On the Existence of Immigration Proof Partition into Countries in Multidimensional Space

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Abstract. The existence of immigration proof partition for communities (countries) in a multidimensional space is studied. This is a Tiebout type equilibrium its existence previously was stated only in one-dimensional setting. The migration stability means that the inhabitants of a frontier have no incentives to change jurisdiction (an inhabitant at every frontier point has equal costs for all possible adjoining jurisdictions). It means that inter-country boundary is represented by a continuous curve (surface).

Provided that the population density is measurable two approaches are suggested: the first one applies an one-dimensional approximation, for which a fixed point (via Kakutani theorem) can be found after that passing to limits gives the result; the second one employs a new generalization of Krasnosel'skii fixed point theorem for polytopes. This approach develops [8] and extends the result to an arbitrary number of countries, arbitrary dimension, possibly continuous dependence on additional parameters and so on.

Keywords: Country formation · Alesina and Spolaore's world · Migration · Stable partitions · Multidimensional space · Krasnosel'skii fixed point theorem

1 Introduction

In the seminal paper [1] a basic model of country formation was offered. In this model, the cost of the population individuum is described as the sum of the two values—the ratio of total costs and the total weight of the population plus transportation costs to the center of the state. This model has been investigated in a number of subsequent studies, but in each of them deals with the case of one-dimensional region and the interval-form countries (country formation on the interval $[0, 1]$).

A progress in the resolution of the problem of existence was made in [2], where the well known Gale–Nikaido–Debreu lemma was applied to state the existence of *nontrivial* immigration proof partition for interval countries, i.e. such that no one has incentive to change their country of residence. In [2] rather strong

Y. Kochetov et al. (Eds.): DOOR 2016, LNCS 9869, pp. 494–508, 2016.
DOI: 10.1007/978-3-319-44914-2_39

assumptions were made on the distribution of the population—continuous density, separated from zero. Next, in [5] the mathematical part of the approach was significantly strengthened and extended to the case of distribution of the population, described as a Radon measure (probability measure defined on the Borel σ-algebra). In [8] a new significant advancement was suggested; it disseminates the result (existence theorem) to the case of 2 or higher dimensional region. The proof in [8] is very elegant and is based on the application of KKM-lemma (Knaster–Kuratowski–Mazurkiewicz), but the result is essentially limited by the presence of capitals with fixed positions in the space. In this paper, I intend to take the next step and let capitals (or other relevant parameters) be changed continuously in space, which is important for example in the context of party formation. The proof is based on a new original generalization of Krasnosel'skii fixed point theorem, which is extended to the case of a convex polytope (bounded polyhedron) that is interesting in its own right.

In the second section, we consider a particular case of division of a rectangular area into two countries at a given measurable random distribution of the population. Here a basic one-dimensional approximation is described, for which a fixed point (via Kakutani theorem) can be found, and then the limit process gives the result.

The third section provides further generalization of the existence result which is extended to an arbitrary number of countries, arbitrary dimension, and possibly continuous dependence on a finite number of significant parameters for country formation (capitals and so on).

2 The Partition into Two Countries on the Plane via One-Dimensional Approximation

The division of the one-dimensional world on countries surely cannot be considered as a satisfactory solution of the problem. However, 2-dimensional formulation seems to be a fundamentally more difficult problem. Now, for a particular example of division of a rectangular area in two countries, we consider an approximating design allowing to find a solution by passing to the limit.

First, we define the principle of stability which is applied to countries located on the plane. As in the case of one-dimensional world, it must be such division that boundary residents have no incentive to change their jurisdiction. Thus, the costs for any boundary resident should be the same with respect to any of the possible for her/him adjoining jurisdictions. It is assumed that the boundaries between two countries allow continuous parametrization, i.e. they are an image of the interval from $\mathbb{R}$ for some *continuous one-to-one* mapping. As a result, as in the one-dimensional case, the function of individual costs of inhabitants should be continuous on *the whole field* of country division, that is, country partition must implement continuous "gluing" of country-depended individual costs.

For the sake of simplicity, we consider now a particular case of a rectangular area of possible settlement represented by rectangle $\square ABCD$ in the Fig. 1. We assume that $c_i(\cdot)$, $i = 1, \ldots, n$, are functions of individual costs, depending on the place of individual location, given by coordinates $(x, y) \in \square ABCD$, the weight of the resident jurisdiction $\mu_i(S_i)$, the location of its center $r_c(S_i)$, metrics $\rho(\cdot,\cdot)$ (to specify the distance to the center) and so on. The basic model representation of these cost functions is

$$c_i(x, y, \delta_i, r_c(S_i)) = \frac{g_i}{\delta_i} + \rho((x, y), r_c(S_i)), \quad g_i > 0, \quad i \in N = \{1, 2, \ldots, n\}. \quad (1)$$

Here scalar variables $\delta_i > 0$ are associated with the i-th country mass of population, i.e. $\delta_i = \mu_i(S_i)$; $g_i > 0$ is an expenditure (costs) on the maintenance of government and they are uniformly distributed among the country citizens. The second summand $\rho((x, y), r_c(S_i))$ presents an individual expenditure specified by inhabitant location at the point $(x, y) \in \square ABCD$. In general, cost functions may have sufficiently general form but they always continuously depend on certain country parameters and obey some other specific assumptions (see Sect. 3). Everywhere below we assume
(**P**) *The distribution of population is described by an absolutely continuous probability measure* μ *such that* $\text{supp}(\mu) = \square ABCD$.[1]

The idea of approach is that given coordinate system (potentially curved), a stable partition, specified for one-dimensional world, must be implemented along every coordinate line. At the same time, the function of individual costs must be calculated relative to the position of "center" of the country and the general population distributed in *two-dimensional space*. It is not easy to find such a partition. To solve the problem we apply a special "one-dimensional approximation", relatively which a country partition can be found by a fixed point theorem (Brouwer or Kakutani).

The construction is as follows: specify $m-2$ straight lines parallel to the base of the rectangle, $m \geq 3$. Let the lower base have the number m, the top one—the number 1, and all others be numbered from the top to the bottom. Each i-th segment is divided into two parts by the point x_i, which can be considered the point from interval $[0, 1]$ (length of the base $\square ABCD$), $i = 1, \ldots, m$. Straight line segments connecting consecutive points $x_1, \ldots, x_m$, form a polygon line, which we accept as the boundary between the left and right countries. Now, if density $f(x, y)$ is presented then it is possible to integrate it over each of the country area, finding the weights (size) $\mu(S)$ of their populations.

Within each country its "center" (the capital) $r_c(S) \in S$ is specified. We assume these positions *depend continuously* from a given country settings $\mathbf{x} = (x_1, \ldots, x_m) \in [0, 1]^m$. Thus we have:

$$\mu(\mathbf{S_{left}}) = \int_{\mathbf{S_{left}}} f(x, y)dxdy \geq 0, \quad r_c(\mathbf{S_{left}}) = r_{\mathbf{left}}(x_1, \ldots, x_m) \in \mathbf{S_{left}}$$

$$\mu(\mathbf{S_{right}}) = \int_{\mathbf{S_{right}}} f(x, y)dxdy \geq 0, \quad r_c(\mathbf{S_{right}}) = r_{\mathbf{right}}(x_1, \ldots, x_m) \in \mathbf{S_{right}}.$$

[1] This being combined means that $\mu(A) > 0 \iff \int_A dxdy > 0$ for every measurable $A \subseteq \square ABCD$.

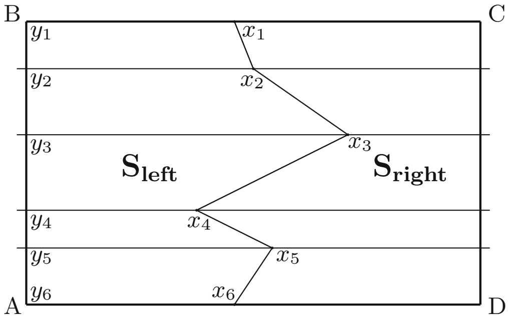

Fig. 1. Possible division into two countries of the rectangular area $ABCD$, $m = 6$.

Moreover, without loss of generality

$$\mu(\mathbf{S_{left}}) + \mu(\mathbf{S_{right}}) = 1.$$

The fact that we talk about the "mass of the population" and the "distance to the center" (transport availability of capital) as the main parameters determining the costs of individuals in a country is only an interpretation of the cost function in the context of the main model variant. The same can be said about the property of the center of the country be located on its territory—it is just a natural variant of content, from a mathematical point of view, the center could be anywhere. The really important fact is (described below) certain specific properties of individual costs.

Next we consider a point-to-set mapping, whose fixed point gives the desired country partition. The construction of mapping applies the ideas borrowed from the one-dimensional case, see [6]. Define

$$X = [0, 1]^m.$$

Now we specify a point-to-set mapping of X into itself.

Let $c_1(\cdot)$, $c_2(\cdot)$ be the functions of individual costs depending on the weight of the jurisdiction population $\mu_1(\mathbf{x})$, $\mu_2(\mathbf{x})$, location of its center $r_c(S_1)$, $r_c(S_2)$, metrics $\rho(\cdot, \cdot)$ (to determine the distance to the center) and a place of the individual location specified by coordinates $(x, y) \in \Box ABCD$. The basic model representation of these functions is (1). Now we shall consider that they are functions of a general form continuously depending on $\mathbf{x} = (x_1, \ldots, x_m) \in [0, 1]^m$ for $\mu(S_k(\mathbf{x})) > 0$, $k = 1, 2$. Additionally, we assume that

(i) $c_k(x, y, \mathbf{x}) > 0$ for $\mu(S_k) \neq 0$ and
(ii) $c_k(x, y, \mathbf{x}) \to +\infty$ if $\mu(S_k) \to 0$, $k = 1, 2$.

For the functions of (1) this condition is always satisfied. At the same time, if the density $f(\cdot)$ of the population is so that $\int_A dxdy > 0$ implies $\int_A f(x, y)dxdy > 0$ for every measurable subset $A \subset \Box ABCD$ (i.e. each subset of nonzero area

(Lebesgue measure) has a population of non-zero mass), the latter requirement is equivalent to

$$\begin{cases} c_1(x, y, \mathbf{x}) \to +\infty \iff \mathbf{x} \to (0, \dots, 0), \\ c_2(x, y, \mathbf{x}) \to +\infty \iff \mathbf{x} \to (1, \dots, 1). \end{cases} \tag{2}$$

For the boundary points $x_1, \dots, x_m$ of country areas let us find an excess cost of possible (two) jurisdictions (constants $y_1, \dots, y_m$ in the argument are excluded)

$$h_i(\mathbf{x}) = c_1(x_i, \mathbf{x}) - c_2(x_i, \mathbf{x}), \quad i = 1, \dots, m.$$

Notice that (2) implies that for all $i = 1, \dots, m$, $h_i(\mathbf{x}) \to +\infty$ for $\mathbf{x} \to 0$, and $\mathbf{x} \to \mathbf{1}$ when $h_i(\mathbf{x}) \to -\infty$.

Next we define the (single-valued) map $\varphi : X \to X = [0, 1]^m$ putting

$$\varphi_i(\mathbf{x}) = \begin{cases} x_i - \frac{x_i}{2} \cdot \frac{h_i(\mathbf{x})}{1+h_i(\mathbf{x})}, & \text{for} \quad h_i(\mathbf{x}) \geq 0, \\ x_i + \frac{1-x_i}{2} \cdot \frac{h_i(\mathbf{x})}{h_i(\mathbf{x})-1}, & \text{for} \quad h_i(\mathbf{x}) \leq 0. \end{cases} \tag{3}$$

By construction, this mapping is well defined everywhere on X with the exception of two points $\mathbf{x} = \mathbf{0} = (0, \dots, 0)$ and $\mathbf{x} = \mathbf{1} = (1, \dots, 1)$, values of which can be defined by continuity:

$$\varphi(\mathbf{0}) = (0, \dots, 0), \quad \varphi(\mathbf{1}) = (1, \dots, 1).$$

It is obvious that according to the construction these points are *trivial* fixed points of $\varphi(\cdot)$, that does not comply with the requirements of the division of rectangular area. Further construction and analysis will focus on finding of the *nontrivial* fixed point corresponding to the division of the area into two countries with non-zero masses of the population.

Now we define a point-to-set mapping Φ from $\mathfrak{X} = X \times \Delta$ to X, where $\Delta = \{(\mu_1, \mu_2) \mid \mu_1 + \mu_2 = 1, \ \mu_1 \geq 0, \ \mu_2 \geq 0\}$, by formula: for $(\mu_1, \mu_2) = (\mu(\mathbf{S_{left}}(\mathbf{x})), \mu(\mathbf{S_{right}}(\mathbf{x})))$ specify

$$\Phi(\mathbf{x}, \nu) = \begin{cases} \{\frac{\nu_1}{\mu_1} \varphi(\mathbf{x})\}, & \text{for } \nu_1 \leq \mu_1, \quad \mu_1 \neq 0, \\ \{\frac{\nu_2}{\mu_2} \varphi(\mathbf{x}) + \frac{\mu_2 - \nu_2}{\mu_2} (1, \dots, 1)\}, & \text{for } \nu_2 \leq \mu_2, \quad \mu_2 \neq 0, \\ X, & \text{for } \nu_1 = \mu_1 = 0 \text{ or } \nu_1 = \mu_1 = 1. \end{cases} \tag{4}$$

The second mapping $\Psi : X \rightrightarrows \Delta$ is specified as follows

$$\Psi(\mathbf{x}) = \underset{\nu \in \Delta}{\operatorname{argmax}} \langle H(\mathbf{x}), \nu \rangle. \tag{5}$$

where $H(\mathbf{x}) = (H_1(\mathbf{x}), H_2(\mathbf{x}))$ and

$$I_+ = \{i \mid h_i(\mathbf{x}) \geq 0, \ i = 1, \dots, n\}, \quad I_- = \{i \mid h_i(\mathbf{x}) \leq 0, \ i = 1, \dots, n\}$$

are defined by formulas[2]

$$\begin{aligned} H_1(\mathbf{x}) &= [\inf_{i=1,\dots,m} h_i(\mathbf{x})]^+ + \sum_{i \in I_+} x_i \frac{h_i(\mathbf{x})}{h_i(\mathbf{x})+1}, & I_+ \neq \emptyset \\ H_2(\mathbf{x}) &= [\sup_{i=1,\dots,m} h_i(\mathbf{x})]^- + \sum_{i \in I_-} (1 - x_i) \frac{h_i(\mathbf{x})}{h_i(\mathbf{x})-1}, & I_- \neq \emptyset. \end{aligned}$$

[2] We use standard notations $z^+ = \sup\{z, 0\}$ and $z^- = \sup\{(-z), 0\}$ for any real z.

If $I_+ = \emptyset$ or $I_- = \emptyset$, then by definition $H_1(\mathbf{x}) = 0$ and $H_2(\mathbf{x}) = 0$, respectively. Constructed map is well defined everywhere excepting $\mathbf{0}$ and $\mathbf{1}$ for which we postulate

$$\Psi(\mathbf{0}) = (1, 0), \qquad \Psi(\mathbf{1}) = (0, 1).$$

Finally, we define the resulting mapping

$$\Upsilon : \mathfrak{X} \rightrightarrows \mathfrak{X}, \qquad \Upsilon(\mathbf{x}, \nu) = \Phi(\mathbf{x}, \nu) \times \Psi(\mathbf{x}, \nu);$$

its fixed points give us the desired result. The following lemma describes the important properties of the mapping $\Upsilon(\cdot)$.

Lemma 1. *The mapping $\Upsilon : \mathfrak{X} \rightrightarrows \mathfrak{X}$ is a Kakutani map, i.e. it has closed graph and for every $\kappa \in \mathfrak{X}$ takes non-empty convex values.*

Proof of Lemma 1. We check the properties of $\Psi(\cdot)$. We need to show that it has a closed graph. First, we establish the continuity of $H = (H_1, H_2)$. To this end, we consider the functions

$$g^-(t) = \begin{cases} \frac{t}{t-1}, & \text{for } \ t \le 0, \\ 0, & \text{for } \ t \ge 0, \end{cases} \qquad g^+(t) = \begin{cases} \frac{t}{t+1}, & \text{for } \ t \ge 0, \\ 0, & \text{for } \ t \le 0, \end{cases}$$

which obviously are continuous on $[-\infty, +\infty]$. From the construction one can now derive

$$H_1(\mathbf{x}) = [\inf_{i=1,\dots,m} h_i(\mathbf{x})]^+ + \sum_{i=1}^{m} x_i \cdot g^+(h_i(\mathbf{x})),$$

$$H_2(\mathbf{x}) = [\sup_{i=1,\dots,m} h_i(\mathbf{x})]^- + \sum_{i=1}^{m} (1 - x_i) g^-(h_i(\mathbf{x})).$$

This form of representation clearly implies the continuity of $H(\cdot)$ at all points except for $\mathbf{0}$ and $\mathbf{1}$. So, everywhere on X, excepting these points, $\Psi(\cdot)$ is closed. It is also closed at zero, since by construction (due to the first term) $H_1(\mathbf{x}) > 0$ and $H_2(\mathbf{x}) = 0$ for all $\mathbf{x}$ sufficiently close to zero. Consequently, in some neighborhood of zero $\Psi(\mathbf{x}) \equiv (1, 0)$, which means that the closure of $\Psi(\cdot)$ at $\mathbf{0}$. Closeness at $\mathbf{1}$ is stated in a similar way.

All other required properties of the mapping $\Upsilon(\cdot)$ are established by a routine checking of definitions. Lemma is proved.

Lemma 2. *Under the above assumptions, the map $\varphi(\cdot)$ has **nontrivial** fixed point in X such that the mass of the population of each country is **nonzero**.*

Proof of Lemma 2. Consider any fixed point

$$(\bar{\mathbf{x}}, \bar{\nu}) \in \Upsilon(\bar{\mathbf{x}}, \bar{\nu}),$$

which does exist due to Lemma 1 and Kakutani fixed point theorem. Let us show that this point satisfies

$$0 < \bar{\nu}_1 < 1 \quad \& \quad \bar{\mathbf{x}} \neq \mathbf{0}, \quad \bar{\mathbf{x}} \neq \mathbf{1}. \tag{6}$$

Suppose that the first country has zero mass of the population, that is $\mu(\mathbf{S}_{\mathbf{left}}(\bar{\mathbf{x}})) = \mu_1 = 0$. This is possible only if $\bar{\mathbf{x}} = 0$ that implies $h_i(\bar{\mathbf{x}}) = +\infty$ $\forall i = 1, \ldots, m \Rightarrow H_1(\bar{\mathbf{x}}) > 0$ and $H_2(\bar{\mathbf{x}}) = 0$. Now by formula (5) and properties of the fixed point we conclude $\bar{\nu} = (\bar{\nu}_1, \bar{\nu}_2) = (1, 0)$ that due to (4) in the case $\nu_1 = 1 \geq 0 = \mu_1$ and $\mu_2 = 1$, $\nu_2 = 0$ implies

$$\frac{\bar{\nu}_2}{\mu_2}\varphi(\mathbf{x}) + \frac{\mu_2 - \bar{\nu}_2}{\mu_2}(1, \ldots, 1) = (1, \ldots, 1) \neq 0 = \bar{\mathbf{x}}.$$

This contradiction proves $\bar{\mathbf{x}} \neq 0$.

The case of the second country with zero population mass is considered in a similar way:

$$\mu(\mathbf{S}_{\mathbf{right}}(\bar{\mathbf{x}})) = \mu_2 = 0 \quad \Longleftrightarrow \quad \bar{\mathbf{x}} = (1, \ldots, 1) \quad \Rightarrow \quad h_i(\bar{\mathbf{x}}) = -\infty \ \ \forall i = 1, \ldots, m.$$

Therefore, $H_2(\bar{\mathbf{x}}) > 0$ and $H_1(\bar{\mathbf{x}}) = 0$, that due to (5) implies $\bar{\nu} = (\bar{\nu}_1, \bar{\nu}_2) = (0, 1)$. By construction (4) in the case $\nu_1 \leq \mu_1$ and $\mu_1 = 1$, $\nu_1 = 0$ one has

$$\frac{\bar{\nu}_1}{\mu_1}\varphi(\bar{\mathbf{x}}) = (0, \ldots, 0) \neq (1, \ldots, 1) = \bar{\mathbf{x}},$$

that proves (6). This due to (5) allows to conclude $H_1(\bar{\mathbf{x}}) = H_2(\bar{\mathbf{x}})$. Let us show now that $H_1(\bar{\mathbf{x}}) = H_2(\bar{\mathbf{x}}) = 0$ is the only possibility.

Suppose $H_1(\bar{\mathbf{x}}) = H_2(\bar{\mathbf{x}}) \neq 0$. Firstly notice that $[\inf_{i=1,\ldots,n} h_i(\bar{\mathbf{x}})]^+ > 0$ is now impossible since otherwise $H_1(\bar{\mathbf{x}}) > 0$ and $H_2(\bar{\mathbf{x}}) = 0$ that is invalid. Likewise, it is impossible $[\sup_{i=1,\ldots,n} h_i(\bar{\mathbf{x}})]^- > 0$. Therefore, both of these terms in the definition of H vanish. Now, from the definition of H one can conclude that there are i, j such that

$$h_i(\bar{\mathbf{x}}) > 0 \quad \& \quad \bar{x}_i \cdot \frac{h_i(\bar{\mathbf{x}})}{h_i(\bar{\mathbf{x}}) + 1} \neq 0 \quad \Rightarrow \quad \bar{x}_i > 0,$$

$$h_j(\bar{\mathbf{x}}) < 0 \quad \& \quad (1 - \bar{x}_j) \cdot \frac{h_j(\bar{\mathbf{x}})}{h_j(\bar{\mathbf{x}}) - 1} \neq 0 \quad \Rightarrow \quad \bar{x}_j < 1.$$

Next, we turn again to the properties of the fixed point and the formula (4). In the first case, for $0 < \nu_1 \leq \mu_1 < 1 \Rightarrow 0 < \lambda = \frac{\nu_1}{\mu_1} \leq 1$, via $\bar{x}_i > 0$ we have

$$\bar{x}_i > \Phi_i(\bar{\mathbf{x}}, \bar{\nu}) = \lambda\varphi_i(\bar{\mathbf{x}}) = \lambda\left[\bar{x}_i - \frac{\bar{x}_i}{2} \cdot \frac{h_i(\bar{\mathbf{x}})}{h_i(\bar{\mathbf{x}}) + 1}\right].$$

In the second case, for $0 < \lambda = \frac{\nu_2}{\mu_2} \leq 1$, via $\bar{x}_j < 1$ we have

$$\bar{x}_j < \Phi_i(\bar{\mathbf{x}}, \bar{\nu}) = \lambda\varphi_j(\bar{\mathbf{x}}) + 1 - \lambda = \lambda\left[x_j + \frac{1 - \bar{x}_j}{2} \cdot \frac{h_j(\bar{\mathbf{x}})}{h_j(\bar{\mathbf{x}}) - 1}\right] + 1 - \lambda.$$

Both cases are impossible. Consequently, it is proved $H_1(\bar{\mathbf{x}}) = H_2(\bar{\mathbf{x}}) = 0$. By construction, this is equivalent to

$$\bar{x}_i \frac{h_i(\bar{\mathbf{x}})}{h_i(\bar{\mathbf{x}}) + 1} = 0, \quad h_i(\bar{\mathbf{x}}) \geq 0 \ \ \forall i = 1, \ldots, m,$$

$$(1 - \bar{x}_j)\frac{h_j(\bar{\mathbf{x}})}{h_j(\bar{\mathbf{x}}) + 1} = 0 \quad h_j(\bar{\mathbf{x}}) \leq 0 \quad \forall j = 1, \ldots, m.$$

Now due to (3) this means that $\bar{\mathbf{x}} \in X$ is a nontrivial fixed point of $\varphi(\cdot)$.

Theorem 1. *Let the individual costs be given by (1) and centers be situated on a line parallel to the axis of abscissa. Then for each positive integer $m \in \mathbb{N}$ there exists the partition of $\square ABCD$ into two countries $\mathbf{S_{left}}(\mathbf{x})$ and $\mathbf{S_{right}}(\mathbf{x})$, with piecewise linear boundary formed by the points $x_k, \ldots, x_l$, $1 < k+1 \leq l-1 < m$ where all $x_{k+1}, \ldots, x_{l-1}$ are immigration proof.*

Corollary 1. *Let the costs in formula (1) be calculated relative to the Euclidean distance. Then in the conditions of Theorem 1 boundary points $x_{k+1}, \ldots, x_{l-1}$ are suited on classical hyperbola. In the case of a more general form of the metric (for example, for p-norm), these points belong to a generalized hyperbola.*

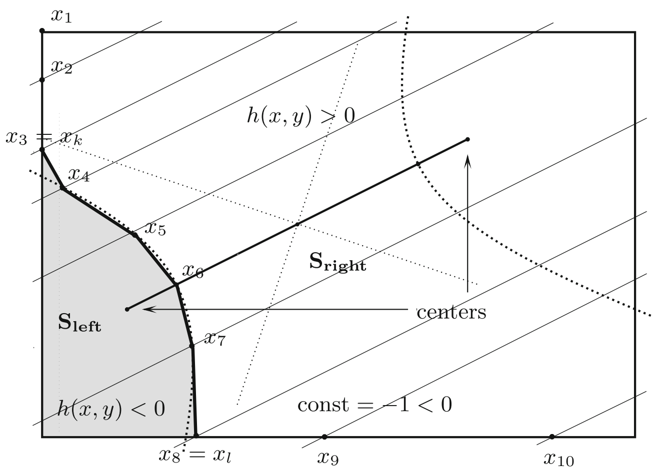

Fig. 2. Partition according to (i)–(ii) for $const < 0 \iff g_2\mu(\mathbf{S_{left}}) < g_1\mu(\mathbf{S_{right}})$.

Proof of Theorem 1. Consider a fixed point

$$\mathbf{x} = (x_1, \ldots, x_n) = \varphi(\mathbf{x}),$$

which satisfies the conclusion of Lemma 2. Note that from the construction of $\varphi(\cdot)$ for each $i = 1, \ldots, m$ (formula (3)) only one of three possibilities is realized:

(i) $h_i(\mathbf{x}) = 0$,
(ii) $h_i(\mathbf{x}) > 0 \Rightarrow x_i = 0$,
(iii) $h_i(\mathbf{x}) < 0 \Rightarrow x_i = 1$.

Indeed, for example consider alternative (*ii*). Assuming the contrary, one concludes $x_i \frac{h_i(\mathbf{x})}{h_i(\mathbf{x})+1} > 0$, that implies $x_i > \varphi_i(\mathbf{x})$—contradiction with fixed point. Alternative (*iii*) is checked out in a similar manner.

Next we consider alternative (*i*) and the corresponding set of points x_i onto coordinate segments. All these points can be described as the intersection of the coordinate segments with the curve described by the equation

$$h(x, y) = c_1(x, y, \mathbf{x}) - c_2(x, y, \mathbf{x}) = 0,$$

where $\mathbf{x}$ can be treated as a constant. Specifically, $\mathbf{x} = (x_1, \ldots, x_m)$ plays the role of parameters defining curve in the most general terms. To illustrate the idea and to formulate concrete result we turn to the analysis of a particular case, given in formula (1), recall:

$$c_k(x, y, \mu(S_k), r_c(S_k)) = \frac{g_k}{\mu(S_k)} + \rho((x, y), r_c(S_k)), \quad g_k > 0, \quad k = 1, 2.$$

Here we are interested in the curve which is completely determined by the population mass $\mu(S_k)$ and centers $r_c(S_k)$ of two countries $k = 1, 2$. Both of these parameters are continuous functions of $\mathbf{x}$. For a given fixed point they are fixed. Therefore, in the case of the Euclidean distance in the plane equation of the curve defines the classic *hyperbola* (geometric definition), which presents the boundary between two countries:

$$h(x, y) = 0 \iff ||(x, y) - r_c(S_1)||_2 - ||(x, y) - r_c(S_2)||_2 = \frac{g_2}{\mu(S_2)} - \frac{g_1}{\mu(S_1)} = const.$$

The sign of the constant determines which of two branches one must take: negative constant corresponds to the branch which is nearest to the first center and vice versa. The described situation is illustrated in Fig. 2 which represents the hyperbolic boundary case with a negative right hand side. The alternatives (*i*)–(*ii*) are implemented now and (*i*), (*iii*)—for the polar case. Of course, case (*i*) is possible in a pure form. Notice also that options (*ii*) and (*iii*) do not occur simultaneously: this follows from the convexity of the rectangular area and the convexity of one of the areas bounded by the hyperbola.

Finally, as soon as the centers of the country are located on a common straight line parallel to the base, this line is parallel to the coordinate segments and therefore each of these segments has the *only* point of intersection with the hyperbola or do not intersect it at all (the cases (*ii*) and (*iii*)). It establishes the existence of numbers k and l from the theorem statement. Theorem 1 is proved.

Theorem 2. *Let for the rectangle $\square ABCD$ individual costs be defined by (1) and centers of the country be located on a line parallel to the axis of abscissa. Then there is immigration proof division into two countries $\mathbf{S_{left}}$ and $\mathbf{S_{right}}$ with a continuous boundary.*

Remark 1. It is an immaterial fact that the considered area is a rectangular. This result holds for any convex closed bounded domain. So, this result can be generalized, and the continuous dependence on parameters defining the country center is the only requirement, but it will require substantial transformation of presented proof. The simplest method is to consider moving coordinate lines parallel to the line passing through the centers of countries. $\square$

Proof of Theorem 2. Let us consider an increasing family

$$Y_\xi \subset Y_{\xi+1} \subset [0,1], \quad \xi = 1, 2, \ldots$$

of points on the y-axis defining intercountry piecewise–linear boundary. We choose a family so that

$$\mathrm{cl}\left(\bigcup_{\xi\in\mathbb{N}} Y_\xi\right) = [0,1].$$

For every $\xi \in \mathbb{N}$, Lemma 1 takes place, that implies: for every ξ hyperbola is specified by the parameters of the country centers (focuses) $r_c(\mathbf{S}^\xi_{\mathbf{left}})$, $r_c(\mathbf{S}^\xi_{\mathbf{right}})$ and "population masses" $\mu(\mathbf{S}^\xi_{\mathbf{left}})$, $\mu(\mathbf{S}^\xi_{\mathbf{right}})$. These parameters vary under limits and therefore they contain convergent subsequences. Without loss of generality we can assume that already presented sequences are converged. Limit values

$$\bar{\mathbf{r}}_{\mathbf{k}} = \lim_\xi r_c(\mathbf{S}^\xi_{\mathbf{k}}), \quad \bar{\mu}_{\mathbf{k}} = \lim_\xi \mu_k(\mathbf{S}^\xi_{\mathbf{k}}), \quad k = 1, 2$$

define a limit hyperbola. For this hyperbola one can easily prove two key facts that give the desired result:

(i) $\bar{\mu}_{\mathbf{k}} \neq 0$, $k = 1, 2$, proof by contradiction with the fixed point property $\mathbf{x}_\xi \in \varphi(\mathbf{x}_\xi)$ for all $\xi \in \mathbb{N}$.
(ii) Let $\bar{\xi} \in \mathbb{N}$ and $y_{\bar{\xi}} \in Y_{\bar{\xi}}$ be fixed. As soon as $Y_{\bar{\xi}} \subset Y_\xi$ $\forall \xi \geq \bar{\xi}$, then a sequence $(x_\xi, y_{\bar{\xi}})$, $\xi \geq \bar{\xi}$ of points is defined; they satisfy all fixed point relations. In the rectangle they are, starting with some number, either points located on the left or on the right hand side, or couple $(x_\xi, y_{\bar{\xi}})$ is placed on ξ-th hyperbola (the intersection of $\bar{\xi}$-th segment with the hyperbola). Since hyperbola converge to the limit option, then their (the only!) points of intersection with a fixed line will be convergent, i.e. $(x_\xi, y_{\bar{\xi}}) \to (\bar{x}_{\bar{\xi}}, y_{\bar{\xi}})$, $\xi \to \infty$. Consequently, the limit values of the population $\bar{\mu}_{\mathbf{k}} = \lim_\xi \mu_k(\mathbf{S}^\xi_{\mathbf{k}})$, $k = 1, 2$ for countries with piecewise-linear boundaries *coincide* with the value (mass) of the population of marginal hyperbola areas.

Thus, we have found a nontrivial fixed point this is the map (in the space of continuous functions) whose graph consists of a (non-empty) intersection of the hyperbola with the area, and possibly two vertical segments. This fragment of the hyperbola is the desired boundary between two countries. Theorem 2 is proved.

3 General Partition into Three or More Countries

Now we consider a general method that allows us to establish the existence of immigration proof division into n countries not only on the plane, but in any finite-dimensional space. It is not a possible generalization only, but an opportunity in its context to consider more general problems, e.g. partition according to party affiliation.

The initial construction is similar to the one proposed in [8]. We need to divide the area $\mathcal{A} \subset \mathbb{R}^l$ into n counties, $N = \{1, \ldots, n\}$. The difference is that the cost function $c_i(\cdot)$ may depend not only on the mass $\delta_i \in [0, 1]$ of country, individual location $x \in \mathcal{A}$, but also additional parameters $y \in Y$, which can be changed according to a partition configuration. In particular, y can be used as a center of the country as well as other important for country formation parameters. It is assumed that the cost functions depend continuously on $\delta \in \Delta^{(n-1)}$ and $y \in Y$; moreover Y (the range of y) is convex and compact. More specifically, in addition to assumption (**P**) (page 3) we impose

(**C**) *For each $i \in N$ costs $c_i(\cdot)$ are defined and continuous on*

$$\mathcal{A} \times Y \times (\Delta^{(n-1)} \setminus F_i), \text{ where } F_i = \{\delta \in \Delta^{(n-1)} \mid \delta_i = 0\},$$

and obey

(i) $c_i(x, y, \delta_1, \ldots, \delta_n) \to +\infty$ *when* $(x, y, \delta_i, \delta_{-i}) \to (\bar{x}, \bar{y}, 0, \bar{\delta}_{-i})$, *i.e.* $\bar{\delta}_i = 0$;

(ii) *the set of indifferent agents*

$$A_{ij}(y, \delta) = \{x \in \mathcal{A} \mid c_i(x, y, \delta) = c_j(x, y, \delta)\}$$

has zero Lebesgue measure $\forall j \neq i$, *and for all fixed* $(y, \delta) \in Y \times \Delta^{(n-1)}$.

Note the difference between our assumption and the one in [8]: the continuity relative to all variables and for item (*ii*)—the set $A_{ij}(y, \delta)$ may depend on $y \in Y$ and masses of other jurisdictions δ_k, $k \neq i, j$.

The idea of the proof is that for a collection $(\delta_1, \ldots, \delta_n, y)$ of *nominal* parameters one can put into correspondence a similar collection of *real* parameters, calculated for an immigration stable partition defined by nominal ones. While doing so, we define a mapping with a nontrivial fixed point which obeys all requirements of country partition we seek for. Now we consider this construction in more details.

Let us consider a standard simplex $\Delta^{(n-1)} = \{\delta \in \mathbb{R}^n \mid \sum \delta_i = 1,\ \delta_i \geq 0\ \forall i\}$, the mappings $S_i : (\delta, y) \to S_i(\delta, y) \subset \mathcal{A}$, $(\delta, y) \in \Delta^{(n-1)} \times Y$, $i \in N$, and $\mathcal{M} : (S_i)_{i \in N} \to (\mu_i)_{\in N}$ defined by formulas[3]:

$$S_i(\delta, y) = \{x \in \mathcal{A} \mid c_i(x, \delta, y) = \min_{j \in N} c_j(x, \delta, y)\}, \quad \mu_i(\delta, y) = \mu(S_i(\delta, y)), \quad i \in N.$$

[3] Here as above $\mu(\cdot)$ is absolutely continuous measure on $\mathcal{A}$, specifying the resettlement of the population.

Assuming also that there is a continuous $\mathcal{F} : \Delta^{(n-1)} \times Y \to Y$, we obtain the resulting map

$$[\mathcal{M} \times \mathcal{F}](\delta, y) = \mathcal{M}(\delta, y) \times \mathcal{F}(\delta, y), \quad (\delta, y) \in \Delta^{(n-1)} \times Y.$$

Clearly, it suffices to find a *nontrivial* fixed point $\bar{\delta} = (\bar{\delta}_1, \ldots, \bar{\delta}_n) \in \Delta^{(n-1)}$, $\bar{y} \in Y$ of this map, i.e.

$$\bar{y} = \mathcal{F}(\bar{\delta}, \bar{y}), \quad \mu_i(\bar{\delta}, \bar{y}) = \bar{\delta}_i, \ \forall i \in N, \ \text{ such that } \ \bar{\delta} = (\bar{\delta}_1, \ldots, \bar{\delta}_n) \gg 0.$$

It is proven in [8] that[4] that: for some $0 < \varepsilon < 1$

(i) the map $\mathcal{M}(\cdot)$ is continuous on $\Delta_{\varepsilon}^{(n-1)}$ and
(ii) $\mathcal{M}(\cdot)$ maps the ε-sub-simplex

$$\Delta_{\varepsilon}^{(n-1)} = \{\delta \in \mathbb{R}^n \mid \sum \delta_i = 1, \ \delta_i \geq \varepsilon \ \forall i \in N\}$$

so that the faces of $\Delta_{\varepsilon}^{(n-1)}$ pass into the corresponding faces of initial simplex, i.e.

$$\delta = (\delta_1, \ldots, \delta_n) \in \Delta_{\varepsilon}^{(n-1)} \ \& \ \delta_i = \varepsilon \ \Rightarrow \ \mu_i(\delta) = 0, \ \mathcal{M}(\delta) = (\mu_1(\delta), \ldots, \mu_n(\delta)).$$

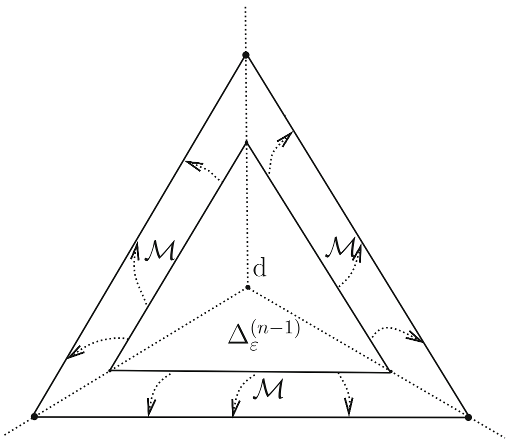

Fig. 3. Initial and embedded sub-simplexes $\Delta_{\varepsilon}^{(n-1)}$ and the mapping $\mathcal{M}(\cdot)$.

The properties (i), (ii) can be easily extended to our case although now we need to find a fixed point of the map $\mathcal{M} \times \mathcal{F}$. So, the existence of the required fixed point can be proved via (i), (ii). Note that the Brouwer fixed point theorem (and similar theorems) cannot be applied in the case, because $\mathcal{M}(\cdot)$ being defined on

[4] This is Lemma 1 from [7], where its comprehensive proof is also presented.

$\Delta_\varepsilon^{(n-1)} \times Y$ *is not* a mapping *into itself*, i.e. requirement $\mathcal{M}(\Delta_\varepsilon^{(n-1)} \times Y) \subseteq \Delta_\varepsilon^{(n-1)}$ *is not true*. Moreover, I do not know any other workable theorem for our case. In [8] further argumentation is based on the application of KKM-lemma (Knaster–Kuratowski–Mazurkiewicz), which is rather elegant solution to the issue, but it is limited to a particular case of fixed (unchanged) capitals. In our case this means the lack of postulated dependence of $\mathcal{M}$ on $y \in Y$.

The foregoing reasons yield the need in additional analysis and proving of the following theorem, which can be viewed as a (new) generalization of Krasnosel'skii's theorem in case of a bounded polyhedron (simplex) see [3] and its generalizations in [4].

Let $M \subset \mathbb{R}^n$ be a *convex bounded* polyhedron and $A(M)$ be its *affine hull*. Let $d \in \mathrm{ri}M$ be a point in the relative interior of a polyhedron M, and F_t, $t = 1, \ldots, m$ its non-trivial faces of a maximum dimension (one less than M). With every facet associate cone $K_t \subset A(M)$ with a vertex at d:

$$K_t = \{d + \lambda(\kappa - d) \mid \kappa \in F_t,\ \lambda \geq 0\} \quad \Rightarrow \quad A(M) = \bigcup_{t=1,\ldots,m} K_t.$$

Theorem 3. *Let $f : M \to A(M)$ be a continuous mapping defined on a polyhedron M and $d \in \mathrm{ri}M$, $A(M)$, F_t, K_t be defined as described above. Let one of the conditions hold:*

(i) *Compressive form*

$$f(F_t) \subset M, \quad \forall t = 1, \ldots, m. \tag{7}$$

(ii) *Expansive form*

$$f(F_t) \subset K_t \setminus \mathrm{ri}M, \quad \forall t = 1, \ldots, m. \tag{8}$$

Then $f(\cdot)$ has a fixed point in M.

Proof of Theorem 3. Consider the following parametrization in the affine space $A(M)$, spanned by a polyhedron M. As $A(M) = \cup_{t=1,\ldots,m} K_t$ point $x \in A(M)$ can be specified as $x = d + \lambda(\kappa - d)$, where real $\lambda > 0$ and, for $x \neq d$, the vector $\kappa \in \cup_{t=1,\ldots,m} F_t$ on the boundary of the polyhedron are defined one-to-one. Now the points of the polyhedron can be associated with pairs (λ, κ) for $0 \leq \lambda \leq 1$ and a continuous map can be unambiguously extended onto pairs (λ, κ). Next we consider the alternatives of the theorem.

(*i*) *Compressive form.* Let $f(\lambda, \kappa) = (\lambda', \kappa')$. We now define a new mapping[5] $g(\lambda, \kappa) = (1 \wedge \lambda', \kappa')$. Obviously, $g : M \to M$ is continuous and due to Brouwer theorem it has a fixed point $\bar{x} = (\bar{\lambda}, \bar{\kappa}) = g(\bar{\lambda}, \bar{\kappa})$. Let us show that this point is also a fixed point of f. Indeed, the difference in the values of f and g can be revealed only if $\bar{\lambda} < 1$ and $\bar{\lambda}' > 1$. But then $\bar{\lambda} = 1 \wedge \bar{\lambda}' = 1$, that cannot be true for the fixed point.

[5] Here $a \wedge b = \min\{a, b\}$.

(ii) *Expansive form.* Without loss of generality we can assume that $f(\lambda, \kappa) = (\lambda', \kappa')$ and $\lambda' \leq 2$. Otherwise, consider the new mapping $f'(\lambda, \kappa) = (2 \wedge \lambda', \kappa')$, which has the same fixed points on M as the original one. Next, we define $g(\lambda, \kappa) = (2\lambda - \lambda', \kappa')$. For $(\lambda, \kappa) \in F_t$ we have $\lambda = 1$, $1 \leq \lambda' \leq 2$ and therefore $0 \leq 2\lambda - \lambda' \leq 1$, which implies $g(F_t) \subset M$ $\forall t$. By the above item (i), $g(\cdot)$ has in M a fixed point $(\bar{\lambda}, \bar{\kappa})$ i.e. there is $(\bar{\lambda}, \bar{\kappa}) = (2\bar{\lambda} - \lambda', \kappa')$. Writing this componentwise we have $\bar{\kappa} = \kappa'$ and $\bar{\lambda} = 2\bar{\lambda} - \lambda' \Rightarrow \bar{\lambda} = \lambda'$, but this means that $(\bar{\lambda}, \bar{\kappa})$ is a fixed point of f. Theorem 3 is proved.

Remark 2. Note that we apply the parametrization $A(M)$ via (λ, κ) only to specify a transformation of the initial function f which does not change fixed points. A new function defined in this way is continuous and maps M into itself.

Notice that the assumption $d \in \mathrm{ri}M$ is essential—without it the theorem statement becomes wrong, appropriate examples can be easily constructed. The analysis of the proof shows that Theorem 3 can be generalized to the case of the Cartesian product of maps provided that the first satisfies the condition of Theorem 3 and the second map obeys the conditions of Brouwer theorem or it can be reducible to it. □

So now we can formulate the main result. In the case of our interest we have

$$\mathcal{M} : \Delta_{\varepsilon}^{(n-1)} \times Y \rightarrow \Delta^{(n-1)}.$$

If as a central point $d \in M = \Delta_{\varepsilon}^{(n-1)}$ one considers the center of simplex $(\frac{1}{n}, \ldots, \frac{1}{n}) = d$ then, by expansive property (ii) of the map $\mathcal{M}$, condition (ii) of Theorem 3 is fulfilled. Now if

$$\mathcal{F} : \Delta_{\varepsilon}^{(n-1)} \times Y \rightarrow Y$$

is any continuous map, then the map $\mathcal{M} \times \mathcal{F}$ has a fixed point in $X = \Delta_{\varepsilon}^{(n-1)} \times Y$. As a result we proved the following

Theorem 4. *Let $\mathcal{A}$ be a compact subset of a finite dimensional linear space and μ be a measure on $\mathcal{A}$. If assumptions* (**P**), (**C**) *are satisfied, then the area $\mathcal{A}$ can be nontrivially partitioned into any number of immigration proof communities. This partition can also obey any consistent continuous requirements.*

Notice that this result *does not imply* Theorem 2, in which we did not require restrictive assumption $\mathbf{C}(ii)$. Thus Theorems 2 and 4 complement each other.

The mapping $\mathcal{F}$, introduced into the design of the search of a fixed point, expresses some additional requirements for cross-country division. For example, one can impose requirements on the centers (the capital) of countries. In particular, one can require the capital be located in the center of gravity of the countries and so on.

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