On the Existence of Immigration Proof Partition into Countries in Multidimensional Space

Valeriy M. MARAKULIN*

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Abstract

The existence of immigration proof partition for communities (countries) in a multidimensional space is studied. This is a Tiebout type equilibrium which existence previously was stated only in one-dimensional setting. The migration stability suggests that the inhabitants of frontier have no incentives to change jurisdiction (an inhabitant at every frontier point has equal costs for all possible adjoining jurisdictions). It means that inter-country boundary is represented by a continuous curve (surface).

Assuming measurable population density two approaches are suggested: the first is based on an one-dimensional approximation, for which a fixed point (via Kakutani's theorem) can be found and passing to limits gives the result; the second applies a new generalization of Krasnosel'skii fixed point theorem for polytopes. This develops [8] and extends the result to an arbitrary number of countries, arbitrary dimension, possibly continuous dependence on additional parameters and so on.

Keywords and Phrases: country formation, Alesina and Spolaore's world, migration, stable partitions, multidimensional space, Krasnosel'skii fixed point theorem
JEL Classification Numbers: D70, H20, H73

1 Introduction

In the seminal paper [1] a basic model of country formation was offered. In this model, the cost of the population is described as the sum of the two values—the ratio of total costs on the total weight of the population plus transportation costs to the center of the state. This model has been studied in a number of subsequent studies, but in each of them deals with the case of one-dimensional region and the interval-form countries (country formation on the interval $[0, 1]$).

The first progress in the resolution of the problem of existence was obtained in [2], where well known Gale–Nikaido–Debreu lemma was applied to state the existence of *nontrivial* immigration proof partition for interval countries, i.e. such that no one has incentive to change their country of residence. In [2] were made rather strong assumptions on the distribution of the population—continuous density, separated from zero. Next in [5] mathematical part of the approach was significantly strengthened and extended to the case of distribution of the population, described as a Radon measure (probability measure defined on the Borel σ-algebra). In [8] a new significant advancement was suggested, it disseminates the result (existence theorem) to the case of 2-dimensional (or more) region. The proof of [8] is very elegant and is based on the application of KKM-lemma (Knaster–Kuratowski–Mazurkiewicz), but the result is essentially limited by the presence of fixed in the space positions of capitals. In this paper, I intend to take the next step and let capitals (or other relevant parameters) be changed continuously in space, which is important

*Valeriy M. MARAKULIN, Sobolev Institute of Mathematics, Russian Academy of Sciences, 4 Acad. Koptyug avenue, Novosibirsk, Russia, 630090; Novosibirsk State University, 2 Pirogova, Novosibirsk, Russia, 630090; **e-mail**: marakul@math.nsc.ru

for example in the context of party formation. The proof is based on a new original generalization of Krasnosel'skii's fixed point theorem, which is extended to the case of a convex polytope (bounded polyhedron) that is interesting in its own right.

In the first section, we consider a particular case of division of the rectangular area into two countries at a given measurable random distribution of the population. There will be described a basic one-dimensional approximation, for which a fixed point (via Kakutani's theorem) can be found, and then the limit process gives the result.

The second section provides a further generalization of the existence result which is extended to an arbitrary number of countries, arbitrary dimension, and possibly continuous dependence on a finite number of significant parameters for country formation (capitals and so on).

2 The partition into two countries on the plane

Of course, the division of the one-dimensional world on countries can not be considered as a satisfactory solution of the problem. However, two-dimensional formulation seems fundamentally more difficult problem. Now for a particular example of division of the rectangular area in two countries we consider an approximating design allowing to find a solution by passing to the limit.

First, we define the principle of stability applied for the country located on the plane. As in the case of one-dimensional world, it must be such division that boundary residents have no incentive to change their jurisdiction. Thus, the costs for any boundary resident should be the same with respect to any of the possible for her/him adjoining jurisdictions. It is assumed that the boundaries between the two countries allow continuous parametrization, i.e. they are the image of the interval from $\mathbb{R}$ for some *continuous one-to-one* mapping. As a result, as in the one-dimensional case, the function of individual costs of individuals should be continuous on *the whole field* of country division, that is, country partition must implement continuous "gluing" of country-depended individual costs.

For simplicity, we consider now a particular case of a rectangular area of possible settlement represented in the Figure 2 rectangle $\square ABCD$. We assume that $c_i(\cdot)$, $i = 1, \dots, n$ are the functions of individual costs, depending on the place of individual location—defined via coordinates $(x, y) \in \square ABCD$, the weight of the resident jurisdiction $\mu_i(S_i)$, the location of its center $r_c(S_i)$, metrics $\rho(\cdot, \cdot)$ (to specify the distance to the center) and so on. The basic model representation of these cost functions is

$$c_i(x, y, \delta_i, r_c(S_i)) = \frac{g_i}{\delta_i} + \rho((x, y), r_c(S_i)), \quad g_i > 0, \quad i \in N = \{1, 2, \dots, n\}. \tag{1}$$

Here scalar variables $\delta_i > 0$ are associated with the i-th country mass of population, i.e. $\delta_i = \mu_i(S_i)$; $g_i > 0$ is an expenditure (costs) on the maintenance of government which are uniformly distributed among the country citizens. The second summand $\rho((x, y), r_c(S_i))$ presents an individual expenditure specified by inhabitant location at the point $(x, y) \in \square ABCD$. In general cost functions may have sufficiently general form but they always continuously depend on certain country parameters and obey some other specific assumptions (see Section 3). Everywhere below we shall assume

(**P**) *The distribution of population is described by an absolutely continuous probability measure* μ *such that* $\text{supp}(\mu) = \square ABCD$.[1]

2.1 Particular solution for hyperbolic boundaries

For the beginning we consider a simple proof of immigration stable division into *two* countries with fixed centers. The proof is based on the boundary specific properties due to the representation of the cost functions $c_1(\cdot)$, $c_2(\cdot)$ by (1) and therefore having the boundaries defined by equation

$$||(x, y) - r_c(S_1)||_2 - ||(x, y) - r_c(S_2)||_2 = \frac{g_2}{\mu(S_2)} - \frac{g_1}{\mu(S_1)} = const. \tag{2}$$

[1]This combined means that $\mu(A) > 0 \iff \int_A dxdy > 0$ for every measurable $A \subseteq \square ABCD$.

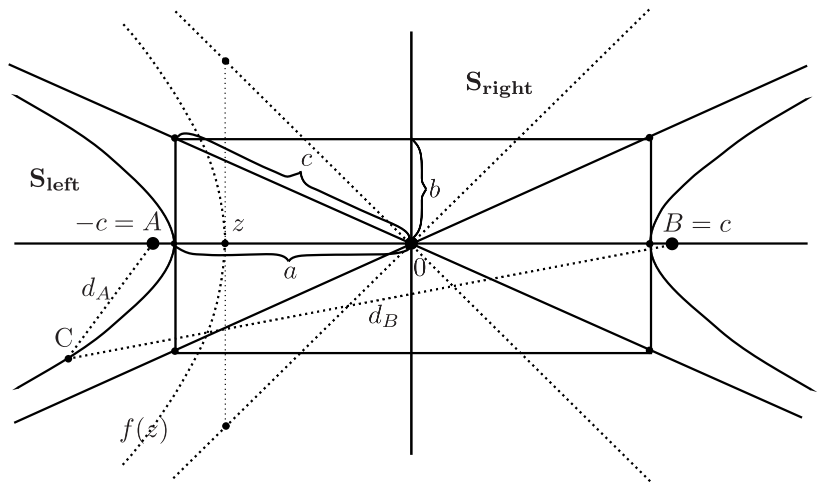

Figure 1: Hyperbola $f(z)$ with focuses at A and B

Thus a possible boundary has hyperbolic form and, for Euclidean distance, (2) defines the classic hyperbola. Now applying the specific properties of this curve, we offer a simple short proof. Next we first recall some well-known facts, see Figure 1.

Hyperbola focuses are located at the points $A = (-c, 0)$ and $B = (c, 0)$, vertices at $(-a, 0)$ and $(a, 0)$ on transverse axis for a real $a > 0$ and for conjugate axis $b > 0$; these values are related by $c^2 = a^2 + b^2$. A current point C on the left hyperbola branch satisfies the property $\rho(C, A) - \rho(C, B) = d_A - d_B = -2a$. Note that for fixed focal points of hyperbola its specific branch can be fully determined by the vertex: in the Figure 1 a point $z \in [-c, c]$ specifies the left branch with parameters $a = -z$, $b = \sqrt{c^2 - z^2}$ (for $z \geq 0$ and $a = z$ the right branch is considered).

In the context of the problem of area division into the countries with the presented centers (foci of the hyperbola) for the segment connecting them one can define the following map:

$$H(z) = \frac{\mathbf{g_{right}}}{\mu(\mathbf{S_{right}}(z))} - \frac{\mathbf{g_{left}}}{\mu(\mathbf{S_{left}}(z))} - 2z, \quad z \in (-c, c).$$

Here $\mu(\mathbf{S_{right}}(z))$ and $\mu(\mathbf{S_{left}}(z))$ are masses of citizens of the countries, whose boundary is the hyperbola. Obviously, that this is a positive-valued continuous functions of $z \in (-c, c)$. It is also evident that

$$\lim_{z \to -c} H(z) = -\infty \quad \& \quad \lim_{z \to c} H(z) = +\infty.$$

Now the continuity of $H(z)$ implies that there exists a point $\bar{z} \in (-c, c)$ such that $H(\bar{z}) = 0$ and therefore

$$\frac{\mathbf{g_{right}}}{\mu(\mathbf{S_{right}}(\bar{z}))} - \frac{\mathbf{g_{left}}}{\mu(\mathbf{S_{left}}(\bar{z}))} = 2\bar{z}.$$

However, for the points of hyperbola f with a vertex $(\bar{z}, 0)$ we have

$$\forall (x, y) \in f \quad ||(x, y) - A||_2 - ||(x, y) - B||_2 = 2\bar{z} = \frac{\mathbf{g_{right}}}{\mu(\mathbf{S_{right}}(\bar{z}))} - \frac{\mathbf{g_{left}}}{\mu(\mathbf{S_{left}}(\bar{z}))} \quad \Rightarrow$$

$$||(x, y) - r_c(\mathbf{S_{left}})||_2 + \frac{\mathbf{g_{left}}}{\mu(\mathbf{S_{left}})} = c_1(x, y) = c_2(x, y) = ||(x, y) - r_c(\mathbf{S_{right}})||_2 + \frac{\mathbf{g_{right}}}{\mu(\mathbf{S_{right}})},$$

which means immigration consistency for the presented country division. We have thus proved

Theorem 1 *Under the above assumptions for any given convex bounded domain and any fixed centers there is immigration proof division into* ***two countries****.*

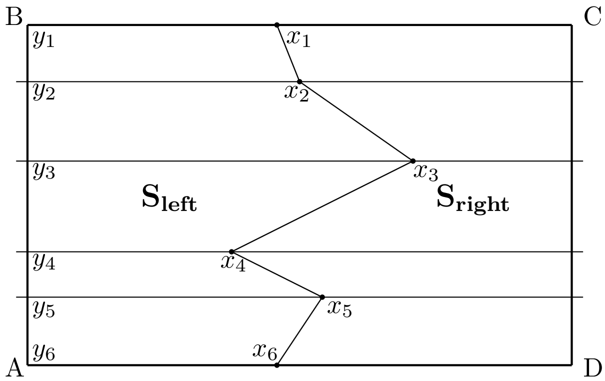

Figure 2: Possible division into two countries of the rectangular area $ABCD$, $m = 6$.

2.2 General partition via one-dimensional approximation

The idea of approach is that given coordinate system (potentially curved), a stable partition relative to one-dimensional world appeared along every coordinate line. At the same time, the function of individual costs must be calculated relative to the position of "center" of the country and the general population distributed in *two-dimensional space.* Finding such a partition is not an easy task, to solve this we shall apply a special "one-dimensional approximation", relative which a country partition can be found by a fixed point theorem (Brouwer or Kakutani).

The construction as follows: specify $m - 2$ straight lines parallel to the base of the rectangle, $m \geq 3$. Let the lower base has a number m, top one—the number 1 and all others are numbered from top to bottom. Each i-th segment is divided into two parts by the point x_i, which can be considered the point from interval $[0, 1]$ (length of the base $\Box ABCD$), $i = 1, \ldots, m$. Straight line segments connecting consecutive points $x_1, \ldots, x_m$, form a polygon line, which we accept as the boundary between the left and right countries. Now, if density $f(x, y)$ is presented then it is possible to integrate it over each of the country area, finding the weights (size) $\mu(S)$ of their populations.

Within each country its "center" (the capital) $r_c(S) \in S$ is specified, the position of which we will consider as *depending continuously* from a given country settings $\mathbf{x} = (x_1, \ldots, x_m) \in [0, 1]^m$. Thus we have:

$$\mu(\mathbf{S_{left}}) = \int_{\mathbf{S_{left}}} f(x, y) dx dy \geq 0, \quad r_c(\mathbf{S_{left}}) = r_{\mathbf{left}}(x_1, \ldots, x_m) \in \mathbf{S_{left}}$$

$$\mu(\mathbf{S_{right}}) = \int_{\mathbf{S_{right}}} f(x, y) dx dy \geq 0, \quad r_c(\mathbf{S_{right}}) = r_{\mathbf{right}}(x_1, \ldots, x_m) \in \mathbf{S_{right}}.$$

Moreover, without loss of generality

$$\mu(\mathbf{S_{left}}) + \mu(\mathbf{S_{right}}) = 1.$$

The fact that we are talking about the "mass of the population" and the "distance to the center" (transport availability of capital) as the main parameter determining the costs of individuals in a country is only an interpretation of the cost function in the context of the main model variant. The same can be said about the property of the center of the country be located on its territory—it's just a natural variant of content, from a mathematical point of view, the center could be anywhere. The really important fact is (described below) certain specific properties of individual costs.

Next, consider a point-to-set mapping, whose fixed point gives the desired country partition. The construction of mapping applies the ideas borrowed from the one-dimensional case, see [6]. Let

$$X = [0, 1]^m,$$

and define a point-to-set mapping of X into itself.

Let $c_1(\cdot)$, $c_2(\cdot)$ be the functions of individual costs depending on the weight of the jurisdiction population $\mu_1(\mathbf{x})$, $\mu_2(\mathbf{x})$, location of its center $r_c(S_1)$, $r_c(S_2)$, metrics $\rho(\cdot,\cdot)$ (to determine the distance to the center) and a place of the individual location specified by coordinates $(x,y) \in \Box ABCD$. The basic model representation of these functions is (1). Now we shall think that they are functions of general form continuously depending on $\mathbf{x} = (x_1,\dots,x_m) \in [0,1]^m$ for $\mu(S_k(\mathbf{x})) > 0$, $k = 1,2$. Additionally assume that

(*i*) $c_k(x,y,\mathbf{x}) > 0$ for $\mu(S_k) \neq 0$ and

(*ii*) $c_k(x,y,\mathbf{x}) \to +\infty$ if $\mu(S_k) \to 0$, $k = 1,2$.

For the functions of (1) this condition is always satisfied. At the same time, if the density $f(x,y)$ of the population is so that $\int_A dxdy > 0$ implies $\int_A f(x,y)dxdy > 0$ for every measurable subset $A \subset \Box ABCD$ (i.e. each subset of nonzero area (Lebesgue measure) has a population of non-zero mass), the latter requirement is equivalent to

$$c_1(x,y,\mathbf{x}) \to +\infty \iff \mathbf{x} \to (0,\dots,0) \;\; \& \;\; c_2(x,y,\mathbf{x}) \to +\infty \iff \mathbf{x} \to (1,\dots,1). \quad (3)$$

For the boundary points $x_1,\dots,x_m$ of country areas let us find an excess cost of possible (two) jurisdictions (constants $y_1,\dots,y_m$ in the argument are excluded)

$$h_i(\mathbf{x}) = c_1(x_i,\mathbf{x}) - c_2(x_i,\mathbf{x}), \quad i = 1,\dots,m.$$

Notice that (3) implies that for all $i = 1,\dots,m$, $h_i(\mathbf{x}) \to +\infty$ for $\mathbf{x} \to 0$, and $\mathbf{x} \to \mathbf{1}$ when $h_i(\mathbf{x}) \to -\infty$.

Next we define the (single-valued) map $\varphi : X \to X = [0,1]^m$ putting

$$\varphi_i(\mathbf{x}) = \begin{cases} x_i - \frac{x_i}{2} \cdot \frac{h_i(\mathbf{x})}{1+h_i(\mathbf{x})}, & \text{for } h_i(\mathbf{x}) \geq 0, \\ x_i + \frac{1-x_i}{2} \cdot \frac{h_i(\mathbf{x})}{h_i(\mathbf{x})-1}, & \text{for } h_i(\mathbf{x}) \leq 0. \end{cases} \quad (4)$$

By construction, this mapping is well defined everywhere on X with the exception of two points $\mathbf{x} = \mathbf{0} = (0,\dots,0)$ and $\mathbf{x} = \mathbf{1} = (1,\dots,1)$, which values can be defined by continuity:

$$\varphi(\mathbf{0}) = (0,\dots,0), \quad \varphi(\mathbf{1}) = (1,\dots,1).$$

It is obvious that according to the construction these points are *trivial* fixed points of $\varphi(\cdot)$, that does not comply with the requirements of the division of rectangular area. Further construction and analysis will focus on the finding of the *nontrivial* fixed point corresponding to the division of the area into two countries with non-zero masses of the population. Let $\Delta = \{(\mu_1,\mu_2) \mid \mu_1 + \mu_2 = 1, \mu_1 \geq 0, \mu_2 \geq 0\}$.

Now we define a point-to-set mapping Φ from $\mathfrak{X} = X \times \Delta$ to X, by formula: for $(\mu_1,\mu_2) = (\mu(\mathbf{S_{left}}(\mathbf{x})), \mu(\mathbf{S_{right}}(\mathbf{x})))$ specify

$$\Phi(\mathbf{x},\nu) = \begin{cases} \{\frac{\nu_1}{\mu_1}\varphi(\mathbf{x})\}, & \text{for } \nu_1 \leq \mu_1, \;\; \mu_1 \neq 0, \\ \{\frac{\nu_2}{\mu_2}\varphi(\mathbf{x}) + \frac{\mu_2-\nu_2}{\mu_2}(1,\dots,1)\}, & \text{for } \nu_2 \leq \mu_2, \;\; \mu_2 \neq 0, \\ X, & \text{for } \nu_1 = \mu_1 = 0, \;\; \text{or } \;\; \nu_1 = \mu_1 = 1. \end{cases} \quad (5)$$

The second mapping $\Psi : X \rightrightarrows \Delta$ is specified as follows

$$\Psi(\mathbf{x}) = \operatorname*{argmax}_{\nu \in \Delta} \langle H(\mathbf{x}), \nu \rangle. \quad (6)$$

where $H(\mathbf{x}) = (H_1(\mathbf{x}), H_2(\mathbf{x}))$ and

$$I_+ = \{i \mid h_i(\mathbf{x}) \geq 0, \; i = 1,\dots,n\}, \quad I_- = \{i \mid h_i(\mathbf{x}) \leq 0, \; i = 1,\dots,n\}$$

are defined by formulas[2]

$$H_1(\mathbf{x}) = [\inf_{i=1,\dots,m} h_i(\mathbf{x})]^+ + \sum_{i\in I_+} x_i \tfrac{h_i(\mathbf{x})}{h_i(\mathbf{x})+1}, \qquad I_+ \neq \emptyset$$
$$H_2(\mathbf{x}) = [\sup_{i=1,\dots,m} h_i(\mathbf{x})]^- + \sum_{i\in I_-} (1-x_i) \tfrac{h_i(\mathbf{x})}{h_i(\mathbf{x})-1}, \qquad I_- \neq \emptyset.$$

If $I_+ = \emptyset$ or $I_- = \emptyset$, then by definition $H_1(\mathbf{x}) = 0$ and $H_2(\mathbf{x}) = 0$ respectively. Constructed map is well defined everywhere except at zero and one for which we postulate

$$\Psi(\mathbf{0}) = (1,0), \qquad \Psi(\mathbf{1}) = (0,1).$$

Finally, we define the resulting mapping

$$\Upsilon : \mathfrak{X} \rightrightarrows \mathfrak{X}, \qquad \Upsilon(\mathbf{x},\nu) = \Phi(\mathbf{x},\nu) \times \Psi(\mathbf{x},\nu),$$

which fixed points give us the desired result. The following lemma describes the important properties of the mapping $\Upsilon(\cdot)$.

Lemma 1 *The mapping $\Upsilon : \mathfrak{X} \rightrightarrows \mathfrak{X}$ is a Kakutani map, i.e. it has closed graph and for every $\kappa \in \mathfrak{X}$ takes non-empty convex values.*

Proof of Lemma 1. Check the properties of $\Psi(\cdot)$. We need to show that it has a closed graph. First, we establish the continuity of $H = (H_1, H_2)$. To this end, we consider the function

$$g^-(t) = \begin{cases} \frac{t}{t-1}, & \text{for } t \le 0, \\ 0, & \text{for } t \ge 0, \end{cases} \qquad g^+(t) = \begin{cases} \frac{t}{t+1}, & \text{for } t \ge 0, \\ 0, & \text{for } t \le 0, \end{cases}$$

which obviously are continuous on $[-\infty, +\infty]$. From the construction one can now write

$$H_1(\mathbf{x}) = [\inf_{i=1,\dots,m} h_i(\mathbf{x})]^+ + \sum_{i=1}^{m} x_i \cdot g^+(h_i(\mathbf{x})),$$

$$H_2(\mathbf{x}) = [\sup_{i=1,\dots,m} h_i(\mathbf{x})]^- + \sum_{i=1}^{m} (1-x_i) g^-(h_i(\mathbf{x})).$$

This form of representation clearly implies the continuity of $H(\cdot)$ at all points except $\mathbf{0}$ and $\mathbf{1}$. So, everywhere on X, except these points, $\Psi(\cdot)$ is closed. It is also closed at zero, since by construction (due to the first term) $H_1(\mathbf{x}) > 0$ and $H_2(\mathbf{x}) = 0$ for all $\mathbf{x}$ sufficiently close to zero. Consequently, in some neighborhood of zero $\Psi(\mathbf{x}) \equiv (1,0)$, which means that the closure of $\Psi(\cdot)$ at the origin. Closeness at the unit is stated in a similar way.

All other required properties of the mapping $\Upsilon(\cdot)$ are established by a routine checking of definitions. Lemma is proved. ■

Lemma 2 *Under the above assumptions, the map $\varphi(\cdot)$ has **nontrivial** fixed point in X such that the mass of the population of each country is **nonzero**.*

Remark 1 The map $\varphi(\cdot)$ can be extended from a finite-dimensional space to the space of continuous functions as follows. Consider the map $\mathfrak{F} : C([0,1]) \to [0,1]$, defined by the formula: for continuous $f : [0,1] \to [0,1]$ define

$$\mathfrak{F}(f)(y) = \begin{cases} f(y) - \frac{f(y)}{2} \cdot \frac{h(f,x,y)}{1+h(f,x,y)}, & \text{if } h(f,x,y) \ge 0, \\ f(y) + \frac{1-f(y)}{2} \cdot \frac{h(f,x,y)}{h(f,x,y)-1}, & \text{if } h(f,x,y) \le 0. \end{cases}$$

Here the boundary between the countries is the graph of f, i.e.

$$\mathbf{S_{left}} = \{(x,y) \in \Box ABCD \mid x \le f(y)\} \quad \& \quad \mathbf{S_{right}} = \Box ABCD \setminus \mathbf{S_{left}}.$$

However, to prove the existence of a fixed point of $\mathfrak{F}$ in the above abstract form is problematic one due to the characterization of compact sets in the space of continuous functions (equicontinuity is required) and the absence of significant motives for compact localization of the fixed point. ■

[2] We use standard notations $z^+ = \sup\{z, 0\}$ and $z^- = \sup\{(-z), 0\}$ for any real z.

Proof of Lemma 2. Consider any fixed point

$$(\bar{\mathbf{x}}, \bar{\nu}) \in \Upsilon(\bar{\mathbf{x}}, \bar{\nu}),$$

which does exist due to Lemma 1 and Kakutani fixed point theorem. Let us show that this point satisfies

$$0 < \bar{\nu}_1 < 1 \quad \& \quad \bar{\mathbf{x}} \neq \mathbf{0}, \quad \bar{\mathbf{x}} \neq \mathbf{1}. \tag{7}$$

Suppose that the first country has zero mass of the population, that is $\mu(\mathbf{S_{left}}(\bar{\mathbf{x}})) = \mu_1 = 0$. This is only possible if $\bar{\mathbf{x}} = 0$ that implies $h_i(\bar{\mathbf{x}}) = +\infty \ \forall i = 1, \ldots, m \Rightarrow H_1(\bar{\mathbf{x}}) > 0$ and $H_2(\bar{\mathbf{x}}) = 0$. Now by formula (6) and properties of the fixed point we conclude $\bar{\nu} = (\bar{\nu}_1, \bar{\nu}_2) = (1, 0)$ that due to (5) in the case $\nu_1 = 1 \geq 0 = \mu_1$ and $\mu_2 = 1$, $\nu_2 = 0$ implies

$$\frac{\bar{\nu}_2}{\mu_2}\varphi(\mathbf{x}) + \frac{\mu_2 - \bar{\nu}_2}{\mu_2}(1, \ldots, 1) = (1, \ldots, 1) \neq 0 = \bar{\mathbf{x}}.$$

This contradiction proves $\bar{\mathbf{x}} \neq 0$.

The case of the second country with zero mass of the population is considered in a similar way:

$$\mu(\mathbf{S_{right}}(\bar{\mathbf{x}})) = \mu_2 = 0 \quad \Longleftrightarrow \quad \bar{\mathbf{x}} = (1, \ldots, 1) \quad \Rightarrow \quad h_i(\bar{\mathbf{x}}) = -\infty \quad \forall i = 1, \ldots, m.$$

Therefore, $H_2(\bar{\mathbf{x}}) > 0$ and $H_1(\bar{\mathbf{x}}) = 0$, that due to (6) implies $\bar{\nu} = (\bar{\nu}_1, \bar{\nu}_2) = (0, 1)$. By construction (5) in the case $\nu_1 \leq \mu_1$ and $\mu_1 = 1$, $\nu_1 = 0$ one has

$$\frac{\bar{\nu}_1}{\mu_1}\varphi(\bar{\mathbf{x}}) = (0, \ldots, 0) \neq (1, \ldots, 1) = \bar{\mathbf{x}},$$

that proves (7). This due to (6) allows to conclude $H_1(\bar{\mathbf{x}}) = H_2(\bar{\mathbf{x}})$. Let us show now that $H_1(\bar{\mathbf{x}}) = H_2(\bar{\mathbf{x}}) = 0$ is the only possibility.

Suppose $H_1(\bar{\mathbf{x}}) = H_2(\bar{\mathbf{x}}) \neq 0$. Now first notice that $[\inf_{i=1,\ldots,n} h_i(\bar{\mathbf{x}})]^+ > 0$ is impossible since otherwise $H_1(\bar{\mathbf{x}}) > 0$ and $H_2(\bar{\mathbf{x}}) = 0$ that is invalid. Likewise, it is impossible $[\sup_{i=1,\ldots,n} h_i(\bar{\mathbf{x}})]^- > 0$. Therefore, both of these terms in the definition of H are vanished. Now, from the definition of H one can conclude that there are i, j such that

$$h_i(\bar{\mathbf{x}}) > 0 \quad \& \quad \bar{x}_i \cdot \frac{h_i(\bar{\mathbf{x}})}{h_i(\bar{\mathbf{x}}) + 1} \neq 0 \quad \Rightarrow \quad \bar{x}_i > 0,$$

$$h_j(\bar{\mathbf{x}}) < 0 \quad \& \quad (1 - \bar{x}_j) \cdot \frac{h_j(\bar{\mathbf{x}})}{h_j(\bar{\mathbf{x}}) - 1} \neq 0 \quad \Rightarrow \quad \bar{x}_j < 1.$$

Next, we turn again to the properties of the fixed point and the formula (5). In the first case, for $0 < \nu_1 \leq \mu_1 < 1 \Rightarrow 0 < \lambda = \frac{\nu_1}{\mu_1} \leq 1$, via $\bar{x}_i > 0$ we have

$$\bar{x}_i > \Phi_i(\bar{\mathbf{x}}, \bar{\nu}) = \lambda\varphi_i(\bar{\mathbf{x}}) = \lambda\left[\bar{x}_i - \frac{\bar{x}_i}{2} \cdot \frac{h_i(\bar{\mathbf{x}})}{h_i(\bar{\mathbf{x}}) + 1}\right].$$

In the second case, for $0 < \lambda = \frac{\nu_2}{\mu_2} \leq 1$, via $\bar{x}_j < 1$ we have

$$\bar{x}_j < \Phi_i(\bar{\mathbf{x}}, \bar{\nu}) = \lambda\varphi_j(\bar{\mathbf{x}}) + 1 - \lambda = \lambda\left[x_j + \frac{1 - \bar{x}_j}{2} \cdot \frac{h_j(\bar{\mathbf{x}})}{h_j(\bar{\mathbf{x}}) - 1}\right] + 1 - \lambda.$$

Both cases are impossible. Consequently, it is proved $H_1(\bar{\mathbf{x}}) = H_2(\bar{\mathbf{x}}) = 0$. By construction, this is equivalent to

$$\bar{x}_i\frac{h_i(\bar{\mathbf{x}})}{h_i(\bar{\mathbf{x}}) + 1} = 0, \quad h_i(\bar{\mathbf{x}}) \geq 0 \quad \forall i = 1, \ldots, m,$$

$$(1 - \bar{x}_j)\frac{h_j(\bar{\mathbf{x}})}{h_j(\bar{\mathbf{x}}) + 1} = 0 \quad h_j(\bar{\mathbf{x}}) \leq 0 \quad \forall j = 1, \ldots, m.$$

However due to (4) the latter means that $\bar{\mathbf{x}} \in X$ is a nontrivial fixed point of $\varphi(\cdot)$. ■

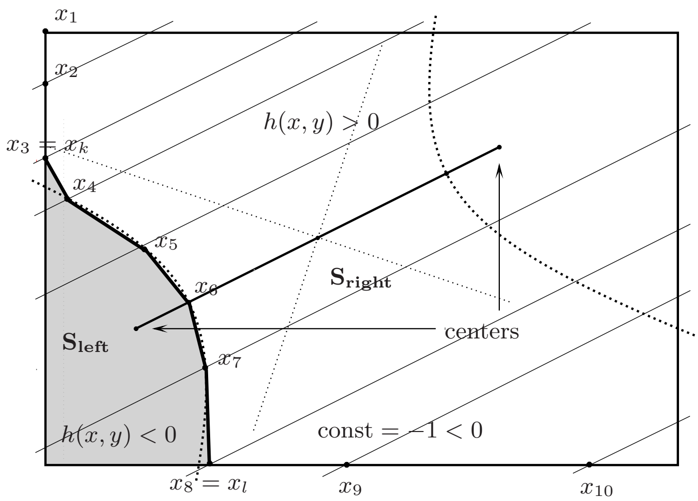

Figure 3: Partition according to (i)–(ii) for $const < 0 \iff g_2\mu(\mathbf{S_{left}}) < g_1\mu(\mathbf{S_{right}})$.

Theorem 2 *Let the individual costs be given by (1) and centers be situated on a line parallel to the axis of abscissa. Then for each positive integer $m \in \mathbb{N}$ there exists the partition of $\square ABCD$ into two countries $\mathbf{S_{left}}(\mathbf{x})$ and $\mathbf{S_{right}}(\mathbf{x})$, with piecewise linear boundary formed by the points $x_k, \ldots, x_l$, $1 < k+1 \le l-1 < m$ where all $x_{k+1}, \ldots, x_{l-1}$ are immigration proof.*

Corollary 1 *Assume that the costs in formula (1) are calculated relative to the Euclidean distance. Then in the conditions of Theorem 2 boundary points $x_{k+1}, \ldots, x_{l-1}$ are suited on classical hyperbola defined by equation (2). In the case of a more general form of the metric (for example, for p-norm), these points belong to a generalized hyperbola.*

Proof of Theorem 2. Consider a fixed point

$$\mathbf{x} = (x_1, \ldots, x_n) = \varphi(\mathbf{x}),$$

which satisfies the conclusion of Lemma 2. Note that from the construction of $\varphi(\cdot)$ for each $i = 1, \ldots, m$ (formula (4)) only one of three possibilities is realized:

(i) $h_i(\mathbf{x}) = 0$,

(ii) $h_i(\mathbf{x}) > 0 \Rightarrow x_i = 0$,

(iii) $h_i(\mathbf{x}) < 0 \Rightarrow x_i = 1$.

Indeed, for example consider alternative (ii). Assuming the contrary, one concludes $x_i \frac{h_i(\mathbf{x})}{h_i(\mathbf{x})+1} > 0$, that implies $x_i > \varphi_i(\mathbf{x})$—contradiction with fixed point. Alternative (iii) is set in a similar manner.

We next consider alternative (i) and the corresponding set of points x_i onto coordinate segments. All these points can be described as the intersection of the coordinate segments with the curve described by the equation

$$h(x,y) = c_1(x,y,\mathbf{x}) - c_2(x,y,\mathbf{x}) = 0,$$

where $\mathbf{x}$ can be viewed as a constant. Specifically, $\mathbf{x} = (x_1, \ldots, x_m)$ plays the role of parameters defining curve in the most general terms. To illustrate the idea and to formulate concrete result we turn to the analysis of a particular case, given in formula (1), recall:

$$c_k(x,y,\mu(S_k), r_c(S_k)) = \frac{g_k}{\mu(S_k)} + \rho((x,y), r_c(S_k)), \quad g_k > 0, \quad k = 1, 2.$$

Here we are interested in the curve which is completely determined by the mass of the population $\mu(S_k)$ and centers $r_c(S_k)$ of two countries $k = 1, 2$. Both of these parameters are continuous functions of $\mathbf{x}$. For a given fixed point they are fixed. Therefore, in the case of the Euclidean distance in the plane equation of the curve defines the classic *hyperbola* (geometric definition), which presents the boundary between the two countries:

$$h(x, y) = 0 \quad \Longleftrightarrow \quad ||(x, y) - r_c(S_1)||_2 - ||(x, y) - r_c(S_2)||_2 = \frac{g_2}{\mu(S_2)} - \frac{g_1}{\mu(S_1)} = const.$$

The sign of the constant determines which of the two branches one must take—nearest to the first center branch is used for a negative constant and vice versa. Described situation is illustrated in Figure 3 which represents the boundary–hyperbola case with a negative right hand side. The alternatives (*i*)–(*ii*) are implemented now and (*i*), (*iii*)—for the polar case. Of course, case (*i*) is possible in a pure form. Notice also that options (*ii*) and (*iii*) do not occur simultaneously: this follows from the convexity of the rectangular area and the convexity of one of the areas bounded by the hyperbola.

Finally. As soon as the centers of the country are located on a common straight line parallel to the base, this line is parallel to the coordinate segments and therefore each of these segments has the *only* point of intersection with hyperbola or do not intersect it at all (the cases (*ii*) and (*iii*)). It establishes the existence of numbers k and l from the theorem statement. Theorem 2 is proved. ■

Theorem 3 *Let for the rectangle* $\square ABCD$ *individual costs be defined by (1) and centers of the country be located on a line parallel to the axis of abscissa. Then there is immigration proof division into two countries* $\mathbf{S_{left}}$ *and* $\mathbf{S_{right}}$ *with a continuous boundary.*

Remark 2 It is immaterial fact that the considered area is a rectangular. This result holds for any convex closed bounded domain. It also follows that the centers can be located on any fixed line (turn the area so that the line is parallel to the axis of abscissa), and it can be *any pair of fixed points.*[3]

In fact, this result can be generalized and the continuous dependence on the parameters defining the country center is the only requirement, but it will require substantial transformation of presented proof. The simplest method is to consider moving coordinate lines parallel to the line passing through the centers of countries. ■

Proof of Theorem 3. Let us consider an increasing family

$$Y_\xi \subset Y_{\xi+1} \subset [0, 1], \quad \xi = 1, 2, \ldots$$

of points on the y-axis defining intercountry piecewise–linear boundary. We choose a family so that

$$\mathrm{cl}\left(\bigcup_{\xi \in \mathbb{N}} Y_\xi\right) = [0, 1].$$

For every $\xi \in \mathbb{N}$ Lemma 1 takes place, that implies: for every ξ hyperbola (2) is specified by the parameters of the country centers (focuses) $r_c(\mathbf{S}^\xi_{\mathbf{left}})$, $r_c(\mathbf{S}^\xi_{\mathbf{right}})$ and "masses of populations" $\mu(\mathbf{S}^\xi_{\mathbf{left}})$, $\mu(\mathbf{S}^\xi_{\mathbf{right}})$ (it defines the right hand side in (2)). These parameters vary under limits and therefore they contain convergent subsequences. Without loss of generality we can assume that already presented sequences are converged. Limit values

$$\bar{\mathbf{r}}_{\mathbf{k}} = \lim_\xi r_c(\mathbf{S}^\xi_{\mathbf{k}}), \quad \bar{\mu}_{\mathbf{k}} = \lim_\xi \mu_k(\mathbf{S}^\xi_{\mathbf{k}}), \quad k = 1, 2$$

[3]Note that this is one of the fundamental differences between the two-dimensional setting and one-dimensional one: it is possible to divide the country for fixed capitals and subject to the capital of country is located in its territory (i.e. the capital is not an enclave). This happens for the reason that the hyperbolic boundary with fixed focuses can allocate an arbitrarily small area.

define limit hyperbola. For this hyperbola one can easily prove two key facts that give the desired result:

(i) $\bar{\mu}_{\mathbf{k}} \neq 0$, $k = 1, 2$, proof by contradiction with the fixed point property $\mathbf{x}_\xi \in \varphi(\mathbf{x}_\xi)$ for all $\xi \in \mathbb{N}$.

(ii) Let $\bar{\xi} \in \mathbb{N}$ and $y_{\bar{\xi}} \in Y_{\bar{\xi}}$ be fixed. As soon as $Y_{\bar{\xi}} \subset Y_\xi$ $\forall \xi \geq \bar{\xi}$, then a sequence $(x_\xi, y_{\bar{\xi}})$, $\xi \geq \bar{\xi}$ of points is defined; they satisfy all fixed point relations. In the rectangle they are, starting with some number, either points located on the left or on the right hand side, or couple $(x_\xi, y_{\bar{\xi}})$ is placed on ξ-th hyperbola (the intersection of $\bar{\xi}$-th segment with the hyperbola). Since hyperbola converge to the limit option, then their (the only!) points of intersection with a fixed line will be convergent, i.e. $(x_\xi, y_{\bar{\xi}}) \to (\bar{x}_{\bar{\xi}}, y_{\bar{\xi}})$, $\xi \to \infty$. Finally note that the obtained pairs form a dense subset in the limit curve, which is a non-trivial fixed point of the map $\mathfrak{F} : C([0, 1]) \to [0, 1]$, described in Remark 1. Consequently, the limit values of the population $\bar{\mu}_{\mathbf{k}} = \lim_\xi \mu_k(\mathbf{S}^\xi_{\mathbf{k}})$, $k = 1, 2$ for countries with piecewise-linear boundaries *coincide* with the value (mass) of the population of marginal hyperbola areas.

Thus, we have found a nontrivial fixed point of $\mathfrak{F}$, this is the map whose graph consists of a (non-empty) intersection of the hyperbola with the area, and possibly two vertical segments. This fragment of hyperbola is the desired boundary between the two countries. Theorem 3 is proved. ■

3 General partition into three or more countries

In the previous section we considered the case of dividing the bounded convex domain into two countries. Now consider the case of the three and more countries. We consider a general method that allows us to establish the existence of immigration proof division into n countries not only on the plane, but in any finite-dimensional space—this is not only possible generalization, but also an opportunity to consider in its context more general problems, e.g. partition according to party affiliation.

Initial construction is similar to proposed in [8]. We need to divide the area $\mathcal{A} \subset \mathbb{R}^l$ into n counties, $N = \{1, \ldots, n\}$. The difference is that the cost function $c_i(\cdot)$ may depend not only on the mass $\delta_i \in [0, 1]$ of country, individual location $x \in \mathcal{A}$, but also from additional parameters $y \in Y$, which can be changed according to a partition configuration. In particular, y can be used as a center of the country as well as other important for country formation parameters. It is assumed that the cost functions depend continuously on $\delta \in \Delta^{(n-1)}$ and $y \in Y$; moreover Y—the range of y—is convex and compact. More specifically in addition to assumption (**P**) (page 2) we impose[4]

(**C**) *For each $i \in N$ costs $c_i(\cdot)$ are defined and continuous on*

$$\mathcal{A} \times Y \times (\Delta^{(n-1)} \setminus F_i), \text{ where } F_i = \{\delta \in \Delta^{(n-1)} \mid \delta_i = 0\},$$

and obey

(i) $c_i(x, y, \delta_1, \ldots, \delta_n) \to +\infty$ *when* $(x, y, \delta_i, \delta_{-i}) \to (\bar{x}, \bar{y}, 0, \bar{\delta}_{-i})$*, i.e.* $\bar{\delta}_i = 0$*;*

(ii) *the set of indifferent agents*

$$A_{ij}(y, \delta) = \{x \in \mathcal{A} \mid c_i(x, y, \delta) = c_j(x, y, \delta)\}$$

has zero Lebesgue measure $\forall j \neq i$*, and for all fixed* $(y, \delta) \in Y \times \Delta^{(n-1)}$.

Note the difference of our assumption with presented in [8]: the continuity relative to all variables and for item *(ii)*—the set $A_{ij}(y, \delta)$ may depends on $y \in Y$ and masses of other jurisdictions δ_k, $k \neq i, j$.

[4]For beginning one can assume as in [8] that the costs are measurable with respect to $x \in \mathcal{A}$, although in all reasonable examples this is continuous dependence and certainly it is so if one wants to get a continuous boundary.

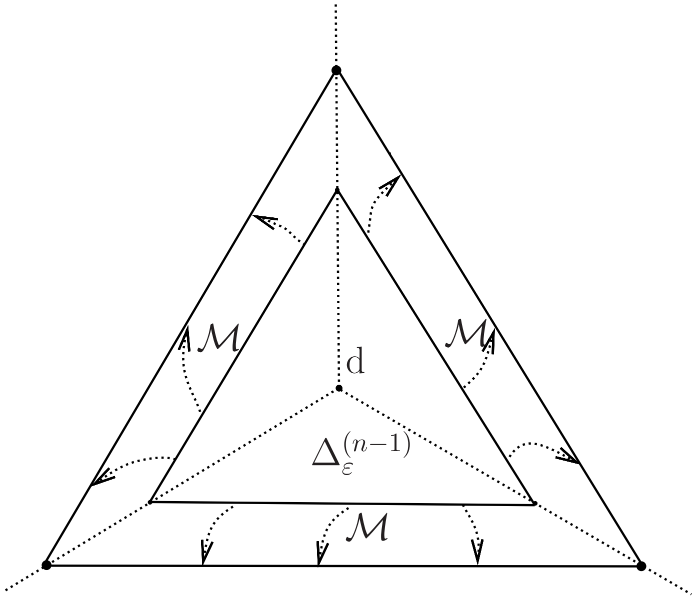

Figure 4: Initial and embedded sub-simplexes $\Delta_\varepsilon^{(n-1)}$ and the mapping $\mathcal{M}(\cdot)$.

An idea of the proof is that for a collection $(\delta_1, \dots, \delta_n, y)$ of *nominal* parameters one can put into correspondence similar collection of *real* parameters, calculated for an immigration stable partition defined by nominal ones. In so doing a mapping is defined which nontrivial fixed point obeys all requirements of country partition that we are looking for. Now we consider this construction in more details.

Let us consider a standard simplex $\Delta^{(n-1)} = \{\delta \in \mathbb{R}^n \mid \sum \delta_i = 1,\ \delta_i \geq 0\ \forall i \in N\}$ and the mappings $S_i : (\delta, y) \to S_i(\delta, y) \subset \mathcal{A}$, $(\delta, y) \in \Delta^{(n-1)} \times Y$, $i \in N$ and $\mathcal{M} : (S_i)_{i\in N} \to (\mu_i)_{\in N}$, defined by formulas:

$$S_i(\delta, y) = \{x \in \mathcal{A} \mid c_i(x, \delta, y) = \min_{j\in N} c_j(x, \delta, y)\}, \quad \mu_i(\delta, y) = \mu(S_i(\delta, y)), \quad i \in N.^5$$

Assuming also there is a continuous $\mathcal{F} : \Delta^{(n-1)} \times Y \to Y$ and we obtain the resulting map

$$[\mathcal{M} \times \mathcal{F}](\delta, y) = \mathcal{M}(\delta, y) \times \mathcal{F}(\delta, y), \quad (\delta, y) \in \Delta^{(n-1)} \times Y.$$

Clearly, it suffices to find a *nontrivial* fixed point $\bar{\delta} = (\bar{\delta}_1, \dots, \bar{\delta}_n) \in \Delta^{(n-1)}$, $\bar{y} \in Y$ of this map, i.e.

$$\bar{y} = \mathcal{F}(\bar{\delta}, \bar{y}), \quad \mu_i(\bar{\delta}, \bar{y}) = \bar{\delta}_i,\ \forall i \in N, \ \text{ such that }\ \bar{\delta} = (\bar{\delta}_1, \dots, \bar{\delta}_n) \gg 0.$$

In [8] there were proved[6] that: for some $0 < \varepsilon < 1$

(*i*) the map $\mathcal{M}(\cdot)$ is continuous on $\Delta_\varepsilon^{(n-1)}$ and

(*ii*) $\mathcal{M}(\cdot)$ maps the ε-sub-simplex

$$\Delta_\varepsilon^{(n-1)} = \{\delta \in \mathbb{R}^n \mid \sum \delta_i = 1,\ \delta_i \geq \varepsilon\ \forall i \in N\}$$

so that the faces of $\Delta_\varepsilon^{(n-1)}$ pass into the corresponding faces of initial simplex, i.e.

$$\delta = (\delta_1, \dots, \delta_n) \in \Delta_\varepsilon^{(n-1)} \ \ \&\ \ \delta_i = \varepsilon \ \ \Longrightarrow \ \ \mu_i(\delta) = 0, \ \ \mathcal{M}(\delta) = (\mu_1(\delta), \dots, \mu_n(\delta)).$$

The properties (*i*), (*ii*) can be easily extended to our case although now we need to find a fixed point of map $\mathcal{M} \times \mathcal{F}$. So, the existence of fixed point we are interesting in can be proved

[5]Here as above $\mu(\cdot)$ is absolutely continuous measure on $\mathcal{A}$, specifying the resettlement of the population.

[6]This is Lemma 1 from [7], where its comprehensive proof is also presented.

via (i), (ii). However note that the Brouwer fixed point theorem and others like it cannot be applied in the case, since $\mathcal{M}(\cdot)$ being defined on $\Delta_{\varepsilon}^{(n-1)} \times Y$ *is not* a mapping *into itself*, i.e. $\mathcal{M}(\Delta_{\varepsilon}^{(n-1)} \times Y) \subseteq \Delta_{\varepsilon}^{(n-1)}$ *is invalid.* Moreover I do not know any other workable theorem for our case. In [8] further argumentation is based on the application of KKM-lemma (Knaster–Kuratowski–Mazurkiewicz), which is rather elegant solution to the issue, but it is limited to a particular case of fixed (unchanged) capitals. In our case this means the lack of postulated dependence of $\mathcal{M}$ on $y \in Y$.

The foregoing has caused the need for additional analysis and the proving of the following theorem, which can be viewed as a (new) generalization of Krasnosel'skii's theorem in case of a bounded polyhedron (simplex), see [3], and its generalizations in [4].

Let $M \subset \mathbb{R}^n$ be a *convex bounded* polyhedron and $A(M)$ is its *affine hull.* Let $d \in \mathrm{ri}M$ be some point in the relative interior of a polyhedron M, and F_t, $t = 1, \dots, m$ its non-trivial faces of a maximum dimension (one less than the dimension of M). With every facet associate cone $K_t \subset A(M)$ with a vertex at d:

$$K_t = \{d + \lambda(\kappa - d) \mid \kappa \in F_t,\ \lambda \geq 0\} \quad \Rightarrow \quad A(M) = \underset{t=1,\dots,m}{\cup} K_t.$$

Theorem 4 *Let $f : M \to A(M)$ be a continuous mapping defined on a polyhedron M and $d \in \mathrm{ri}M$, $A(M)$, F_t, K_t be defined as described above. Let one of the conditions hold:*

(i) *Compressive form*

$$f(F_t) \subset M, \quad \forall t = 1, \dots, m. \tag{8}$$

(ii) *Expansive form*

$$f(F_t) \subset K_t \setminus \mathrm{ri}M, \quad \forall t = 1, \dots, m. \tag{9}$$

Then $f(\cdot)$ has a fixed point in M.

Proof of Theorem 4. Consider the following parametrization in affine space $A(M)$, spanned by a polyhedron M. Since $A(M) = \cup_{t=1,\dots,m} K_t$, then a point $x \in A(M)$ can be specified as $x = d + \lambda(\kappa - d)$, where real $\lambda > 0$ and, for $x \neq d$, the vector $\kappa \in \cup_{t=1,\dots,m} F_t$ on the boundary of the polyhedron are defined one-to-one. Now the points of the polyhedron can be associated with pairs (λ, κ) for $0 \leq \lambda \leq 1$ and a continuous map can be unambiguously extended onto pairs (λ, κ). Next we consider the alternatives of the theorem.

(i) *Compressive form.* Let $f(\lambda, \kappa) = (\lambda', \kappa')$. We now define a new mapping[7] $g(\lambda, \kappa) = (1 \wedge \lambda', \kappa')$. Obviously $g : M \to M$ is continuous and due to Brouwer theorem it has a fixed point $\bar{x} = (\bar{\lambda}, \bar{\kappa}) = g(\bar{\lambda}, \bar{\kappa})$. Let us show that this point is also a fixed point of f. Indeed, the difference in the values of f and g can be revealed only if $\bar{\lambda} < 1$ and $\bar{\lambda}' > 1$. But then $\bar{\lambda} = 1 \wedge \bar{\lambda}' = 1$, that cannot be true for fixed point.

(ii) *Expansive form.* Without loss of generality we can assume that $f(\lambda, \kappa) = (\lambda', \kappa')$ and $\lambda' \leq 2$. Otherwise, consider the new mapping $f'(\lambda, \kappa) = (2 \wedge \lambda', \kappa')$, that on M has the same fixed points as the original. Next, we define $g(\lambda, \kappa) = (2\lambda - \lambda', \kappa')$. For $(\lambda, \kappa) \in F_t$ we have $\lambda = 1$, $1 \leq \lambda' \leq 2$ and therefore $0 \leq 2\lambda - \lambda' \leq 1$, which implies $g(F_t) \subset M$ $\forall t$. By the above item (i) $g(\cdot)$ has in M a fixed point $(\bar{\lambda}, \bar{\kappa})$, i.e. there is $(\bar{\lambda}, \bar{\kappa}) = (2\bar{\lambda} - \lambda', \kappa')$. Writing this componentwise we have $\bar{\kappa} = \kappa'$ and $\bar{\lambda} = 2\bar{\lambda} - \lambda' \Rightarrow \bar{\lambda} = \lambda'$, but this means $(\bar{\lambda}, \bar{\kappa})$ is a fixed point of f. Theorem 4 is proved. ■

Remark 3 Note that we apply parametrization $A(M)$ via (λ, κ) only to specify a transformation of initial function f which does not change fixed points. A new function this way defined is continuous and maps M into itself. ■

[7] Here $a \wedge b = \min\{a, b\}$.

Remark 4 Notice that the assumption $d \in \mathrm{ri}M$ is essential—otherwise the theorem statement is invalid, appropriate examples can be easily constructed. Also the analysis of the proof shows that Theorem 4 can be generalized to the case of the Cartesian product of maps, the first of which satisfies the condition of Theorem 4, and the second—to the conditions of Brouwer theorem or it can be reducible to it—for example, conditions (i) or (ii) are fulfilled for its components. ■

So now we can formulate the main result. In the case of interest we have

$$\mathcal{M} : \Delta_{\varepsilon}^{(n-1)} \times Y \to \Delta^{(n-1)}.$$

If as a central point $d \in M = \Delta_{\varepsilon}^{(n-1)}$ one considers the center of simplex $(\frac{1}{n}, \dots, \frac{1}{n}) = d$, then by expansive property (ii) of the map $\mathcal{M}$ condition (ii) of Theorem 4 is fulfilled. Now if

$$\mathcal{F} : \Delta_{\varepsilon}^{(n-1)} \times Y \to Y$$

is any continuous map, then the map $\mathcal{M} \times \mathcal{F}$ has a fixed point in $X = \Delta_{\varepsilon}^{(n-1)} \times Y$. As a result we have proven the following

Theorem 5 *Let $\mathcal{A}$ be a compact subset of a finite dimensional linear space and μ be a measure on $\mathcal{A}$. If assumptions* (**P**), (**C**) *are satisfied then the area $\mathcal{A}$ can be nontrivially partitioned into any number of immigration proof communities. This partition can also obey any consistent continuous requirements.*

Notice that this result *does not imply* Theorem 3, where we did not required restrictive assumption **C**(ii). Thus Theorems 3 and 5 complement each other.

The mapping $\mathcal{F}$, introduced into the design of the search of a fixed point, expresses some additional requirements for cross-country division. For example, one can impose any requirements on the centers (the capital) of countries. In particular, one can require that the capital located in one of the centers of gravity of the countries, and so on. Further we consider the last topic in more detail.

3.1 The division into countries with capitals in the center of mass and the Euclidean metric

In the seminal works devoted to the analysis of Tiebout equilibrium, special attention is paid to the case of inter-country division under the individual costs $c_i(\cdot)$ defined in (1)—they depend on the weight of the residents $\mu(S_i)$ of jurisdiction, the location of its center $r_c(S_i)$, and the place of individual residence—set by the coordinates $x \in \mathcal{A}$ and Euclidean distance $\rho(\cdot, \cdot)$ to the center of the country. So, this is

$$c_i(x, \mu(S_i), r_c(S_i)) = \frac{g_i}{\mu(S_i)} + \rho(x, r_c(S_i)), \quad g_i > 0, \quad i \in N.$$

For multi-dimensional figures S_i the center of mass (barycenter of i-th country) is defined as

$$r_c(S_i) = \frac{1}{\mu(S_i)} \int_{S_i} x d\mu(x), \quad x \in \mathcal{A} \subset \mathbb{R}^l.$$

It will be so, if the distribution of population is putted in the basis of the concept. However, if we want the center of the territory will be understood without taking into account the population, then we arrive at the concept defined by formulas

$$r_c(S_i) = \frac{1}{m(S_i)} \int_{S_i} x dx, \quad m(S_i) = \int_{S_i} dx.$$

Here $m(S_i)$ is Lebesgue measure of S_i, $i \in N$.

Now to include a requirement that the capitals should be in any kind of barycenters one just need to put

$$\mathcal{F}(\delta_1, \ldots, \delta_n, y_1, \ldots, y_n) = (r_c(S_1(\delta, y)), \ldots, r_c(S_n(\delta, y))) \in Y^n.$$

Here it is appropriate to apply the convex hull of $\mathcal{A}$ as the set Y.

Of course, we must care that considered point-to-set mappings are continuous in some sense (not necessarily by Hausdorff metric, but it is also possible) in order to ensure the continuity of $r_c(S_i(\delta, y))$. Moreover, it is necessary to carefully consider the case of a potentially possible match of capitals at the domain of $\mathcal{F}$. This is not trivial, because now the key assumption $(\mathbf{C})(ii)$ can be violated. Apparently, this problem can be sorted out by a passing to the limit over the family of the individual costs functions that approximate the original ones and such that $(\mathbf{C})(ii)$ holds.

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