Uniqueness theorems for the exponential X-ray transform

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Abstract — We consider the exponential X-ray transform with absorption which does not depend on the point and arbitrarily depends on the direction. The problem of the smallest set of projections sufficient for determining a function is investigated. We prove, for a finite distribution $J$, that any countable family of projections is sufficient if the projection centers do not accumulate to the convex hull of the support of $J$. A similar result is established for incomplete projections.

1. INTRODUCTION

Let

$$
\Omega = \Omega^{n-1} = \{\xi = (\xi^1, \ldots, \xi^n) \in \mathbb{R}^n | |\xi| = 1\} 
$$

be a unit sphere in the space $\mathbb{R}^n$. We consider a fixed (complex-valued) function $\varepsilon \in C^\infty(\Omega)$ which is referred to as the absorption function. Let $\mathbb{R}_0^n = \mathbb{R}^n \setminus \{0\}$. A function $E \in C^\infty(\mathbb{R}_0^n)$ defined as

$$
E(x) = \exp[-|x|\varepsilon(x/|x|)]
$$

is called attenuation corresponding to absorption $\varepsilon$.

Let a function $J$ be finite and summable in $\mathbb{R}^n$. The exponential X-ray transform of $J$ is a function $I_x^\varepsilon J(\xi)$ defined by

$$
(I_x^\varepsilon J)(\xi) = \int_0^\infty E(t\xi) J(\alpha + t\xi) \, dt
$$

for $(\alpha, \xi) \in \mathbb{R}^n \times \Omega$. This transform is of considerable interest for emission tomography. When $\varepsilon \equiv 0$, we use the term X-ray transform (without the attribute exponential).

For a fixed $\alpha \in \mathbb{R}^n$, the exponential X-ray transform $(I_x^\varepsilon J)(\xi)$, as a function of $\xi \in \Omega$, is called the projection of function $J(\alpha)$ centered at a (proper) point $\alpha$ (an alternative term is the divergent projection). Along with these projections, we can consider projections with improper centers (or parallel projections). Let us first give a suitable definition of the improper point.

We use the compactification $\tilde{\mathbb{R}}^n$ of the space $\mathbb{R}^n$ which is defined as follows. Let $\Omega_\infty$ denote a specimen of the sphere $\Omega$ which is considered as nonintersecting with $\mathbb{R}^n$. For example, we can assume $\Omega_\infty = \Omega \times \{\infty\}$, where $\{\infty\}$ is a set containing a single element $\infty$. For $\xi \in \Omega$, we denote the corresponding element of $\Omega_\infty$ by $\xi_\infty$ [i.e. $\xi_\infty = (\xi, \infty)$]. Let $\tilde{\mathbb{R}}^n = \mathbb{R}^n \cup \Omega_\infty$. The points of $\mathbb{R}^n$ are called the proper points of $\tilde{\mathbb{R}}^n$, and those of $\Omega_\infty$ the improper points.

We introduce topology in $\tilde{\mathbb{R}}^n$ so that (1) it induces on $\mathbb{R}^n$ and $\Omega_\infty$ the ordinary topologies of these spaces, (2) $\Omega_\infty$ is closed in $\tilde{\mathbb{R}}^n$, (3) a sequence of proper points $x_k$ converges to an improper point $\xi_\infty$ if and only if $|x_k| \to \infty$ and $x_k/|x_k| \to \xi$.

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For $\xi \in \Omega$, a function $(I_{\xi,\infty}^*J)(x)$ defined on a hyperplane $\xi^\perp = \{x \in \mathbb{R}^n | \langle x, \xi \rangle = 0\}$ by

$$(I_{\xi,\infty}^*J)(x) = \int_{-\infty}^{\infty} E(t\xi)J(x + t\xi)\,dt, \quad x \in \xi^\perp$$

(1.2)
is called a projection of the function $J$ centered at an improper point $\xi_{\infty}$. Hereafter $\langle x, \xi \rangle = \sum x^i\xi^i$ denotes the scalar product.

This compactification of $\mathbb{R}^n$ is necessary by the following. For $\varepsilon \equiv 0$ we have $I_{\xi,\infty}^0J = I_{\xi_{\infty}^0}^\varepsilon J$, and therefore we can consider the two end points of each diameter in the sphere $\Omega_{\infty}$ to be identical. In doing so we transform $\mathbb{R}^n$ into a projective space $\mathbb{R}P^n$, which is the traditional compactification of $\mathbb{R}^n$. For the general case $\varepsilon \neq 0$ the integral in (1.2) depends not only on the line $y = x + t\xi$ but also on its orientation, and consequently we have to use the compactification $\mathbb{R}^n$.

The integrals in (1.1) and (1.2) are defined for a finite summable function $J$. We show in Section 2 that these definitions can be extended to finite distributions $J$ so that $I_{\xi}^*J$ is a distribution on the sphere $\Omega$, for $\alpha \notin \text{supp } J$, and $I_{\xi,\infty}^*J$ is a distribution on a hyperplane $\xi^\perp$.

What is the smallest set of centers of projections which uniquely determine a finite distribution $J$? According to Theorem 2.1, this is any countable set of points belonging to $\mathbb{R}^n \setminus U$, where $U$ is an arbitrary neighbourhood of the convex hull of the support of $J$.

Of appreciable interest is the problem of determining $J$ by a family of projections $I_{\xi}^*J$ which are known only in a certain domain $\omega \subset \Omega$, and not on the whole sphere $\Omega$. In what follows, this case is described by Theorem 2.2.

Theorem 2.1 has been proved by the author in [6, 7] for $\varepsilon \equiv 0$, and independently by Hamacer, Smith, Solmon, and Wagner [4] for finite summable functions. The following statement due to Derevtsov [2, 6] shows that Theorem 2.1 cannot be improved, i.e. for $\varepsilon \equiv 0$ and any finite set of points in $\mathbb{R}^n \setminus B$, where $B$ is a unit ball in $\mathbb{R}^n$, there exists a nonzero distribution $J$ with support in $B$ and zero projections at all points in this set. We have formulated Theorem 2.2 by analogy with Lemma 2.11 in [5].

Our proofs of Theorems 2.1 and 2.2 are based on the so-called method of moments; its idea is as follows. Assuming that the projections of $J$ with centers $\alpha_k$, $k = 1, 2, \ldots$, are zero and $\alpha$ is an accumulation point of the sequence $\alpha_k$, we obtain by induction on $m$ that the distribution $J$ has zero integral moments of degree $m$ along all the straight lines passing through $\alpha$. This method was used by Hadamard for proving the statement that a continuous function in the half-plane $y > 0$ is uniquely determined by its integrals along all semicircles whose centers are in the line $y = 0$. Hadamard’s proof can be found in [1].

In Section 2 we extend definitions (1.1) and (1.2) to finite distributions and formulate the main results of this work, Theorems 2.1 and 2.2. Section 3 is devoted to some topics in the theory of distributions. We mention the basic concepts of this theory only to introduce the notation and terminology and concentrate on the problems which are rarely considered in textbooks, such as the direct image of a distribution or parameter-dependent distributions. Section 4 contains the proofs of Theorems 2.1 and 2.2.

2. FORMULATION OF THE MAIN CONCEPTS AND RESULTS

Let us recall the ordinary designations for distributions.

Here, the term manifold implies a smooth finite-dimensional manifold which may have a boundary; smooth is used as a synonym of the term infinitely differentiable. The complex vector space of smooth functions on a manifold $N$ is denoted by $C^\infty(N)$, and its
subspace containing finite functions by $C^\infty_0(N)$. Let $K \subset N$ be a compact, then $C^\infty_K(N)$ designates a subspace of $C^\infty_0(N)$ containing finite functions $\varphi$ for which $\text{supp} \varphi \subset K$. The space $C^\infty(N)$ with the topology of uniform convergence on each compact is denoted by $\mathcal{E}(N)$, and its subspace $C^\infty_K(N)$ with the topology induced by $\mathcal{E}(N)$ is denoted by $\mathcal{D}_K(N)$. Let $\mathcal{D}(N)$ denotes the subspace $C^\infty_0(N)$ with the inductive limit topology of subspaces $\mathcal{D}_K(N)$ (with respect to all compacts $K \subset N$). Let $\mathcal{D}'(N)$ and $\mathcal{E}'(N)$ be the respective spaces of continuous linear functionals on $\mathcal{D}(N)$ and $\mathcal{E}(N)$. Their elements are referred to as distributions (or finite distributions) on $N$. The value of $F \in \mathcal{D}'(N)$ [$\mathcal{E}'(N)$] on the function $\varphi \in \mathcal{D}(N)$ [$\mathcal{E}(N)$] is denoted by $\langle F, \varphi \rangle$ or $\langle F(x), \varphi(x) \rangle$. The restriction $F \big|_U \in \mathcal{D}'(U)$ is defined for any $F \in \mathcal{D}'(N)$ and an open $U \subset N$.

Let $J \in \mathcal{E}'(\mathbb{R}^n)$, $\alpha \in \mathbb{R}^n$, $\alpha \notin \text{supp} J$. The distribution $I^\alpha_\omega J \in \mathcal{E}'(\Omega)$ defined by the equality

$$\langle I^\alpha_\omega J, \varphi \rangle = \left\langle J(x), \frac{e^{-|x-\alpha|}}{|x-\alpha|^{n-1}} \varphi(x) \right\rangle, \quad \varphi \in \mathcal{E}(\Omega)$$

(2.1)

for $\xi \in \Omega$, let $p_\xi: \mathbb{R}^n \rightarrow \xi^\perp$ be an orthogonal projection. The distribution $I^\omega_\alpha J \in \mathcal{E}'(\xi^\perp)$ defined by

$$\langle I^\omega_\alpha J, \varphi \rangle = \langle J(x), e^{-\langle \xi, x \rangle} \varphi(p_\xi(x)) \rangle, \quad \varphi \in \mathcal{E}(\xi^\perp)$$

(2.2)

is called a projection of distribution $J$ centered at an improper point $\xi_\alpha$.

In order to relate (1.1), (1.2) with (2.1), (2.2) we recall that any function $\mathcal{F}$ locally summable on $\mathbb{R}^n$ can be put into correspondence with a distribution $F = \mathcal{F}(x) \, dx \in \mathcal{D}'(\mathbb{R}^n)$ defined by

$$\langle F, \varphi \rangle = \int_{\mathbb{R}^n} \mathcal{F}(x) \varphi(x) \, dx, \quad \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Similarly, any function $\mathcal{F}$ summable on $\Omega$ corresponds to a distribution $F = \mathcal{F}(\xi) \, d\xi \in \mathcal{E}'(\Omega)$ defined by

$$\langle F, \varphi \rangle = \int_{\Omega} \mathcal{F}(\xi) \varphi(\xi) \, d\xi, \quad \varphi \in \mathcal{E}(\Omega)$$

where $d\xi$ is an angular measure on $\Omega$.

Let $J$ be a finite function summable on $\mathbb{R}^n$, and $J = \mathcal{J}(x) \, dx$. It follows from (1.1) and (2.1) that $I^\alpha_\omega J = (I^\alpha_\omega J) \, d\xi$. Similarly, (2.1) and (2.2) imply that $I^\omega_\alpha J = (I^\omega_\alpha J) \, dx$.

**Theorem 2.1.** Assume that $n \geq 2$, $\varepsilon \in C^\infty(\Omega)$, $J \in \mathcal{E}'(\mathbb{R}^n)$ is a nonzero distribution, and $K$ is a convex hull of the support of $J$. Then the set of points $\alpha \in \mathbb{R}^n \setminus U$ at which $I^\alpha_\omega J = 0$ is finite for any open set $U \subset \mathbb{R}^n$ such that $K \subset U$.

For $\omega \subset \Omega$ and $\alpha \in \mathbb{R}^n$, $C(\alpha, \omega) = \{x \in \mathbb{R}^n| x = \alpha + t\xi, \xi \in \omega, t > 0\}$ denotes a cone with a vertex $\alpha$ and a generatrix set $\omega$. We say that the cone $C(\alpha, \omega)$ is in a free position with respect to a set $K \subset \mathbb{R}^n$ if it contains a ray which begins at $\alpha$ and does not intersect $K$.

**Theorem 2.2.** Assume that $n \geq 2$, $\varepsilon \in C^\infty(\Omega)$, $J \in \mathcal{E}'(\mathbb{R}^n)$, $\alpha_0 \notin \text{supp} J$, and $\omega$ is a domain (an open connected set) in $\Omega$ such that the cone $C(\alpha_0, \omega)$ is in a free position with respect to $\text{supp} J$. If $I^\alpha_\omega J \big|_\omega = 0$ for all $\alpha$ in a certain neighbourhood of $\alpha_0$, then $J \big|_{C(\alpha_0, \omega)} = 0$. 

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Comparing Theorems 2.1 and 2.2 we can ask whether the condition \((I_\alpha^*)J\big|_\varphi = 0\) for all \(\alpha\) in a certain neighbourhood of \(\alpha_0\) in Theorem 2.2 can be replaced by a weaker condition \((I_\alpha^*)J\big|_\alpha_k = 0\) for a sequence of points \(\alpha_k\) which are different from \(\alpha_0\) and convergent to \(\alpha_0\). In case of \(\omega = \Omega\) the answer is positive in accordance with Theorem 2.1. In case of \(\omega \neq \Omega\) we can show that the answer is positive for \(n = 2\) and negative for \(n \geq 3\).

3. SOME CONCEPTS OF DISTRIBUTION THEORY

The spaces \(\mathcal{D}'(N)\) and \(\mathcal{E}'(N)\) are considered as spaces with weak topology. Therefore, a sequence \(F_k \in \mathcal{E}'(N), k = 1, 2, \ldots,\) converges to zero if and only if the sequence \((F_k, \varphi)\) converges to zero for any \(\varphi \in \mathcal{E}(N)\). Distributions can be multiplied by functions from \(\mathcal{E}(N)\) according to the formula \((\varphi F, \psi) = (F, \varphi \psi)\).

**Lemma 3.1.** Let \(U\) be a domain in a smooth manifold \(N\) and \(F \in \mathcal{E}'(N \times \mathbb{R})\); we denote points in \(N \times \mathbb{R}\) by pairs \((x, t)\), where \(x \in N\) and \(t \in \mathbb{R}\). If \((F, t^k \varphi(x)) = 0\) for any integer \(k \geq 0\) and any \(\varphi \in \mathcal{D}(U)\), then \(F\big|_{U \times \mathbb{R}} = 0\).

This lemma is a corollary of the following two statements.

1. The set of functions which have the form
   \[
   \sum_{i=1}^{k} \varphi_i(x) \psi_i(t), \quad \varphi_i \in \mathcal{D}(U), \quad \psi_i \in \mathcal{E}(\mathbb{R})
   \]
   is dense in \(\mathcal{D}(U) \otimes \mathcal{E}(\mathbb{R})\) [3].

2. It is evident that the set of polynomials is dense in \(\mathcal{E}(\mathbb{R})\).
According to the usual rule of differentiation of a product,
\[
\frac{\partial(\varphi F)}{\partial x^i} = \frac{\partial \varphi}{\partial x^i} F + \varphi \frac{\partial F}{\partial x^i}, \quad \varphi \in \mathcal{E}(U).
\]

By convention, the same subscript and superscript in a single term imply summation from 1 to \(n\).

**Lemma 3.2.** Assume that \(F \in \mathcal{E}'(\mathbb{R}^n)\) and \(m \geq 1\) is an integer. If there exist numbers \(a^1, \ldots, a^n, b \in \mathbb{R}\) such that at least one of them is not zero and
\[
(a^{i_1} + bx^{i_1}) \cdots (a^{i_m} + bx^{i_m}) \frac{\partial^m F}{\partial x^{i_1} \cdots \partial x^{i_m}} = 0
\]
then \(F = 0\).

**Proof.** We are to consider two possible cases.

1. \(b = 0, a = (a^1, \ldots, a^n) \neq 0\). It is enough to consider \(m = 1\) since the proof for the general case can be obtained by induction. Here (3.3) reduces to \(a^i \partial F/\partial x^i = 0\). We choose an affine coordinate system \((y^1, \ldots, y^n)\) in \(\mathbb{R}^n\) so that \(a^i\) is the first basic vector in the system. Then \(\partial F/\partial y^1 = a^i \partial F/\partial x^i = 0\), and hence \(F = 0\).

2. \(b \neq 0\). If we divide (2.3) by \(b^n\), we obtain
\[
(x^{i_1} + c^{i_1}) \cdots (x^{i_m} + c^{i_m}) \frac{\partial^m F}{\partial x^{i_1} \cdots \partial x^{i_m}} = 0
\]
where \(c^i = a^i/b\). Let \(G = f' F\), where \(f: \mathbb{R}^n \to \mathbb{R}^n\) is defined by the formula \(f(x) = x + c\).

It is enough to establish that \(G = 0\). Applying \(f'\) to (3.4) and using (3.2) we obtain
\[
x^{i_1} \cdots x^{i_m} \frac{\partial^m G}{\partial x^{i_1} \cdots \partial x^{i_m}} = 0.
\]

We first show that (3.5) implies the inclusion
\[
\text{supp } G \subset \{0\}.
\]

To do this we have to prove that \((G, \varphi) = 0\) for any \(\varphi \in \mathcal{E}(\mathbb{R}^n)\) such that \(0 \notin \text{supp } \varphi\). Assume this \(\varphi\) to be fixed. Then there exists \(\varepsilon > 0\) such that the set \(\{x \in \mathbb{R}^n \mid |x| < \varepsilon\}\) and \(\text{supp } \varphi\) do not intersect. We define \(\psi_t \in \mathcal{E}(\mathbb{R}^n)\) for \(t \in \mathbb{R}\) by putting \(\psi_t(x) = \varphi(e^t x)\). Note that \(\text{supp } \psi_t \subset e^{-t} \text{supp } \varphi\). The function
\[
f(t) = (G, \psi_t)
\]
is smooth on \(\mathbb{R}\). If we denote \(R = \sup \{|x| \mid x \in \text{supp } G\}\) and choose \(t_0\) so that \(\varepsilon e^{-t_0} > R\), then \(\text{supp } \psi_t \cap \text{supp } G = \emptyset\) for \(t < t_0\), and therefore
\[
f(t) = 0, \quad t < t_0.
\]

Let us show that the equality
\[
(-1)^k \left( x^{i_1} \cdots x^{i_k} \frac{\partial^k G}{\partial x^{i_1} \cdots \partial x^{i_k}}, \psi_t \right) = f^{(k)}(t) + \sum_{p=0}^{k-1} c_p^k f^{(p)}(t)
\]

(3.9)
is valid for any integer \(k \geq 0\), with constants \(c_p^k\) dependent only on \(k\) and \(p\). We prove (3.9) by induction on \(k\). For \(k = 0\), we see that (3.9) follows from (3.7). Assuming that (3.9) is valid for some \(k\), we differentiate this equation with respect to \(t\)

\[
 f^{(k+1)}(t) + \sum_{p=1}^{k} c_p^k f^{(p)}(t) = (-1)^k \left( x^{i_1} \ldots x^{i_k} \frac{\partial^k G}{\partial x^{i_1} \ldots \partial x^{i_k}} \frac{\partial \psi_t}{\partial t} \right).
\]  

(3.10)

According to the definition of \(\psi_t\) we have \(\partial \psi_t / \partial t = x^i \partial \psi_t / \partial x^i\). Substituting this expression in (3.10) we find

\[
 f^{(k+1)}(t) + \sum_{p=1}^{k} c_p^k f^{(p)}(t) = (-1)^k \left( x^{i_1} \ldots x^{i_k} \frac{\partial^k G}{\partial x^{i_1} \ldots \partial x^{i_k}} x^{i_{k+1}} \frac{\partial \psi_t}{\partial x^{i_{k+1}}} \right)
\]

\[
 = (-1)^{k+1} \left( \frac{\partial}{\partial x^{i_{k+1}}} \left( x^{i_1} \ldots x^{i_k+1} \frac{\partial^k G}{\partial x^{i_1} \ldots \partial x^{i_k}} \right), \psi_t \right)
\]

\[
 = (-1)^{k+1} \left( x^{i_1} \ldots x^{i_k+1} \frac{\partial^{k+1} G}{\partial x^{i_1} \ldots \partial x^{i_k+1}}, \psi_t \right)
\]

\[
 - (-1)^k (n + k) \left( x^{i_1} \ldots x^{i_k} \frac{\partial^k G}{\partial x^{i_1} \ldots \partial x^{i_k}}, \psi_t \right).
\]

If we replace the second term by its expression using the induction hypothesis (3.9), we obtain

\[
 (-1)^{k+1} \left( x^{i_1} \ldots x^{i_k+1} \frac{\partial^{k+1} G}{\partial x^{i_1} \ldots \partial x^{i_k+1}}, \psi_t \right) = f^{(k+1)}(t) + \sum_{p=0}^{k} [c_p^k + (n + k)c_p^k] f^{(p)}(t)
\]

which completes the proof of (3.9).

Taking \(k = m\) in (3.9) we use (3.5) to verify that \(f(t)\) satisfies the homogeneous equation

\[
 f^{(m)}(t) + \sum_{p=0}^{m-1} c_p^m f^{(p)}(t) = 0.
\]

This and (3.8) yield \(f(t) \equiv 0\), in particular, \(f(0) = \langle G, \varphi \rangle = 0\). We have thus proved (3.6).

It is known that a distribution whose support is the point \(0\) can be represented as a finite linear combination of derivatives of the \(\delta\)-function, i.e.

\[
 G = \sum_{|\alpha| \leq k} c_\alpha D^\alpha \delta.
\]  

(3.11)

We can easily verify by induction on \(m\) that

\[
 x^{i_1} \ldots x^{i_m} \frac{\partial^m (D^\alpha \delta)}{\partial x^{i_1} \ldots \partial x^{i_m}} = a(n, m, \alpha) D^\alpha \delta
\]  

(3.12)

for any multiindex \(\alpha\) and \(a(n, m, \alpha) \neq 0\). Substituting (3.11) in (3.5) and using (3.12) we obtain

\[
 \sum_{|\alpha| \leq k} a(n, m, \alpha) c_\alpha D^\alpha \delta = 0.
\]

(3.13)
Different derivatives of the δ-function are linearly independent. Therefore (3.13) implies that all \( c_a \) are zeros, i.e. \( G = 0 \). The lemma is proved.

We are to consider distributions dependent on several real parameters (e.g. the distribution \( I_\alpha J \) defined on \( \Omega \) by (2.1) depends on \( \alpha \in \mathbb{R}^n \)), and, in particular, to differentiate distributions with respect to parameters. Here we give the general rules for these calculations.

Let \( A \) be a domain of the space \( \mathbb{R}^k \). The space \( \mathbb{R}^k \) is considered as a space of parameters; therefore we designate its points by Greek letters, e.g. \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k \). We call a mapping \( F: A \to \mathcal{E}'(N) \) a distribution dependent on a parameter \( \alpha \in A \) on the manifold \( N \). We say that the distribution \( F \) is \( m \) times differentiable with respect to the parameter \( \alpha \in A \) (\( m \leq \infty \)) if the function \( F \) has continuous partial derivatives of order \( m \) and lower order. Recall that \( \mathcal{E}'(N) \) is considered for weak topology. Thus, if \( F, G: A \to \mathcal{E}'(N) \) are two distributions dependent on a parameter \( \alpha \in A \), the equality \( \partial F / \partial \alpha^i = G \) is equivalent to \( \partial (F(\alpha), \varphi) / \partial \alpha^i = (G(\alpha), \varphi) \) for any \( \varphi \in \mathcal{E}(N) \).

Distributions dependent on a parameter can be multiplied by functions dependent on a parameter. To be precise, if \( F: A \to \mathcal{E}'(N) \) and \( \varphi = \varphi(\chi, \alpha) \) is a smooth function on \( N \times A \), the distribution \( \varphi F: A \to \mathcal{E}'(N) \) is defined by the equation \( (\varphi F)(\alpha) = \langle \varphi(\chi, \alpha), F(\alpha) \rangle \), where \( \varphi_\alpha \in \mathcal{E}(N) \) is given by \( \varphi_\alpha(\chi) = \varphi(\chi, \alpha) \). The standard rule of differentiation of a product with respect to a parameter holds.

Let \( F: A \to \mathcal{E}'(M) \) be a distribution dependent on a parameter and \( f: M \times A \to N \) be a smooth mapping. For \( \alpha \in A \), we introduce a mapping \( f_\alpha: M \to N \) which is defined by the formula \( f_\alpha(\chi) = f(\chi, \alpha) \). The distribution \( f^* F: A \to \mathcal{E}'(N) \) defined by the equality \( (f^* F)(\alpha) = f_{\alpha}^* F(\alpha) \) is called the direct image of the distribution \( F \) in the mapping \( f \).

The following statement is used below for \( L = \mathcal{E}'(N) \).

**Lemma 3.3.** Let \( L \) be a linear topological space. Assume that an \( m \)-times differentiable function \( f: U \to L \) is given in a domain \( U \subset \mathbb{R}^n \) and, for some \( x_0 \in U \), there exists a sequence \( x_k, k = 1, 2, \ldots, \), convergent to \( x_0 \) so that \( x_k \neq x_0 \) and \( f(x_k) = 0 \). Suppose that the sequence \( (x_k - x_0)/|x_k - x_0| \) converges to a vector \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \), and

\[
\frac{\partial^p f}{\partial x^{i_1} \cdots \partial x^{i_p}}(x_0) = 0, \quad 1 \leq i_1, \ldots, i_p \leq n
\]

for any \( 0 \leq p \leq m \). Then

\[
a_1^1 \cdots a_m^m \frac{\partial^m f}{\partial x^{i_1} \cdots \partial x^{i_m}}(x_0) = 0.
\]

**Proof.** By condition, the Taylor expansion of \( f \) in the neighbourhood of \( x_0 \) is

\[
f(x) = (x_1^1 - x_0^1) \cdots (x_m^m - x_0^m) \frac{\partial^m f}{\partial x^{i_1} \cdots \partial x^{i_m}}(x_0) + o(|x - x_0|^m).
\]

Consequently

\[
0 = f(x_k) = (x_k^1 - x_0^1) \cdots (x_k^m - x_0^m) \frac{\partial^m f}{\partial x^{i_1} \cdots \partial x^{i_m}}(x_0) + o(|x_k - x_0|^m).
\]

If we divide this equation by \( |x_k - x_0|^m \) and take the limit \( k \to \infty \), we obtain the statement of the lemma.
4. PROOFS OF THEOREMS 2.1 AND 2.2

It is more convenient to prove Theorem 2.1 in the following formulation which is evidently equivalent to that of Section 2.

**Lemma 4.1.** Assume that $J \in E'(\mathbb{R}^{n+1})$, $\varepsilon \in C^\infty(\Omega^n)$, $n \geq 1$, and $K$ is a convex compact set in $\mathbb{R}^{n+1}$ such that $\text{supp} \, J \subset K$. Suppose that there exists a point $\beta_0 \in \mathbb{R}^{n+1} \setminus K$ and a sequence $\beta_k \in \mathbb{R}^{n+1}$, $k = 1, 2, \ldots$, convergent to $\beta_0$ so that $\beta_k \neq \beta_0$ and $I_{\beta_k} J = 0$. Then $J = 0$.

Before proving this statement we are to give auxiliary constructions and lemmas.

Let us first suppose that $\beta_0$ in Lemma 4.1 is a proper point. For $\beta \in \mathbb{R}^{n+1}$, we denote the central projection with its center at $\beta$ by $p_\beta: \mathbb{R}^{n+1} \setminus \{\beta\} \to \Omega^n$. This projection is defined by the formula $p_\beta(y) = (y - \beta)/|y - \beta|$. Then, it follows from (2.1) that, for $\beta \notin \text{supp} \, J$,

$$I_{\beta} J = p_\beta \left( \frac{E(y - \beta)}{|y - \beta|^n} J \right).$$

We choose a rectangular Cartesian coordinate system $(y^1, \ldots, y^{n+1}) = (x, t) = (x^1, \ldots, x^n, t)$ in $\mathbb{R}^{n+1}$ so that the origin of coordinates is at $\beta_0$ and the plane $t = 0$ and $K$ do not intersect. We also choose $\delta > 0$ such that the domain

$$U = \{(x, t) \in \mathbb{R}^{n+1} | \delta < t < 1/\delta\}$$

contains $K$. Let

$$A = \{((\alpha, \tau) \in \mathbb{R}^{n+1} | \tau < \delta\}.$$

We designate points of the domain $A$ either by $(\alpha, \tau)$ or by $\beta = (\beta^1, \ldots, \beta^{n+1})$. We can assume that all the points $\beta_k = (\alpha_k, \tau_k)$ in Lemma 4.1 belong to $A$.

Let us introduce the mappings

$$p: U \to \mathbb{R}^n, \quad p(x, t) = x \quad (4.1)$$

$$f: U \times A \to U, \quad f((x, t), (\alpha, \tau)) = \left(\frac{x - \alpha}{t - \tau}, t\right) \quad (4.2)$$

$$g: U \times \mathbb{R}^n \to U, \quad g((x, t), \alpha) = \left(\frac{x - \alpha}{t}, t\right) \quad (4.3)$$

$$h: U \to U, \quad h(x, t) = \left(\frac{x}{t}, t\right). \quad (4.4)$$

We define for $(\alpha, \tau) \in A$ a diffeomorphism of $U$ onto itself by the formula $f_{(\alpha, \tau)}(x, t) = f((x, t), (\alpha, \tau))$ and designate it by $f_{(\alpha, \tau)}: U \to U$. Similarly, we define for $\alpha \in \mathbb{R}^n$ a diffeomorphism of $U$ onto itself by the formula $g_{\alpha}(x, t) = g((x, t), \alpha)$ and designate it by $g_\alpha: U \to U$. Note that

$$g_\alpha = f_{(\alpha, 0)}, \quad h = g_0. \quad (4.5)$$

Using these mappings we define a distribution $F: A \to E'(\mathbb{R}^n)$ dependent on the parameter $(\alpha, \tau) \in A$. We put

$$F = p'(\frac{E(tx, t)}{(t - \tau)^n} f' J). \quad (4.6)$$
The distribution $G: \mathbb{R}^n \to \mathcal{E}'(\mathbb{R}^n)$ depends on the parameter $\alpha \in \mathbb{R}^n$ and is defined as

$$G = p'(Etx, t)\frac{g'J}{t^n}.$$  

(4.7)

Finally, for each integer $m \geq 0$ we introduce a distribution $M_m \in \mathcal{E}'(\mathbb{R}^n)$ by the formula

$$M_m = p'(Etx, t)\frac{h'J}{t^{n+m}}.$$  

(4.8)

Note that both $F$ and $G$ are infinitely differentiable with respect to the whole sets of their parameters. It follows from (4.5) that

$$G(\alpha) = F(\alpha, 0), \quad M_0 = G(0).$$  

(4.9)

According to the definition of function $E$ in Section 1,

$$E(tx, t) = e^{-\mu(x)}$$

where the function $\mu \in C^\infty(\mathbb{R}^n)$ is defined by the formula

$$\mu(x) = \sqrt{|x|^2 + 1} \left( \frac{x}{\sqrt{|x|^2 + 1}}, \frac{1}{\sqrt{|x|^2 + 1}} \right).$$  

(4.10)

Using the rule in (3.1) we see that (4.6)-(4.8) can be rewritten as

$$F(\alpha, \tau) = e^{-\mu(x)} p'f'_{(\alpha, \tau)}(E(x - \alpha, t - \tau)\frac{J}{(t - \tau)^n})$$  

(4.11)

$$G(\alpha) = p'g'_{\alpha} \left( E(x - \alpha, t)\frac{J}{t^n} \right)$$  

(4.12)

$$M_m = p'h' \left( E(x, t)\frac{J}{t^{n+m}} \right).$$  

(4.13)

**Lemma 4.2.** Assume that $J \in \mathcal{E}'(U)$ and a distribution $F: A \to \mathcal{E}'(\mathbb{R}^n)$ is defined for $J$ by (4.6). The equality $I_\beta J = 0$ is equivalent to $F(\alpha, \tau) = 0$ for any point $\beta = (\alpha, \tau) \in A$.

**Proof.** According to the choice of the coordinate system

$$\text{supp } I_\beta J \subset \Omega^n_+ = \{ y = (x, t) \in \mathbb{R}^{n+1} | |y| = 1, \ t > 0 \}.$$  

Therefore, we can assume that $I_\beta J \in \mathcal{E}'(\Omega^n_+)$. We define a diffeomorphism $q: \Omega^n_+ \to \mathbb{R}^n$ by $q(x, t) = x/t$. According to (2.1)

$$\langle q'I_\beta^*J, \varphi \rangle = \left< J(x, t), \frac{E(x - \alpha, t - \tau)}{(x - \alpha)^2 + (t - \tau)^2}^{n/2} \varphi \left( \frac{x - \alpha}{t - \tau} \right) \right>$$

$$= \left< E(x - \alpha, t - \tau) J(x, t), \left( 1 + \left| \frac{x - \alpha}{t - \tau} \right|^2 \right)^{-n/2} \varphi \left( \frac{x - \alpha}{t - \tau} \right) \right>.$$
for any $\varphi \in \mathcal{E}(\mathbb{R}^n)$. Hence, if we define $\tilde{\varphi} \in \mathcal{E}(\mathbb{R}^n)$ by the formula

$$\tilde{\varphi}(x) = (1 + |x|^2)^{-n/2} \varphi(x)$$

for $\varphi \in \mathcal{E}(\mathbb{R}^n)$, then

$$\langle q' J_\beta, \varphi \rangle = \left\langle \frac{E(x - \alpha, t - \tau)}{(t - \tau)^n} J(x, t), \varphi \circ p \circ f_{(\alpha, \tau)} \right\rangle$$

$$= \left\langle p' f_{(\alpha, \tau)} \left( \frac{E(x - \alpha, t - \tau)}{(t - \tau)^n} J \right), \tilde{\varphi} \right\rangle.$$

Comparing this equation with (4.11) we arrive at the statement of the lemma.

Lemma 4.3. Assume that $J \in \mathcal{E}'(U)$ and distributions $F(\alpha, \tau)$, $G(\alpha)$ and $M_m$ are defined by (4.6)-(4.8). Then

$$\frac{\partial^k F}{\partial \tau^k} (\alpha, 0) = (-1)^k x_{i_1} \ldots x_{i_k} \frac{\partial^k G(\alpha)}{\partial \alpha^{i_1} \ldots \partial \alpha^{i_k}}$$

$$\frac{\partial^m G}{\partial \alpha^{i_1} \ldots \partial \alpha^{i_m}} (0) = \frac{\partial^m M_m}{\partial x_{i_1} \ldots \partial x_{i_m}}$$

$$+ \sum_{k=0}^{m-1} \sum_{|\beta| \leq k} \mu_{i_1 \ldots i_m, \beta} (x) D^\beta M_k, \quad 1 \leq i_1, \ldots, i_m \leq n \tag{4.15}$$

for any integers $k \geq 0$ and $m \geq 0$ and some $\mu_{i_1 \ldots i_m, \beta} \in C^\infty(\mathbb{R}^n)$ which are dependent only on $\varepsilon$.

We will prove this lemma later. We are now to complete the proof of Lemma 4.1 using (4.14) and (4.15) for a proper point $\beta_0$.

The coordinate system is chosen so that $\beta_0 = 0$. Without loss of generality we can suppose that the sequence $\beta_k = (\alpha_k, \tau_k)$ is such that the sequence $\beta_k/|\beta_k|$ converges to some unit vector $c = (c_1, \ldots, c^{n+1}) = (a^1, \ldots, a^n, b)$. According to Lemma 4.2

$$F(\alpha_k, \tau_k) = 0. \tag{4.16}$$

Taking the limit $k \to \infty$ we obtain $F(0, 0) = 0$. In view of (4.9) this implies

$$G(0) = M_0 = 0. \tag{4.17}$$

Then, we are to prove by induction on $p$ that

$$M_p = 0$$

$$\frac{\partial^p G}{\partial \alpha^{i_1} \ldots \partial \alpha^{i_p}} (0) = 0, \quad 1 \leq i_1, \ldots, i_p \leq n \tag{4.19}$$

for any integer $p \geq 0$. Indeed, suppose that (4.18) and (4.19) have already been proved for all $p$ such that $0 \leq p < m$. Then, because of (4.14) and (4.15)

$$\frac{\partial^p F}{\partial \beta^{i_1} \ldots \partial \beta^{i_p}} (0, 0) = 0, \quad 0 \leq p < m, \quad 1 \leq i_1, \ldots, i_p \leq n + 1 \tag{4.20}$$

$$\frac{\partial^m G}{\partial \alpha^{i_1} \ldots \partial \alpha^{i_m}} (0) = \frac{\partial^m M_m}{\partial x_{i_1} \ldots \partial x_{i_m}}, \quad 1 \leq i_1, \ldots, i_m \leq n.$$
Applying Lemma 3.3 we obtain
\[ \sum_{i_1, \ldots, i_m = 1}^{n+1} a^{i_1} \cdots a^{i_m} \frac{\partial^m F}{\partial \beta_{i_1} \cdots \partial \beta_{i_m}} (0, 0) = 0 \]
which can be rewritten in terms of \((\alpha, \tau)\) as
\[ \sum_{k=0}^{m} \binom{m}{k} b^{k} a^{i_{k+1}} \cdots a^{i_{m}} \frac{\partial^m F}{\partial \tau^k \partial \alpha^{i_{k+1}} \cdots \partial \alpha^{i_m}} (0, 0) = 0. \]
Substituting the expressions for the partial derivatives of \(F\) from (4.14) and (4.20), we get
\[ (a^{i_1} - bx^{i_1}) \cdots (a^{i_m} - bx^{i_m}) \frac{\partial^m M_m}{\partial x^{i_1} \cdots \partial x^{i_m}} = 0. \]
Using Lemma 3.2, we verify that \(M_m = 0\). Then, using (4.20) we obtain (4.18) and (4.19) for \(p = m\). Therefore, the induction step is completed and (4.18), (4.19) are proved for all \(p\).

In view of (4.13), equation (4.18) can be rewritten as
\[ p' h' \left( \frac{E(x, t)}{t^{n+m}} J \right) = 0, \quad m = 0, 1, \ldots \]  \hspace{1cm} (4.21)
Let \(r: U \to U\) denote a diffeomorphism which is defined by \(r(x, t) = (x, t^{-1})\). Taking into account that \(r^{-1} = r\), we can rewrite (4.21) as
\[ p' r' r h' \left( \frac{E(x, t)}{t^{n+m}} J \right) = 0, \quad m = 0, 1, \ldots \]
Using (3.1) we rearrange this formula to get
\[ 0 = p' r' \left[ (t^{-m} \circ h^{-1} \circ r) r' h' \left( \frac{E(x, t)}{t^n} J \right) \right] = p' r' \left[ t^m r' h' \left( \frac{E(x, t)}{t^n} J \right) \right]. \]
This means that
\[ 0 = \left< p' r' \left[ t^m r' h' \left( \frac{E(x, t)}{t^n} J \right) \right], \varphi \right> = \left< r' h' \left( \frac{E(x, t)}{t^n} J \right), t^m \varphi(x) \right> \]
for any integer \(m \geq 0\) and any \(\varphi \in \mathcal{E}(\mathbb{R}^n)\). Hence, using Lemma 3.1 we obtain that
\[ r' h' \left( \frac{E(x, t)}{t^n} J \right) = 0. \]
Since \(r \circ h\) is a diffeomorphism, this equation implies that \(t^{-n} E(x, t) J = 0\). The function \(t^{-n} E(x, t)\) does not vanish on \(U\), therefore the latter relation shows that \(J = 0\).

Proof of Lemma 4.3. Recall that the distributions \(F(\alpha, \tau)\), \(G(\alpha)\) and \(M_m\) are defined for the distribution \(J \in \mathcal{E}'(U)\) by formulae (4.6)--(4.8). The left- and right-hand sides of (4.14) and (4.15), if considered with respect to \(J\) for fixed \(\alpha\), are continuous operators acting from \(\mathcal{E}'(U)\) into \(\mathcal{E}'(\mathbb{R}^n)\). Since the set of distributions \(J = \mathcal{J}(x, t) dx dt\), with \(\mathcal{J} \in C^\infty_0(U)\), is dense in \(\mathcal{E}'(U)\), we see that it is enough to prove the lemma only for these \(J\).
Thus, let $J = J(x,t) \, dx \, dt$, where $J \in C_0^\infty(U)$. Then, it follows from definitions (4.6)-(4.8) that

$$F(\alpha, \tau) = F(\alpha, \tau, x) \, dx, \quad G(\alpha) = G(\alpha, x) \, dx, \quad M_m = M_m(x) \, dx$$

and the functions $F, G$ and $M_m$ can be expressed in terms of $J$ by the formulae

$$F(\alpha, \tau, x) = \int_0^\infty e^{-t\mu(x)} J(tx + \alpha - \tau x, t) \, dt, \quad x \in \mathbb{R}^n, \quad (\alpha, \tau) \in A$$

$$G(\alpha, x) = \int_0^\infty e^{-t\mu(x)} J(tx + \alpha, t) \, dt, \quad x, \alpha \in \mathbb{R}^n$$

$$M_m(x) = \int_0^\infty t^{-m} \, e^{-t\mu(x)} J(tx, t) \, dt, \quad x \in \mathbb{R}^n, \quad m = 0, 1, \ldots$$

where the function $\mu(x)$ is defined in (4.10).

Comparing (4.22) with (4.23), we see that $F(\alpha, \tau, x) = G(\alpha - \tau x, x)$. We differentiate this equation to obtain

$$\frac{\partial^k F}{\partial \tau^k}(\alpha, 0, x) = (-1)^k x^1 \ldots x^k \frac{\partial^k G(\alpha, x)}{\partial \alpha^1 \ldots \partial \alpha^k}$$

which is equivalent to (4.14).

Differentiating (4.23), we get

$$\frac{\partial^m G}{\partial \alpha^1 \ldots \partial \alpha^m}(0, x) = \int_0^\infty e^{-t\mu(x)} \frac{\partial^m J}{\partial x^1 \ldots \partial x^m}(tx, t) \, dt.$$  \hspace{1cm} (4.25)

Then, if we differentiate (4.24) we obtain (using multiindices $\alpha, \beta$ and $\gamma$ of length $n$)

$$D^\alpha M_m(x) = \int_0^\infty t^{(|\alpha| - m)} e^{-t\mu(x)} (D^\alpha_x J)(tx, t) \, dt$$

$$+ \sum_{\beta + \gamma = \alpha, \gamma < \alpha} \frac{\alpha! \beta! \gamma!}{\beta! \gamma!} \int_0^\infty t^{|\gamma| - m} D^\beta_x e^{-t\mu(x)} (D^\gamma_x J)(tx, t) \, dt.$$  \hspace{1cm} (4.26)

By induction on $|\beta|$, it is easy to verify that

$$D^\beta_x e^{-t\mu(x)} = e^{-t\mu(x)} \sum_{p=1}^{\beta} t^p \mu_{\beta p}(x), \quad |\beta| > 0$$

for some smooth functions $\mu_{\beta p}(x)$ which are determined by $\mu(x)$. Substituting this expression in (4.26) we obtain

$$D^\alpha M_m(x) = A_m^\alpha(x) + \sum_{\gamma < \alpha} \sum_{p=1}^{|\alpha| - |\gamma|} \mu_{\alpha \gamma p}(x) A_m^\gamma(x)$$  \hspace{1cm} (4.27)

where

$$A_m^\alpha(x) = \int_0^\infty t^{(|\alpha| - k)} e^{-t\mu(x)} (D^\alpha_x J)(tx, t) \, dt.$$  \hspace{1cm} (4.28)

If equations (4.28) are considered as a system of linear equations with respect to $A_m^\alpha(x)$, $|\alpha| \leq m$, then the matrix of this system is triangular, and therefore the system has a solution

$$A_m^\alpha(x) = D^\alpha M_m(x) + \sum_{\gamma < \alpha} \sum_{p=1}^{|\alpha| - |\gamma|} \bar{\mu}_{\alpha \gamma p}(x) D^\gamma M_{m-p}(x).$$

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If we put here $|a| = m$ and recall the definition of $A^m_m(x)$ in (4.29), we obtain

$$
\int_0^\infty e^{-\mu(x)} \frac{\partial^m \mathcal{J}}{\partial x^1 \ldots \partial x^m}(tx, t) \, dt = \frac{\partial^m \mathcal{M}_m}{\partial x^1 \ldots \partial x^m} + \sum_{\gamma < \alpha} \sum_{p=1}^{[m]-[\gamma]} \mu_{i_1 \ldots i_m, \gamma_p} D^\gamma \mathcal{M}_{m-p}.
$$

This and (4.25) yield

$$
\frac{\partial^m G}{\partial x^1 \ldots \partial x^m}(0, x) = \frac{\partial^m \mathcal{M}_m}{\partial x^1 \ldots \partial x^m} + \sum_{k=0}^{m-1} \sum_{\gamma_k \leq k} \mu_{i_1 \ldots i_m, \gamma_k} D^\gamma \mathcal{M}_k
$$

which is equivalent to (4.15). Lemma 4.3 is proved.

We have thus proved Lemma 4.1 provided that $\beta_0$ is a proper point.

Let us now examine the case when $\beta_0$ in Lemma 4.1 is an improper point, i.e. $\beta_0 = \xi_\infty$ for $\xi \in \Omega^n$, and the sequence $\beta_k$ contains an infinite number of proper points. Without loss of generality we can consider all $\beta_k$ to be proper points.

We choose a coordinate system in $\mathbb{R}^{n+1}$ so that the vector $\xi$ has coordinates $(0, \ldots, 0, -1)$, and the set $K$ is inside the above domain $U$. Let $(\alpha_k/\tau_k, 1/\tau_k)$ be the coordinates of $\beta_k$. By condition, $\beta_k \to \xi_\infty$ as $k \to \infty$, which implies that $\tau_k < 0$ while $\alpha_k \to 0$. Besides, by condition we have $f_{\beta_k}^1 J = 0$, which in view of Lemma 4.2 is equivalent to the equalities

$$
F\left(\frac{\alpha_k}{\tau_k}, \frac{1}{\tau_k}\right) = 0, \quad k = 1, 2, \ldots. \tag{4.30}
$$

According to (4.6) and (4.10), we have for $\alpha \in \mathbb{R}^n$ and $\tau < 0$

$$
F\left(\frac{\alpha}{\tau}, \frac{1}{\tau}\right) = (-\tau)^n p'\left(\frac{e^{-\mu(x)}}{(1-\tau t)^n} f_{(\alpha/\tau, 1/\tau)}^1 J\right). \tag{4.31}
$$

In order to simplify this expression we introduce the diffeomorphism $q_{(\alpha, \tau)}: \mathbb{R}^n \to \mathbb{R}^n,

\begin{equation}
q_{(\alpha, \tau)}(x) = (x - \alpha)/\tau.
\end{equation}

Applying $q'_{(\alpha, \tau)}$ to (4.31), we have

$$
(-\tau)^{-n} q_{(\alpha, \tau)} F\left(\frac{\alpha}{\tau}, \frac{1}{\tau}\right) = q'_{(\alpha, \tau)} p'\left(\frac{e^{-\mu(x)}}{(1-\tau t)^n} f_{(\alpha/\tau, 1/\tau)}^1 J\right).
$$

Note that $q_{(\alpha, \tau)} \circ p = p \circ r_{(\alpha, \tau)}$, where the diffeomorphism $r_{(\alpha, \tau)}: U \to U$ is defined by the formula $r_{(\alpha, \tau)}(x, t) = ((\alpha - x)/\tau, t)$. Therefore, the above equation can be rewritten as

$$
(-\tau)^{-n} q_{(\alpha, \tau)} F\left(\frac{\alpha}{\tau}, \frac{1}{\tau}\right) = \alpha' r_{(\alpha, \tau)} p'\left(\frac{e^{-\mu(x)}}{(1-\tau t)^n} f_{(\alpha/\tau, 1/\tau)}^1 J\right).
$$

The diffeomorphism inverse to $r_{(\alpha, \tau)}$ is given by $r_{(\alpha, \tau)}^{-1}(x, t) = (\alpha - \tau x, t)$. Applying (3.2), we rearrange the previous equation:

$$
(-\tau)^{-n} q_{(\alpha, \tau)} F\left(\frac{\alpha}{\tau}, \frac{1}{\tau}\right) = \alpha' p'\left(\frac{e^{-\mu(x-\tau \alpha)}}{(1-\tau t)^n} f_{(\alpha, \tau)}^1 J\right) \tag{4.32}
$$

where the diffeomorphism

$$
\tilde{f}_{(\alpha, \tau)} = r_{(\alpha, \tau)} \circ f_{(\alpha/\tau, 1/\tau)}: U \to U
$$

is defined by

$$
\tilde{f}_{(\alpha, \tau)}(x, t) = \frac{(x - \alpha t)}{1 - \tau t}, t.
$$
The inverse diffeomorphism has the form

\[ f^{-1}_{(\alpha, \tau)}(x, t) = (x + (\alpha - \tau x)t, t). \]

Let us again rearrange (4.32) to simplify the function \( \mu \). We make a substitution \( J = \exp[(c, x) + c_0 t] J \), where \( c = (c^1, \ldots, c^n) \) and \( c_0 \) are constants which will be specified later. Using the rule (3.2) we obtain

\[ (-\tau)^{-n} e^{-(c, x)} q_{(\alpha, \tau)} F(\alpha, \frac{1}{\tau}) = p' \left( \frac{e^{-i\mu(\alpha - \tau x)}}{(1 - \tau t)^n} f'_{(\alpha, \tau)} J \right) \]  

where \( \mu(x) = \mu(x) - (c, x) - c_0 \). Let us choose the constants \( c \) and \( c_0 \) so that

\[ \mu(0) = 0, \quad \frac{\partial \mu}{\partial x^i}(0) = 0, \quad i = 1, \ldots, n. \]

Comparing (4.30) with (4.33) we can conclude that

\[ \tilde{F}(\alpha_k, \tau_k) = 0 \]

where the distribution \( \tilde{F} : A \to \mathcal{E}'(\mathbb{R}^n) \) is dependent on the parameter \( (\alpha, \tau) \in A \) and defined by the equations

\[ \tilde{F} = p' \left( \frac{e^{-i\mu(\alpha - \tau x)}}{(1 - \tau t)^n} f' J \right) \]

\[ \tilde{f} : U \times A \to U, \quad \tilde{f}((x, t), (\alpha, \tau)) = \left( \frac{x - \alpha t}{1 - \tau t}, t \right) \]  

(the domain \( A \) was defined after Lemma 4.1).

By analogy with (4.1)-(4.8), we define the mapping

\[ \tilde{g} : U \times \mathbb{R}^n \to U, \quad \tilde{g}((x, t), \alpha) = (x - \alpha t, t) \]

and the distribution \( \tilde{G} : \mathbb{R}^n \to \mathcal{E}(\mathbb{R}^n) \) dependent on a parameter \( \alpha \in \mathbb{R}^n \):

\[ \tilde{G} = p'(e^{-i\tilde{\mu}(x)} \tilde{g}' J) . \]

For integer \( m \geq 0 \), we define \( \tilde{M}_m \in \mathcal{E}'(\mathbb{R}^n) \) by

\[ \tilde{M}_m = p'(t^m J) . \]

**Lemma 4.4.** Assume that \( \tilde{\mu} \in C^\infty(\mathbb{R}^n) \) satisfies (4.34) and the distributions \( \tilde{F}(\alpha, \tau) \), \( \tilde{G}(\alpha) \) and \( \tilde{M}_m \) are defined in (4.36)-(4.40) for \( J \in \mathcal{E}'(\mathbb{R}^n) \). Then

\[ \frac{\partial^k \tilde{F}}{\partial \tau^k}(\alpha, 0) = (-1)^k x^{i_1} \ldots x^{i_k} \frac{\partial^k \tilde{G}(\alpha)}{\partial \alpha^{i_1} \ldots \partial \alpha^{i_k}} \]

\[ \frac{\partial^m \tilde{G}}{\partial \alpha^{i_1} \ldots \partial \alpha^{i_m}}(0) = \frac{\partial^m \tilde{M}_m}{\partial x^{i_1} \ldots \partial x^{i_m}} \]

\[ + \sum_{k=0}^{m-1} \sum_{|\beta| \leq k} \tilde{\mu}_{i_1 \ldots i_m, \beta k}(x) D^\beta \tilde{M}_k, \quad 1 \leq i_1, \ldots, i_m \leq n \]

for any \( k \geq 0 \) and \( m \geq 0 \) and some \( \tilde{\mu}_{i_1 \ldots i_m, \beta k} \in C^\infty(\mathbb{R}^n) \) which depend only on \( \tilde{\mu} \).
If we assume that this lemma is valid and repeat the argument given after the statement of Lemma 4.3, we can derive from (4.35) that $\tilde{J} = 0$ and hence $J = \exp[(c, x) + c_0]\tilde{J} = 0$.

**Proof of Lemma 4.4.** Similar to the proof of Lemma 4.3, we are to verify that it is enough to consider the case when $J = J(x, t) dt$, where $J \in C_0^\infty(U)$. In this case

$$F(\alpha, \tau) = F(\alpha, \tau, x) dx, \quad G(\alpha) = G(\alpha, x) dx, \quad \tilde{M}_m = M_m(x) dx$$

where

$$F(\alpha, \tau, x) = \int_0^\infty e^{-t\tilde{\mu}(\alpha - \tau x)} J(x + (\alpha - \tau x)t, t) dt \quad (4.43)$$

$$G(\alpha, x) = \int_0^\infty e^{-t\tilde{\mu}(\alpha)} J(x + \alpha t, t) dt \quad (4.44)$$

$$M_m(x) = \int_0^\infty t^m J(x, t) dt. \quad (4.45)$$

Comparing (4.43) with (4.44) we see that $F(\alpha, \tau, x) = G(\alpha - \tau x, x)$ which implies (4.41).

We differentiate (4.44) with respect to $\alpha$ (using a variable $\alpha \in \mathbb{R}^n$ and multiindices $\beta, \gamma$ and $\delta$) to get

$$D_\alpha^\beta G(\alpha, x) = \sum_{\gamma + \delta = \beta} \frac{\beta!}{\gamma! \delta!} \int_0^\infty t^{\|\gamma\|} D_\alpha^\gamma (e^{-t\tilde{\mu}(\alpha)})(D_\alpha^\delta J)(x + \alpha t, t) dt. \quad (4.46)$$

It is easy to prove by induction on $|\gamma|$ that

$$D_\alpha^\gamma (e^{-t\tilde{\mu}(\alpha)}) = e^{-t\tilde{\mu}(\alpha)} \left[ (-t)^{\|\gamma\|} \left( \frac{\partial \tilde{\mu}(\alpha)}{\partial \alpha} \right) + \sum_{k<|\gamma|} t^k \tilde{\mu}_{\gamma k}(\alpha) \right] \quad (4.47)$$

where

$$\left( \frac{\partial \tilde{\mu}(\alpha)}{\partial \alpha} \right)^\gamma = \left( \frac{\partial \tilde{\mu}(\alpha)}{\partial \alpha^1} \right)^{\gamma_1} \cdots \left( \frac{\partial \tilde{\mu}(\alpha)}{\partial \alpha^n} \right)^{\gamma_n}$$

and $\tilde{\mu}_{\gamma k}(\alpha)$ are smooth functions determined by the function $\tilde{\mu}$. We substitute (4.47) in (4.46) and put $\alpha = 0$ in the equation obtained. Taking into account (4.34) we have

$$(D_\alpha^\beta G)(0, x) = D_x^\beta \int_0^\infty t^{\|\gamma\|} J(x, t) dt + \sum_{k<|\beta|} \sum_{|\gamma|\leq k} \tilde{\mu}_{\gamma k}(x) D_x^\gamma \int_0^\infty t^k J(x, t) dt$$

which, in view of (4.45), can be rewritten as

$$(D_\alpha^\beta G)(0, x) = D_x^\beta \mathcal{M}_m(x) + \sum_{k<|\beta|} \sum_{|\gamma|\leq k} \tilde{\mu}_{\gamma k}(x) D_x^\gamma \mathcal{M}_k(x).$$

This equation is equivalent to (4.42). The lemma is proved.

Finally, let us consider the case in which all $\beta_k$ in Lemma 4.1 are improper points. We assume that

$$\beta_k = (\xi_k, \eta_k)_{k=0}^\infty, \quad k = 0, 1, \ldots$$

where $(\xi_k, \eta_k) = (\xi_1, \ldots, \xi_k, \eta_k) \in \Omega^n$. We choose the coordinate system so that the vector $(\xi_0, \eta_0)$ has coordinates $(0, \ldots, 0, 1)$, and the set $K$ is inside the above domain $U$.

Then

$$|\xi_k|^2 + \eta_k^2 = 1, \quad \xi_k \to 0, \quad \eta_k \to 1, \quad k \to \infty. \quad (4.48)$$
According to the definition given in (2.2), we have for $(\xi, \eta) = (\xi^1, \ldots, \xi^n, \eta) \in \Omega^n$

$$I_{(\xi, \eta)}^e J = p'_{(\xi, \eta)}[\exp(-((\xi, x) + \eta t)) \epsilon(\xi, \eta) J]$$  \hspace{1cm} (4.49)

where $p_{(\xi, \eta)} : \mathbb{R}^{n+1} \to (\xi, \eta)^\perp$ denotes an orthogonal projection. For $(\xi, \eta) \in \Omega^n$, $\eta > 0$, we define the diffeomorphism $q_{(\xi, \eta)} : (\xi, \eta)^\perp \to \mathbb{R}^n$, $q_{(\xi, \eta)}(x, t) = x - t\xi/\eta$. Applying the operator $q'_{(\xi, \eta)}$ to (4.49), we get

$$q'_{(\xi, \eta)}(I_{(\xi, \eta)}^e J) = q'_{(\xi, \eta)}p'_{(\xi, \eta)}[\exp(-((\xi, x) + \eta t)) \epsilon(\xi, \eta) J].$$

It is easy to verify that $q'_{(\xi, \eta)} \circ p_{(\xi, \eta)} = p \circ \tilde{g}_{\xi/\eta}$, where $\tilde{g}$ is given by (4.48). Consequently, the previous equation can be written as

$$q'_{(\xi, \eta)}(I_{(\xi, \eta)}^e J) = p'\tilde{g}'_{\xi/\eta}[\exp(-(\xi, x) - \eta t) \epsilon(\xi, \eta) J].$$  \hspace{1cm} (4.50)

We can readily obtain

$$[\exp(-(\xi, x) - \eta t) \epsilon(\xi, \eta)] \circ \tilde{g}_{\xi/\eta}^{-1} = \exp[-(\xi, x) \epsilon(\xi, \eta) - t\mu(\xi/\eta)].$$

Therefore, using (2.1) and (2.2) we can rearrange (4.50) into the form

$$e^{-(\xi, x)\epsilon(\xi, \eta)} q'_{(\xi, \eta)}(I_{(\xi, \eta)}^e J) = p'[e^{-t\mu(\xi/\eta)} \tilde{g}'_{\xi/\eta} J].$$  \hspace{1cm} (4.51)

Then, using the above technique we substitute $J = \exp[(c, x) + c_0] \tilde{J}$, which yields

$$e^{-(\xi, x)\epsilon(\xi, \eta)} q'_{(\xi, \eta)}(I_{(\xi, \eta)}^e J) = p'[e^{-t\mu(\xi/\eta)} \tilde{g}'_{\xi/\eta} J]$$  \hspace{1cm} (4.52)

where $\tilde{\mu} \in C^\infty(\mathbb{R}^n)$ satisfies (4.34). Comparing (4.52) with (4.39) we see that

$$e^{-(\xi, x)\epsilon(\xi, \eta)} q'_{(\xi, \eta)}(I_{(\xi, \eta)}^e J) = \tilde{G}(\xi/\eta).$$  \hspace{1cm} (4.53)

By condition, $I_{(\xi, \eta)}^e J = 0$. In view of (4.53), this implies

$$\tilde{G}(\alpha_k) = 0$$  \hspace{1cm} (4.54)

where $\alpha_k = \xi_k/\eta_k$. According to (4.48), we have $\alpha_k \to 0$ for $k \to \infty$.

Similar to the proof of (4.18), we begin with relations (4.42) and (4.54) and verify by induction on $m$ that $\tilde{M}_m = 0$ for all $m$. Hence, $\tilde{J} = 0$, and therefore $J = \exp[(c, x) + c_0] \tilde{J} = 0$.

This completes the proof of Theorem 2.1.

**Proof of Theorem 2.2.** We say that a set $\omega \subset \Omega^{n-1}$ lies in a hemisphere if there exists $\xi_0 \in \Omega^{n-1}$ and $a$, $0 < a < 1$, such that $(\xi, \xi_0) > a$ for all $\xi \in \omega$. Note that it is enough to prove Theorem 2.2 for domains which lie in hemispheres. Indeed, suppose that Theorem 2.2 is valid for domains which lie in a hemisphere and $\omega$ is an arbitrary domain in $\Omega^{n-1}$ which satisfies the conditions of Theorem 2.2. Let us represent $\omega$ in the form $\omega = \bigcup_{i=1}^k \omega_i$, where each $\omega_i$ lies in a hemisphere, the cone $C(\alpha, \omega_1)$ is in free position with respect to $\text{supp} J$, and the intersection of $\omega_i$, $i = 2, \ldots, k$, and $\omega_1 \cup \ldots \cup \omega_{i-1}$ is not empty. According to the assumption, $J\big|_{C(\alpha_0, \omega_1)} = 0$. Since $\omega_1 \cap \omega_2 \neq \emptyset$, then $C(\alpha_0, \omega_2)$ is in free position with respect to $\text{supp} J$, and therefore $J\big|_{C(\alpha_0, \omega_2)} = 0$. 


We can verify by induction that $J_{|C(α_0,ω_i)} = 0$, $i = 1, \ldots, k$. Since $C(α_0,ω) = \cup_i C(α_0,ω_i)$, $J_{|C(α_0,ω)} = 0$.

Suppose that

$$J' \in \mathcal{E}'(\mathbb{R}^{n+1}), \quad β_0 \in \mathbb{R}^{n+1}, \quad β_0 \notin \text{supp } J$$

and $ω \in \Omega^n$ is a domain lying in a hemisphere such that the cone $C(β_0,ω)$ is in a free position with respect to $\text{supp } J'$. Let $(I_β^*J^h)_{|ω} = 0$ for all $β$ in a certain neighbourhood of $β_0$. Let us choose a coordinate system $(x,t)$ in $\mathbb{R}^{n+1}$ so that $β_0 = (0, \ldots, 0)$ and the inequality $η > a > 0$ be valid for $(ξ^1, \ldots, ξ^n, η) ∈ ω$. We fix a small $δ > 0$ and suppose that $λ_δ(τ) = 0$ for $τ ≤ δ$ and $λ_δ(τ) = 1$ for $τ ≥ 2δ$. Let $J = λ_δ(τ)J'$. Hence $\text{supp } J \subset U$, where $U$ is the domain introduced after the formulation of Lemma 4.1. Then, if $δ$ is sufficiently small, the inequality

$$I_β^*J_{|ω} = I_β^*J'_{|ω} = 0 \quad (4.55)$$

is valid for all $β$ in a certain neighbourhood of $β_0$.

Using $J \in \mathcal{E}'(U)$, we define the distributions $F(α, τ), G(α)$ and $M_m$ as after Lemma 4.1. Repeating the argument in the proof of Lemma 4.2, we verify that the equation $I_β^*J_{|ω} = 0$ is equivalent to $F(α, τ)_{|D} = 0$ for any $β = (α, τ) ∈ A$. Here $D$ is the image of $ω$ in the diffeomorphism $q: \Omega^+ \rightarrow \mathbb{R}^n$ which was defined in this proof. Consequently, it follows from (4.55) that $F(α, τ)_{|D} = 0$ for all $(α, τ)$ in a certain neighbourhood of zero. Therefore, by using (4.9) and (4.15) we can verify that the equations

$$\frac{∂^mM_m}{∂x^1 \ldots ∂x^m} + \sum_{k=0}^{m-1} \sum_{|β|≤ k} β_k(x)D^βM_k = 0 \quad (4.56)$$

are valid in $D$.

According to the condition of the theorem, the cone $C(0,ω)$ is in free position with respect to $\text{supp } J$. This implies that there exists a domain $ω' \subset ω$ such that $C(0,ω') \cap \text{supp } J = \emptyset$. Suppose that $D'$ is the image of $ω'$ in the diffeomorphism $q: \Omega^+ \rightarrow \mathbb{R}^n$. Let us show that

$$M_m_{|D'} = 0, \quad m = 0, 1, \ldots . \quad (4.57)$$

Indeed, let $φ ∈ \mathcal{D}(D')$. In view of (4.13)

$$\langle M_m, φ \rangle = \left\langle p'h\left(\frac{E(x,t)}{t^{n+m}}J\right), φ \right\rangle = \left\langle E(x,t), φ\left(\frac{x}{t}\right) \right\rangle = 0$$

because the support of the function $ψ(x,t) = φ(x/t)$ is in $C(0,ω')$.

Since $D' \subset D$, then (4.56) and (4.57) imply that $M_m_{|D} = 0$, $m = 0, 1, \ldots$. Repeating the argument given after (4.21) we find that

$$\langle r'h(E(x,t)J/t^n), t^mφ(x) \rangle = 0$$

for any $φ ∈ \mathcal{D}(D)$, where the diffeomorphism $r: U → U$ is defined by the formula $r(x,t) = (x, t^{-1})$. Applying Lemma 3.1, we obtain $r'h(E(x,t)J/t^n)_{|D} = 0$. It is easy to see that the diffeomorphism $r o h$ maps $C(0,ω)$ onto $D$. Therefore, the previous equation implies that $(E(x,t)J/t^n)_{|C(0,ω)} = 0$, and consequently $J'_{|C(0,ω)} = 0$. The theorem is proved.
REFERENCES