Equilibria in infinite dimensional commodity spaces revisited*

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Abstract

The aim of the paper is to provide a new proof of the Mas-Colell–Richard existence of equilibrium result when preferences are non-transitive and incomplete. Our proof generalizes the main ideas of the Negishi approach to the case of unordered preferences.

Keywords and Phrases: Vector lattices, Competitive equilibrium, Uniformly proper preferences.

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1 Introduction

In their paper Mas-Colell and Richard [11] (cf., Aliprantis [1]) proved the existence of equilibria in vector lattices which are not necessarily locally solid. They assumed that the commodity space is a vector lattice endowed with a locally convex topology such that the positive cone is closed. They also required the topological dual of this space be a sub-lattice of it's order dual. This enabled them to extend the seminal result of Mas-Colell [10] to a broader set of models, e.g., Huang & Kreps [8] and Jones [9]. In their proof, Mas-Colell and Richard [11] use an extension of well known Negishi approach. They also consider individual supporting prices for weakly Pareto allocations and their supremum, that allows them to avoid uniform continuity of lattice operations in using the decomposition property of vector lattices. As in earlier Mas-Colell [10], the quasi-equilibrium existence proof is based on fixed-point argument in the utility space (later Yannelis and Zame [14] generalized the Mas-Colell theorem from [10] to unordered preferences, while the commodity space is a Banach lattice).

The purpose of this paper is to extend Mas-Colell-Richard's theorem to the case of unordered preferences. It is clear that the direct application of Negishi approach is not suitable in this setting since it requires the preferences be representable by utility functions. Our proof modifies the Negishi approach, and as in Bewley [6], Yannelis-Zame [14], and Podczeck [12] our result is obtained by considering a direct set of truncations of the economy. The result is proved under specific assumption of "uniform properness" of preferences, which is slightly weaker than Yannelis and Zame's "uniform properness" (see [14]). In a transitive context, it is also slightly weaker than Mas-Colell's uniform properness. Related results can be found in Podczeck [12] who proves the existence of equilibrium in Mas-Colell and Richard's setting without ordered preferences. Also related results of Tourky [13] and Deghdak-Florenzano [7]. They prove core equivalence theorems in our setting. Their results, when combined with an existence theorem for Edgeworth equilibria, also imply the existence of Walrasian equilibria with unordered preferences. Notice, however, that our properness notion is different from the ones used in Podczeck [12] and in Tourky [13]. Moreover, Tourky [13] allows for more general consumption sets than the positive cone of the commodity space.

Riesz spaces were first introduced into general equilibrium theory in the paper of Aliprantis-Brown [2]. The lattice structure of the commodity space was then used in the paper of Mas-Colell [10] to prove his remarkable theorem on the existence of equilibrium. The basic structure of the infinite dimensional analysis in a context of economic model was developed in [3]. For a general overview of the literature the reader can consult the book of Aliprantis-Brown-Burkinshaw [4].

The paper is organized as follows. In Section 2, we describe the model and state the main result. Section 3 is devoted to the strategy of proof and to auxiliary results. The fourth section contains the detailed proofs.

2 The exchange model and the main result

We consider a typical exchange economy in which the commodity space L is a partially ordered vector space equipped with a Hausdorff, locally convex topology τ. Let $N = \{1, ..., n\}$ denote the set of economic agents, whose consumption sets coincide with the positive cone of the space $L_+ = \{x \in L \mid x \geq 0\}$. The agent's preferences are described by point-to-set mappings $P_i : L_+ \to L_+$, so that $P_i(x_i)$ denotes the set of all consumption bundles strictly preferred by the i-th agent to the bundle x_i. We also will use the notation $y_i \underset{i}{\succ} x_i$ which is equivalent to $y_i \in P_i(x_i)$. Each consumer i is endowed with an initial endowment $\omega_i \in L_+$. We require the prices π to be chosen in the topological dual of L, denoted by L'. Thus, the model under study is a triplet

$$\mathcal{E} = (N, \langle L, L' \rangle,\ \{P_i(\cdot), \omega_i\}_{i \in N}).$$

Let $\omega = \sum_N \omega_i$. In what follows, we assume $\omega \neq 0$. The other assumptions on the economy are divided into two groups. The first one consists of

Structural Assumptions (SA)

(i) L *is a linear vector lattice (or Riesz space) ;*

(ii) L_+ *is closed in the* τ*-topology of* L *;*

(iii) L' *is a sublattice of the order dual to* L*;*

(iv) the order interval $[0, \omega]$[1] *is* $\sigma(L, L')$[2] *- compact.*

It is worth noticing that *(i)* and *(ii)* do not imply that the lattice operations $x \vee y, x \wedge y, x, y \in L$ are *continuous* with respect to topology τ. In other words, we do not assume the topology τ to be locally solid.

If τ were locally solid, then requirements *(ii), (iii)* would be automatically valid. Since we avoid this hypothesis, we need to require them directly. Almost the same can be said about *(iv)* : if the topology of the space guarantees that every order interval is $\sigma(L, L')$-compact (for example, if L is Dedekind complete and $\sigma(L, L')$-order continuous, see [4]), then we avoid this assumption, otherwise not. For more specific explanations and references, the reader is referred to [4].

The second group of assumptions consists of the properties of agents' characteristics. All these assumptions are well known in the literature and only one of them requires special explanations, the so-called *uniform properness of preferences.* In this paper, properness will be defined as follows.

Definition 2.1 *The preference* $P(\cdot)$ *is said to be* ω*–uniformly* τ*-proper on* $Y \subset L_+$ *if there exists a* τ*-neighborhood* V *of the origin such that*

$$y - \alpha\omega + z \in conv\ P(y)^3,\ y \in Y,\ \alpha > 0 \implies z \notin \alpha V.$$

[1]The notation $[a, b]$ denotes the order interval, i.e. $[a, b] = \{c \in L \mid a \leq c \leq b\}$.

[2]$\sigma(L, L')$ denotes weak topology on L.

[3]In this paper, *conv* A denotes the convex hull of the set A, clA is its closure and $\setminus$ is set for the set-theoretical difference.

Originally a slightly stronger notion was introduced by Mas-Colell [10], in order to overcome the emptiness of the interior of the positive cone for many interesting commodity spaces. The Mas-Colell definition was extended by Yannelis and Zame [14] to unordered preferences.

Assumptions on Preferences (PA)

For each $i \in N$,

(i) upper semicontinuity:

$$P_i(x) \ \ is \ \ \tau - open \ in \ L_+ \ for \ each \ x \in L_+;$$

(ii) lower semicontinuity: for each $x \in L_+$, the set

$$P_i^{-1}(x) = \{y \in L_+ \mid x \in P_i(y)\} \ \ is \ \ \sigma(L, L') - open \ in \ L_+;$$

(iii) weak convexity, irreflexivity and local nonsatiation: for each $x \in [0, \omega]$

$$x \in cl(conv\ P_i(x)) \backslash conv\ P_i(x);$$

(iv) monotonicity: for each $x \in L_+$

$$P_i(x) + L_+ \subset cl(conv\ P_i(x));$$

(v) $P_i(\cdot)$ is ω–uniformly τ-proper on $[0, \omega]$.

Note that there is no loss of generality to assume in *(v)* that for each i, the individual properness τ- neighborhood of the origin V_i is convex and circled (i.e. $V_i = -V_i$).

Let

$$\chi = \{x \in L_+^n \mid \sum_N x_i = \omega\}$$

be the set of feasible allocations.

Definition 2.2 *A couple (x, π) is said to be a quasi-equilibrium if $x \in \chi$, $\pi \in L'_+, \pi(\omega) > 0$ and for each $i \in N$, it holds:*

$$\langle P_i(x_i), \pi\rangle \geq \pi(x_i) = \pi(\omega_i).^4 \tag{1}$$

The quasi-equilibrium is an equilibrium if the inequalities in (1) are strict.

The main result of this paper is

Theorem 2.1 *If $\mathcal{E}$ satisfies structural assumptions SA and if agents' preferences satisfy PA, then there exists a quasi-equilibrium (x, π).*

Theorem 2.1 will be first proved under an additional assumption on individual initial endowments in the following preliminary result.

[4] $\langle A, \pi\rangle$ denotes the set $\{\langle a, \pi\rangle \mid a \in A\}$ and $A \geq b$ means $a \geq b$ for all $a \in A$.

Theorem 2.2 *Under the same assumptions and if*

$$\exists\ h > 0 \text{ such that } \omega_i \geq h\omega, \quad i \in N \tag{2}$$

then there exists a quasi-equilibrium (x, π)*. Moreover,* $\pi(\omega) > 0$ *and* $\pi = \vee_{i\in N} p_i$ *with for each* $i \in N$, $\langle p_i, \omega + V\rangle \geq\ 0$*, where* $V = \bigcap_{i\in N} V_i$ *and* V_i *is an open, convex and circled neighborhood of zero taken from the* ω*–uniform* τ*-properness condition for* i*-th agent's preferences.*

3 Strategy of proof, auxiliary results and discussion

Mas-Colell and Richard suggested the attractive idea of representing the equilibrium price as the *supremum of "individual" supporting prices.* They used such an approach in their Lemma 1 and Proposition. They also constructed compact sets, containing "individual" supporting prices for any given weak Pareto-optimal allocation, using explicitly the ω-uniform properness of preferences. We are borrowing these ideas but applying them in a different way. Our method is based on a direct usage of mappings which define the sets of continuous functionals, supporting all the i-th agent preferred points to a given i-th consumption bundle.

Let $x_i \neq 0$, $x_i \in L_+$ and $\lambda_i > 0$ be fixed. If V_i is an open, convex, circled neighborhood of zero taken from the definition of the uniform properness, let

$$\Gamma_i(x_i) = conv\ P_i(x_i) + K, \tag{3}$$

where

$$K = \{\alpha(\omega + V) \mid \alpha > 0\}\ \ \&\ \ V = \bigcap_{i\in N} V_i.$$

The main properties of the mapping $\Gamma_i(.)$ are summarized in the following easy proposition.

Proposition 3.1 *If* $P_i : L_+ \to L_+$ *satisfies PA then*
(i) $\Gamma_i(x)$ *is open, convex and nonempty at each* $x \in [0, \omega]$*;*
(ii) if $x \in [0, \omega]$ *then* $x \in cl(\Gamma_i(x) \bigcap L_+)$ *and* $x \notin \Gamma_i(x)$*;*
(iii) $conv\ P_i(x) \subset cl(\Gamma_i(x) \bigcap L_+)$*;*
(iv) $cl(\Gamma_i(x)) + L_+ \subset cl(\Gamma_i(x))$*;*
(v) $\Gamma_i^{-1}(y)$ *is* $\sigma(L, L')$*-open in* L_+ *for every* $y \in L$*.*

Now for $x_i \in [0, \omega]$ and real $\lambda_i > 0$, let us define the mappings of "individual supporting prices" by

$$\Pi_i(x_i, \lambda_i) = \{p_i \in L'_+ \mid \langle\Gamma_i(x_i), p_i\rangle \geq \lambda_i = p_i(x_i)\}. \tag{4}$$

The mappings, defined by (4), will play a crucial role below. Indeed, by (3), (4) we have

$$\mid \langle p_i, V\rangle \mid \leq \langle p_i, \omega\rangle, \quad p_i \in \Pi_i(x_i, \lambda_i),$$

which, under the additional condition $x_i \geq \varepsilon\omega$ for some $\varepsilon > 0$, implies, by Alaoglu's theorem, the compactness of $\Pi_i(x_i, \lambda_i)$.

We also would like the values λ_i to satisfy

$$\lambda_i = (\vee_{j\in N}\ p_j)(\omega_i) = \sup_{x_j \geq 0, \sum x_j = \omega_i} \sum p_j(x_j).$$

Then following Mas-Colell-Richard's arguments (see Proposition from [11]) we would state that $\pi = \vee p_j$ is an equilibrium price. However any attempt to construct a point-to-set mapping, the fixed points of which satisfy the previous conditions, encounters many problems.

The first one is that $\Pi_i(\cdot)$ may have an unclosed graph with respect to the weak* topology $\sigma(L', L)$, due to the lack of joint continuity of the inner product $\langle p, x\rangle = p(x),\ p \in L', x \in L$. For this reason, we will confine in a first step our considerations to *finite-dimensional* subspaces of the commodity space.

Another one is that each $\Pi_i(x_i, \lambda_i)$, being defined on a convex compact domain, should take values in a convex compact set. The previous ideas are summarized in the following lemma and in the subsequent constructions.

Lemma 3.1 *Let assumptions PA (i)-(v) hold for some preference mapping $P_i(\cdot)$ and Y be a subset of some finite-dimensional subspace of L such that $Y \subset [0, \omega]$. Then*

(i) for each $x \in [0, \omega]$ and $\lambda > 0$, if $x \geq \varepsilon\omega$ for some $\varepsilon > 0$ then the set $\Pi_i(x, \lambda)$, constructed by (3) and (4), is nonempty, convex and $\sigma(L', L)$-compact,

(ii) the map $(x, \lambda) \to \Pi_i(x, \lambda)$ with the domain $Y \times [\alpha, \beta]$ for some real $\beta > \alpha > 0$ has a closed graph in $Y \times [\alpha, \beta] \times L'_+$ with respect to the $\sigma(L', L)$-topology on L'_+.

Suppose now that for some fixed real $h > 0$, we have $h\omega \leq \omega_i$ for all i. Take a fixed ε such that $h > \varepsilon > 0$. Fix also $\mathcal{L}$, a finite dimensional subspace of L containing all ω_i. Given $\mathcal{L}$ and ε, we define the following sets. First,

$$\Psi_\varepsilon = \{p \in L'_+ \mid\ |\langle p, V\rangle| \ \leq \frac{1}{\varepsilon}, \quad p(\omega) \geq h\}.$$

By Alaoglu's theorem, Ψ_ε is $\sigma(L', L)$ - compact. It is also clear that Ψ^n_ε is convex, $\sigma(L'^n, L^n)$-compact and nonempty provided $\varepsilon > 0$ is small enough[5].

For allocations, we consider:

$$X^{\mathcal{L}}_\varepsilon = \{(x_1, ..., x_n) \in \mathcal{L}^n_+ \mid \quad \varepsilon\omega \leq x_i \leq \omega, \quad i \in N, \quad \sum_{i\in N} x_i = \omega\}$$

and note that $(\omega_1, \ldots, \omega_n) \in X^{\mathcal{L}}_\varepsilon$.

The domain of variables λ_i is defined as follows

$$\Delta = \{(\lambda_1, ..., \lambda_n) \in \mathbf{R}^n_+ \mid \quad \sum_N \lambda_i \leq 1, \quad \lambda_i \geq h, \quad i \in N\},$$

[5]From Lemma 3.1 *(i)*, we have $\Pi_i(\omega, h) \neq \emptyset$. Take $\varepsilon \leq \frac{1}{h}$ and to check the non-emptyness observe that $\Pi_i(\omega, h) \subset \Psi_\varepsilon$.

this set being nonempty for $h \leq \frac{1}{n}$. Then, assembling specified sets, we have the nonempty convex compact set

$$Z_\varepsilon^{\mathcal{L}} = X_\varepsilon^{\mathcal{L}} \times \Delta \times \Psi_\varepsilon^n.$$

We now construct a point-to-set mapping from $Z_\varepsilon^{\mathcal{L}}$ into itself. This mapping is represented as a product of three maps. One of them was almost specified above. It is

$$s : (x, \lambda) \longrightarrow \prod_{i \in N} \Pi_i \ (x_i, \lambda_i).$$

The second map is specified by

$$r_\varepsilon^{\mathcal{L}} : q = (p_1, ..., p_n) \longrightarrow \underset{x' \in X_\varepsilon^{\mathcal{L}}}{argmax} \ q(x').$$

The third mapping has a more complex construction. Let

$$X_{\mathcal{L}}^i = \{x' \in \mathcal{L}_+^n \mid \sum_{j \in N} x'_j = \omega_i\}, \quad X_{\mathcal{L}} = \{x' \in \mathcal{L}_+^n \mid \sum_{j \in N} x'_j = \omega\}.$$

Determine

$$\lambda_i'^{\mathcal{L}}(q) = \max_{x' \in X_{\mathcal{L}}^i} \ q(x'), \quad \lambda_\omega^{\mathcal{L}}(q) = \max_{x' \in X_{\mathcal{L}}} \ q(x'), \tag{5}$$

then the required map

$$\lambda^{\mathcal{L}} \ : q = (p_1, ..., p_n) \longrightarrow (\lambda_1^{\mathcal{L}}(q), ..., \lambda_n^{\mathcal{L}}(q)) = \lambda^{\mathcal{L}}(q)$$

is specified by $\lambda_i^{\mathcal{L}}(q) = \lambda_i'^{\mathcal{L}}(q)/\lambda_\omega^{\mathcal{L}}(q)$, $i \in N$. The resulting mapping is the following

$$\varphi_\varepsilon^{\mathcal{L}} : (x, \lambda, q) \longrightarrow r_\varepsilon^{\mathcal{L}} \ (q) \times \{\lambda^{\mathcal{L}} \ (q)\} \times s \ (x, \lambda).$$

Lemma 3.2 *Assume $h \leq 1/n$. Then for every ε such that $0 < \varepsilon < h$, the set $Z_\varepsilon^{\mathcal{L}}$ is nonempty convex and compact and $\varphi_\varepsilon^{\mathcal{L}} : Z_\varepsilon^{\mathcal{L}} \longrightarrow 2^{\mathcal{L}_+^n} \times 2^{\mathbf{R}_+^n} \times 2^{(L'_+)^n}$ has a closed graph with nonempty compact convex values in $Z_\varepsilon^{\mathcal{L}}$. Hence $\varphi_\varepsilon^{\mathcal{L}}$ has a fixed point.*

Keep $\mathcal{L}$ fixed and denote by $(x_\varepsilon^{\mathcal{L}}, \lambda_\varepsilon^{\mathcal{L}}, q_\varepsilon^{\mathcal{L}})$ a current fixed point of $\varphi_\varepsilon^{\mathcal{L}}$. As it is observed in the following lemma, for $\varepsilon > 0$ small enough, all fixed points of this map may be included into a common compact set.

Lemma 3.3 *Assume $\varepsilon \in (0, 1/2n]$. Then all fixed points $(x_\varepsilon^{\mathcal{L}}, \lambda_\varepsilon^{\mathcal{L}}, q_\varepsilon^{\mathcal{L}})$ of $\varphi_\varepsilon^{\mathcal{L}}(.)$ can be included in some common compact set which does not depend on ε. Moreover for each such point, if $q_\varepsilon^{\mathcal{L}} = (p_1, \ldots, p_n)$ then*

$$2 \geq p_i(\omega) \geq h, \quad \mid \langle p_i, V \rangle \mid \leq 2.$$

It allows us to let $\varepsilon \to 0$ and to pass to the limits. As a result, we then have

Lemma 3.4 *(MAIN AUXILIARY LEMMA). Assume that* $\omega_i \geq h \cdot \omega$ *for some* $h > 0$ *and let* $\mathcal{L} \subset L$ *be a finite-dimensional subspace, such that* $\omega_i \in \mathcal{L},\ i \in N$. *Then there exists a feasible allocation* $x^{\mathcal{L}} = (x_1^{\mathcal{L}}, ..., x_n^{\mathcal{L}}) \in \mathcal{L}_+^n$, *prices* $q^{\mathcal{L}} = (p_1^{\mathcal{L}}, ..., p_n^{\mathcal{L}}) \in (L'_+)^n$, *and numbers* $\lambda_i^{\mathcal{L}} \geq h,\ i \in N,\ \sum_N \lambda_i^{\mathcal{L}} \leq 1$, *such that for each* $i \in N$ *the following conditions hold :*

(i) $\langle p_i^{\mathcal{L}}, \Gamma_i(x_i^{\mathcal{L}})\rangle \geq p_i^{\mathcal{L}}(x_i^{\mathcal{L}}) = \lambda_i^{\mathcal{L}}, \quad \Gamma_i(x_i^{\mathcal{L}}) = conv\ P_i(x_i^{\mathcal{L}}) + \{\alpha(\omega + V) \mid \alpha > 0\}$;

(ii) $p_i^{\mathcal{L}} \in \{p \in L'_+ \mid\ |\langle p, V\rangle| \leq 2,\ h \leq p\omega \leq 2\}$;

(iii) $\lambda_i^{\mathcal{L}} \geq max\ \{q^{\mathcal{L}}(y) \mid \quad y \in \mathcal{L}_+^n, \quad \sum_{i\in N} y_j = \omega_i\}$.

We prove Theorem 2.2 using a limiting process on the couples $(x^{\mathcal{L}}, q^{\mathcal{L}})_{\mathcal{L}\subset L}$, obtained in the previous lemma. A second limiting process letting $h \to 0$ for an appropriate approximation ω_i^h of initial endowments $\omega_i,\ i \in N$ allows us to prove Theorem 2.1.

It is worth noticing an observation used in the proof. If $q = (p_1, ..., p_n)$ is a bundle of limit individual supporting prices, then all variables $z^i \in L_+$ satisfying

$$q(z^i) = \max_{x_j \geq 0, \sum x_j = \omega_i,} \sum p_j(x_j) = (\vee p_j)(\omega_i)$$

may be included into a common finite-dimensional subspace that gives us an opportunity to realize crucial estimates and to use the equilibrium properties of the couples $(x^{\mathcal{L}}, q^{\mathcal{L}})_{\mathcal{L}\subset L}$.

4 Proofs

Proof of Proposition 3.1. *(i)–(iv)* are immediate from the definition of Γ_i and assumptions *PA* on preferences. To verify *(v)*, choose any $x \in L_+$, $y \in L$ such that $y \in \Gamma_i(x) \iff x \in \Gamma_i^{-1}(y)$. By definition (see (3)), we can write $y = \sum_{r=1}^m t_r z_r + h$ with $h \in K,\ z_r \in P_i(x)$ and $t_r \geq 0, \quad r = 1, \ldots m, \quad \sum_{r=1}^m t_r = 1$. Since due to *PA(ii)* the set $P_i^{-1}(z_r)$ is a $\sigma(L, L')$-open neighborhood of x in L_+, the set $V_x = \bigcap_{r=1}^{r=m} P_i^{-1}(z_r)$ is also a $\sigma(L, L')$-open neighborhood of x in L_+. Then $x' \in V_x \Longrightarrow z_r \in P_i(x'),\ r = 1, \ldots, m \implies y \in \Gamma_i(x')$, i.e. $V_x \subset \Gamma_i^{-1}(y)$. Since x was chosen arbitrary, we see that $\Gamma_i^{-1}(y)$ is the neighborhood of each own point and therefore is $\sigma(L, L')$-open in L_+. **Q.E.D.**

Proof of Lemma 3.1. To check *(i)*, we first show that $\Pi_i(x, \lambda) \neq \emptyset$. Observe that due to *PA(iii),(v)*, we have

$$(x - K) \bigcap conv P_i(x) = \emptyset.$$

Due to Proposition 3.1 *(i)*, applying the first separation theorem, we find nonzero $p \in L'$ such that

$$\langle p, x - K\rangle < p(x) \leq \langle p, conv P_i(x)\rangle.$$

From *PA(iv)*, we conclude that $p \in L'_+$. Since $\langle p, K\rangle > 0$, we have $p(\omega) > 0$ which implies $p(x) > 0$ and we can renormalize p putting $p(x) = \lambda$. We see that this $p \in \Pi_i(x, \lambda)$.

To prove the compactness of $\Pi_i(x, \lambda)$, note that since $p \geq 0$ we have

$$\lambda = p(x) \geq \varepsilon p(\omega) \Longrightarrow p(\omega) \leq \frac{\lambda}{\varepsilon},$$

that together with $\langle p, \Gamma_i(x) \rangle \geq \langle p, x \rangle$ imply

$$\langle p, K \rangle \geq 0 \Longrightarrow \langle p, V + \omega \rangle \geq 0 \Longrightarrow | \langle p, V \rangle | \leq \langle p, \omega \rangle,$$

since V was chosen to be circled. This and the latter one implies

$$| \langle p, V \rangle | \leq \frac{\lambda}{\varepsilon}$$

for each $p \in \Pi_i(x, \lambda)$ and $i \in N$. Therefore by Alaoglu's theorem, $\Pi_i(x, \lambda)$ is a compact set. The convexity and closeness of $\Pi_i(x, \lambda)$ is trivial due to specification.

To prove item *(ii)*, let us take some directed net $(x^\alpha, p^\alpha, \lambda^\alpha)_{\alpha \in \Lambda}$ such that $x^\alpha \in Y$, $p^\alpha \in \Pi_i(x^\alpha, \lambda^\alpha)$ and $x^\alpha \underset{\Lambda}{\longrightarrow} y \in Y$, $\lambda^\alpha \underset{\Lambda}{\longrightarrow} \lambda$ and $p^\alpha \underset{\Lambda}{\longrightarrow} p$ in the $\sigma(L', L)$-topology. It is enough to show that $p \in \Pi_i(y, \lambda)$, i.e. to check

$$\langle p, \Gamma_i(y) \rangle \geq p(y) = \lambda.$$

In fact, let $x' \in conv\ P_i(y) + \{\gamma(\omega + V) \mid \gamma > 0\}$ be fixed. In view of Proposition 3.1 *(v)*, there is $\bar{\alpha} \in \Lambda$ such that $x' \in \Gamma_i(x^\alpha)$ for all $\alpha \geq \bar{\alpha}$, that implies

$$p^\alpha(x') \geq p^\alpha(x^\alpha) = \lambda^\alpha.$$

We assumed $\mathcal{L}$ to be finite-dimensional, therefore choosing and fixing some finite linear basis (z_t) of $\mathcal{L}$, we can write $x^\alpha = \sum_t \beta_t^\alpha z_t$. Without loss of generality, we can assume that $\beta_t^\alpha \underset{\Lambda}{\longrightarrow} \beta_t$, and $y = \sum \beta_t z_t$, that entails

$$\lambda^\alpha = p^\alpha(x^\alpha) = \sum_t \beta_t^\alpha p^\alpha(z_t) \Longrightarrow \sum_t \beta_t p(z_t) = p(y) = \lim_\Lambda p^\alpha(x^\alpha)$$

because $p^\alpha(z_t) \underset{\Lambda}{\longrightarrow} p(z_t)$ for each t. Hence $p(x') \geq p(y) = \lambda$, that we wanted to have. **Q.E.D.**

Proof of Lemma 3.2. We already noticed that $Z_\varepsilon^{\mathcal{L}}$ is convex and compact. Let us first verify that $Z_\varepsilon^{\mathcal{L}} \neq \emptyset$. Obviously $X_\varepsilon^{\mathcal{L}} \neq \emptyset$. The condition $h \leq \frac{1}{n}$ supposed in this lemma immediately implies that $\Delta \neq \emptyset$. It also implies that $\varepsilon \leq \frac{1}{h}$ and that $\Psi_\varepsilon^n \neq \emptyset$ (see footnote 5).

Let us now consider the point-to-set mapping $\varphi_\varepsilon^{\mathcal{L}} = r_\varepsilon^{\mathcal{L}} \times \lambda^{\mathcal{L}} \times s$. The fact that $r_\varepsilon^{\mathcal{L}}(\cdot)$ and $\lambda^{\mathcal{L}}(\cdot)$ have closed graphs and nonempty convex compact values may be proved quite standardly. Using Lemma 3.1, we see that $s(\cdot, \cdot)$ has also a closed graph and nonempty convex compact values. In order to apply Kakutani-Fan's fixed point theorem [6], we only need to show that $\varphi_\varepsilon^{\mathcal{L}}(z) \subset Z_\varepsilon^{\mathcal{L}},\ z \in Z_\varepsilon^{\mathcal{L}}$. Let us do it.

[6]We refer here as Kakutani–Fan's theorem the fact, known since a long time, that Kakutani's theorem still holds true in locally convex topological vector spaces (see for example Sect.5 Ch.9 from [5])

Obviously $r_\varepsilon^{\mathcal{L}}(q) \subset X_\varepsilon^{\mathcal{L}}$. By construction of $\lambda^{\mathcal{L}}(q)$ (see (5)), we have

$$\lambda_i'^{\mathcal{L}} = \max_{x' \in X_{\mathcal{L}}^i} q(x') \geq \max_{y \in \mathcal{L}_+^n,\ \sum_{j \in N} y_j = h\omega} q(y) = h \cdot \lambda_\omega^{\mathcal{L}},$$

$$\sum_{i \in N} \lambda_i'^{\mathcal{L}} = \sum_{i \in N} \max_{x' \in X_{\mathcal{L}}^i} q(x') \leq \max_{y \in X_{\mathcal{L}}} q(y) = \lambda_\omega^{\mathcal{L}},$$

that implies $\lambda^{\mathcal{L}}(q) \subset \Delta$.

Furthermore, if $p_i \in \Pi_i(x_i, \lambda_i)$, where $x \in X_\varepsilon^{\mathcal{L}}$ and $\lambda \in \Delta$, then

$$p_i(\omega) \geq p_i(x_i) = \lambda_i \geq h \tag{6}$$

and

$$\lambda_i = p_i(x_i) \geq \varepsilon p_i(\omega) \Longrightarrow p_i(\omega) \leq 1/\varepsilon.$$

By definition, we have

$$\langle p_i, \Gamma_i(x_i) \rangle \geq p_i(x_i) = \lambda_i,$$

which, having in mind that V is circled, implies

$$\langle p_i, \omega + V \rangle \geq 0 \Longrightarrow \mid \langle p_i, V \rangle \mid \leq p_i(\omega). \tag{7}$$

Combining the latter and the former ones, we get

$$\mid \langle p_i, V \rangle \mid \leq 1/\varepsilon$$

that gives $s(x, \lambda) \subset \Psi_\varepsilon^{\mathcal{L}}$. So, for every $\varepsilon > 0$ such that $\varepsilon < h$, the set $Z_\varepsilon^{\mathcal{L}}$ and the map $\varphi_\varepsilon^{\mathcal{L}}$ satisfy the conditions of Kakutani-Fan's fixed point theorem. **Q.E.D.**

Proof of lemma 3.3. Let

$$(x_\varepsilon^{\mathcal{L}}, \lambda_\varepsilon^{\mathcal{L}}, q_\varepsilon^{\mathcal{L}}) \in \varphi_\varepsilon^{\mathcal{L}}(x_\varepsilon^{\mathcal{L}}, \lambda_\varepsilon^{\mathcal{L}}, q_\varepsilon^{\mathcal{L}})$$

be a fixed point. Obviously, $x_\varepsilon^{\mathcal{L}}$ and $\lambda_\varepsilon^{\mathcal{L}}$ belong to the compact sets $X^{\mathcal{L}}$ and Δ. For easier writing, we drop indices $\mathcal{L}$ and ε in the following calculations. Because of $\lambda_i = p_i(x_i)$ we have

$$\sum_{i \in N} \lambda_i = \sum_{i \in N} p_i(x_i) = q(x) = \max_{x' \in X_\varepsilon} q(x') = \max_{y \in \mathcal{L}_+^n, \sum_N (y_i + \varepsilon\omega) = \omega} \langle q, (y_1 + \varepsilon\omega, ..., y_n + \varepsilon\omega) \rangle$$

$$= \max_{y \in \mathcal{L}_+^n, \sum_N y_i = (1 - \varepsilon n)\omega} q(y) + \varepsilon q(\tilde{\omega}) = (1 - \varepsilon n)\lambda_\omega + \varepsilon q(\tilde{\omega}),$$

where $\tilde{\omega} = (\omega, ..., \omega)$, that (see also (5)) implies

$$\lambda_\omega \geq \sum_{i \in N} \lambda_i \geq (1 - \varepsilon n)\lambda_\omega. \tag{8}$$

Let us now choose $\varepsilon \in (0, 1/2n]$. Using (8) and remembering $\sum \lambda_i \leq 1$, we conclude $\lambda_\omega \leq 2$. Since $\lambda_\omega \geq p_i(\omega) \geq h$ (see (5),(6)), we have

$$2 \geq p_i(\omega) \geq h.$$

The latter one, due to (7), gives us the result. **Q.E.D.**

Proof of Lemma 3.4. Applying Lemma 3.2 for each $\varepsilon \in (0, 1/2n]$, we may find $z^{\mathcal{L}}_\varepsilon \in Z^{\mathcal{L}}_\varepsilon$ such that $z^{\mathcal{L}}_\varepsilon \in \varphi^{\mathcal{L}}_\varepsilon(z^{\mathcal{L}}_\varepsilon)$. In view of Lemma 3.3, letting $\varepsilon \to 0$, we can assume without loss of generality that

$$z^{\mathcal{L}}_\varepsilon = (x^{\mathcal{L}}_\varepsilon, \lambda^{\mathcal{L}}_\varepsilon, q^{\mathcal{L}}_\varepsilon) \longrightarrow (x^{\mathcal{L}}, \lambda^{\mathcal{L}}, q^{\mathcal{L}}),\ \lambda^{\mathcal{L}} \in \lambda^{\mathcal{L}}(q^{\mathcal{L}}),\ q^{\mathcal{L}} \in s(x^{\mathcal{L}}, \lambda^{\mathcal{L}}).$$

Obviously $x^{\mathcal{L}}$ is a feasible allocation in $\mathcal{L}$ and $\lambda^{\mathcal{L}} \in \Delta$. $q^{\mathcal{L}} \in s(x^{\mathcal{L}}, \lambda^{\mathcal{L}})$ proves *(i)*. We have at the limit

$$h \le p^{\mathcal{L}}_i(\omega) \le 2, \quad \mid \langle p^{\mathcal{L}}_i, V\rangle \mid \le 2,$$

that proves *(ii)*.

Condition *(iii)* for the limit points is easily verified in view of the finite-dimensionality of $\mathcal{L}$ and therefore the joint continuity of the map $(q, x) \to q(x)$. We can pass to the limit when $\varepsilon \to 0$ in relations (8) and get $\lambda^{\mathcal{L}}_{\varepsilon\omega} \to \lambda^{\mathcal{L}}_\omega = \sum_{i\in N} \lambda^{\mathcal{L}}_i \le 1$ which yields

$$\lambda^{\mathcal{L}}_i = \lambda'^{\mathcal{L}}_i / \lambda^{\mathcal{L}}_\omega \ge \lambda'^{\mathcal{L}}_i, \quad i \in N,$$

that by the definition of values $\lambda'^{\mathcal{L}}_i$ (see (5)) gives us *(iii)*. **Q.E.D.**

Proof of Theorem 2.2. It is based on a limiting process on the net of the finite dimensional subspaces $\mathcal{L}$ of L containing all ω_i, directed by inclusion. Let us consider triplets $(x^{\mathcal{L}}, \lambda^{\mathcal{L}}, q^{\mathcal{L}})$ obtained in Lemma 3.4. Note that $x^{\mathcal{L}} \in \chi$ which is $\sigma(L^n, L'^n)$-compact and $\lambda^{\mathcal{L}} \in \Delta$ which is compact. In view of Lemma 3.4 *(ii)*, by Alaoglu's theorem we can assume without loss of generality that $(x^{\mathcal{L}}, \lambda^{\mathcal{L}}, q^{\mathcal{L}}) \longrightarrow (\bar{x}, \bar{\lambda}, \bar{q})$. We now define $\pi = \vee \bar{p}_i$.

Let us show that

$$\bar{p}_i(y) \ge \bar{\lambda}_i = \pi(\omega_i), \quad y \in \Gamma_i(\bar{x}). \tag{9}$$

To begin with we specify the values $x^{(i)} \in L^n_+$ from the condition

$$\bar{q}(x^{(i)}) = \max_{x' \in L^n_+, \sum x'_j = \omega_i} \bar{q}(x') = \pi(\omega_i), \quad i \in N.$$

We can realize that $x^{(i)} \in X^i_{\mathcal{L}}$ for every $\mathcal{L} \supset \bar{\mathcal{L}}$ for some $\bar{\mathcal{L}}$. Since $\Gamma^{-1}_i(y)$ is $\sigma(L, L')$-open in L_+ (see Proposition 3.1 *(v)*), we can assume that $y \in \Gamma_i(\bar{x}^{\mathcal{L}}_i)$ for every $\mathcal{L} \supset \bar{\mathcal{L}}$ that implies

$$p^{\mathcal{L}}_i(y) \ge \lambda^{\mathcal{L}}_i = p^{\mathcal{L}}_i(x^{\mathcal{L}}_i) \ge \max_{x' \in X^i_{\mathcal{L}}} q^{\mathcal{L}}(x').$$

Since from $x^{(i)} \in X^i_{\mathcal{L}}$ we have $\max_{x' \in X^i_{\mathcal{L}}} q^{\mathcal{L}}(x') \ge q^{\mathcal{L}}(x^{(i)})$, we conclude

$$p^{\mathcal{L}}_i(y) \ge q^{\mathcal{L}}(x^{(i)}).$$

Passing to the limit in the latter inequality and by the choice of $x^{(i)}$ we have

$$\bar{p}_i(y) \ge \bar{q}(x^{(i)}) = (\vee \bar{p}_j)(\omega_i) = \pi(\omega_i)$$

that proves (9). This, being applied for $y \in P_i(\bar{x}_i)$, in view of $y \geq 0$ & $\pi \geq 0$ and Proposition 3.1 *(iii)*, yields $\pi(y) \geq \pi(\omega_i)$. In addition, we have $\pi(\omega) \geq \bar{p}_i(\omega) \geq h$ that, together with the latter one, (9) and due to *PA(ii)*, gives us the result. **Q.E.D.**

Proof of Theorem 2.1. We now drop condition (2) of Theorem 2.2. Let us define

$$\omega_i^h = (1 - h \cdot n)\omega_i + h\omega, \quad i \in N$$

for some real $h \in (0, 1/n)$. Clearly (2) holds for the initial endowments (ω_i^h) since $\sum_{i\in N} \omega_i^h = \sum_{i\in N} \omega_i = \omega$. Let (x^h, q) be an "equilibrium" couple satisfying the conclusion of Theorem 2.2 for the given endowments. Since $\pi(\omega) > 0$ where $\pi = \vee_{i\in N} p_i$, the functional q can be normalized as

$$q^h : \ = q/\pi(\omega), \quad q^h = (p_1^h, ..., p_n^h),$$

that obviously gives $(\vee p_i^h)(\omega) = 1$. To this moment we have : for every h, $0 < h < 1/n$ there exists some "equilibrium" couple (x^h, q^h) such that if $\pi^h = \vee_i p_i^h$ then

$$\langle p_i^h, \Gamma_i(x^h)\rangle \geq \pi^h(\omega_i^h),$$
$$1 = \pi^h(\omega) \geq p_i^h(\omega), \quad i \in N.$$

Besides,

$$1 = \pi^h(\omega) = \max_{y\geq 0, \sum y_j = \omega} q^h(y) \Longrightarrow \sum_N p_i^h(\omega) \geq 1$$

and by Theorem 2.2

$$\langle p_i^h, \omega + V\rangle \geq 0 \Longrightarrow |\ \langle p_i^h, V\rangle\ | \leq p_i^h(\omega) \leq 1.$$

Therefore,

$$q^h \in \{q = (p_1, ..., p_n) \mid \ \ |\ \langle p_i, V\rangle\ | \leq 1, \quad \forall i \in N, \quad \sum_N p_i^h(\omega) \geq 1\} := M,$$

where the set M is $\sigma(L'^n, L^n)$-compact. Again, without loss of generality, we can assume that

$$(x^h, q^h) \underset{\text{weak}}{\longrightarrow} (x, q), \quad \langle q, \tilde{\omega}\rangle \geq 1, \quad \tilde{\omega} = (\omega, ..., \omega).$$

Let us show that (x, π) with $\pi = \vee_i p_i$ is a quasi-equilibrium. For this purpose we choose $x^{(0)} \geq 0, \quad x^{(i)} \geq 0$ satisfying the conditions :

$$q(x^{(i)}) = \max_{y\geq 0, \sum_N y_j = \omega_i} q(y), \quad i \in N,$$

$$q(x^{(0)}) = \max_{y\geq 0, \sum_N y_j = \omega} q(y).$$

By construction we have

$$\pi^h(\omega_i^h) = (1 - h \cdot n)\pi^h(\omega_i) + h \cdot \pi^h(\omega) \geq$$

$$(1-h\cdot n)q^h(x^{(i)})+h\cdot q^h(x^{(0)})\geq(1-h\cdot n)q^h(x^{(i)})\underset{h\downarrow 0}{\longrightarrow}q(x^{(i)})=\pi(\omega_i).$$

Given $y\in\Gamma_i(x_i)\bigcap L_+$, by Proposition 3.1 *(v)*, we have $y\in\Gamma_i(x_i^h)$ for h small enough, that entails

$$p_i^h(y)\geq\pi^h(\omega_i^h).$$

Now we can pass to the limit letting $h\to 0$, which in view of the previous relation gives

$$p_i(y)\geq\pi(\omega_i),\ \ y\in\Gamma_i(x_i)\cap L_+,\ \ i\in N.$$

Finally, since $\pi\geq p_i$,

$$\pi(y)\geq\pi(\omega_i),\quad i\in N,\quad y\in\Gamma_i(x_i)\cap L_+. \tag{10}$$

To finish the proof note that by Proposition 3.1 *(ii) (iii)* for each $i\in N$ we can find the net $x_i^\xi\in\Gamma_i(x_i)\cap L_+,\ \xi\in\Xi$ such that $x_i^\xi\underset{weak}{\longrightarrow}x_i$. Then substituting x_i^ξ to y in (10) and then passing to limit, we get

$$\pi(x_i)\geq\pi(\omega_i),\quad i\in N.$$

Now, if $\pi(x_i)>\pi(\omega_i)$ for some i then $\sum_{i\in N}\pi(x_i)>\pi(\omega)$, which contradicts $\sum_{i\in N}x_i=\omega$. Hence $\pi(x_i)=\pi(\omega_i),\ i\in N$, that together with (10), *SA(iii)* and Proposition 3.1 *(iii)* gives us the required result. **Q.E.D.**

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