \).

Let us specify (4.15) for \( k = 0 \). As is seen from (2.3), the function \( (J^0 f)(x, \xi) \) is positively homogeneous of zero degree in the second argument. By the Euler equation for homogeneous functions,
\[
\xi^p \frac{\partial (J^0 f)}{\partial \xi^p} = 0.
\] (4.16)
Besides this, \( J^0 f \) satisfies (see (2.4))
\[
(J^0 f)(x + t \xi, \xi) = (J^0 f)(x, \xi) \quad (t \in \mathbb{R}).
\]
Differentiating this identity with respect to \( t \) and then setting \( t = 0 \), we obtain
\[
\xi^p \frac{\partial (J^0 f)}{\partial x^p} = 0.
\] (4.17)
In virtue of (4.16)–(4.17), equalities (4.15) for \( k = 0 \) are simplified to the following ones:
\[
\frac{\partial (J^0 f)}{\partial x^i} = X_i (I^0 f), \quad \frac{\partial (J^0 f)}{\partial \xi^i} = \Xi_i (I^0 f) \quad \text{on} \quad T\mathbb{S}^{n-1}.
\] (4.18)
Let us specify (4.15) for \( k = 1 \). As is seen from (2.3), the function \((J^1 f)(x,\xi)\) is positively homogeneous of degree \(-1\) in the second argument. By the Euler equation for homogeneous functions,
\[
\xi^p \frac{\partial (J^1 f)}{\partial \xi^p} = -J^1 f. \tag{4.19}
\]
Besides this, \(J^1 f\) satisfies (see (2.4))
\[
(J^1 f)(x + t\xi, \xi) = -t(J^0 f)(x + t\xi, \xi) + (J^1 f)(x, \xi) \quad (t \in \mathbb{R}).
\]
Differentiating this identity with respect to \( t \) and then setting \( t = 0 \), we obtain
\[
\xi^p \frac{\partial (J^1 f)}{\partial x^p} = -(J^0 f). \tag{4.20}
\]
In virtue of (4.19)–(4.20), equalities (4.15) for \( k = 1 \) are simplified to the following ones:
\[
\frac{\partial (J^1 f)}{\partial x^i} = X_i (I^1 f) - \xi_i (I^0 f), \quad \frac{\partial (J^1 f)}{\partial \xi^i} = (\Xi_i - \xi_i) (I^1 f) - x_i (I^0 f) \quad \text{on } T\mathbb{S}^{n-1}. \tag{4.21}
\]
Inserting value (4.18) of \( \partial (J^0 f)/\partial \xi^i \) and value (4.21) of \( \partial (J^1 f)/\partial x^i \) into (4.1), we arrive to (4.13).

Recall that \( x_i \) is the Fourier dual variable of \( x_i \). The formulas for commuting the Fourier transform with operators \( X_i \) and \( \Xi_i \) are described by the following

**Lemma 4.4.** The equalities
\[
\begin{align*}
\hat{X}_i \varphi = i y_i \hat{\varphi}, & \quad \Xi_i \hat{\varphi} = \hat{\Xi}_i \varphi, & \quad \hat{x}_i \bar{\varphi} = i X_i \hat{\varphi} \tag{4.22} \\
\end{align*}
\]
hold for every function \( \varphi \in \mathcal{S}(T\mathbb{S}^{n-1}) \) and for every \( i, \ 1 \leq i \leq n \).

**Proof.** Given a function \( \varphi \in \mathcal{S}(T\mathbb{S}^{n-1}) \), define \( \psi \in \mathcal{C}^\infty (\mathbb{R}^n \times (\mathbb{R}^n \setminus \{ 0 \}) \) by
\[
\psi(x, \xi) = \varphi\left(x - \frac{\langle \xi, x \rangle}{|\xi|^2} \xi, \frac{\xi}{|\xi|}\right).
\]
The function satisfies the identities
\[
\psi(x, r\xi) = \psi(x, \xi) \ (0 \neq r \in \mathbb{R}), \quad \psi(x, x + r\xi) = \psi(x, \xi) \ (r \in \mathbb{R})
\]
which imply (by the same arguments that were used for deriving (4.16)–(4.17))
\[
\xi^p \frac{\partial \psi}{\partial x^p} = 0, \quad \xi^p \frac{\partial \psi}{\partial \xi^p} = 0. \tag{4.23}
\]
For \((y, \xi) \in T\mathbb{S}^{n-1}\), the definition (2.8) of the Fourier transform can be written in terms of \( \psi\):  
\[
\hat{\varphi}(y, \xi) = (2\pi)^{(1-n)/2} \int_{\xi} e^{-i(y, x)} \psi(x, \xi) \, d^{n-1}x. \tag{4.24}
\]
This formula makes sense for any \((y, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{ 0 \}) \) sufficiently close to the submanifold \( T\mathbb{S}^{n-1} \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus \{ 0 \}) \). Therefore the partial derivatives \( \partial \hat{\varphi} / \partial y^i \) and \( \partial \hat{\varphi} / \partial \xi^i \) make sense.

By (4.3),
\[
X_i \varphi = \hat{X}_i \psi = \frac{\partial \psi}{\partial x^i} - \xi^p \frac{\partial \psi}{\partial x^p}. \tag{4.25}
\]
With the help of (4.23), this is simplified to
\[
X_i \varphi = \frac{\partial \psi}{\partial x^i}.
\]
Applying the Fourier transform to this equality, we obtain
\[ \hat{X}_i \varphi(y, \xi) = (2\pi)^{(1-n)/2} \int_{\xi^\perp} e^{-i(y,x) \partial \varphi \over \partial x^i} (x, \xi) \, d^{n-1}x. \]

From this with the help of integration by parts
\[ \hat{X}_i \varphi(y, \xi) = i y_i (2\pi)^{-(n-1)/2} \int_{\xi^\perp} e^{-i(y,x) \varphi(x, \xi)} \, d^{n-1}x = i y_i \hat{\varphi}(y, \xi). \]

This proves the first of formulas (4.22).

Differentiate equality (4.24) with respect to \( y^i \)
\[ \frac{\partial \hat{\varphi}}{\partial y^i}(y, \xi) = -i (2\pi)^{(1-n)/2} \int_{\xi^\perp} e^{-i(y,x)} x_i \varphi(x, \xi) \, d^{n-1}x. \] (4.25)

From this
\[ \xi^p \frac{\partial \hat{\varphi}}{\partial y^p}(y, \xi) = -i (2\pi)^{(1-n)/2} \int_{\xi^\perp} e^{-i(y,x)} \langle \xi, x \rangle \varphi(x, \xi) \, d^{n-1}x = 0. \] (4.26)

By (4.3),
\[ \Xi_i \hat{\varphi} = \frac{\partial \hat{\varphi}}{\partial \xi^i} - y_i \xi^p \frac{\partial \hat{\varphi}}{\partial y^p} - \xi_i \xi^p \frac{\partial \hat{\varphi}}{\partial \xi^p}. \]

In virtue of (4.26), this is simplified to
\[ \Xi_i \hat{\varphi} = \frac{\partial \hat{\varphi}}{\partial \xi^i} - \xi_i \xi^p \frac{\partial \hat{\varphi}}{\partial \xi^p}. \] (4.27)

The differentiation of equality (4.24) with respect to \( \xi^i \) is not very easy because the integration hyperplane \( \xi^\perp \) depends on \( \xi \). We first transform integral (4.24) into an integral over a hyperplane independent of \( \xi \). Fix a vector \( \xi_0 \in \mathbb{S}^{n-1} \) sufficiently close to \( \xi_0 \), the orthogonal projection
\[ \xi_0^\perp \to \xi^\perp, \quad x' \mapsto x = x' - \langle \xi, x' \rangle \xi \] (4.28)

is one-to-one. We change the integration variable in (4.24) according to (4.28). The Jacobian of the change is \( \langle \xi_0, \xi \rangle^{-1} \). After the change, formula (4.24) takes the form
\[ \hat{\varphi}(y, \xi) = \frac{(2\pi)^{(1-n)/2}}{\langle \xi_0, \xi \rangle} \int_{\xi_0^\perp} e^{-i(y,x'-(\xi,x')\xi)} \varphi(x'-(\xi, x')\xi, \xi) \, d^{n-1}x'. \]

We can now differentiate this equality with respect to \( \xi^i \)
\[
\begin{align*}
\frac{\partial \hat{\varphi}}{\partial \xi^i}(y, \xi) &= -\frac{\xi_0, i}{\langle \xi_0, \xi \rangle^2} (2\pi)^{(1-n)/2} \int_{\xi_0^\perp} e^{-i(y,x'-(\xi,x')\xi)} \varphi(x'-(\xi, x')\xi, \xi) \, d^{n-1}x' \\
&+ \frac{1}{\langle \xi_0, \xi \rangle} (2\pi)^{(1-n)/2} \int_{\xi_0^\perp} \frac{\partial e^{-i(y,x'-(\xi,x')\xi)}}{\partial \xi^i} \varphi(x'-(\xi, x')\xi, \xi) \, d^{n-1}x' \\
&+ \frac{1}{\langle \xi_0, \xi \rangle} (2\pi)^{(1-n)/2} \int_{\xi_0^\perp} e^{-i(y,x'-(\xi,x')\xi)} \varphi(x'-(\xi, x')\xi, \xi) \, d^{n-1}x'.
\end{align*}
\]
Substituting the values

\[ \frac{\partial e^{-i(y,x'-(\xi,x')\xi)}}{\partial \xi^i} = ie^{-i(y,x'-(\xi,x')\xi)}((\xi,y)x_i' + (\xi,x')y_i) \]

and

\[ \frac{\partial \psi(x'-(\xi,x')\xi)}{\partial \xi^i} = \frac{\partial \psi}{\partial \xi^i}(x'-(\xi,x')\xi) - (\xi,x')\frac{\partial \psi}{\partial x^i}(x'-(\xi,x')\xi,\xi) \]

we obtain

\[ \frac{\partial \hat{\varphi}}{\partial \xi^i}(y,\xi) = -\frac{\xi_0,i}{(\xi_0,\xi)^2} (2\pi)^{(1-n)/2} \int_{\xi^i_0} e^{-i(y,x'-(\xi,x')\xi)} \psi(x'-(\xi,x')\xi,\xi) d^{n-1}x' \]

\[ + \frac{i}{(\xi_0,\xi)} (2\pi)^{(1-n)/2} \int_{\xi^i_0} e^{-i(y,x'-(\xi,x')\xi)}((\xi,y)x_i' + (\xi,x')y_i) \psi(x'-(\xi,x')\xi,\xi) d^{n-1}x' \]

\[ + \frac{1}{(\xi_0,\xi)} (2\pi)^{(1-n)/2} \int_{\xi^i_0} e^{-i(y,x'-(\xi,x')\xi)}(\xi,x') \frac{\partial \psi}{\partial x^i}(x'-(\xi,x')\xi,\xi) d^{n-1}x' \]

\[ - \frac{1}{(\xi_0,\xi)} (2\pi)^{(1-n)/2} \int_{\xi^i_0} e^{-i(y,x'-(\xi,x')\xi)}x_i' \xi^p \frac{\partial \psi}{\partial x^p}(x'-(\xi,x')\xi,\xi) d^{n-1}x'. \]

On assuming \((y,\xi_0) \in T\mathbb{S}^{n-1}\), we set \(\xi = \xi_0\) in the latter formula. The formula simplifies to the following one:

\[ \frac{\partial \hat{\varphi}}{\partial \xi^i}(y,\xi) = -\xi_0,i(2\pi)^{(1-n)/2} \int_{\xi^i_0} e^{-i(y,x') \psi(x',\xi_0) d^{n-1}x'} \]

\[ + (2\pi)^{(1-n)/2} \int_{\xi^i_0} e^{-i(y,x')} \frac{\partial \psi}{\partial \xi^i}(x',\xi_0) d^{n-1}x' \]

\[ - (2\pi)^{(1-n)/2} \int_{\xi^i_0} e^{-i(y,x')} x_i' \xi^p \frac{\partial \psi}{\partial x^p}(x',\xi_0) d^{n-1}x'. \]

By (4.23), the integrand of the last integral is identically equal to zero. Thus, replacing the notations \(\xi_0\) and \(x'\) with \(\xi\) and \(x\) respectively, we obtain

\[ \frac{\partial \hat{\varphi}}{\partial \xi^i}(y,\xi) = (2\pi)^{(1-n)/2} \int_{\xi^i} e^{-i(y,x)} \frac{\partial \psi}{\partial \xi^i}(x,\xi) d^{n-1}x = -\xi_i \hat{\varphi}(y,\xi) \quad \text{for } (y,\xi) \in T\mathbb{S}^{n-1}. \]

With the help of (4.23), we obtain from (4.29)

\[ \xi^p \frac{\partial \hat{\varphi}}{\partial \xi^p}(y,\xi) = (2\pi)^{(1-n)/2} \int_{\xi^i} e^{-i(y,x)} \xi^p \frac{\partial \psi}{\partial \xi^p}(x,\xi) d^{n-1}x - \hat{\varphi}(y,\xi) = -\hat{\varphi}(y,\xi) \quad \text{for } (y,\xi) \in T\mathbb{S}^{n-1}. \]

(4.30)
Substitute values (4.29) and (4.30) into (4.27) to obtain

\[
(\hat{\Xi}_i \hat{\phi})(y, \xi) = (2\pi)^{(1-n)/2} \int_{\xi} e^{-i\langle y, x \rangle} \frac{\partial \psi}{\partial \xi_i} (x, \xi) d^{n-1}x \quad \text{for} \quad (y, \xi) \in TS^{n-1}.
\] (4.31)

By (4.23),

\[
(\hat{\Xi}_i \psi)(x, \xi) = \frac{\partial \psi}{\partial \xi_i} (x, \xi) \quad \text{for} \quad (x, \xi) \in TS^{n-1}.
\]

Therefore formula (4.31) can be written as

\[
(\hat{\Xi}_i \hat{\phi})(y, \xi) = (2\pi)^{(1-n)/2} \int_{\xi} e^{-i\langle y, x \rangle} (\hat{\Xi}_i \psi)(x, \xi) d^{n-1}x = \hat{\Xi}_i \phi(y, \xi) \quad \text{for} \quad (y, \xi) \in TS^{n-1}.
\]

This proves the second of formulas (4.22).

By the definition of \(X_i\),

\[
X_i \hat{\phi} = \frac{\partial \hat{\phi}}{\partial y^i} - \xi_i \xi_p \frac{\partial \hat{\phi}}{\partial y^p}.
\]

In virtue of (4.26), this is simplified to

\[
X_i \hat{\phi} = \frac{\partial \hat{\phi}}{\partial y^i}.
\]

With the help of (4.25), this gives

\[
X_i \hat{\phi} = -i(2\pi)^{(1-n)/2} \int_{\xi} e^{-i\langle y, x \rangle} x_i \varphi(x, \xi) d^{n-1}x = -i \hat{x}_i \hat{\phi}.
\]

This proves the last of the formulas in (4.22). \(\square\)

4.2. Reshetnyak formula for vector fields.

**Theorem 4.5.** Given a vector field \(f \in S(\mathbb{R}^n; \mathbb{C}^n)\), the equality

\[
\|f\|_{H^s(\mathbb{R}^n; \mathbb{C}^n)}^2 = a_n \left[ \sum_{i=1}^{n} \|\Xi_i (I^0 f)\|_{H^{s+1/2}(TS^{n-1})}^2 + \|I^0 f\|_{H^{s+1/2}(TS^{n-1})}^2 \right] + \|I^1 f\|_{H^{s+1/2}(TS^{n-1})}^2 + 2\Re (i Z(I^0 f), I^1 f)_{H^{s+1/2}(TS^{n-1})}
\]

(4.32)

holds for every \(s \in \mathbb{R}\) and every \(t > -n/2\), where the constant \(a_n\) is defined by (2.13) and \(Z\) is the first order pseudodifferential operator

\[
Z : S(TS^{n-1}) \to C^\infty(TS^{n-1})
\]

defined by

\[
\hat{Z} \varphi(y, \xi) = \frac{y^i}{|y|} (\Xi_i \hat{\varphi})(y, \xi).
\]

(4.33)

**Proof.** Applying the Fourier transform to formula (4.13), we get

\[
\hat{I^i f} = -X_i(\hat{I^0 f}) + \Xi_i(\hat{I^0 f}) + \xi_i \hat{I^0 f}.
\]

With the help of Lemma 4.4, this gives

\[
\hat{I^i f} = -iy_i \hat{I^1 f} + \Xi_i \hat{I^0 f} + \xi_i \hat{I^0 f}.
\]

(4.34)
For a vector field \( f = (f_i) \),
\[
\|f\|_{H^s_t(\mathbb{R}^n;\mathbb{C}^n)}^2 = \sum_{i=1}^n \|f_i\|_{H^s_t(\mathbb{R}^n)}^2.
\]

Applying the Reshetnyak formula (2.12) for scalar functions, we obtain
\[
\|f\|_{H^s_t(\mathbb{R}^n;\mathbb{C}^n)}^2 = a_n \sum_{i=1}^n \|\hat{f}_i\|_{H^{s+1/2}_{t+1/2}(T^{S^n-1})}^2.
\]

On using the definition (2.11) of the \( H^{s+1/2}_{t+1/2}(T^{S^n-1}) \)-norm, this is written as
\[
\|f\|_{H^s_t(\mathbb{R}^n;\mathbb{C}^n)}^2 = \frac{a_n}{2\pi} \int_{S^{n-1}} \int \xi \int \xi H^{s+1/2}_{t+1/2}(T^{S^n-1}) \sum_{i=1}^n \int_{S^{n-1}} |\hat{f}_i(y, \xi)|^2 d^n x d\xi.
\]

Now we calculate the integrand on (4.35). By (4.34),
\[
\sum_{i=1}^n |\hat{f}_i|^2 = \sum_{i=1}^n (\xi_i \hat{\rho}_f + \xi_t \hat{\rho}_f - i y_i \hat{\tau}_f) \left( \hat{\xi}_i \hat{\rho}_f + \xi_t \hat{\rho}_f + i y_t \hat{\tau}_f \right).
\]

After opening parentheses and using the equalities \(|\xi|^2 = 1, (\xi, y) = 0\), this gives
\[
\sum_{i=1}^n |\hat{f}_i|^2 = \sum_{i=1}^n (\xi_i \hat{\rho}_f^2 + \hat{\rho}_f^2 + |y|^2 \hat{\tau}_f^2 + 2(\hat{\xi} \hat{\rho}_f \cdot \hat{\tau}_f) + 2(\hat{\xi} \hat{\rho}_f \cdot \hat{\tau}_f)).
\]

By (4.7), \( \hat{\xi} \hat{\rho}_f = 0 \), and the formula takes the form
\[
\sum_{i=1}^n |\hat{f}_i|^2 = \sum_{i=1}^n (\xi_i \hat{\rho}_f^2 + \hat{\rho}_f^2 + |y|^2 \hat{\tau}_f^2 + 2(\hat{\xi} \hat{\rho}_f \cdot \hat{\tau}_f)).
\]

On using the operator (4.33), this can be written as
\[
\sum_{i=1}^n |\hat{f}_i|^2 = \sum_{i=1}^n (\xi_i \hat{\rho}_f^2 + \hat{\rho}_f^2 + |y|^2 \hat{\tau}_f^2 + 2 |y| \Re (\hat{\xi} \hat{\rho}_f \cdot \hat{\tau}_f)).
\]

Substituting the value (4.36) for \( \sum_{i=1}^n |\hat{f}_i|^2 \) into (4.35), we get,
\[
\sum_{i=1}^n \int_{S^{n-1}} \int \xi \int \xi H^{s+1/2}_{t+1/2}(T^{S^n-1}) \sum_{i=1}^n \int_{S^{n-1}} |\hat{f}_i(y, \xi)|^2 d^n x d\xi
\]
\[
+ \frac{1}{2\pi} \int_{S^{n-1}} \int \xi \int \xi H^{s+1/2}_{t+1/2}(T^{S^n-1}) \sum_{i=1}^n \int_{S^{n-1}} |\hat{f}_i(y, \xi)|^2 d^n x d\xi
\]
\[
+ \frac{1}{2\pi} \int_{S^{n-1}} \int \xi \int \xi H^{s+1/2}_{t+1/2}(T^{S^n-1}) \sum_{i=1}^n \int_{S^{n-1}} |\hat{f}_i(y, \xi)|^2 d^n x d\xi
\]
\[
+ 2\Re \left( \int_{S^{n-1}} \int \xi \int \xi H^{s+1/2}_{t+1/2}(T^{S^n-1}) \sum_{i=1}^n \int_{S^{n-1}} |\hat{f}_i(y, \xi)|^2 \hat{\xi} \hat{\rho}_f(y, \xi) \hat{\tau}_f(y, \xi) d^n x d\xi \right).
\]
On using the statement \( \Xi_i \mathcal{T} f = \mathcal{T} ( \mathcal{I}_0 f ) \) of Lemma 4.4, we rewrite the latter formula as

\[
\| f \|_{\mathcal{H}_t^r (\mathbb{R}^n; \mathbb{C}^n)} = \alpha_n \left[ \sum_{i=1}^n \frac{1}{2\pi} \int_{\mathbb{S}^{n-1}} |y|^{2t+1}(1 + |y|^2)^{s-t}|\Xi_i (\mathcal{I}_0 f)(y, \xi)|^2 \, d\xi \right. \\
\left. \quad + \frac{1}{2\pi} \int_{\mathbb{S}^{n-1}} |y|^{2t+1}(1 + |y|^2)^{s-t}|\mathcal{T} f(y, \xi)|^2 \, d\xi \right. \\
\quad + \frac{1}{2\pi} \int_{\mathbb{S}^{n-1}} |y|^{2t+3}(1 + |y|^2)^{s-t}|\mathcal{T} f(y, \xi)|^2 \, d\xi \\
\quad + 2\Re \left( \frac{1}{2\pi} \int_{\mathbb{S}^{n-1}} |y|^{2t+2}(1 + |y|^2)^{s-t}|\Xi_i (\mathcal{T} f)(y, \xi)|^2 \, d\xi \right] \\
\]

where the constant \( \alpha_n \) is defined by (2.13). In view of (2.11), this is equivalent to (4.32).

Formula (4.32) suggests the idea of introducing the norm

\[
\| (\varphi, \psi) \|_{\mathcal{H}_t^{r+1} (\mathbb{S}^{n-1})} = \alpha_n \left[ \sum_{i=1}^n \| \Xi_i \varphi \|_{\mathcal{H}_t^{r+1/2} (\mathbb{S}^{n-1})}^2 + \| \varphi \|_{\mathcal{H}_t^{r+1/2} (\mathbb{S}^{n-1})}^2 \\
\quad + \| \psi \|_{\mathcal{H}_t^{r+3/2} (\mathbb{S}^{n-1})}^2 + 2\Re \left( i \mathcal{Z} \varphi, \psi \right)_{\mathcal{H}_t^{r+1} (\mathbb{S}^{n-1})} \right] \\
\tag{4.37}
\]

on the space \( \mathcal{S}(\mathbb{S}^{n-1}) \oplus \mathcal{S}(\mathbb{S}^{n-1}) \), where the constant \( \alpha_n \) is defined by (2.13). Let us show that the right-hand side of (4.37) is positive for \( (\varphi, \psi) \neq (0, 0) \). Indeed, the inequality

\[
\left| (\varphi, \psi)_{\mathcal{H}_t^{r+1} (\mathbb{S}^{n-1})} \right| \leq \| \varphi \|_{\mathcal{H}_t^{r+1/2} (\mathbb{S}^{n-1})} \| \psi \|_{\mathcal{H}_t^{r+3/2} (\mathbb{S}^{n-1})} \\
\tag{4.38}
\]
easily follows from the definition of \( \mathcal{H}_t^r \)-norms with the help of the Schwartz inequality. In particular,

\[
\left| (i \mathcal{Z} \varphi, \psi)_{\mathcal{H}_t^{r+1} (\mathbb{S}^{n-1})} \right| \leq \| \mathcal{Z} \varphi \|_{\mathcal{H}_t^{r+1/2} (\mathbb{S}^{n-1})} \| \psi \|_{\mathcal{H}_t^{r+3/2} (\mathbb{S}^{n-1})} \\
\leq \frac{1}{2} \| \mathcal{Z} \varphi \|_{\mathcal{H}_t^{r+1/2} (\mathbb{S}^{n-1})}^2 + \frac{1}{2} \| \psi \|_{\mathcal{H}_t^{r+3/2} (\mathbb{S}^{n-1})}^2.
\]
The inequality

\[
\| \mathcal{Z} \varphi \|_{\mathcal{H}_t^{r+1/2} (\mathbb{S}^{n-1})}^2 \leq \sum_{i=1}^n \| \Xi_i \varphi \|_{\mathcal{H}_t^{r+1/2} (\mathbb{S}^{n-1})}^2 \\
\]
easily follows from definition (4.33) of the operator \( \mathcal{Z} \). Together with the previous inequality, it gives

\[
2 \left| (i \mathcal{Z} \varphi, \psi)_{\mathcal{H}_t^{r+1} (\mathbb{S}^{n-1})} \right| \leq \sum_{i=1}^n \| \Xi_i \varphi \|_{\mathcal{H}_t^{r+1/2} (\mathbb{S}^{n-1})}^2 + \| \psi \|_{\mathcal{H}_t^{r+3/2} (\mathbb{S}^{n-1})}^2. \\
\tag{4.39}
\]

On using the last inequality, we derive from (4.37)

\[
\| (\varphi, \psi) \|_{\mathcal{H}_t^{r+1} (\mathbb{S}^{n-1})} \geq \alpha_n \| \varphi \|_{\mathcal{H}_t^{r+1/2} (\mathbb{S}^{n-1})}.
\]
The right-hand side of this inequality is non-negative and equals zero only if \( \varphi = 0 \). In the latter case the right-hand side of (4.37) is non-negative and equals zero only if \( \psi = 0 \).
We define the Hilbert space $\mathcal{H}^{1,s}_{t}(T\Sigma^{n-1})$ as the completion of $\mathcal{S}(T\Sigma^{n-1}) \oplus \mathcal{S}(T\Sigma^{n-1})$ with respect to the norm (4.37).

**Corollary 4.6.** The map
\[ \mathcal{S}(\mathbb{R}^{n};\mathbb{C}^{n}) \to \mathcal{S}(T\Sigma^{n-1}) \oplus \mathcal{S}(T\Sigma^{n-1}), \quad f \mapsto (I^{0}f, I^{1}f) \]
extends to the linear isometric embedding
\[ H^{s}_{t}(\mathbb{R}^{n};\mathbb{C}^{n}) \to \mathcal{H}^{1,s}_{t}(T\Sigma^{n-1}) \]
(4.40)
for every $s \in \mathbb{R}$ and every $t > -n/2$. In other words, the equality
\[ \|f\|_{H^{s}_{t}(\mathbb{R}^{n};\mathbb{C}^{n})} = \|(I^{0}f, I^{1}f)\|_{\mathcal{H}^{1,s}_{t}(T\Sigma^{n-1})} \]
(4.41)
holds for all $f \in H^{s}_{t}(\mathbb{R}^{n};\mathbb{C}^{n})$.

The embedding (4.40) is not surjective because the function $(I^{0}f)(x,\xi)$ is odd in the second argument and the function $(I^{1}f)(x,\xi)$ is even in the second argument as is seen from (2.3). The interesting open question is about a description of the range of the operator (4.40).

Formula (4.41) gives the best stability estimate for the problem of recovering a vector field $f$ from the data $(I^{0}f, I^{1}f)$, although a rather exotic norm is involved. The stability estimate involving more traditional norms can be also easily derived from the Reshetnyak formula.

**Corollary 4.7.** For every $s \in \mathbb{R}$ and for every $t > -n/2$, the stability estimate
\[ \|f\|_{H^{s}_{t}(\mathbb{R}^{n};\mathbb{C}^{n})}^{2} \leq 2a_{n} \left( \sum_{i=1}^{n} \|\Xi_{i}(I^{0}f)\|_{H^{s+1/2}_{t}}^{2} + \|I^{0}f\|_{H^{s+1/2}_{t}}^{2} + \|I^{1}f\|_{H^{s+3/2}_{t}}^{2} \right) \]
(4.42)
holds for all $f \in H^{s}_{t}(\mathbb{R}^{n};\mathbb{C}^{n})$.

The abbreviated notation $\| \cdot \|_{H^{s}_{t}}$ for the norm $\| \cdot \|_{\mathcal{H}^{1,s}_{t}(T\Sigma^{n-1})}$ is used on the right-hand side of (4.42). Inequality (4.42) follows from (4.32) with the help of (4.39).

### 4.3. Reshetnyak formula for second rank symmetric tensor fields

Along with the operator $Z$ defined by (4.33), we need two similar operators
\[ Q, W : \mathcal{S}(T\Sigma^{n-1}) \to C^{\infty}(T\Sigma^{n-1}) \]
that are defined by the formulas
\[ (y'y'\Xi,\Xi_{j}\hat{\phi})(y,\xi) = |y|^{2}Q\hat{\phi}(y,\xi) \]
and
\[ (y'X_{i}\hat{\phi})(y,\xi) = |y|W\phi(y,\xi) \]
respectively. We will sometimes write $\Xi^{i}$ instead of $\Xi_{i}$ in order to adopt our formulas to the Einstein summation rule.
Theorem 4.8. Given a tensor field \( f \in \mathcal{S}(\mathbb{R}^n; S^2 \mathbb{R}^n) \), the equality
\[
\|f\|_{H^t(\mathbb{R}^n; S^2 \mathbb{R}^n)}^2 = \frac{a_n}{4} \left[ \|\mathcal{I}\mathcal{F}f\|_{H^{t+5/2}}^2 + 4\Im(\mathcal{I}\mathcal{F}f, Z(\mathcal{I}f))_{H^{t+2}} + 2 \sum_{i} \|\Xi_i(\mathcal{I}f)\|_{H^{t+3/2}}^2 \right.
\]
\[
+ 2\|\|\|\mathcal{I}\mathcal{F}f\|_{H^{t+3/2}}^2 + 2\|Z(\mathcal{I}f)\|_{H^{t+3/2}}^2 + 2\Re(\mathcal{I}\mathcal{F}f, W(\mathcal{I}f))_{H^{t+2}}
\]
\[
- 4\Re(\mathcal{I}\mathcal{F}f, \mathcal{I}f)_{H^{t+3/2}} - 2\Re(\mathcal{I}\mathcal{F}f, Q(\mathcal{I}f))_{H^{t+3/2}}
\]
\[
- 4\Re(\Xi'(\mathcal{I}f), x_i\mathcal{I}f)_{H^{t+3/2}} + 8\Im(Z(\mathcal{I}f), \mathcal{I}f)_{H^{t+1}}
\]
\[
+ 4\Im(\mathcal{I}f, Z(\mathcal{I}f))_{H^{t+1}} + 4\Im(\Xi'(\mathcal{I}f), Z\Xi_i(\mathcal{I}f))_{H^{t+1}}
\]
\[
+ \sum_{i} \|x_i\mathcal{I}f\|_{H^{t+3/2}}^2 - 2\Im(x^i\mathcal{I}f, Z\Xi_i(\mathcal{I}f))_{H^{t+1}}
\]
\[
- 4\Re(W(\mathcal{I}f), \mathcal{I}f)_{H^{t+1}} + \sum_{i,j} \|\Xi_i\Xi_j(\mathcal{I}f)\|_{H^{t+1/2}}^2
\]
\[
+ 4\Re(\Xi'\Xi_i(\mathcal{I}f), \mathcal{I}f)_{H^{t+1/2}} + \sum_{i} \|\Xi_i(\mathcal{I}f)\|_{H^{t+1/2}}^2 + 4n\|\mathcal{I}f\|_{H^{t+1/2}}^2 \right]
\]
holds for every \( s \in \mathbb{R} \) and every \( t > -n/2 \), where the constant \( a_n \) is defined by (2.13).

As before, the abbreviated notation \( \| \cdot \|_{H^t(\mathbb{R}^n; S^2 \mathbb{R}^n)} \) for the norm \( \| \cdot \|_{H^t(\mathbb{T}^n; S^{n-1})} \) is used on the right-hand side of (4.43).

Sketch of the proof. The proof follows the same line as the proof of Theorem 4.5 although all calculations are more cumbersome. We will present key formulas only.

The Reshetnyak formula (2.12) for scalar functions implies
\[
\|f\|_{H^t(\mathbb{R}^n; S^2 \mathbb{R}^n)}^2 = \frac{a_n}{2\pi} \sum_{i,j} \int \int_{\mathbb{T}^n} \left| y^{2n+1}(1 + |y|^2)^{s-t}|\mathcal{I}\mathcal{F}_{ij}(y, \xi)|^2 \right| d^{n-1}y d\xi. \tag{4.44}
\]

Applying the Fourier transform to equation (4.44) and using Lemma 4.4, we obtain the following analog of formula (4.34):
\[
\mathcal{I}\mathcal{F}_{ij} = \frac{1}{2} \sigma(ij) \left( -y_i y_j \mathcal{I}\mathcal{F}f - 2i y_i \Xi_j \mathcal{I}\mathcal{F}f + 2i \xi_i \mathcal{I}\mathcal{F}f + \Xi_i \Xi_j \mathcal{I}\mathcal{F}f - y_i X_j \mathcal{I}\mathcal{F}f + 3\xi_i \Xi_j \mathcal{I}\mathcal{F}f + 2\delta_{ij} \mathcal{I}\mathcal{F}f \right).
\]

This implies
\[
|\sum_{i,j} \mathcal{F}_{ij}(y, \xi)|^2 = \frac{1}{4} \sum_{i,j=1}^n \left| \sigma(ij) \left( -y_i y_j \mathcal{I}\mathcal{F}f(y, \xi) - 2i y_i \Xi_j \mathcal{I}\mathcal{F}f(y, \xi) - 2i \xi_i \mathcal{I}\mathcal{F}f(y, \xi)
\right.
\]
\[
\left. + (\Xi_i \Xi_j \mathcal{I}\mathcal{F}f)(y, \xi) - y_i (X_j \mathcal{I}\mathcal{F}f)(y, \xi) + 3\xi_i (\Xi_j \mathcal{I}\mathcal{F}f)(y, \xi) + 2\delta_{ij} \mathcal{I}\mathcal{F}f(y, \xi) \right) \right|^2.
\]
Substituting this expression into (4.44), we get,
\[
\|f\|_{H^1_\mathbb{T}(\mathbb{R}^n;\mathbb{S}^2\mathbb{R}^n)}^2 = \frac{a_n}{4} \frac{1}{2\pi} \int_{\mathbb{S}^{n-1}} \int_{\xi} |y|^{2r+1}(1 + |y|)^{s-t} \\
\times \sum_{i,j=1}^n \left| \sigma(ij) \left( -y_i y_j \hat{\nabla}_I f(y, \xi) - 2i y_i (\Xi_j \hat{\nabla}_I f)(y, \xi) - 2i \xi_i y_j \hat{\nabla}_f(y, \xi) \\
+ (\Xi_i \Xi_j \hat{\nabla}_f)(y, \xi) - y_i (X_j \hat{\nabla}_f)(y, \xi) + 3\xi_i (\Xi_j \hat{\nabla}_f)(y, \xi) + 2\delta_{ij} \hat{\nabla}_f(y, \xi) \right) \right|^2 \, d^{n-1}y d\xi.
\]
(4.45)

By pure algebraic calculations, on using the identities $|\xi|^2 = 1$ and $\langle \xi, y \rangle = 0$, combined with Lemma 4.1, we obtain the following analog of (4.36):
\[
\sum_{i,j=1}^n \left| \sigma(ij) \left( -y_i y_j \hat{\nabla}_f(y, \xi) - 2i y_i (\Xi_j \hat{\nabla}_f)(y, \xi) - 2i \xi_i y_j \hat{\nabla}_f(y, \xi) \\
+ (\Xi_i \Xi_j \hat{\nabla}_f)(y, \xi) - y_i (X_j \hat{\nabla}_f)(y, \xi) + 3\xi_i (\Xi_j \hat{\nabla}_f)(y, \xi) + 2\delta_{ij} \hat{\nabla}_f(y, \xi) \right) \right|^2 \\
= |y|^4 |\hat{\nabla}_f|^2 + 2|y|^2 \sum_i |\Xi_i \hat{\nabla}_f|^2 + 2|y|^2 |\hat{\nabla}_f|^2 + 2 |y|^4 \Xi_i \hat{\nabla}_f|^2 \\
+ |y|^2 \sum_i (\Xi_i \hat{\nabla}_f)(y, \xi) |X_i \hat{\nabla}_f(y, \xi) + |y|^2 \sum_i (\Xi_i \hat{\nabla}_f)(y, \xi) |X_i \hat{\nabla}_f(y, \xi) \\
+ 4|y|^2 \Im(\hat{\nabla}_f \cdot y^* \Xi \hat{\nabla}_f) - 2 \Re(\hat{\nabla}_f \cdot y^* \Xi \hat{\nabla}_f) \\
+ 2 |y|^2 \Re(\hat{\nabla}_f \cdot y^* \hat{\nabla}_f) - 4|y|^2 \Re(\hat{\nabla}_f \cdot \hat{\nabla}_f) + 4\Im(\Xi \hat{\nabla}_f \cdot y^* \hat{\nabla}_f) - 4|y|^2 \Im(\Xi \hat{\nabla}_f \cdot \hat{\nabla}_f) \\
+ 8 \Im(y^* \Xi \hat{\nabla}_f \cdot \hat{\nabla}_f) + 4 \Im(\hat{\nabla}_f \cdot y^* \hat{\nabla}_f) \\
- 2 \Re(X^* \hat{\nabla}_f \cdot y^* \Xi \hat{\nabla}_f) + 4 \Re(X^* \Xi \hat{\nabla}_f \cdot \hat{\nabla}_f) - 4 \Re(y^* \hat{\nabla}_f \cdot \hat{\nabla}_f).
\]

Substitute this value into (4.45)
\[
\frac{4}{a_n} \|f\|_{H^1_\mathbb{T}(\mathbb{R}^n;\mathbb{S}^2\mathbb{R}^n)}^2 = \frac{1}{2\pi} \int_{\mathbb{S}^{n-1}} \int_{\xi} |y|^{2r+1}(1 + |y|)^{s-t} \left[ |y|^4 |\hat{\nabla}_f|^2(y, \xi) + 2|y|^2 |\hat{\nabla}_f|^2(y, \xi) \right] \\
+ 4n |\hat{\nabla}_f|^2(y, \xi)^2 - 4|y|^2 \Re(\hat{\nabla}_f \cdot \hat{\nabla}_f)(y, \xi) \\
+ 2|y|^2 \sum_i |(\Xi_i \hat{\nabla}_f)(y, \xi)|^2 + \sum_i |(\Xi_i \hat{\nabla}_f)(y, \xi)|^2 \\
+ \sum_{i,j} |(\Xi_i \Xi_j \hat{\nabla}_f)(y, \xi)|^2 + 2 |(y^* \Xi \hat{\nabla}_f)(y, \xi)|^2 + |y|^2 \sum_i |(X_i \hat{\nabla}_f)(y, \xi)|^2 \\
+ 4|y|^2 \Im(\hat{\nabla}_f(y, \xi) \cdot y^* \Xi \hat{\nabla}_f(y, \xi)) - 2 \Re(\hat{\nabla}_f(y, \xi) \cdot y^* \Xi \hat{\nabla}_f(y, \xi)) \\
+ 2 |y|^2 \Re(\hat{\nabla}_f(y, \xi) \cdot y^* \hat{\nabla}_f(y, \xi)) + 4 \Im(\Xi \hat{\nabla}_f(y, \xi) \cdot y^* \Xi \hat{\nabla}_f(y, \xi)) \\
- 4|y|^2 \Im(\Xi \hat{\nabla}_f(y, \xi) \cdot X_i \hat{\nabla}_f(y, \xi)) + 8 \Im(y^* \Xi \hat{\nabla}_f(y, \xi) \cdot \hat{\nabla}_f(y, \xi)) \\
+ 4 \Im(\hat{\nabla}_f(y, \xi) \cdot y^* \hat{\nabla}_f(y, \xi)) - 2 \Re(\hat{\nabla}_f(y, \xi) \cdot y^* \Xi \hat{\nabla}_f(y, \xi)) \\
+ 4 \Re(\Xi \Xi \hat{\nabla}_f(y, \xi) \cdot \hat{\nabla}_f(y, \xi)) - 4 \Re(y^* \hat{\nabla}_f(y, \xi) \cdot \hat{\nabla}_f(y, \xi)) \\
\right] \, d^{n-1}y d\xi.
\]
(4.46)
There are 19 terms in the brackets on the right-hand side of (4.46) which are in one-to-one correspondence with terms on the right-hand side of (4.43). On using Lemma 4.4 and operators $Z, Q, W$, one checks that the integral of each term in the brackets on the right-hand side of (4.46) (with the weight written before the brackets) is equal to the corresponding term on the right-hand side of (4.43). □

Repeating arguments of Section 4.2, we introduce the following norm on the space $S(TS^{n-1}) \oplus S(TS^{n-1}) \oplus S(TS^{n-1})$:

$$
\|[(\varphi, \psi, \chi)]\|_{H^{2,s}(T\mathbb{S}^{n-1})}^2 = \frac{a_n}{4} \left[ \|\chi\|^2_{H^{r+3/2}_t} + 4 \Re(\chi, Z\psi)_{H^{r+3/2}_t} + 2 \| \Xi_i \psi \|^2_{H^{r+3/2}_t} 
+ 2 \|\psi\|^2_{H^{r+3/2}_t} + 2 \| Z\psi \|^2_{H^{r+3/2}_t} + 2 \Re(\chi, W\varphi)_{H^{r+3/2}_t} 
- 4 \Re(\chi, \varphi)_{H^{r+3/2}_t} - 2 \Re(\chi, Q\varphi)_{H^{r+3/2}_t} 
- 4 \Re(\Xi_i \psi, x_i \varphi)_{H^{r+3/2}_t} + 8 \Re(Z\psi, \varphi)_{H^{r+1}_t} 
+ 4 \Re(\psi, Z\varphi)_{H^{r+1}_t} + 4 \Re(\Xi_i \psi, Z\Xi_i \varphi)_{H^{r+1}_t} 
+ \sum_i \| x_i \varphi \|^2_{H^{r+3/2}_t} - 2 \Re(x^i \varphi, Z\Xi_i \varphi)_{H^{r+1}_t} 
- 4 \Re(W\varphi, \varphi)_{H^{r+1}_t} + \sum_{i,j} \| \Xi_i \Xi_j \varphi \|^2_{H^{r+3/2}_t} 
+ 4 \Re(\Xi_i \Xi_j \varphi, \varphi)_{H^{r+3/2}_t} + \sum_{i,j} \| \Xi_i \varphi \|^2_{H^{r+1}_t} + 4n \| \varphi \|^2_{H^{r+1}_t} \right].
$$

(4.47)

**Lemma 4.9.** The right-hand side of (4.47) is positive unless $(\varphi, \psi, \chi) = (0, 0, 0)$.

This statement was actually proved before. Indeed, the integral

$$
A[\varphi, \psi, \chi] = \frac{a_n}{4} \frac{1}{2\pi} \int \int_{\mathbb{S}^{n-1}} |y|^{2r+1}(1 + |y|^2)^{s-t} \sum_{i,j=1}^n \sigma(ij) \left( - y_i y_j \tilde{\chi}(y, \xi) - 2i y_i (\Xi_j \tilde{\psi})(y, \xi) 
- 2i \xi_i y_j \tilde{\psi}(y, \xi) + (\Xi_i \Xi_j \tilde{\varphi})(y, \xi) - y_i (X_j \tilde{\varphi})(y, \xi) + 3 \xi_i (\Xi_j \tilde{\varphi})(y, \xi) + 2 \delta_{ij} \tilde{\varphi}(y, \xi) \right)^2 d^{n-1}y d\xi
$$

(4.48)

is positive unless $\varphi = \psi = \chi = 0$. The right-hand side of (4.45) is equal to $A[I^0 f, I^1 f, I^2 f]$. After formula (4.45), the proof of Theorem 4.8 consists of some transformations of the right-hand side of (4.45). No specific property of the functions $(I^0 f, I^1 f, I^2 f)$ is used in the transformations, i.e., the right-hand side of (4.48) can be transformed in the same manner. In this way, we demonstrate that $A[\varphi, \psi, \chi]$ is equal to the right-hand side of (4.47).

We define the Hilbert space $H^{2,s}_t(T\mathbb{S}^{n-1})$ as the completion of $S(TS^{n-1}) \oplus S(TS^{n-1}) \oplus S(TS^{n-1})$ with respect to the norm (4.47).

**Corollary 4.10.** The map

$$
S(\mathbb{R}^n, S^2 \mathbb{R}^n) \to S(TS^{n-1}) \oplus S(TS^{n-1}) \oplus S(TS^{n-1}), \quad f \mapsto (I^0 f, I^1 f, I^2 f)
$$

extends to the linear isometric embedding

$$
H^{2,s}_t(\mathbb{R}^n; \mathbb{C}^n) \to H^{2,s}_t(T\mathbb{S}^{n-1})
$$
for every \(s \in \mathbb{R}\) and every \(t > -n/2\). In other words, the equality
\[
\|f\|_{H_t^s(\mathbb{R}^n; S^2 \mathbb{R}^n)} = \|(I^0 f, I^1 f, I^2 f)\|_{H_t^{2,s}(T^2 \mathbb{S}^{n-1})}
\]
holds for all \(f \in H_t^s(\mathbb{R}^n; S^2 \mathbb{R}^n)\).

Estimating scalar products on the right-hand side of (4.43) with the help of (4.38), we also obtain

**Corollary 4.11.** For every \(s \in \mathbb{R}\) and every \(t > -n/2\), the stability estimate
\[
\|f\|_{H_t^s(\mathbb{R}^n; S^2 \mathbb{R}^n)} \leq b_n \left[ \|I^2 f\|_{H_t^{2,s}+n/2} + \sum_i \|\Xi_i(I^1 f)\|_{H_t^{2,s}+3/2} + \|Z(I^1 f)\|_{H_t^{2,s}+3/2} \right]
\]
holds for all \(f \in H_t^s(\mathbb{R}^n; S^2 \mathbb{R}^n)\) with some constant \(b_n\) depending on \(n\) only.

In principle, our approach works for every \(m\), i.e., the Reshetnyak formula can be obtained which expresses the norm \(\|f\|_{H_t^s(\mathbb{R}^n; S^m \mathbb{R}^n)}\) through some norm of \((I^0 f, \ldots, I^m f)\) for a rank \(m\) symmetric tensor field \(f\). But the length of the formula grows fast with \(m\). For example, the Reshetnyak formula contains more than 250 terms in the case of \(m = 3\). Unfortunately, we are not able to bring the formula into a compact form.

5. **The two-dimensional case.** Due to the existence of natural coordinates on \(T^1 \mathbb{S}^1 = \mathbb{S}^1 \times \mathbb{R}\), all our formulas are simplified in the 2D-case. In particular, we do not need to use the operators \(X_1, \Xi_1\) as well as the operators \(Z, Q, W\) participating in (4.43). Actually our work on the subject was started with considering the 2D-case.

The coordinates \((p, \theta)\) on the two-dimensional manifold \(T^1 \mathbb{S}^1\) are defined by
\[
x = p(-\sin \theta, \cos \theta), \quad \xi = (\cos \theta, \sin \theta) \quad \text{for} \quad (x, \xi) \in T^1 \mathbb{S}^1.
\]
The operators \(X_1\) and \(\Xi_1\) are easily expressed through \(\frac{\partial}{\partial p}\) and \(\frac{\partial}{\partial \theta}\)
\[
X_1 = -\sin \theta \frac{\partial}{\partial p}, \quad X_2 = \cos \theta \frac{\partial}{\partial p}, \quad \Xi_1 = -\sin \theta \frac{\partial}{\partial \theta}, \quad \Xi_2 = \cos \theta \frac{\partial}{\partial \theta}.
\]
In particular, the first term on the right-hand side of (4.37) can be written as
\[
\sum_{i=1}^{2} \|\Xi_i(I^0 f)\|_{H_t^{2,s+1/2}(T^1 \mathbb{S}^1)} = \left\| \frac{\partial(I^0 f)}{\partial \theta} \right\|_{H_t^{2,s+1/2}(T^1 \mathbb{S}^1)}.
\]
The Fourier transform \(S(T^1 \mathbb{S}^1) \rightarrow S(T^1 \mathbb{S}^1), \varphi \mapsto \hat{\varphi}\) is just the one-dimensional Fourier transform in the variable \(p\) with the Fourier dual variable \(q\), where \(\theta\) stands as a parameter
\[
\hat{\varphi}(q, \theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iqp} \varphi(p, \theta) \, dp.
\]
The **Hilbert transform** \(H : H_t^s(T^1 \mathbb{S}^1) \rightarrow H_t^s(T^1 \mathbb{S}^1)\) is defined by
\[
\hat{H}\varphi(q, \theta) = \text{sgn}(q) \hat{\varphi}(q, \theta).
\]
As is seen from (4.33) and (5.1),

$$Z = H \frac{\partial}{\partial \theta}. \quad (5.3)$$

With the help of (5.2) and (5.3), the Reshetnyak formula (4.32) for vector fields takes the following form in the two-dimensional case:

$$\|f\|^2_{H^s_1(\mathbb{R}^2)} = \frac{1}{2} \left[ \left\| \partial \theta (I^f) \right\|^2_{H^{s+1/2}_{t+1/2}(T \mathbb{S}^1)} + \left\| I^f \right\|^2_{H^{s+1/2}_{t+1/2}(T \mathbb{S}^1)} \right. \right.$$

$$\left. + \left. \| I^f \|^2_{H^{s+3/2}_{t+1/2}(T \mathbb{S}^1)} + 2 \Re (iH \partial \theta (I^f), I^f) \right] \right].$$

and the stability estimate (4.42) takes the form

$$\|f\|^2_{H^s_1(\mathbb{R}^2)} \leq \left\| \partial \theta (I^f) \right\|^2_{H^{s+1/2}_{t+1/2}(T \mathbb{S}^1)} + \left\| I^f \right\|^2_{H^{s+1/2}_{t+1/2}(T \mathbb{S}^1)} + \| I^f \|^2_{H^{s+3/2}_{t+1/2}(T \mathbb{S}^1)}.$$

Formulas (4.43) and (4.49) admit similar simplifications in the 2D-case. Omitting details, we present the result. the Reshetnyak formula (4.43) for second rank tensor fields takes the following form in the two-dimensional case:

$$8 \|f\|^2_{H^s_1(\mathbb{R}^2)} = \| I^f \|^2_{H^{s+5/2}_{t+5/2}} + 2 \| \partial \theta (I^f) \|^2_{H^{s+3/2}_{t+3/2}} + 2 \| I^f \|^2_{H^{s+3/2}_{t+3/2}}$$

$$+ \| \partial \theta (I^f) \|^2_{H^{s+1/2}_{t+1/2}} - 2 \| \partial \theta (I^f) \|^2_{H^{s+1/2}_{t+1/2}} + \| pI^f \|^2_{H^{s+3/2}_{t+3/2}} + 8 \| I^f \|^2_{H^{s+1/2}_{t+1/2}}$$

$$- 4 \Re (iH (I^f), \partial \theta (I^f))_{H^{s+1/2}_{t+1/2}} + 2 \Re (iH (I^f), I^f)_{H^{s+1/2}_{t+1/2}} - 2 \Re (I^f, \partial \theta (I^f))_{H^{s+3/2}_{t+3/2}}$$

$$- 4 \Re (I^f, I^f)_{H^{s+3/2}_{t+3/2}} - 4 \Re (\partial \theta (I^f), I^f)_{H^{s+3/2}_{t+3/2}} - 4 \Re (iH \partial \theta (I^f), \partial \theta (I^f))_{H^{s+1/2}_{t+1/2}}$$

$$+ 4 \Re (iH (I^f), \partial \theta (I^f))_{H^{s+1/2}_{t+1/2}} - 2 \Re (iH \partial \theta (I^f), pI^f)_{H^{s+1/2}_{t+1/2}} + 4 \Re (iH (pI^f), I^f)_{H^{s+1/2}_{t+1/2}}.$$

Here the shorter notation $H^s_{t+1}$ is used instead of $H^s_{t+1}(T \mathbb{S}^1)$ on the right-hand side. The corresponding stability estimate is

$$\|f\|^2_{H^s_1(\mathbb{R}^2)} \leq 6 \left[ \| I^f \|^2_{H^{s+5/2}_{t+5/2}} + \| \partial \theta (I^f) \|^2_{H^{s+3/2}_{t+3/2}} + \| I^f \|^2_{H^{s+3/2}_{t+3/2}}$$

$$+ \| pI^f \|^2_{H^{s+3/2}_{t+3/2}} + \| \partial \theta (I^f) \|^2_{H^{s+1/2}_{t+1/2}} + \| \partial \theta (I^f) \|^2_{H^{s+1/2}_{t+1/2}} + \| I^f \|^2_{H^{s+1/2}_{t+1/2}} \right].$$

Even in the 2D-case, the Reshetnyak formula for rank 3 symmetric tensor fields is too long to be presented here.

REFERENCES

E-mail address: vkrishnan@tifrbng.res.in, ramesh@tifrbng.res.in, suman@math.tifrbng.res.in, sharafut@list.ru

*Corresponding author
†TIFR Centre for Applicable Mathematics, Sharada Nagar, Chikkabommasandra, Yelahanka New Town, Bangalore, India
‡Sobolev Institute of Mathematics; 4 Koptyug Avenue, Novosibirsk, 630090, Russia; Novosibirsk State University, 2 Pirogov street, 630090, Russia