Equilibria with nonstandard prices in vector lattice overlapping generations economies

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Abstract

The paper investigates overlapping generations (OLG) economies in vector lattices framework. Agents' preferences are assumed uniformly proper, though they may be nontransitive and incomplete. Existence is stated for the "equilibrium with nonstandard prices", a notion that may be looked upon as a particular (or generalized, in other aspect) case of known "compensated equilibria" of OLG-economies. The difference is that compensated values are described via explicit formula given in nonstandard analysis terms. This approach enables more clear economic interpretation, and shows some new properties of compensated values, such as their linearity over agents' endowments. Also it allows easy to prove the existence of equilibria under classical additional assumptions on agents' endowments.

Keywords and Phrases: overlapping generations, exchange economy, competitive equilibrium, compensated equilibrium, non-standard analysis.
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Introduction

The problem of existence of general equilibrium in infinite dimensional economic models of different kinds was intensively investigated during last thirty years time period. One of directions of these investigations was, starting from seminal Peleg–Yaari [23] and Bewley [8] papers, the study of models with infinite dimensional commodity space and the finite number of consumers. Due to many authors results the importance of order structures for commodity spaces have then understood and other requirements to the economic models and space were steady relaxed. Commodity spaces being Riesz spaces were first introduced into equilibrium theory in Aliprantis–Brown [2]. The lattice properties of the commodity space were then used in the paper of Mas-Colell [18] to prove his remarkable theorem on the existence of equilibrium. The basic structure of the infinite dimensional analysis in a context of economic model was developed in [3], where was postulated the commodity space has to be *topological vector lattice* — the space equipped with locally convex solid topology (see also [20] for general overview). However this assumption (together with other model requirements) was also then relaxed and in modern theory this space is described as vector lattice (Riesz space), endowed with a linear topology in which lattice operations *may not be continuous* (i.e. topology is not assumed to be locally solid) but the positive cone is closed and the space of linear continuous functionals is the sublattice of order dual. A price in these models is identified with a *linear continuous* functional defined *on the whole* commodity space. It is commonly presumed in economy also that agents' preferences, reflecting their tastes, are *proper*. This specific notion of *properness* compensates the possible negative properties of commodity space in which positive cone may have empty interior. The assumption of this kind was first introduced by Mas-Colell [18] and then was relaxed and reconsidered in other papers. Now this is somewhat abstract but in some of particular cases it means that with any attainable consumption bundle an open convex cone with vertex in this bundle may be associated, which does not contain preferred points. In [19] Mas-Colell and Richard (cf., Aliprantis [1], see also [25]) proved the first result under described structural assumptions, extending existence theorem from [18] to a broader set of models, e.g., Huang & Kreps [16] and Jones [17]. Then Mas-Colell–Richard's theorem in linear vector lattice settings was generalized by many authors in several directions, see novelist results [24], [22], [11], [27]. Now the most powerful theorems state the existence of quasi-equilibria for non-ordered preferences and under very weak properness assumption for preferences and consumption sets, which in difference with Mas-Colell–Richard's theorem has pointwise characterization. Analogous results for production economies were proved in [28], [14].

In this paper we are studying an intertemporal model of special type, it is the so-called OLG-economies (overlapping generations) in vector lattices structural framework. It is a well known class of models characterized by countably many time periods of the economy life and by countably many agents, while only a finite number of agents are living during every time period. Assuming commodity space at each time period is finite dimensional Wilson [29] has proved the existence of equilibrium prices (the sequence of time-period prices) if either every agent is endowed with the finite-living assets or there is a finite set of agents owning a positive fraction of the total endowments (see also Burke [9]). He has demonstrated that if the economy does not satisfy these assumptions then only the *compensated* equilibria existence can be guaranteed. The difference of this notion with equilibrium one is that it is postulated the existence of non-negative real numbers, called as compensated values, which are added to the right hand side of agents' budget constrains in their consumption tasks.

Dealing with the *finite-dimensional* commodity spaces for every time period (so as in Wilson [29] and others), Danilov [12] has generalized the notion of compensated equilibrium and proved its existence theorem. He has suggested a novel notion of generalized prices as a *linear* functional having *finite values* only for the initial endowments of each agent while its value may be infinite for the total endowments. The equilibrium prices by Danilov have better economic interpretation than the previous ones and allow many of their useful properties to be derived. In particular, if the endowment of an agent is represented as a linear combination of the finite number of other agents' endowments, then the same takes place for their compensated values; if somebody has a finite-living endowment then his/her compensated value is zero and so on. It also allows one to state the existence of equilibria under the additional assumptions such as given in [29], [9], since either all compensated values are equal to zero (by the property mentioned above) or the functional is continuous. Another line of generalizations dealt with the consideration of infinite-dimensional spaces of commodities for every time period. For the first time lattice structures were used extensively for OLG-economy context in [4], [5] (see also [6], [10] and [15] for general overview). All these results have assured the existence of equilibria under the traditional for infinite-commodities equilibrium theory structural assumptions, such as local solid topologies and so on.

The aim of the paper is to study existence equilibrium problem for OLG-economies under modern weakest structural assumptions mentioned above. We introduce the concept of *equilibrium with non-standard prices* — an analog of generalized compensated equilibria by Danilov, saving its positive properties, for which compensated values are described via explicit formula given in non-standard analysis terms. This is the only reason why we use non-standard prices in equilibrium definition instead of customary standard one — they are more informative. Assuming every time period commodity space to be a *linear vector lattice*, and preferences to be *nontransitive, incomplete* and *uniformly proper*, we prove equilibrium with non-standard prices existence theorem. It is shown that this theorem allows easy to conclude the existence of equilibria under classical additional model assumptions — if either every agent is endowed with the finite-living assets or there is a finite set of agents owning a positive fraction of the total endowments. Our arguments are based on generalized Mas-Colell–Richard's theorem from [22] applied to the perturbed economies with the finite number of agents constructed with respect to the special commodity-price duality. Then passing to the limits employing non-standard analysis methods and using modified Danilov's [12] arguments we achieve the existence of equilibrium prices.

The paper is organized as follows. In section 1 we describe the model, give assumptions and formulate main results. Section 2 is devoted to the description of general strategy, auxiliary results and ideas of their proofs and to discussion. Third section contains the detailed proofs.

1 Model and main result

We consider a traditional pure exchange economy with overlapping generations, assuming the countable number of overlapping generations, each of them consisting of a finite number of finitely living agents.

The index t is used to denote some time period and $T = \{1, 2, ..., t, ...\}$ denotes the set of all time periods. The commodity space for each time period is a linear partial ordered space E_t equipped with a Hausdorff, locally convex topology τ_t. $\mathcal{I} = \{1, 2, ..., i, ...\}$ denotes the set of agents' numbers (countably many). Each agent $i \in \mathcal{I}$ is living finitely many time periods denoted by $T_i \subset T$. The latter means that his/her point-to-set preference mapping $\mathcal{P}_i$ is defined on $X_i = \prod\limits_{i \in T_i} E_t^+$ so that $\mathcal{P}_i(x_i) \subset X_i$ is the set of all strictly preferable to x_i consumption bundles. Below we also will use traditional notation $y_i \succ_i x_i$ which is equivalent to $y_i \in \mathcal{P}_i(x_i)$. The set X_i is interpreted as the consumption set of the i-th agent and $E_t^+ = \{x \in E_t \mid x \geq 0\}$ denotes the positive cone of the space E_t. The i-th agent initial endowments are denoted by $\omega_i = (\omega_i^t)_{t \in T}$, $\omega_i^t \in E_t^+$. We require the prices for each time period t to be chosen in *topological dual* $E_t' = (E_t, \tau_t)'$ of E_t. Summing up that has been said above, the economy under study is specified as a quadruple

$$\mathcal{E} = \langle\ \mathcal{I},\ T,\ \{\langle E_t, E_t' \rangle\}_{t \in T},\ \{\mathcal{P}_i, \omega_i, T_i\}_{i \in \mathcal{I}}\ \rangle.$$

To consider the equilibrium notions which may be applied for OLG-economy we need some mathematical parameters make sense. This is why we first impose the assumptions for $\mathcal{E}$ and then give the definitions. These assumptions are divided into several groups.

First, to obtain interesting results we need to impose some restrictions on the population of consumers and on the structure of their endowments.

STRUCTURAL OLG-FINITENESS (SF) :

(i) the set of all alive agents for each $t \in T$ is finite and nonempty, i.e.

$$I(t) = \{i \in \mathcal{I} \mid t \in T_i\} \neq \emptyset, \quad | I(t) | < +\infty;$$

(ii) the number of owners for each $t \in T$ is finite and nonempty, i.e.

$$| J(t) | < +\infty, \quad J(t) = \{i \in \mathcal{I} \mid \omega_i^t \neq 0\} \neq \emptyset.$$

The economic meaning of (SF) has to be clear, both of these assumptions are well known in literature. In particular $(SF)(ii)$ entails the existence of total (society) endowments for each time period $\omega^t = \sum\limits_{\mathcal{I}} \omega_i^t$ that we will use in next group of structural assumptions. Also this allows us to consider the ω-uniform properness of preferences and to define feasible allocations correctly. $(SF)(i)$ plays another role, we will use it explicitly in the construction of the perturbed model that will be described in next section, for more see [15].

The second group of assumptions consists of the structural properties of commodity spaces at each time period. We will require the commodity-price duality $\langle (E_t, \tau_t), (E_t, \tau_t)' \rangle$ for every $t \in T$ satisfies

STRUCTURAL ASSUMPTIONS (SA) :

(i) E_t *is a linear vector lattice (or Riesz space);*

(ii) E_t^+ *is closed in the* τ_t*-topology of* E_t*;*

(iii) E_t' *is a sublattice of the order dual to* E_t*;*

(iv) the order interval $[0, \omega^t]$[1] *is* $\sigma(E_t, E_t')$[2]*-compact.*

Let for a moment $(L, \tau) = (E_t, \tau_t)$. One can see the assumptions (SA) for duality $\langle (L, \tau), (L, \tau)' \rangle$ are exactly the same that were done in modern novelist results, mentioned in introduction (see [24], [22], [11], [27], [28], [14]). It is worth noticing that *(i) – (iii) do not imply* that the lattice operations $x \vee y,\ x \wedge y,\ x, y \in L$ are *continuous* with respect to topology τ. In other words, we do not assume the topology τ to be locally solid. If τ were locally solid, then requirements *(ii), (iii)* would be automatically valid. Since we avoid this hypothesis, we need to require them directly. Almost the same can be said about *(iv)*: if the topology of the space guarantees that every order interval is $\sigma(L, L')$-compact (for example, if L is Dedekind complete and $\sigma(L, L')$ is order continuous, see [6], [7]), then we avoid this assumption, otherwise not. For more specific explanations and references, the reader is referred to [6], [7].

The next group of assumptions consists of the properties of agents' characteristics. All these assumptions are well known in the literature but one of them requires explanations, it is so-called the *uniform properness of preferences.* In this paper the notion of properness is borrowed from [22] and is defined as follows (see also [20] for explanations and motivations).

To simplify the notations in the definition and assumptions below we denote (L, τ) some topological vector space. By convention the requirement in which (L, τ) is used being applied for a given consumer $i \in \mathcal{I}$ means that it has to be fulfilled for $L = L_i = \prod_{t \in T_i} E_t$ and relative to duality $\langle L_i, L_i' \rangle$. The same can be said about vector $\omega \in L$ — always it means that $\omega = \omega_{|_{L_i}} = (\omega^t)_{t \in T_i}$. Below we shall also often identify, without special saying, the elements of L_i with vectors from $\prod_{t \in T} E_t$, supplementing the vector by zero components.

Definition 1.1 *The preference relation* $\mathcal{P}(\cdot) : L^+ \Rightarrow L^+$ *is said to be* ω*-uniformly* τ*-proper on* $Y \subset L^+$ *if there exists a* τ*-neighborhood* V *of the origin such that*

$$y - \alpha\omega + z \in conv\mathcal{P}(y)^3,\ y \in Y,\ \alpha > 0 \implies z \notin \alpha V.$$

[1] The notation $[a, b]$ denotes the order interval, i.e. $[a, b] = \{c \in E \mid a \leq c \leq b\}$.

[2] $\sigma(E_t, E_t')$ denotes weak topology on E_t.

[3] In this paper, $convA$ denotes the convex hull of the set A, clA is its closure and $\setminus$ is set for the set-theoretical difference.

Originally a slightly stronger notion was suggested by Mas-Colell [18], in order to overcome the emptiness of the interior of the positive cone for many interesting commodity spaces.

Let $\mathcal{R} : L \Rightarrow L$ be some point-to-set mapping (correspondence). Recall that the *graph* of $\mathcal{R}$ is the set

$$Gr\mathcal{R} = \{(x, y) \in L \times L \mid y \in \mathcal{R}(x)\}.$$

ASSUMPTIONS ON PREFERENCES (PA) :

(i) preference relation $\mathcal{P} : L^+ \Rightarrow L^+$ has $\sigma(L, L')$-open graph in $L^+ \times L^+$;

(ii) weak convexity and irreflexivity: for each $x \in [0, \omega]$

$$x \in cl(conv\mathcal{P}(x)) \backslash conv\mathcal{P}(x)\,;$$

(iii) monotonicity: for each $x, y \in L^+$ such that $x \leq y$

$$\mathcal{P}(x) + L^+ \subset cl(conv\mathcal{P}(x)) \quad \& \quad \mathcal{P}(y) \subseteq cl(conv\mathcal{P}(x))\,;$$

(iv) $\mathcal{P}(\cdot)$ is ω–uniformly τ-proper on $[0, \omega]$.

Condition *(PA)(i)* means the continuity of preferences and is stronger then the modern weakest assumptions applied for economy with finite many agents, where it is required the openness of lower sections in weak topology and the openness of upper sections in initial one. The monotonicity of preferences *(PA)(iii)* is also a little bit stronger in comparison with given in [22], where only the first part of given requirement is used (to obtain the positive equilibrium price functional). Clear that the first part of *(PA)(iii)* implies the second one for ordered convex preferences, such as considered in [19], but not for our case. The uniform properness used in *(PA)(iv)* may be relaxed to the pointwise condition (see [24], [14], [27]) for finitely many agents economy, but in our case to realize passing to limits we need to have the uniform kind. Note that there is no loss of generality to assume in *(PA)(iv)* that the (individual) properness τ-neighborhood of the origin V is convex and circled, i.e. $V = -V$.

We need also to impose an additional property of the preferences being nonsatiated for each time period. Let $i \in \mathcal{I},\ t \in T$ and $y \in \prod_{t' \in T_i \backslash \{t\}} E^+_{t'}$ be fixed. We define *a partial preference* $\mathcal{P}^t_i(., y)$ by means of the equivalence

$$z' \in \mathcal{P}^t_i(z, y) \Longleftrightarrow (z', y) \in \mathcal{P}_i(z, y), \quad z' \in E^+_t.$$

PARTIAL NONSATIATION (PN). If E_t is infinite dimensional and $t \in T_i$ for some $i \in \mathcal{I}$ then the partial preference $\mathcal{P}^t_i(., y)$ onto $[0, \omega^t] \subset E^+_t$ is τ_t-locally nonsatiated for each $y \in \prod_{t' \in T_i \backslash \{t\}} E^+_{t'}$, i.e. $x \in cl(conv\mathcal{P}^t_i(x, y))$ for every $x \in [0, \omega^t]$.

Note that *(PN)* is stronger than the simple local nonsatiation, that we have already by *(PA)(ii)* for $x \in [0, \omega_{|L_i}] \subset L_i$. We require this condition only for infinite dimensional time period commodity spaces since for finite dimensional case the inner product $\langle \cdot, \cdot \rangle$ is continuous by both variables simultaneously.

An allocation $x \in (L_i^+)^{\mathcal{I}}$ is called *feasible* if $\sum_{j:t \in T_j} x_j^t = \omega^t$ for each $t \in T$. Denote

$$\mathcal{A}(\mathcal{E}) = \{(x_i)_{i \in \mathcal{I}} \mid \; x_i \in L_i^+, \;\; i \in \mathcal{I}, \;\; \sum_{j:t \in T_j} x_j^t = \omega^t \;\; \forall \; t \in T\}$$

the set of all feasible allocations.

IRREDUCIBILITY (IR). If $x = (x_i)_{i \in \mathcal{I}}$ is a feasible allocation and I is any proper and nonempty subset of $\mathcal{I}$, then there exists $i \in I$ and $y \in \prod_{t \in T_i} E_t^+$ such that $y \leq \sum_{j \in \mathcal{I} \setminus I} \omega_j$ and $x_i + y \in \mathcal{P}_i(x_i)$.

This or some another form of irreducibility *(IR)* is necessary to pass from a quasi-equilibrium to the strict equilibrium and it means a form of interconnection of the agents' interests and endowments. In view of *(SF)(ii)* the condition implies the existence of *a finite* subset $S \subset \mathcal{I} \setminus I$ so that $0 \leq y \leq \sum_{j \in S} \omega_j$ et cetera as in (IR).

Let us pass now to the equilibrium concepts which are reasonable to consider in overlapping generations economy framework. We start from the classical compensated equilibrium definition.

Any sequence $p = (p^t)_{t \in T}$ of *positive linear continuous* functionals $p^t \in (E_t, \tau_t)'$ may be considered to be as OLG-price functional. For $L = \prod_T E_t$ and $x \in L^+$, $x_i \in L_i^+$ for some $i \in \mathcal{I}$ let us define

$$\langle p, x \rangle = \sup_{S \subset T, \; |S| < +\infty} \sum_{t \in S} p^t x^t, \quad \langle p, x_i \rangle = \sum_{t \in T_i} p^t x_i^t. \tag{1}$$

Clear that both values are correctly defined but the first one may not be finite for some p and x.

Definition 1.2 *A couple (x, p) is said to be a compensated equilibrium if $p = (p^t)_{t \in T}$ is OLG-price functional, $x \in \mathcal{A}(\mathcal{E})$ and*

$$\langle \mathcal{P}_i(x_i), p \rangle > \langle p, x_i \rangle^4 \tag{2}$$

and

$$\langle p, x_i \rangle \geq \langle p, \omega_i \rangle \tag{3}$$

for each $i \in \mathcal{I}$ holds. The compensated equilibrium is an equilibrium if the inequalities in (3) are turned into equalities for all $i \in \mathcal{I}$.

[4] $\langle A, p \rangle$ denotes the set $\{\langle a, p \rangle \mid a \in A\}$ and $A > b$ means $a > b$ for all $a \in A$, symetrically for $\geq$.

The values $\alpha_i = \langle p, x_i\rangle - \langle p, \omega_i\rangle$, associated with consumers in some compensated equilibrium may not be zero and they are called *compensated* ones. Note that in the classical version of last definition it is required commonly in addition $\alpha_i = 0$ as soon as the set $supp\,(\omega_i) = \{t \in T \mid \omega_i^t \neq 0\}$ is finite.

Applying *Definition 1.2* to the case of *finite* sets T & $\mathcal{I}$ one can derive the classical notion of competitive equilibrium. If in addition we relax condition (2) requiring the non-strict form of inequality together with $\langle p, \omega\rangle > 0$, we are coming to the notion of *quasi-equilibrium.*

Danilov [12] has generalized the price-functional notion for OLG-economy, assuming the functional to be finite for agents' endowments and their consumption bundles but it may be infinite for the total endowments. In fact due to the first part of (1) an OLG-price may be considered to be the functional defined on a subspace L_p of $L = \prod E_t$. This is a subspace of componentwise ordered L and it may be defined via its generating positive cone[5] L_p^+ — the set of all $x \in L^+$ such that $\langle p, x\rangle$ is finite. One can see this correctly defines a cone and p is also additive on L_p^+ and therefore it may be unambiguously extended onto $L_p = L_p^+ - L_p^+$. In view of (3) clearly $\omega_i \in L_p$ for every $i \in \mathcal{I}$. Consequently L_p contains (order) ideal $H \subset L$ generated by the family $\{\omega_i\}_{i\in\mathcal{I}}$. On the other hand $\omega = (\omega^t)_{t\in T}$ may not belong to H hence $\omega \notin L_p$ is possible, i.e. p may take infinite value on ω. In other words if $\omega^S = \sum_S \omega_i$ for finite $S \subset \mathcal{I}$ then $\omega^S \uparrow \omega$ but the monotonic net $p\omega^S = \sum_S p\omega_i$ may not have upper bound. In our approach we use explicitly a non-standard price-functional (an internal sequence of non-standard prices), for which its value is well defined on ω (in some sense) though being non-standard it may be not *finite* (standard bounded). This way is more convenient from mathematical point of view and yields an economic interpretation of the compensated values as some prices of infinite-living assets[6].

Let $p = (p^t)_{t\in {}^*T}$ be an internal sequence of positive non-standard functionals $p^t \in {}^*(E_t'),\ t \in {}^*T$. We will call p *non-standard prices* for OLG-model if $p^t(y) \approx f^t(y)$ for some τ_t-continuous linear functional f^t on E_t and every $y \in E_t,\ t \in T$ and $p\omega_i < +\infty$ for each $i \in \mathcal{I}$. This requirement may be reformulated in an equivalent form, it means that functionals p^t and $p\omega_i$ are near-standard, i.e. values $st(p^t),\ t \in T$ & $st(p\omega_i),\ i \in \mathcal{I}$ do exist[7], where the standard parts $st(p^t)$ are taken relative to the weak-$*$ topology[8] (see [13], [31] for more on non-standard analysis). Moreover, the functionals $st(p^t) = f^t$

[5]The positive cone H of a partially ordered space A is said to be generating if $H - H = A$.

[6]Danilov also used non-standard methods in arguments but did not define the notion of non-standard price functional explicitly.

[7]The notation ${}^*A$ means $*$-image of A, $st(x)$ means the standard part of non-standard x, i.e. $st(x)$ is a standard one such that $st(x) \approx x$.

[8]Notice that the monad of zero in topology $\sigma(E_t', E_t)$ may be describe by

$$\mu(0) = \{p \in {}^*E_t' \mid py \approx 0\ \forall y \in E_t\}.$$

may be explicitly described by formula

$$st(p^t)y = st(p^t y) \quad \forall\, y \in E_t.$$

Using this formula one may equivalently require that for every $y \in E_t$ the value $p^t y$ is *finite* (i.e. $|p^t y|$ is bounded by a standard value) and the last functional is τ_t-continuous. By the way, (E, τ) being a topological vector space, the sufficient condition for some functional $h \in {}^*E'$ to be weak-$*$ near-standard, is to require $hy \approx 0$ for every $y \underset{\tau}{\approx} 0$, $y \in {}^*E$. To see this it is enough to find such a neighborhood of zero U that the functional value of ${}^*U$ is standard bounded (apply Theorem 1.1.1 from [31]).

Since the notation $st(p)$ is commonly used with respect to a given topological space, to be specific we will denote $\bar{p} = (st(p^t))_{t \in T}$.

For some non-standard OLG-price π put

$$\langle \bar{\pi}, \omega_i \rangle = \sup_{S \subset T,\ |S| < +\infty} \sum_{t \in S} st(\pi^t)\omega_i^t, \quad \langle \bar{\pi}, x_i \rangle = \sum_{t \in T_i} st(\pi^t) x_i^t, \quad x_i \in L_i,\ i \in \mathcal{I}.$$

Below we will use also the notations $\langle \bar{\pi}, \omega_i \rangle = \bar{\pi}\omega_i$ and $\langle \bar{\pi}, x_i \rangle = \bar{\pi} x_i$.

Definition 1.3 *A couple (x, π) is said to be an equilibrium with non-standard prices if $\pi \geq 0$ is a non-standard OLG-price, $x \in \mathcal{A}(\mathcal{E})$ and there exist infinite $t_i \in {}^*T \setminus T,\ i \in \mathcal{I}$ so that*

$$\langle \bar{\pi}, \mathcal{P}_i(x_i) \rangle > \langle \bar{\pi}, x_i \rangle = \langle \bar{\pi}, \omega_i \rangle + st(\sum_{t \geq t_i} \pi^t \omega_i^t)$$

holds for each $i \in \mathcal{I}$. Moreover, for each $k \in {}^*T \setminus T$ and $i \in \mathcal{I}$ if $k < t_i$ then $st(\sum_{t=k}^{t=t_i} \pi^t \omega_i^t) = 0$, i.e. $\sum_{t=k}^{t=t_i} \pi^t \omega_j^t \approx 0$.

Note that if a couple (x, π) satisfies *Definition 1.3* and $st(\sum_{t \geq t_i} \pi^t \omega_i^t) = 0$ for each $i \in \mathcal{I}$ then $(x, \bar{\pi})$ is an equilibrium (strict). Note also that to every equilibrium (x, π) by *Definition 1.3* a compensated equilibrium $(x, \bar{\pi})$ corresponds, which satisfies *Definition 1.2* for $p = \bar{\pi}$, but equilibrium (x, π) is more informative. In fact now the compensated values are specified as $st(\sum_{t \geq t_i} \pi^t \omega_i^t)$. They are likely to be the prices of the infinite "tails" of endowments. Many of the useful properties of equilibrium prices can be derived from this explicit representation and the first among them is the "linearity" of compensated values with respect to the agents' endowments. It is also clear that if $\omega_i^t = 0$ for all infinite t and some $i \in \mathcal{I}$, i.e. if $|supp(\omega_i)| < +\infty$ then for this i the value is zero. Moreover if conditions, the model satisfies, are such that $\sum_{k > t} \pi^t \omega_i^t \approx 0$ can be shown for each $t \in {}^*T \setminus T,\ i \in \mathcal{I}$, then an equilibrium is realized automatically.

Now we are ready to formulate the following main result.

Theorem 1.1 *Let the duality $\langle E_t, E_t' \rangle$ satisfy (SA) for each $t \in T$, preferences satisfy (PA) & (PN) and (SF) & (IR) hold. Then an equilibrium with non-standard prices does exist.*

As a corollary of *Theorem 1.1* we state

Theorem 1.2 *In the conditions of Theorem 1.1 if either the initial endowments of each agent have a finite support, or there exists a finite subset* $I \subset \mathcal{I}$ *and real* $\sigma > 0$ *such that* $\sum_I \omega_i \geq \sigma \cdot \omega$*, then an equilibrium does exist.*

2 Strategy of proof and auxiliary results

Our starting point to obtain the results on equilibria existence consists in constructing some perturbed economy with finitely many agents and a commodity-price duality in such a way that each agent earns a profit in any quasi-equilibrium. This economy is such that the existence of quasi-equilibrium theorem from [22] can be applied and if the initial OLG-model $\mathcal{E}$ satisfies the condition of irreducibility *(IR)*, then so does the perturbed model and therefore it has the equilibrium[9]. This allows us to realize the non-standard passing to the limits and then study the limit points.

Let us choose and fix any finite subset M of $\mathcal{I}$ and construct the perturbed economy $\mathcal{E}_M$. We equip this model with the consumers from M, saving all their characteristics, and a special agent with number "0". Following Richard–Srivastava's idea [26] we define initial endowments of this agent as follows $\omega_0 = \omega - \sum_{i \in M} \omega_i$. To specify his/her preferences we use the special construction. Let

$$T_0 = T_M = \bigcup_{j \in \mathcal{I} \setminus M} T_j$$

be the set of his/her lifetime periods (infinite). Let us take any linear continuous functional

$$r_i^t : E_t \to I\!R$$

such that $r_i^t(\omega_i^t) = 1$ holds whenever $\omega_i^t \neq 0$ and $r_i^t(.) \equiv 0$ otherwise. The existence of such functionals must be clear in view of Hahn–Banach theorem. Let us define

$$s_t = \bigvee_{i \in M} r_i^t, \quad s_t^+ = s_t \vee 0.$$

We assumed E_t' to be a vector lattice, hence s_t, s_t^+ are continuous and s_t^+ is also positive functionals and in so doing they satisfy

$$\text{if } \ \omega_i^t \neq 0, \ \text{ for some } i \in M \implies s_t(\omega_i^t) > 0.$$

Finishing this construction let us determine a sequence of numbers $\alpha_t > 0, \ t \in T_M$ satisfying the condition

$$\sum_{t \in T_M} \alpha_t \cdot s_t^+ \omega^t < +\infty$$

[9] It is well known if *(PA)* and *(IR)* are fulfilled then every quasi-equilibrium turns into equilibrium.

and define the 0-agent's utility function as follows

$$u_0(x) = \sum_{t \in T_M} \alpha_t \cdot s_t^+ x^t, \quad x = (x^t)_{t \in T_M}, \quad x^t \in E_t^+.$$

There is no doubt that this function cannot be well defined for every sequence $x = (x^t)_{t \in T}$. Now we construct some subspace L_ω of $L = \prod\limits_T E_t$ which we will use as a commodity space in the perturbed model (it is adopted from Cherif *et al.* [10], see also seminal Aliprantis *et al.* [4]). Let

$$P = \{p \in \prod_T E_t' \mid \sum_T |p^t \omega^t| < +\infty\}$$

be the space of continuous functionals on the principal ideal of L, generated by $\omega = (\omega^t)_{t \in T}$. Let us put

$$L_\omega = \{x \in \prod_T E_t \mid \sum_T |p^t x^t| < +\infty \quad \forall\, p \in P\}$$

and equip this space with the topology β defined as the weakest topology among those that are stronger than the product of topologies τ_t induced onto L_ω and such that each functional $p \in P$ is continuous. This topology may be described in terms of seminorms in the following way. If $\{\rho_\xi\}_{\xi \in \Xi}$ is a family of seminorms specifying the product topology $\otimes \tau_t$ onto L and the seminorms $\{\rho_p\}_{p \in P}$ are determined by

$$\rho_p(x) = \sum_T |p^t x^t|, \quad x \in L_\omega, \quad p \in P$$

then the joined family $\{\rho_\xi\}_{\xi \in \Xi} \cup \{\rho_p\}_{p \in P}$ specifies the topology β. It must be clear by construction that (L_ω, β) is a vector lattice the componentwise defined positive cone of which L_ω^+ is closed with respect to β. Moreover, $L_\omega' = P$ takes place, that can be checked in a routine way. Note also that by construction this duality does not depend on the choice of $M \subset \mathcal{I}$ and the order interval $[0, \omega]$ is a compact in $\sigma(L_\omega, L_\omega')$-topology. The latter one can be easily shown due to $(SA)(iv)$, by definition and construction.

So, we consider the commodity-price duality $\langle L_\omega, L_\omega' \rangle$ and study the perturbed model

$$\mathcal{E}_M = \langle M_0, \langle L_\omega, L_\omega' \rangle, \quad \{\mathcal{P}_i(\cdot),\ \omega_i\}_{i \in M_0} \rangle$$

of the economy $\mathcal{E}$, where $M_0 = M \cup \{0\}$ and preferences for $i \in M$ are trivially extended onto L_ω^+.[10]

One can see the duality $\langle L_\omega, L_\omega' \rangle$ satisfies structural assumptions *(SA)* and if each $\mathcal{P}_i(\cdot)$ satisfies *(PA)* with respect to $\langle L_i, L_i' \rangle$, then the extension of $\mathcal{P}_i(\cdot)$ satisfies *(PA)* with respect to $\langle L_\omega, L_\omega' \rangle$. *(PA)* is also fulfilled for the utility $u_0(\cdot)$ of the 0-agent (there are no problems to check it). One can see that now economy $\mathcal{E}_M$ satisfies the existence quasi-equilibrium theorem from [22]. Moreover, analyzing the proof of main theorem

[10]By formula: $x \overset{M}{\underset{i}{\succ}} y, \quad x, y \in L_\omega^+ \iff x_i \underset{i}{\succ} y_i, \quad x_i = (x^t)_{t \in T_i},\ y_i = (y^t)_{t \in T_i} \in L_i^+$; to simplify notations we omit upper index M below.

from [22] it is easy to conclude the existence of a quasi-equilibrium (x^M, π^M) such that there are linear continuous $p_i^M \in L'_\omega$, $i \in M_0$ such that

$$\pi^M = \bigvee_{i \in M_0} p_i^M, \quad \langle \pi^M, \omega \rangle > 0$$

with for each $i \in M_0$

$$\langle p_i^M, \omega + U_i \rangle \geq \ 0, \tag{4}$$

where U_i is an open, convex and circled neighborhood of zero taken from the ω-uniform β-properness condition for i-th agent's preferences in (L_ω, β). Note that for $i \neq 0$ these neighborhoods may be define as follows

$$U_i = \{y \in L_\omega \mid y = (y_i, y_{-i}),\ y_i \in V_i,\ y_{-i} \in \prod_{T \setminus T_i} E_t\},$$

where the neighborhood $V_i \subset L_i$ is chosen due to *(PA)(iv)*. Notice also that this implies $(p_i^M)_t = 0$ for all $t \in T \setminus T_i$, $i \in M$. Similarly notice that by u_0 specification the properness neighborhood of zero may be chosen so that $(p_0^M)_t = 0$ for all $t \notin T_M$.

Finally, if $\mathcal{E}$ is irreducible *(IR)* then by $u_0(\cdot)$ construction the model $\mathcal{E}_M$ is also irreducible and this, together with that was said above, gives us

Proposition 2.1 *In Theorem 1.1 conditions for every finite $M \subset \mathcal{I}$ the model $\mathcal{E}_M$ has an equilibrium couple (x^M, π^M) such that $\pi^M = \vee_{i \in M_0} p_i^M$ while functionals $p_i^M \geq 0$, $p_i^M \in L'_\omega$ satisfy*

$$\langle p_i^M, \mathcal{P}_i(x_i^M) \rangle \geq \pi^M x_i = \pi^M \omega_i,\ \langle p_i^M{}_{|L_i},\ \omega_{|L_i} + V_i \rangle \geq 0, \tag{5}$$

where V_i is chosen from properness assumption $(PA)(iv)$, $\langle \pi^M, \omega_i \rangle > 0,\ i \in M$ and $\langle \pi^M, \omega \rangle = 1$. Moreover, each bundle $x_i^M = (x_i^t)_{t \in T}$ and each functional p_i^M satisfies $x_i^t = 0$ and $(p_i^M)_t = 0$ for $t \notin T_i$, $i \in M_0$.

We apply this proposition to realize non-standard passing to the limits by the net of equilibrium couples (x^M, π^M). In fact, by transfer principle (see [13]) we can rewrite the latter proposition making $*$-images of all mathematical parameters (in other words we are adding asterisks for all constants). Then we choose and take a "finite" internal $M \subset {}^*\mathcal{I}$ as fixed so that $\mathcal{I} \subset M$ holds (it does exist due to $(SF)(i)$ and transfer) and study the non-standard equilibrium couple $(x, \pi) = (x^M, \pi^M)$. The following results allow us to "standardize" this equilibrium and to show that "standardized point" is *an equilibrium with non-standard prices.*

Proposition 2.2 *In Theorem 1.1 conditions for every "finite" internal $M \subset {}^*\mathcal{I}$, $\mathcal{I} \subset M$ each non-standard equilibrium couple (x, π) of $\mathcal{E}_M$ satisfies $st(\pi\omega_i / \pi\omega_j) > 0$, $i, j \in \mathcal{I}$.*

It is to be noticed that to prove this fact we certainly need assumption *(IR)* to be fulfilled and the preferences of agents from $\mathcal{I}$ should have open graphs in a suitable product of $\sigma(E_t, E'_t)$-topologies.

Since the duality $\langle L_\omega, L'_\omega \rangle$ is common for all perturbed models and the consumption bundles of agents belong to ${}^*[0,\omega]$, and $[0,\omega]$ is $\sigma(L_\omega, L'_\omega)$-compact, then, in view of non-standard characterization of compact sets[11], these bundles can be standardized. The non-standard price-functional can not be standardized yet, but if we normalize it putting $\pi\omega_i = 1$ for some $i \in \mathcal{I}$, that is possible in view of *Proposition 2.2*, then each π^t has a standard part for $t \in T$.

Lemma 2.1 *In Proposition 2.2 conditions for each $t \in T$ if the equilibrium price π satisfies $\pi\omega_i = 1$ for some $i \in \mathcal{I}$, then $st(\pi^t)$ does exist and $st(\pi^t) \geq 0,\ t \in T$, where $st(.)$ is taken relative to $\sigma(E'_t, E_t)$ topology.*

We prove *Theorem 1.1* using the latter and the former results. The representation of $st(\pi\omega_i)$ in a form required in this theorem gives

Lemma 2.2 *In conditions of Theorem 1.1 every non-standard equilibrium price π normalized by $\pi\omega_1 = 1$ satisfies the following property: for each $i \in \mathcal{I}$ there exists $t_i \in {}^*T \backslash T$ such that equilibrium prices satisfy*

$$st(\pi\omega_i) = \bar{\pi}\omega_i + st(\sum_{t \geq t_i} \pi^t \omega_i^t).$$

Moreover, if $k < t_i,\ k \in {}^*T \setminus T$, then $\sum_{t=k}^{t=t_i} \pi^t \omega_i^t \approx 0$.

Recall, that we denote

$$\bar{\pi}\omega_i = \sup_{S \subset T,\ |S|\ < +\infty} (\sum_{t \in S} \bar{\pi}^t \omega_i^t).$$

Danilov's arguments to prove the result similar to our *Theorem 1.1* (but in finite-dimensional setting, see [12]) are close to ours. The main difference is that he exploited the existence theorem of *semi-equilibrium* with non-standard prices proved in [21]. This theorem avoids any form of Slater's condition in the consumer's problem (our irreducibility condition plays its role here). He constructed a perturbed model with the finite number of agents similar to ours but did not equip it with any analog of Slater's condition, that we are doing here. Then, after non-standard passing to the limits and standardization of necessary parameters he used "adding up" arguments. This idea can not be applied directly in our structural framework in view of the absence of the existence theorem analogous to that was obtained in [21] (of cause there are also specific mathematical difficulties appearing in infinite dimensional commodity spaces). That is why we first study the model $\mathcal{E}_M$ satisfying irreducibility condition *(IR)*.

[11] The set A is a compact iff each point $a \in {}^*A$ has a standard part, see [13].

3 Proofs

Proof of Proposition 2.1. What we really need to prove is to show that the model $\mathcal{E}_M$ is irreducible *(IR)*. Really, let I be a proper nonempty subset of $M_0 = M \cup \{0\}$ and $x = (x_i)_{i \in M_0}$ be a feasible allocation. If $I \subset M$, then we can find the required $i \in I$ and $0 \leq y \leq \sum_{j \in M_0 \setminus I} \omega_j$ in view of $\sum_{j \in M_0 \setminus I} \omega_j = \sum_{j \in \mathcal{I} \setminus I} \omega_j$ and the irreducibility of $\mathcal{E}$. Let $0 \in I \subset M_0$. Let us take any feasible allocation $x' = (x'_i)_{i \in \mathcal{I}}$ of the model $\mathcal{E}$ such that $x'_j = x_j$ for $j \in M$. Then, owing to the irreducibility of $\mathcal{E}$ there are $i \in (I \setminus \{0\}) \cup (\mathcal{I} \setminus M)$ and $z_i \in \prod_{t \in T_i} E_t^+$ such that $0 \leq z_i \leq \sum_{M \setminus I} \omega_j$ and $z_i + x'_i \in \mathcal{P}_i(x'_i)$ holds. If this $i \in I \setminus \{0\}$ then there are no problems, using the i-th agent we can check the definition directly. If $i \in \mathcal{I} \setminus M$, then $T_i \subset T_M$ and

$$x'_i + \sum_{M \setminus I} \omega_j \in \mathcal{P}_i(x'_i) \Longrightarrow \sum_{M \setminus I} [\omega_j]_{T_i} > 0,$$

where $[\omega_j]_{T_i} = (\omega_j^t)_{t \in T_i}$ that by $u_0(\cdot)$ construction involves

$$\sum_{j \in M \setminus I, t \in T_i} \alpha_t \cdot s_t^+ \omega_j^t > 0 \Longrightarrow u_0(x_0 + \sum_{M \setminus I} \omega_j) > u_0(x_0).$$

So we can take the "0"-agent to verify *(IR)* and then we see $\mathcal{E}_M$ is irreducible.

Further, we have already seen the assumptions of Theorem 2.1 from [22] are fulfilled therefore we can conclude the existence of quasi-equilibrium couple $(x^M, \pi^M) = (x, \pi)$ for $\mathcal{E}_M$ such that π satisfy the conditions of *Proposition 2.1.* It is well known that due to *(PA)(i),(ii)* to see this couple turns into equilibrium one it is enough to show that each agent earns profit, i.e. $\pi \omega_i > 0 \ \forall i \in M_0$. To do it determine

$$I = \{i \in M_0 \mid \pi \omega_i \neq 0\}$$

and apply property *(IR)*. Assuming $I \neq M_0$ by assumptions *(IR)* & *(PA)(i)* we can find such $0 \leq z \leq \sum_{j \notin I} \omega_j$ and $i \in I$ that

$$\exists \ \ 0 < \delta < 1 \ : \ y = \delta x_i + z \succ_i x_i \ \Longrightarrow \ \pi y \geq \pi x_i,$$

that implies $\pi y = \delta(\pi x_i) + 0 \geq \pi x_i = \pi \omega_i \neq 0$ and we are coming to contradiction.

To show the fact that $x_i^t = 0$ for each $t \notin T_i$, $i \in M_0$ let us assume the contrary and determine the bundles $\bar{x}_i = (\bar{x}_i^t)_{t \in T}$ putting $\bar{x}_i^t = x_i^t$ for $t \in T_i$ and $\bar{x}_i^t = 0$ for $t \notin T_i$. Clear that for equilibrium prices π we have

$$\langle \pi, x_i - \bar{x}_i \rangle = 0, \quad i \in M_0$$

and taking into account $x_i - \bar{x}_i \geq 0$, $i \in M_0$ we can specify the bundle

$$z = \sum_{i \in M_0} (x_i - \bar{x}_i) \geq 0, \quad z = (z^t)_{t \in T}.$$

Now, distributing z^t in any way among consumers living at the moment $t \in T$ and adding these bundles to $\bar{x}_i$, we specify the new bundles, which due to *(PA)(iii)* forms equilibrium allocation with respect to prices π. The last property: $(p_i^M)_t = 0$ for $t \notin T_i$, $i \in M_0$, can be observed from (4) and properness assumption *(PA)(iv)* being applied for extended onto L_ω^+ preferences (for $i = 0$ due to u_0 specification as a linear continuous functional with zero components for $t \notin T_M$ one can easy find an appropriate properness neighborhood of zero which also implies this property). $\square$

Proof of Proposition 2.2. Let (x, π) be a non-standard equilibrium couple existing due to *Proposition 2.1* and to transfer principle for some internal "finite" $M \subset {}^*\mathcal{I}$, $\mathcal{I} \subset M$. Take $i = 1$, normalize π putting $\pi\omega_1 = 1$ and consider the subset

$$I = \{j \in \mathcal{I} \mid \ \pi\omega_j \not\approx 0\}.$$

We want to show that $I = \mathcal{I}$. Assuming contrary we have $\mathcal{I} \setminus I \neq \emptyset$ and $I \neq \emptyset$. Next let us specify the sequence of standard consumption bundles putting

$$\bar{x}_i = st(x_i), \quad i \in \mathcal{I}.$$

The standardization is taken here with respect to the weak topology $\sigma(L_\omega, L'_\omega)$. It is correct because $x_i \in {}^*[0, \omega]$ and since $[0, \omega]$ is $\sigma(L_\omega, L'_\omega)$-compact, then each point of ${}^*[0, \omega]$ has the standard part. In view of *Proposition 2.1* the vector x_i^t may be nonzero only if $t \in T_i$, but each T_i is a finite set. Due to the fact that each $I(t)$ is finite (this is a set of all agents living at the time period t, see $(SF)(i)$) we also have

$$\sum_{\mathcal{I}} \bar{x}_i^t = \sum_{I(t)} st(x_i^t) = st(\sum_{I(t)} x_i^t) = st(\sum_{M} x_i^t) = \omega^t \quad \forall\, t \in T,$$

hence $\bar{x} = (\bar{x}_i)_{i \in \mathcal{I}}$ is a feasible allocation of $\mathcal{E}$. Now let us apply irreducibility (IR) assumption concerning I and $\bar{x}$ and find $i \in I$ & $0 \leq z_i \leq \sum_{\mathcal{I} \setminus I} \omega_j$ such that $z_i + \bar{x}_i \in \mathcal{P}_i(\bar{x}_i)$ holds. We may think $z_i \in L_i = \prod_{T_i} E_t$, that by *(SF)(ii)* entails the existence of *a finite* $S \subset \mathcal{I} \setminus I$ such that $0 \leq z_i \leq \sum_S \omega_j$. Because of *(PA)(i)* we have

$$y = (1 - \varepsilon)\bar{x}_i + z_i \in \mathcal{P}_i(\bar{x}_i)$$

for some standard $\varepsilon > 0$ small enough. Fix this ε. We assumed the graph of $\mathcal{P}_i$ to be $\sigma(L_i, L'_i)$-open, hence for every $y' \approx y,\ x'_i \approx \bar{x}_i$ from ${}^*L_i^+$ we have $\ y' \in {}^*\mathcal{P}_i(x'_i)$. Since $x_i \approx \bar{x}_i$ and for $\tilde{y} = (1 - \varepsilon)x_i + z_i$ we have $\tilde{y} \in {}^*X_i,\ \tilde{y} \approx y$, the last gives

$$(1 - \varepsilon)x_i + z_i \in {}^*\mathcal{P}_i(x_i).$$

Now using the equilibrium properties of the non-standard allocation $x = (x_i)_{i \in M_0}$ we obtain

$$\pi\tilde{y} = (1 - \varepsilon)\pi x_i + \pi z_i > \pi\omega_i,$$

that in view of $\pi x_i = \pi\omega_i$ after routine transformation gives

$$\pi z_i > \varepsilon \cdot \pi\omega_i. \tag{6}$$

However we had $0 \leq z_i \leq \sum_S \omega_j$, where $S \subset \mathcal{I} \setminus I$ is finite. Hence from $\pi \geq 0$ we obtain

$$0 \leq \pi z_i \leq \pi \sum_S \omega_j = \sum_S \pi\omega_j \approx 0$$

by the definition of I that entails $\pi z_i \approx 0$. However $i \in I$ and $0 < \pi\omega_i \not\approx 0$ that contradicts (6), therefore $I = \mathcal{I}$.

To finish the proof we should note only that if we found such $i \in \mathcal{I}$ that $\pi\omega_i$ were not near-standard, then renormalizing π by $\pi\omega_i = 1$ we would determine another subset I such that $1 \in \mathcal{I} \setminus I \neq \emptyset$. However, in view of the thing proved above, the latter one is impossible. □

Proof of Lemma 2.1. Proposition 2.1 and the transfer gives us the representation of non-standard equilibrium price in the form $\pi = \bigvee_{i \in M_0} p_i,\ p_i \in {}^*L'_\omega,\ i \in \mathcal{I}$ for some internal $M \subset {}^*\mathcal{I}$ such that $\mathcal{I} \subset M,\ M_0 = M \cup \{0\}$. Now first we wish to show that $st(p_i^t)$ does exist for each $i \in \mathcal{I},\ t \in T$.

Fix $i \in \mathcal{I}$. Due to *Proposition 2.1* we have

$$\langle p_i, {}^*U_i + \omega\rangle \geq 0.$$

We may think the neighborhood U_i to be represented in the form

$$U_i = (V_i \times \prod_{t \in T \setminus T_i} E_t) \cap L_\omega,$$

where V_i is some circled neighborhood of zero in $L_i = \prod_{T_i} E_i$, chosen by ω-uniform properness of $\mathcal{P}_i$. The former and the latter imply

$$|\langle p_i, {}^*U_i\rangle| \leq \langle p_i, \omega\rangle, \quad p_i^t = 0, \quad t \in {}^*T \setminus T_i. \tag{7}$$

Notice also that by transfer, *(SF)(i)*, from T_M construction and due to $\mathcal{I} \subset M$ one can easy to conclude that $T_M \subset {}^*\mathcal{I} \setminus \mathcal{I}$. But then *Proposition 2.1* and transfer for 0's consumer yields $p_0^t = 0$ for all $t \in T$.

Let $J \subset \mathcal{I}$ be a set of all agents living during at least one time period from T_i, i.e.

$$J = \bigcup_{t \in T_i} I(t).$$

Note that J is a finite set. Taking into account (7) and the property of equilibrium allocations described in *Proposition 2.1* ($x_j^t \neq 0 \Rightarrow t \in T_j$) we can write

$$\langle p_i, \omega\rangle = \langle p_i, \sum_J x_j\rangle = \sum_{j \in J}\langle p_i, x_j\rangle \leq \sum_J \pi x_j = \sum_J \pi\omega_j.$$

In view of *Proposition 2.2*, for normalized π by $\pi\omega_1 = 1$ the value $\sum_J \pi\omega_j$ is standardly bounded because of the finiteness of J. Hence each point of the set

$$K = \{p \in {}^*L'_\omega \mid\ |\langle p, {}^*U_i\rangle|\ \leq \sum_J \pi\omega_j\}$$

is near-standard with respect to $\sigma(L'_\omega, L_\omega)$-topology (by Alaoglu–Bourbaki's theorem and the non-standard characterization of compact sets, see [13]), that means the existence of $st(p_i^t)$ taken relative to $\sigma(E'_t, E_t)$-topology. As a result

$$\pi^t = \bigvee_{i \in M_0} p_i^t = \bigvee_{i \in I(t)} p_i^t$$

takes place where each p_i^t is $\sigma(E_t', E_t)$ near-standard.

To prove $\pi = (\pi^t)$ is non-standard OLG-price we need to check π^t is $\sigma(E_t', E_t)$ near-standard for every $t \in T$. For fixed t and $x \in E_t^+$ due to *(SA)(iii)* and transfer one may apply Riesz–Kantorovich formula explicitly representing the supremum functional

$$\pi^t(x) = sup\ \{ \sum_{i \in I(t)} p_i^t y_i \mid \sum_{i \in I(t)} y_i = x,\ y_i \in {}^*E_t^+,\ i \in I(t)\}.$$

Clear $\pi^t(x)$ is near-standard. Therefore $st(\pi^t)$ does exist in the $*$-image of algebraic dual E_t^* to E_t relative to weak-$*$ topology. In view of *(SA)(iii)* to show this functional is τ_t-continuous it is enough to prove $st(\pi^t) = \bigvee_{i \in I(t)} st(p_i^t)$, i.e. since obviously $st(\pi^t)(x) = st(\pi^t(x))$ we have to check

$$st(\pi^t(x)) = sup\ \{ \sum_{i \in I(t)} st(p_i^t) y_i \mid \sum_{i \in I(t)} y_i = x,\ y_i \in E_t^+,\ i \in I(t)\} \tag{8}$$

for every $x \in E_t^+$. To do it let us restrict attention the order ideal $E(x) = E_t(x)$ of E_t generated by the point x.[12] Equip $E(x)$ with Riesz norm $||\cdot||$, which unit ball is $B(x) = \{y \in E_t \mid |y| \le x\}$. Now $(E(x), ||\cdot||)$ is Riesz norm space and easy to see that $\pi^t_{|E(x)}$, $p^t_{i|E(x)}$ are the elements of ${}^*(E(x), ||\cdot||)'$ and, moreover, every $p^t_{i|E(x)}$ is $||\cdot||^*$-near standard (since the image of ${}^*B(x)$ is standard bounded). But the space $(E(x), ||\cdot||)'$ being equipped with dual norm $||\cdot||^*$ is also Riesz norm (and Banach) space (see [30], p. 64, Davies' theorem (6.12)), in which order operations are continuous. Therefore $st_{||\cdot||^*}(\pi^t_{|E(x)}) = \bigvee_{i \in I(t)} st_{||\cdot||^*}(p^t_{i|E(x)})$ and since $st_{||\cdot||^*}(p^t_{i|E(x)})$ coincides onto $E_t(x)$ with the standard part of p_i^t relative to $\sigma(E_t', E_t)$ (because $\sigma(E_t', E_t)$ is Hausdorff), (8) is true for given x and consequently for every $x \in E_t^+$. The proof is complete. □

Proof of Theorem 1.1. Let us consider the non-standard model $\mathcal{E}_M$ with an internal "finite" $M \subset {}^*\mathcal{I}$ such that $\mathcal{I} \subset M$. Let us take the non-standard equilibrium couple (x, π), existing by *Proposition 2.1* and transfer principle. Due to *Proposition 2.2* we can normalize π putting $\pi\omega_1 = 1$. Let us study the "equilibrium" point $(\bar{x}, \pi)$ specified by $\bar{x}_i = st(x_i),\ i \in \mathcal{I}$. In view of the weak compactness of the interval $[0, \omega]$ in L_ω this specification is correct. Also due to *Lemma 2.1* the values $\bar{\pi}^t = st(\pi^t)$ do exist for all $t \in T$ and by the chosen normalization of π, the property $st(\pi\omega_i) > 0$ holds for each $i \in \mathcal{I}$ by *Proposition 2.2*. So the functional π is a non-standard OLG-price. Now we have to show that the couple $(\bar{x}, \pi)$ is an equilibrium with non-standard prices. Due to *Lemma 2.2* (proved below) and the fact that each agent earns profit, everything will be proved if we show that

$$\sum_{T_i} \bar{\pi}^t y_t \ge \sum_{T_i} \bar{\pi}^t \bar{x}_i^t = st(\pi\omega_i) \quad \forall\, y = (y_t)_{T_i} \in \mathcal{P}_i(\bar{x}_i),\ i \in \mathcal{I}.$$

To do it we first establish

$$\sum_{T_i} \bar{\pi}^t y_t \ge st(\pi\omega_i), \quad y \in \mathcal{P}_i(\bar{x}_i), \quad i \in \mathcal{I}. \tag{9}$$

[12] This subspace can be defined by $E(x) = \{y \in E_t \mid \exists \lambda \in I\!R,\ \lambda \ge 0 :\ |y| \le \lambda x\}$.

Let $y \in \mathcal{P}_i(\bar{x}_i)$ and $i \in \mathcal{I}$ be fixed. We assumed the graph of $\mathcal{P}_i$ to be $\sigma(L_i, L_i')$-open in L_i^+, hence for $x_i \approx \bar{x}_i$ we have $y \in {}^*\mathcal{P}_i(x_i)$, that gives

$$\pi y \geq \pi x_i = \pi \omega_i,$$

because (x, π) is a non-standard equilibrium. Since $y \in {}^*L_i$, where L_i may be identify as a subspace of L_ω, we have $\pi y = \sum_{T_i} \pi^t y_t$ and the latter inequality may be standardized, which gives us (9).

Second we establish

$$\sum_{T_i} \bar{\pi}^t \bar{x}_i^t = st(\pi\omega_i).$$

Let $t \in T_i$ and $i \in \mathcal{I}$ be fixed. Applying transfer to *Proposition 2.1* one can find an appropriate $p_i \in {}^*L_i'$, which has "equilibrium properties" (5) and realize $\pi^t = \bigvee_{j \in I(t)} p_j^t$. Now since $\pi^{t'} \geq p_i^{t'}$, $x_i^{t'} \geq 0$ we have

$$\langle p_i^t, {}^*\mathcal{P}_i^t(x_i)\rangle > \pi\omega_i - \sum_{t' \in T_i, t' \neq t} \pi^{t'} x_i^{t'} = \lambda_i^t, \tag{10}$$

where $\mathcal{P}_i^t(x_i) = \mathcal{P}_i^t(x_i^t, x_i^{-t})$, $x_i^{-t} = (x_i^{t'})_{t' \in T_i \setminus \{t\}}$ is the section of $\mathcal{P}_i(x_i)$ relative to E_t. Let us choose any $z \underset{\tau_t}{\approx} \bar{x}_i^t$, $z \in {}^*conv\mathcal{P}_i^t(x_i^t, x_i^{-t})$, existing by *(PN)* and in view of the weak openness of $\mathcal{P}_i$-graph. Then (10) yields

$$p_i^t z > \lambda_i^t \implies p_i^t \bar{x}_i^t + p_i^t(z - \bar{x}_i^t) > \lambda_i^t.$$

But $(z - \bar{x}_i^t) \underset{\tau_t}{\approx} 0$ and due to (5) it is easy to conclude $p_i^t y \approx 0$ for all $y \underset{\tau_t}{\approx} 0$, $y \in {}^*E_t$. Therefore $p_i^t(z - \bar{x}_i^t) \approx 0$ and we obtain

$$p_i^t \bar{x}_i^t \gtrsim \lambda_i^t \implies \pi^t \bar{x}_i^t \gtrsim \lambda_i^t. \tag{11}$$

Now we show that inverse inequality also holds. In fact, observing that $\sum_{i \in I(t)} \bar{x}_i^t = \omega^t$ we have

$$\sum_{i \in I(t)} \pi^t \bar{x}_i^t = \pi^t \omega^t.$$

But by construction and by equilibrium properties we also have

$$\sum_{I(t)} \lambda_i^t = \pi^t \omega^t, \quad \pi^t x_i^t = \lambda_i^t.$$

Considering (11), the latter and the former relation we conclude $\pi_i^t \bar{x}_i^t \approx \pi_i^t x_i^t$ (to obtain it we need special arguments because of only one-sided continuity of inner product with respect to weak topologies, if E_t is finite dimensional then we have it automatically), that implies

$$\bar{\pi}^t \bar{x}_i^t \approx \pi^t \bar{x}_i^t \approx \pi^t x_i^t = \lambda_i^t \Longrightarrow \bar{\pi}^t \bar{x}_i^t \approx \lambda_i^t.$$

Now, summing up the last relations by $t \in T_i$ we obtain

$$\sum_{T_i} \bar{\pi}^t \bar{x}_i^t \approx |T_i|\pi\omega_i - (|T_i| - 1)\sum_{T_i} \pi^t x_i^t \quad \Longrightarrow \quad \sum_{T_i} \bar{\pi}^t \bar{x}_i^t = st(\pi\omega_i).$$

This together with (9) gives us the result. □

Proof of Lemma 2.2. Fix $i \in \mathcal{I}$ and put $\alpha_k = \sum_{t\le k} \pi^t \omega_i^t,\ k \in {}^*T$ and $\bar{\alpha} = \sup_{k\in T}\ st(\alpha_k)$. Because of this and due to the definition of $\bar{\pi}\omega_i$ we have

$$\bar{\alpha} = \bar{\pi}\omega_i = \sup_{S\subset T,\ |S|<\infty} \sum_{t\in S} \bar{\pi}^t \omega_i^t.$$

Let us specify the following internal sequence

$$\beta_k = (\alpha_k - \bar{\alpha})^+, \quad k \in {}^*T.$$

Since π and ω_i are positive we obtain $\beta_k \approx 0$ for all $k \in T$. Now we can apply the theorem on the extension of an internal sequence (see [13]), which gives the existence of such $t_i \in {}^*T \setminus T$ that $\beta_k \approx 0$ for all $k < t_i$. Hence

$$\sum_{t\le k} \pi^t \omega_i^t \lesssim \bar{\pi}\omega_i \ \ \forall\ k < t_i,\ k \in {}^*T.$$

Furthermore, for each $finite\ S \subset T$ we have

$$\sum_{t<k} \pi^t \omega_i^t \ge \sum_{S} \pi^t \omega_i^t \gtrsim \sum_{S} \bar{\pi}^t \omega_i^t \ \ \forall\ k < t_i,\ k \in {}^*T \setminus T, \quad \Longrightarrow \quad \sum_{t<k} \pi^t \omega_i^t \gtrsim \bar{\pi}\omega_i.$$

Thus, taking into account the former relation we obtain

$$st(\sum_{t<k} \pi^t \omega_i^t) = st(\sum_{t<t_i} \pi^t \omega_i^t) = \bar{\pi}\omega_i \ \ \forall\ k < t_i,\ k \in {}^*T \setminus T.$$

From this we eventually have

$$\pi\omega_i = \sum_{t<t_i} \pi^t \omega_i^t + \sum_{t\ge t_i} \pi^t \omega_i^t \ \Rightarrow\ st(\pi\omega_i) = st(\sum_{t<t_i} \pi^t \omega_i^t) + st(\sum_{t\ge t_i} \pi^t \omega_i^t) = \bar{\pi}\omega_i + st(\sum_{t\ge t_i} \pi^t \omega_i^t)$$

and hit the target. □

Proof of Theorem 1.2. Since the first part of the theorem is obvious, by followed from *Theorem 1.1* we consider the case in which there is a finite group of agents $I \subset \mathcal{I}$ owning a positive fraction of total initial endowments. So we assume

$$\sum_{I} \omega_i \ge \sigma \sum_{\mathcal{I}} \omega_j, \quad \sigma > 0. \tag{12}$$

Let us consider any equilibrium couple (x, π) with non-standard prices existing in accordance with *Theorem 1.1.* In view of (12) the equilibrium prices π can be normalized by $\pi\omega = 1$ so that $st(\pi\omega_i) > 0$ for each $i \in \mathcal{I}$. To prove *Theorem 1.2* on the ground of *Theorem 1.1* it suffices to show that $\sum_{k\ge t} \pi^k \omega_i^k \approx 0$ holds for any arbitrary infinite $t \in {}^*T \setminus T$ and for each $i \in \mathcal{I}$. Let us do it.

Given $t \in T$, we specify the set J_t of all agents having nonzero ownership during the time periods $\{1, ..., t\}$, i.e. we put

$$J_t = \{i \in \mathcal{I} \mid \ \exists k \le t : \omega_i^k \ne 0\}.$$

Clear J_t is a finite set. We may realize that t is chosen in such manner that $I \subset J_t$, where I is taken by (12). This gives us

$$\sum_{J_t} \pi\omega_j \geq \sum_{k\leq t} \pi^k \omega^k + \sigma \sum_{k>t} \pi^k \omega^k.$$

On the other hand if $\bar{t} > t$ is some time period such that $T_i \subset \{1, ..., \bar{t}\}$ for every $i \in J_t$ then we obtain

$$\sum_{k\leq \bar{t}} \pi^k \omega^k \geq \sum_{J_t} \pi x_j = \sum_{J_t} \pi \omega_j.$$

Comparing the former and the latter inequalities we obtain

$$\sigma \sum_{k>t} \pi^k \omega^k \leq \sum_{t<k\leq \bar{t}} \pi^k \omega^k,$$

that implies

$$st(\sum_{k>t} \pi^k \omega^k) \leq \frac{1}{\sigma} \sum_{t<k\leq \bar{t}} \bar{\pi}^k \omega^k. \tag{13}$$

Since $\sum_{k\leq t} \bar{\pi}^k \omega^k \to \bar{\pi}\omega$ for $t \to \infty$, then the right side in (13) may be done arbitrarily small, that gives

$$0 \leq \sum_{k\geq t} \pi^k \omega^k \approx 0 \quad \forall\, t \in {}^*T \setminus T \quad \Longrightarrow \quad st(\pi\omega_i) = \bar{\pi}\omega_i, \quad i \in \mathcal{I}$$

as we wanted to prove. □

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