On the multidimensional permanent and $q$-ary designs

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Let $Q_q = \{0, 1, \ldots, q - 1\}$ and $Q_{q^*} = Q_q \cup \{\ast\}$.

$Q^n_q$ denotes the $n$-dimensional hypercube.

The set of faces of $Q^n_q$ is in one-to-one correspondence with $Q^n_{q^*}$ and each $k$-dimensional face ($k$-face) corresponds to a codeword with $k$ symbols $\ast$.

For example, the set $(0, \ast, 2, \ast) = \{(0, x, 2, y) \mid x, y \in Q_3\}$ is a 2-face of $Q^4_3$. 
Definition

An $H(n, q, w, t)$ design (H-design) is a collection of $(n - w)$-faces of the hypercube $Q^q_n$ that perfectly pierce all $(n - t)$-faces.

Example

The set $\{(0, 0, 0, *), (1, 1, 1, *), (1, 0, *, 0), (0, 1, *, 1), (0, *, 1, 0), (1, *, 0, 1), (*, 1, 0, 0), (*, 0, 1, 1)\}$ is an $H(n = 4, q = 2, w = 3, t = 2)$ design.
**Definition**

An $A(n, q, w, t)$ design (A-design) is a collection of $(n - t)$-faces of $Q_q^n$ that perfectly cover all $(n - w)$-faces.

**Example**

The set $\{(0, 0, 0, *), (1, 1, 1, *), (1, 0, *, 0), (0, 1, *, 1), (0, *, 1, 0), (1, *, 0, 1), (*, 1, 0, 0), (*, 0, 1, 1)\}$ is an $A(n = 4, q = 2, w = 4, t = 3)$ design.
If $q = 1$ then an $H(n, 1, w, t)$ design is just a Steiner system $S(t, w, n)$. Here $*$ is replaced by 0 and 0 is replaced by 1.

$$(*, 0, *, *, *, 0, 0) \Rightarrow (0, 1, 0, 0, 0, 1, 1)$$

$H(7, 1, 3, 2) \Rightarrow S(2, 3, 7)$

Moreover, an $A(n, 1, w, t)$ design is just a Steiner system $S(n - w, n - t, n)$. Here $*$ is replaced by 1.

$$(*, 0, *, 0, 0, *, 0) \Rightarrow (1, 0, 1, 0, 0, 1, 0)$$

$A(7, 1, 5, 4) \Rightarrow S(2, 3, 7)$
A set of 1-faces is called a **precise clique matching** if it is both $H(n, q, n - 1, n - 2)$ design and $A(n, q, n, n - 1)$ design. The precise clique matchings with $n = 2^{t+1}$ and $q = 2^t$ are constructed in


**Example (when $t = 1$) from**

P. Hamburger, R. E. Pippert and W. D. Weakley, On a leverage problem in the hypercube, Networks. 1992


Construction 1

Let $S \subseteq Q^n_{q^*}$ be an $H(n, q, w, t)$ design and let $R \subseteq Q^w_{q'^*}$ be an $H(w, q', w, w - 1)$ design (MDS code). Given $(a^1, \ldots, *, \ldots, a^i, \ldots, *, \ldots, a^w) \in S$ and $(b_1, \ldots, b_w) \in R$ arrange the codeword

$$(((a^1, b_1), \ldots, *, \ldots, (a^i, b_i), \ldots, *, \ldots, (a^w, b_w)) \in Q^n_{qq'^*}."

Proposition 1

The set of all these codewords is an $H(n, qq', w, t)$ design.

As mentioned above, $H(2k, k, 2k - 1, 2k - 2)$ designs exist for $k = 2^t$, $t \geq 1$. Since MDS codes with distance 2 exist for all $q \geq 2$, we get

Corollary 1

For all $s, t \geq 1$ there exist $H(2^{t+1}, s2^t, 2^{t+1} - 1, 2^{t+1} - 2)$ designs.
Construction 2

Let $S \subset Q_{q^*}^n$ be an $A(n, q, w, t)$ design. For each pair of $(a^1, \ldots, *, \ldots, a^i, \ldots, *, \ldots, a^t) \in S$ and $(b_1, \ldots, b_t) \in Q_{q'}^w$ we form the codeword

$$((a^1, b_1), \ldots, *, \ldots, (a^i, b_i), \ldots, *, \ldots, (a^t, b_t)) \in Q_{qq^*}^n.$$

Proposition 2

The set of all these codewords is an $A(n, qq', w, t)$ design.

As mentioned above, each Steiner system $S(n - w, n - t, n)$ is equivalent to an $A(n, 1, w, t)$ design.

Corollary 2

If there exists a Steiner system $S(n - w, n - t, n)$ then for each $q \geq 1$ there exists an $A(n, q, w, t)$ design.
Consider a $k$-partite hypergraph $G_k$ containing $N$ vertices in each part $C_i$, $i = 1, \ldots, k$. Suppose that each $k$-edge of $G_k$ consists of $k$ vertices, with one vertex in each part of the hypergraph. A set of disjoint $k$-edges that matches all vertices of the hypergraph is called a perfect $k$-matching.
Let each part of the hypergraph be enumerated by $1, 2, \ldots, N$. We define the adjacency array $M(G_k) = (a_{i_1 \ldots i_k})$ by the following rule: $a_{i_1 \ldots i_k} = 1$ if there exists a $k$-edge consisting of vertices with numbers $i_1$ from the first part, $i_2$ from the second part and so on and $a_{i_1 \ldots i_k} = 0$ otherwise. A $k$-element subset $I$ of $\{1, \ldots, N\}^k$ is called a diagonal if every pair of elements of $I$ is distinct in each position. We define the $k$-dimensional permanent of $M(G_k)$ as

$$\text{per}_k M(G_k) = \sum_{I \in D_N} \prod_{(i_1, \ldots, i_k) \in I} a_{i_1 \ldots i_k},$$

where $D_N$ is the set of all diagonals.
It is well known that the permanent of the adjacency matrix of a bipartite graph is equal to the number of perfect matchings of the graph. The following statement is straightforward.

Proposition 3

The number of perfect $k$-matchings of a hypergraph $G_k$ is equal to $\text{per}_k M(G_k)$. 
Denote by $Q_q^n(t)$ the set of $(n - t)$-faces of $Q^n_q$. By definition each $A(n, q, w, t)$ design is a subset of $Q_q^n(t)$ such that its faces do not intersect but cover $Q_q^n(w)$. We assume that there exists a partition $A = \{A_1, \ldots, A_m\}$, where $m = \binom{w}{t}$, of $Q_q^n(t)$ into $A(n, q, w, t)$ designs. Define the $m$-part hypergraph $GA$ with parts $A_1, \ldots, A_m$. A collection $\{a_1, \ldots, a_m\}$, where $a_i \in Q^n_q(t)$, is a $m$-edge in $GA$ if there exists $b \in Q^n_q(w)$, $\overline{b} = \bigcap_{i=1}^{m} \overline{a_i}$.

**Proposition 4**

The number of different $H(n, q, w, t)$ designs is equal to $\text{per}_m M(GA)$.

**Proposition 5**

The number of different $A(n, q, w, t)$ designs is equal to $\text{per}_k M(GH)$. 