Equilibrium analysis in Kantorovich spaces.*

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Abstract

The paper presents a survey of new results in general equilibrium theory with linear vector lattice commodity space (Kantorovich space). The importance of order structures and Riesz-Kantorovich formula is clarified. The main novelty of paper is new characterizations of fuzzy core elements in an exchange economy. Then these characterizations are applied to prove new quasi-equilibrium existence theorem for linear vector lattice economy. This theorem, based on E-properness of preferences by Podczeck-Florenzano-Marakulin, develops Florenzano-Marakulin approach [14] and generalizes previous Tourky's results [27].

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Introduction

The discovery and development of the theory of partially ordered vector spaces is one of the brightest achievements of L. V. Kantorovich. This theory originated from the internal needs of functional analysis, which was going through the formative stage in the 1930s; but it is intertwined, in a striking way, with other scientific interests of its discoverer, which manifested themselves later, namely, with his interest to economic problems. At that time, the Arrow–Debreu–McKenzie formal model of economics had not yet been constructed; von Neumann and Morgenstern had not yet written the famous treatise "Theory of games and economic behavior"; in fact, the ideas of game theory had only begun to penetrate into economic theory. Moreover, that was the time of ideological taboos in the USSR, when every deviation from the Marxist doctrine threatened with physical extermination, and, certainly, this did not promote study and propagation of western economic theories. It appears that Kantorovich saw economic problems from his own, independent point of view; apparently, he considered them in the light of another his creation – linear programming. However, the idea of partially ordered linear space (semiordered space according to Kantorovich) is so fundamental that, despite its "abstract" nature, it nevertheless found natural applications in modern economic studies as they became more saturated with mathematics. Indeed, the order relations "greater–less" are very typical for economics; this can be seen even at the real-life level. However, for theoretical purposes, it is the structural lattice properties of linear spaces, which in the framework of an economic model are *commodity spaces*, that are of utmost importance. In this paper, we will consider the theory of economic (competitive) equilibrium in the framework of the well-known Arrow–Debreu model; more precisely, in its simplest realization – a pure exchange economy.

The notion of equilibrium is one of the basic concepts of modern economic theory; it reflects the idea of demand–supply equality, i.e., the balance of all commodity markets. However, the realization of this concept depends substantially on the type of the model and the level of its refinement. In course of development of economic theory, many economic models were suggested, and the notion of equilibrium underwent the corresponding modifications. It took some time before economists realized the necessity to prove the very existence of equilibrium; but how else one can prove that the concept itself, as well as a formal model of economics, is well-defined, i.e., mathematically consistent? The economic model that appeared in the mid 50s due to Kenneth Arrow and Gerard Debreu [7], as well as Lionel McKenzie [23], is one of the most fundamental models in modern economic theory. It includes a finite set of economic agents – producers and consumers – which, solving their own extremal problems, form demand and supply. Consumers are supposed to maximize their utility under budget constraint (there are also other restrictions, of noneconomic nature), accepting the current market prices as prescribed ones. Producers, being guided by the (fixed) market prices, maximize their profit, which is then redistributed among consumers. Equlibrium prices correspond to the situation when the demand (the sum of individual consumer solutions) is equal to the supply (the sum of production solutions and initial endowments). In this setting, it is no longer clear when there exist equlibrium prices and which properties of the model are significant (there are simple examples of economies without equilibrium). The authors of the model found reasonable conditions for the existence of equlibriums; later, these conditions (model assumptions) were weakened by many researchers (see the survey [9]), and the model itself was generalized in many directions. One of these generalizations is the investigation of the Arrow–Debreu–McKenzie model in the case of infinite-dimensional commodity space. A detailed survey of the results in this direction is contained in the next section of this paper. Here we just point out the most essential features of these problems.

The first difference of the infinite-dimensional model from the finite-dimensional one is that instead of a price vector one uses a linear functional defined on the commodity space. In this case, the value of a consumption bundle is given by the value of the price functional calculated at a given vector (bundle of commodities). The value of the price functional at a vector presenting the production program of a producer gives the value of the profit. The second difference is that it is reasonable to require that close consumption bundles should have close values, hence the price functional should be continuous. Thus we arrive at a number of specific mathematical problems related to the compactness of certain subsets, the choice of an appropriate topology, and so on (one uses weak and $*$-weak topologies, the Alaoglu theorem, etc.). In course of development of infinite-dimensional equlibrium theory, the analysis was involving new (known) functional spaces, but by the mid 80s it became clear that the *key role* is played by the *order properties* of these spaces. Namely, commodity spaces must be semiordered spaces according to Kantorovich; in modern terminology, they are called *linear vector lattices* or *Riesz spaces*. The main reason is the Riesz–Kantorovich formula for the supremum functional (the least upper bound of two order-bounded linear functionals), which is valid in this class of spaces. Moreover, the relation between the topological and order structures may be very weak, and the order operations may be discontinuous. The only important requirement is that the positive cone should be closed in the original topology and that the topological dual should be a sublattice of the order dual to the commodity space. It is interesting that in this model, the final equilibrium price functional can be found as the least upper bound of functionals, determined in a special way, corresponding to economic agents. These functionals calculated for an allocation belonging to the fuzzy core of the economy are support functionals for the set of preferred consumer plans for each consumer, and for the production set for each producer. Thus every allocation from the fuzzy core can be decentralized with the help of the supremum price functional found by the Riesz–Kantorovich formula.

The paper is organized as follows. In the first section, we give a brief (and incomplete) historical survey of equlibrium theory with infinite-dimensional commodity space. In the second section, we introduce the notation, describe a model of a pure exchange economy, and establish new characteristic properties of elements of the fuzzy core, which is an important tool for the study described in this paper. The third section is devoted to the problem of existence of equilibrium in a linear lattice exchange economy. Here we formulate and prove the main result – a new existence theorem. In the fourth section, we compare the result obtained with the similar results from Tourky's paper [27].

1 Equlibrium analysis in infinite-dimensional models: a brief historical survey

The foundation of equilibrium theory with infinite-dimensional commodity spaces should be attributed to Truman Bewley, whose paper [8] initiated wide investigations in this direction.[1] In [8], Bewley established the existence of equilibriums in a model of Arrow–Debreu type with finitely many economic agents and commodity space $L_\infty(\mathbb{R}^l)$, the space of essentially bounded measurable functions (with domain $[0, 1]$ and range $\mathbb{R}^l$). This paper highlighted the importance of the Alaoglu theorem and the role of "weak topologies" associated with duality – weak, $*$-weak, and Mackey topologies – in the problem of existence of equilibrium with continuous (in the original topology of the commodity space) prices. It also changed the viewpoint (formally mentioned already by Debreu in [10]) on what are prices; now prices are not just a vector, but a linear functional. The space $L_\infty(\mathbb{R}^l)$ is very convenient for mathematical analysis and enjoys a number of key properties of a finite-dimensional space. One of such properties is that the positive cone has a *nonempty* interior in the original topology (determined by the norm that is defined as the essential supremum of the absolute value of a function). The latter fact made it possible to use the usual mathematical idea of finite-dimensional approximations, in which one establishes the existence of "finite-dimensional equilibriums" by ordinary methods. Then one extends the price functional up to a continuous functional on the whole space and passes to the limit. In this way Bewley obtains prices from the topological dual of $L_\infty(\mathbb{R}^l)$, the space $(ba)^l$. In order to obtain prices from the space $L_1(\mathbb{R}^l) \subset (ba)^l$, one should strengthen the requirements on the continuity of preferences and assume their lower semicontinuity in the Mackey topology (the strongest locally convex topology associated with a duality) in the duality $\langle L_\infty(\mathbb{R}^l), L_1(\mathbb{R}^l)\rangle$.

Another remarkable result that underlies modern equlibrium analysis in economies with infinitely many commodities is the paper by Peleg and Yaari [24]. Peleg and

[1] The paper [24] by Peleg and Yaari appeared somewhat earlier.

Yaari considered a rather special case of an exchange economy, with finitely many consumers and commodity space $\mathbb{R}_\infty$ (i.e., $\mathbb{R}^{\mathbb{N}}$, where $\mathbb{N}$ is the natural series of "time periods"). First of all, the paper is of interest due to the method of proof, which reduces to a generalization of the Debreu–Scarf theorem on the coincidence of core and equilibrium under perfect competition (in "replicated" economies) and a generalization of the Scarf theorem (to the case of infinite-dimensional spaces), the fact that the core is nonempty in balanced cooperative games without side payments. This method leads to the strongest modern results in infinite-dimensional equilibrium theory. In the Peleg–Yaari model (which is a one-commodity model) prices were interpreted as interest rates for passing from one time period to another. It is because of this interpretation that in the Bewley model prices from L_1 are preferable to those from ba (it is difficult to give a meaningful interpretation of a purely finitely additive measure). The Peleg–Yaari model also displayed the typical difficulty of working with infinite-dimensional spaces: the positive cone, which is usually taken as the consumer set, very often has an *empty* interior.

Subsequent investigations revealed the theoretical importance of the order structures of the basic commodity space (which are of little importance in finite-dimensional spaces). Partially ordered spaces (and the corresponding model notions) were first explicitly considered by Kreps [17]. Riesz spaces (linear lattices; in the Russian literature, Kantorovich spaces[2]) were introduced into the theory of competitive equilibrium by Aliprantis and Brown [2]. Later, the lattice structure of the commodity space was used by Mas-Colell [19] to prove a remarkable theorem on the existence of equlibrium (a general survey of the literature on this subject can be found in [22]). In the same paper, Mas-Colell introduced the important notion of *uniform* properness of preferences (a kind of analog of the uniform continuity or Lipschitz property) and extended the analysis to topological vector lattices. Formally, this specific *notion of properness* compensates (possible) negative properties of the commodity space, in which the positive cone may have an empty interior. The notion (assumption) of properness was later weakened and revised by many authors. In modern papers, it is defined in rather abstract form, but in some special cases it means that with every feasible consumption bundle (vector) one can associate an open convex cone, with vertex at a given point, that does not intersect the set of (strictly) preferable consumer plans (for more details, see [22] and [4]).

The monograph by Aliprantis, Brown, and Burkinshaw [4] summarized the achievements of the 80s in the equlibrium analysis of models with infinitely many commodities. It paid a special attention to the duality of vector lattices and their locally solid topologies. It is these topologies that ensure the uniform continuity of

[2]More precisely, the term *Kantorovich space* means a Dedekind-complete (every set bounded from above has a least upper bound) linear lattice. However, in the context of this paper, we interprete the term "Kantorovich spaces" in an extended sense, meaning spaces used in the theory constructed by Kantorovich.

the lattice operations (join and meet). Recently, Mas-Colell and Richard [21] made the next step and established the existence of equlibrium in models in which the commodity space is a linear vector lattice. This term, unlike that of topological vector lattice, *does not assume the continuity of the lattice operations*. It is important that this is not just another generalization, but it allows one to describe meaningful economic models that earlier were not included into the general theory. For example, in the model with differentiable commodities studied by Jones [15], the commodity space is the space of Borel measures defined on a compact space $\mathcal{M}$. This space is a linear lattice but not a topological lattice (in the $*$-weak topology determined by the duality $\langle C(\mathcal{M}), ca(\mathcal{M})\rangle$; see also the close paper [16] by Huang and Kreps). Note that Mas-Colell and Richard explicitly use the order properties of preferences of economic agents (and thus these preferences can be represented as utility functions, which allows one to seek for a fixed point in the space of utilities; this is the so-called Negishi approach). In subsequent studies, many authors (see [25, 27, 11, 14]) delivered the theory from this unnatural assumption, establishing the existence of equilibriums under nontransitive and incomplete preferences of economic agents (of course, this requires a radical revision of the whole proof). Besides, in most of the above-mentioned papers, the requirement of the uniform properness of preferences is replaced by a *pointwise* characterization (Araujo and Monteiro [6] and Duffie and Zame [12] were the first to point out this possibility), which is substantially weaker and, which is more important, better agrees with the economic meaning in a number of applications of the general model (for example, in applications to financial theory, where preferences are not uniformly proper even in the Peleg–Yaari model). In this connection, let us mention the rather interesting paper by Podczeck [25], which substantially generalizes the notion of properness of preferences. Similar results for a model with production sector are proved in [28, 14]. In conclusion, let us repeat that the whole development of infinite-dimensional equilibrium theory was carried out in the framework of Kantorovich spaces, as the key Riesz–Kantorovich formula for the supremum functional holds precisely in linear lattices. Moreover, this formula is the basic element of the further development of the theory, where prices are allowed to be nonlinear functionals (see, for example, [5]).

However, equilibrium theory in linear lattice economies is still incomplete and contains the following gap. Indeed, one can observe that the results by Florenzano and Marakulin [14], being very general, still do not cover the theorems proved by Tourky in [27] and [28], and by now these results are complementary. The reason is that the main existence theorems in [14] were proved under a specific notion of properness (the so-called E-properness), which is formulated relative to the principal ideal of the commodity space generated by the vector of total initial endowments. However, in this case this conception is not comparable with the M-properness used in [27, 28]. At the same time, the E-properness from [14] regarded relative to the *whole* commodity space is substantially weaker than Tourky's M-properness, but in [14] such a theorem was not proved for spaces without order unit. The present paper fills this gap (in the context of an exchange model). The proof is based on new characterizations of elements of the fuzzy core of an economic model. Then, using one of the characterizations, we prove a new theorem on the existence of quasi-equilibrium for linear vector lattice economy under other modern weakest assumptions. This theorem is established under the above structural assumptions and the E-properness assumption from [14] relative to the *whole* commodity space, and does not depend on whether or not it has an order unit.

2 Exchange economy and fuzzy core characterizations

2.1 Notations

In this paper together with standard notations we will apply the following ones. Let L be a vector space over $\mathbb{R}$ and let L^* be its algebraic dual. Then

$\operatorname{co} A$ denotes the *convex hull* of $A \subset L$,

$A + x = \{a + x \mid a \in A\}$ for all $A \subseteq L$, $x \in L$,

$A + B = \{a + b \mid a \in A, b \in B\}$ for all $A \subseteq L$, $B \subseteq L$,

$\langle p, x \rangle = p(x) = px$ denotes the *inner product* of vectors $p \in L^*$, $x \in L$,

$\langle p, A \rangle = \{p(x) \mid x \in A\}$, where $p \in L^*$, $A \subset L$,

$A \geq B \iff a \geq b, \forall\, a \in A, \ \forall\, b \in B$ for all $A \subset \mathbb{R}$, $B \subset \mathbb{R}$,

$A \setminus B = \{x \in A \mid x \notin B\}$ is the set-theoretical *difference*.

The *linear segments* in L with the tips $a, b \in L$ are denoted in the following way.

$\lfloor a, b \rfloor = \operatorname{co}\{a, b\} = \{\lambda a + (1 - \lambda) b \mid 0 \leq \lambda \leq 1\}$,

$(a, b\rfloor = \lfloor a, b \rfloor \setminus \{a\} = \{\lambda a + (1 - \lambda) b \mid 0 < \lambda \leq 1\}$.

Let $x, y \in L^n$, $x = (x_i)_{i=1,\dots,n}$, $y = (y_i)_{i=1,\dots,n}$. Then

$\llbracket x, y \rrbracket = \prod_{i=1}^{n} \lfloor x_i, y_i \rfloor$ denotes the *direct product* of (closed) linear segments.

Notation $[a, b]$ is applied in the case of partially ordered space $(L, \geq)$ and denotes the *order* interval with tips a and b, i.e.,

$[a, b] = \{x \in L \mid a \leq x \leq b\}$.

If L is equipped with a topology, then for $A \subseteq L$

$\overline{A}$ denotes the *closure* of set A, and

$\text{int}A$ is its *interior*.

2.2 Main notions of an exchange economy

The subject of analysis in this paper is a standard pure exchange economy. In this model L denotes a *commodity space* and there is a (finite) set of economic agents (traders or consumers) $\mathcal{I} = \{1, \ldots, n\}$. A consumer $i \in \mathcal{I}$ is standardly characterized by a *consumption set* $X_i \subset L$, a vector of initial endowments $\omega_i \in X_i$ and by *preference* relation described as a point-to-set mapping (correspondence) $\mathcal{P}_i : \mathcal{X} \Rightarrow X_i$, $\mathcal{X} = \prod_{\mathcal{I}} X_j$, where the set $\mathcal{P}_i(x)$ is interpreted as a collection of all strictly preferred consumption bundles to the bundle x_i with respect to allocation $x = (x_j)_{j \in \mathcal{I}} \in \mathcal{X}$. Also notation $y_i \succ_i^x x_i$ will be applied, which by definition is equivalent to $y_i \in \mathcal{P}_i(x)$. Thus an exchange economy can be briefly represented as a triplet

$$\mathcal{E} = \langle \mathcal{I}, L, (X_i, \mathcal{P}_i, \omega_i)_{i \in \mathcal{I}} \rangle.$$

It will always be presumed later that the model $\mathcal{E}$ satisfies the following assumption **(C)**, and in the limits of this section this is the only assumption.

Denote $\omega = (\omega_i)_{i \in \mathcal{I}}$ the full vector of initial endowments of *all* traders and determine the set of all *feasible allocations* by formula

$$\mathcal{A}(\mathcal{X}) = \{x \in \mathcal{X} \mid \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \omega_i\}.$$

(C) For every $x = (x_1, \ldots, x_n) \in \mathcal{A}(\mathcal{X})$ and each $i \in \mathcal{I}$:

(i) *the convex irreflexiveness*: $x_i \notin \text{co}\,\mathcal{P}_i(x)$,

(ii) *the convexity of preferences*: the set $\{x_i\} \cup \mathcal{P}_i(x)$ is convex.

Notice that assumption **(C)** implies the convexity of $\mathcal{P}_i(x)$[3] that of course can be separately postulated (then item (i) will be written in the form $x_i \notin \mathcal{P}_i(x)$). Notice also that preferences may be satiated, i.e., $\mathcal{P}_i(x) = \emptyset$ is possible for some agents. However if $\mathcal{P}_i(x) \neq \emptyset$, then **(C)**(ii) implies the *local non-satiation* of preference at the point x (since $(x_i, y\rfloor \subset \mathcal{P}_i(x),\ \forall\, y \in \mathcal{P}_i(x)$). We recall further the fuzzy core concept and other definitions.

A pair (x, p) is said to be a *quasi-equilibrium* of $\mathcal{E}$ if $x \in \mathcal{A}(\mathcal{X})$ and there exists a linear functional $p \neq 0$ onto L such that

$$\langle p, \mathcal{P}_i(x)\rangle \geq px_i = p\omega_i, \quad \forall\, i \in \mathcal{I}.$$

A quasi-equilibrium such that $x'_i \in \mathcal{P}_i(x)$ actually implies $px'_i > px_i$ is a *Walrasian or competitive equilibrium.*

On the other hand, $x \in \mathcal{A}(\mathcal{X})$ is said to be dominated (blocked) by a nonempty coalition $S \subseteq \mathcal{I}$ if there exists $y^S \in \prod_{i\in S} X_i$ such that $\sum_{i\in S} y_i^S = \sum_{i\in S} \omega_i$ and $y_i^S \in \mathcal{P}_i(x),\ \forall\, i \in S$.

The *core* of $\mathcal{E}$, denoted by $\mathcal{C}(\mathcal{E})$, is the set of all $x \in \mathcal{A}(\mathcal{X})$ that are blocked by no (nonempty) coalition.

Weak Pareto boundary for $\mathcal{E}$, denoted by $\mathcal{PB}^w(\mathcal{E})$, is the set of all $x \in \mathcal{A}(\mathcal{X})$ that cannot be dominated by the coalition $\mathcal{I}$ of all agents.

The next important notion, which fruitfully is used in the theory of economic equilibrium to model perfect competition conditions, is the concept of fuzzy core. Recall that any vector

$$t = (t_1, \ldots, t_n) \neq 0, \quad 0 \leq t_i \leq 1, \quad \forall\, i \in \mathcal{I}$$

may be identified with a fuzzy coalition, the real number t_i being interpreted as the measure of agent i in the coalition. A coalition t is said to dominate (block) an allocation $x \in \mathcal{A}(\mathcal{X})$ if there exists $y^t \in \prod_{\mathcal{I}} X_i$ such that

$$\sum_{i\in\mathcal{I}} t_i y_i^t = \sum_{i\in\mathcal{I}} t_i \omega_i \iff \sum_{i\in\mathcal{I}} t_i (y_i^t - \omega_i) = 0 \tag{2.1}$$

and

$$y_i^t \succ_i^x x_i, \quad \forall\, i \in \operatorname{supp}(t) = \{i \in \mathcal{I} \mid t_i > 0\}. \tag{2.2}$$

The set of all feasible allocations which cannot be dominated by fuzzy coalitions is denoted by $\mathcal{C}^f(\mathcal{E})$ and is called the *fuzzy core* of the economy $\mathcal{E}$.

[3] For $y, z \in \mathcal{P}_i(x)$ from (ii) we have $\lambda y + (1-\lambda)z \in \{x_i\} \cup \mathcal{P}_i(x)$, $\lambda \in [0,1]$, but by (i) $x_i \neq \lambda y + (1-\lambda)z$.

2.3 Fuzzy core characterizations in exchange economies

It is well known that for non-satiated preferences, i.e., when $\mathcal{P}_i(x) \neq \emptyset$, $\forall\, i \in \mathcal{I}$, conditions (2.1), (2.2) may be equivalently rewritten in the form

$$0 \in \sum_{i\in\mathcal{I}} t_i(\mathcal{P}_i(x) - \omega_i).$$

Thus since all $\mathcal{P}_i(x)$ are convex due to assumption **(C)**, condition $x \in \mathcal{C}^f(\mathcal{E})$ is equivalent to[4]

$$0 \notin \mathrm{co}[\bigcup_{\mathcal{I}}(\mathcal{P}_i(x) - \omega_i)], \tag{2.3}$$

which, by the separation theorem, implies that the elements of the fuzzy core are quasiequilibria. Below we propose other useful in applications characterizations of fuzzy core points. To this end, let us consider the sets

$$\Omega_i(x) = \mathrm{co}(\mathcal{P}_i(x) \cup \{\omega_i\}), \quad i \in \mathcal{I}.$$

Due to the convexity of $\mathcal{P}_i(x)$, for $\mathcal{P}_i(x) \neq \emptyset$, we conclude

$$\mathrm{co}(\mathcal{P}_i(x) \cup \{\omega_i\}) = \bigcup_{0\le\lambda\le1}[\lambda\mathcal{P}_i(x) + (1-\lambda)\omega_i] = \bigcup_{0\le\lambda\le1}\lambda(\mathcal{P}_i(x) - \omega_i) + \omega_i, \quad i \in \mathcal{I}.$$

This implies that the condition $z + \omega \in \prod_{\mathcal{I}} \Omega_i(x)$, where $\omega = (\omega_1, \ldots, \omega_n)$, is equivalent to the existence of $0 \le \lambda_i \le 1$ and $[y_i \in \mathcal{P}_i(x) \neq \emptyset$ and $y_i = \omega_i$, if $\mathcal{P}_i(x) = \emptyset]$, $i \in \mathcal{I}$ such that

$$z = (\lambda_1(y_1 - \omega_1), \ldots, \lambda_n(y_n - \omega_n)).$$

Hence, due to (2.1), (2.2)

$$x \in \mathcal{C}^f(\mathcal{E}) \iff \nexists\, z \in L^{\mathcal{I}},\ z \neq 0: \quad z + \omega \in \prod_{\mathcal{I}} \Omega_i(x_i) \quad \& \quad \sum_{i\in\mathcal{I}} z_i = 0 \iff$$

$$\prod_{\mathcal{I}} \Omega_i(x) \bigcap \{(z_1, \ldots, z_n) \in L^{\mathcal{I}} \mid \sum_{i\in\mathcal{I}} z_i = \sum_{i\in\mathcal{I}} \omega_i\} = \{\omega\}. \tag{2.4}$$

In doing so we have proven the following

Proposition 2.1 *An allocation $x \in \mathcal{A}(\mathcal{X})$ is the element of fuzzy core if and only if relation (2.4) is true.*

[4] It is easy to see that domination by the fuzzy coalitions is equivalent to domination by the *normalized* coalitions corresponding to the weight coefficients of a convex combination, i.e., for a dominating coalition t one may always think that $\sum_{i\in\mathcal{I}} t_i = 1$.

Further we consider other fuzzy core characterizations. With this in mind let us consider an open cone $\Gamma_i(x)$, corresponding to the set of all agent i's preferred bundles, setting

$$\Gamma_i(x) = \{\alpha(z_i - x_i) \mid \alpha > 0, z_i \in \mathcal{P}_i(x)\}$$

for $\mathcal{P}_i(x) \neq \emptyset$. By specification $\mathcal{P}_i(x) - x_i \subset \Gamma_i(x)$, that yields $\mathcal{P}_i(x) \subset \Gamma_i(x) + x_i$. Define also

$$\mathcal{A}(L^{\mathcal{I}}) = \{(z_1, \ldots, z_n) \in L^{\mathcal{I}} \mid \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \omega_i\}.$$

Proposition 2.2 *Let $x \in \mathcal{A}(\mathcal{X})$ and $\mathcal{P}_i(x) \neq \emptyset$ for all $i \in \mathcal{I}$. Then $x \in \mathcal{C}^f(\mathcal{E})$ implies each of following conditions*:

(i) $\prod_{\mathcal{I}}(\mathcal{P}_i(x) + \lfloor 0, \omega_i - x_i \rfloor) \bigcap \mathcal{A}(L^{\mathcal{I}}) = \emptyset$,

(ii) $\prod_{\mathcal{I}}(\Gamma_i(x) + \lfloor x_i, \omega_i \rfloor) \bigcap \mathcal{A}(L^{\mathcal{I}}) = \emptyset$.

It is easy to see that conditions (i) and (ii) are equivalent in fact. Notice also that $\Omega_i(x) \setminus \{\omega_i\} \subset \mathcal{P}_i(x) + \lfloor 0, \omega_i - x_i \rfloor$, $\forall\, i \in \mathcal{I}$. Thus conditions (i), (ii) of Proposition 2.2 are very close to be not only necessary but also sufficient.

Proof. Since

$$\mathcal{P}_i(x) + \lfloor 0, \omega_i - x_i \rfloor \subset \Gamma_i(x) + \lfloor x_i, \omega_i \rfloor, \ \forall\, i \in \mathcal{I},$$

then it is sufficient to prove that $x \in \mathcal{C}^f(\mathcal{E}) \Rightarrow$ (ii). Let us do it.

Choose

$$y \in \prod_{\mathcal{I}}(\Gamma_i(x) + \lfloor x_i, \omega_i \rfloor) \bigcap \mathcal{A}(L^{\mathcal{I}}).$$

Then for each i there exists $z_i \in \mathcal{P}_i(x)$, $\lambda_i \in [0, 1]$ and $\alpha_i > 0$ such that

$$y_i = \lambda_i x_i + (1 - \lambda_i)\omega_i + \alpha_i(z_i - x_i). \tag{2.5}$$

Also $y = (y_i)_{\mathcal{I}}$ satisfies $\sum_{i \in \mathcal{I}} y_i = \sum_{i \in \mathcal{I}} \omega_i$. Now consider real $\beta > 0$ small enough. Due to $x \in \mathcal{A}(\mathcal{X}) \subset \mathcal{A}(L^{\mathcal{I}})$, $y \in \mathcal{A}(L^{\mathcal{I}})$, we have

$$h = \beta y + (1 - \beta)x \in \mathcal{A}(L^{\mathcal{I}}).$$

The components of vector $h = (h_i)_{i \in \mathcal{I}}$ can be written now in the form

$$h_i = \beta[\lambda_i x_i + (1 - \lambda_i)\omega_i + \alpha_i(z_i - x_i)] + (1 - \beta)x_i =$$

$$= (1 - \beta + \beta\lambda_i)x_i + (\beta - \beta\lambda_i)\omega_i + (1 - \beta + \beta\lambda_i)\frac{\alpha_i\beta}{1 - \beta + \beta\lambda_i}(z_i - x_i).$$

By the choice of β we may think $\mu_i = \frac{\alpha_i\beta}{1-\beta+\beta\lambda_i} \le 1$, that due to **(C)** implies

$$\mu_i(z_i - x_i) \in \mathcal{P}_i(x) - x_i \;\Rightarrow\; \exists\, \eta_i \in \mathcal{P}_i(x) : \; \mu_i(z_i - x_i) = \eta_i - x_i.$$

Therefore the previous formula gives

$$h_i = (1 - \beta + \beta\lambda_i)\eta_i + (\beta - \beta\lambda_i)\omega_i,$$

that in view of $(1 - \beta + \beta\lambda_i) + (\beta - \beta\lambda_i) = 1$ and $\eta_i \in \mathcal{P}_i(x)$ implies $h_i \in \Omega_i(x)$. This allows us to apply relation (2.4), concluding $h = h(\beta) = \omega$ for *all* real $\beta > 0$ small enough. Write this equality componentwise and find

$$y_i = \frac{(\beta - 1)x_i + \omega_i}{\beta} = x_i + \frac{\omega_i - x_i}{\beta}, \; i \in \mathcal{I}.$$

However these equalities can be true for *different* $\beta > 0$ only if $y_i = x_i = \omega_i$, $i \in \mathcal{I}$, that due to (2.5) implies $x_i = \omega_i \in \mathcal{P}_i(x)$ and contradicts to (**C**)(i). $\square$

Described above characterizations of fuzzy core elements have a clear geometrical presentation in Edgeworth's box. Really, in the case of a *2-agents economy*, condition (2.4) may be rewritten in the form

$$\Omega_1(x) \cap (\bar{\omega} - \Omega_2(x)) = \{\omega_1\}, \quad \bar{\omega} = \omega_1 + \omega_2.$$

Moreover, in this case the fact that fuzzy coalition $(t_1, t_2) > 0$, $t_i \le 1$ dominates allocation (x_1, x_2) can be illustrated as follows. Consider the Edgeworth's box. Due to the definition of domination, in a nontrivial case it can occur only if $t_1 \ne 0$ & $t_2 \ne 0$ and

$$\exists\; y_1, y_2 \in \mathbb{R}^2_+ \; : \; y_1 \succ_1^x x_1, \; y_2 \succ_2^x x_2 \;\; \& \;\; t_1(y_1 - \omega_1) = t_2(\omega_2 - y_2).$$

Let $z_2 = \bar{\omega} - y_2$ be the consumption bundle of the first agent when the second one consumes y_2. This vector represents y_2 in the natural coordinate system of the first agent's consumption bundles. Substituting z_2 in the right hand side of the last relation we obtain

$$t_1(y_1 - \omega_1) = t_2(\omega_2 - (\bar{\omega} - z_2)) = t_2(z_2 - \omega_1) \iff z_2 = \omega_1 + \frac{t_1}{t_2}(y_1 - \omega_1), \; t_2 \ne 0.$$

Geometrically, it means that the points y_1 and z_2 lie on one line with the point ω_1, and on one side of this line, with respect to ω_1 (i.e., they belong to a ray starting at ω_1). Moreover, due to the definition of domination, we have $y_1 \in \mathcal{P}_1(x)$ and $z_2 \in \bar{\omega} - \mathcal{P}_2(x)$. Hence,

> $(x_1, x_2) \notin \mathcal{C}^f(\mathcal{E}) \iff \exists$ *ray* starting at the point ω_1, which intersects both sets, $\mathcal{P}_1(x)$ and $\bar{\omega} - \mathcal{P}_2(x)$.

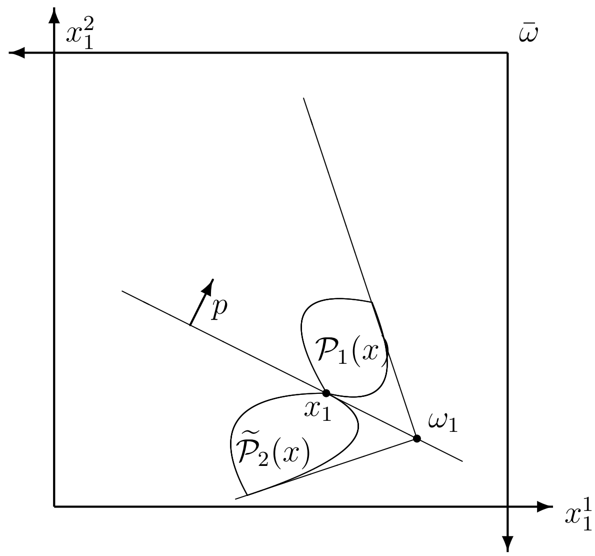

Figure 2.1: The fuzzy core

For graphic illustration of the above analysis in the Edgeworth's box for a 2-goods economy, see Figure 2.1, where $\widetilde{\mathcal{P}}_2(x) = \bar{\omega} - \mathcal{P}_2(x)$. In fact, for this case an allocation x lying in the fuzzy core is equivalent to the convex hulls of $\mathcal{P}_1(x) \cup \{\omega_1\}$ and of $[\bar{\omega} - \mathcal{P}_2(x)] \cup \{\omega_1\}$ having only one point, ω_1, in common (alternatively, in the terms of cones with common vertex ω_1, which are going across the sets of all strictly preferred consumption bundles).

The similar illustrations of requirements (i) and (ii) from Proposition 2.2 are given in Figures 2.2, 2.3.

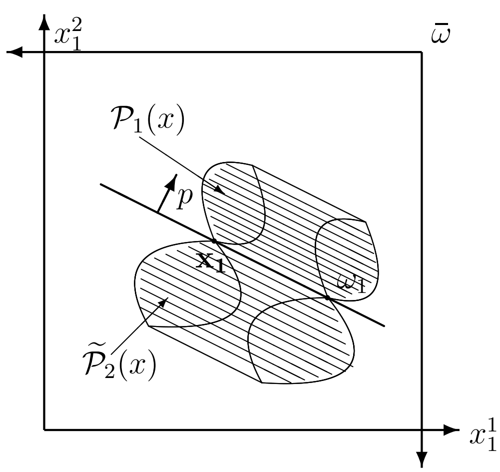

Figure 2.2: The fuzzy core, property (i)

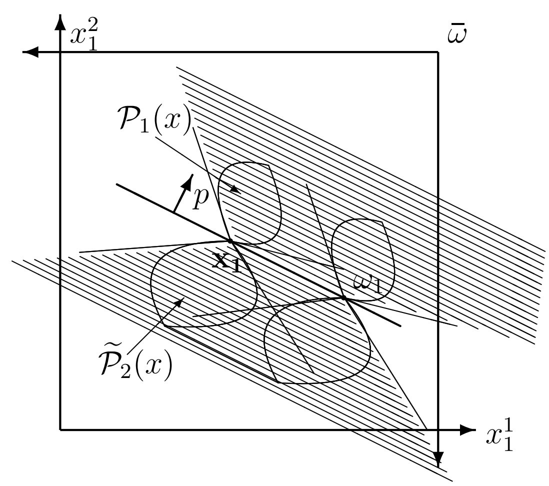

Figure 2.3: The fuzzy core, property (ii)

3 The existence of equilibria in linear vector lattice economy

The equilibrium existence problem in economy with infinite many commodities was intensively investigated in the literature of last twenty years. The history of this subject started from Peleg–Yaari [24] and Bewley [8] papers. The main difficulty in the analysis of these models was that a partially ordered linear space is considered in which an order is defined via a cone of positive elements that may have empty interior. Thus commodity space has two structures: an ordering and a linear topology. In order to solve this problem relative to continuous price functionals, Mas-Colell [19] introduced the notion of uniformly proper preferences. Later other similar (weaker) properness requirements were considered in literature, see, e.g. [29], [25], [27], [14], some of them in a pointwise form. Initially the analysis of these models was realized for commodity spaces, which are topological vector lattices — the filter of neighborhoods of zero has a base of convex-solid sets. In these spaces the lattice operations are uniformly continuous. However after Jones [15] and Huang–Kreps [16] papers the models, which commodity spaces are not so well qualified, began to be investigated. These spaces are called linear vector lattices and they have only a weak link between topological and ordering structures — the cone of positive elements is closed and the dual space of linear continuous functionals is a subspace of order dual (the space of order bounded linear functionals). The analysis of this type models is essentially complicated because lattice operations may be discontinuous. The aim of this paper is to prove the existence theorem of quasiequilibrium with continuous prices for linear vector lattice economy. This theorem generalizes results by Tourky [27] and Florenzano–Marakulin [14]. Further let us consider assumptions.

To this end we will assume that commodity space L is endowed with a Hausdorff linear topology τ, and at the same time this is partially ordered linear space $(L, \geq)$ with a cone of positive elements $L^+ = \{x \in L \mid x \geq 0\}$. In addition we require the following properties of *commodity–price* pairing $\langle (L,\tau), (L,\tau)' \rangle$ to be fulfilled.

SA (structural assumptions):

(i) L is a linear vector lattice (or Riesz space) endowed with a locally convex Hausdorff linear topology τ;

(ii) L^+ is a closed cone in the τ-topology of L;

(iii) $(L,\tau)'$ is a sublattice of the order dual of L.

Further let us consider a concept of proper preferences applied in this paper.

Definition 3.1 *A preference relation $\mathcal{P}_i : \mathcal{X} \Rightarrow X_i$ is said to be F-proper at a point $x \in \mathcal{X}$ if there exists a τ-open convex subset $V_x^i \subset L$ and a sublattice $Z_x^i \subset L$, verifying $Z_x^i + L^+ \subset Z_x^i$, such that $x_i \in \overline{V}_x^i \cap Z_x^i$ and*

$$\emptyset \neq V_x^i \bigcap Z_x^i \subseteq \mathcal{P}_i(x). \tag{3.6}$$

If, moreover,

$$\mathcal{P}_i(x) \subset \overline{V}_x^i \bigcap Z_x^i, \tag{3.7}$$

then the preference relation $\mathcal{P}_i$ is said to be E-proper at $x \in \mathcal{X}$.

In fact this notion of properness is the properness relative to an ideal $K \subseteq L$ by Definition 2.1 from Florenzano–Marakulin [14], in the case $K = L$. The only difference, made here for simplicity, is that (3.6), (3.7) were required in [14] locally, i.e., it was postulated the existence of radial subset $A_x \subset L$[5], such that (3.6), (3.7) are true only in the limits of A_x (intersected with it). This is precisely E-properness relative to the whole commodity space, being a weakest form of E-properness, that is the subject of our investigation.

In a parallel with equilibrium, the concept of *nontrivial* quasiequilibrium has the most theoretical interest, being an intermediate object to state the existence of equilibrium. In order to provide existence of nontrivial quasiequilibrium, E-properness requirement has to be strengthened a little bit that is realized in the following definition.

Definition 3.2 *An economy $\mathcal{E}$ is said to be nontrivially E-proper at a point x, if all agents preferences are E-proper at x and if, in the previous definition, the following additional conditions are true: for each $i \in \mathcal{I}$ there exists a vector $v_i \in L$, such that $x_i + v_i \in V_x^i \cap Z_x^i$,*

$$\omega' - v_i \in \sum_{t \in \mathcal{I}} X_t, \ \forall\, i \in \mathcal{I}, \tag{3.8}$$

and the set

$$\sum_{\mathcal{I}} [Z_x^i \cap L(u^v)] \tag{3.9}$$

is a radial at $\omega' = \sum_{\mathcal{I}} \omega_i$ subset of ideal $L(u^v)$, where $L(u^v)$ denotes the ideal generated in L by

$$u^v = \sum_{\mathcal{I}} |\omega_i| + \sum_{\mathcal{I}} |x_i| + \sum_{\mathcal{I}} |v_i|.$$

Here all sets Z_x^i and V_x^i are chosen from E-properness definition.

The vectors $\{v_i\}$ are called nontrivial properness vectors of $\mathcal{E}$ at x.

[5]Recall that a subset $A \subset L$ is called radial (absorbing) at a point $y \in A$, if for every $u \in L$ there exists a real number $\bar{\lambda}$, $0 < \bar{\lambda} \leq 1$, such that $(1-\lambda)y + \lambda u \in A$ for every $\lambda \geq 0$ with $\lambda \leq \bar{\lambda}$.

Notice that in difference with [14] we do not assume in properness assumptions the inclusions $\omega_i \in Z^i_x$, $i \in \mathcal{I}$ to be true. Below we discuss how this additional assumption can be used.

In Florenzano–Marakulin [14] were noticed conditions (see Lemma 3.1), under which one can guarantee the sets (3.9) to be radial at ω'. These conditions are the following.

Let all properness vectors be chosen from ideal $L(u)$, generated by $u = \sum_{i\in\mathcal{I}} |x_i| + \sum_{i\in\mathcal{I}} |\omega_i|$, i.e., $v_i \in L(u)$, $\forall\, i \in \mathcal{I}$. Thus $L(u^v) = L(u)$. Then if $\omega' = \sum_{\mathcal{I}} \omega_i > 0$ and $\omega_i, 0 \in Z^i_x$, $\forall\, i \in \mathcal{I}$, the set $\sum_{\mathcal{I}}[Z^i_x \cap L(u^v)]$ is radial at ω' in $L(u^v)$. In particular, for $\omega' > 0$ one can assume, similarly to Tourky [27], that each properness vector coincides with the vector of total initial endowments of economy, i.e., $v_i = \omega'$, $\forall\, i \in \mathcal{I}$. In such a case condition (3.8) can be written in the form $\lambda\omega' \in \sum_{\mathcal{I}} X_i$ for some real $\lambda < 1$ (or simply as $0 \in \sum_{\mathcal{I}} X_i$), that is unlikely to be seriously objectionable.

The existence of nontrivial quasiequilibrium follows from the existence (nonemptiness) of fuzzy core and the possibility to decentralize its elements. In other words, when fuzzy core is non-empty it is sufficient to show that every allocation from fuzzy core can be represented as a quasiequilibrium. To realize this, in our analysis we apply the characterization of fuzzy core elements elaborated in previous section, this is item (i) of Proposition 2.2. In fact in such a case this relation can be rewritten in a form[6]

$$\prod_{\mathcal{I}} \mathcal{P}_i(x) \bigcap \left[\left(\prod_{\mathcal{I}} \lfloor 0, x_i - \omega_i \rfloor \right) + \mathcal{A}(L^{\mathcal{I}}) \right] = \emptyset.$$

Due to the fact that preferences are proper, by (3.6), we have

$$V \cap Z \cap [\lfloor\!\lfloor 0, x - \omega \rfloor\!\rfloor + \mathcal{A}(L^{\mathcal{I}})] = \emptyset,$$

where $V = \prod_{\mathcal{I}} V^i_x$, $Z = \prod_{\mathcal{I}} Z^i_x$. Replacing brackets in last relation we obtain

$$V \bigcap \left[Z \cap [\lfloor\!\lfloor 0, x - \omega \rfloor\!\rfloor + \mathcal{A}(L^{\mathcal{I}})] \right] = \emptyset. \tag{3.10}$$

So we yield the crucial property allowing us to decentralize a point $x \in \mathcal{C}^f(\mathcal{E})$. Really, as it will be clear later, it will be enough to separate the sets in (3.10) by a linear functional. Due to imposed assumptions classical separation theorem is applicable to (3.10), but by technical reasons it will be more convenient for us to find separating functional in two stages. First, separating the sets from (2.3) we find linear functional separating the set from (3.10) on subspace $[L(u^v)]^{\mathcal{I}}$. During second stage this functional is extended onto the whole space $L^{\mathcal{I}}$ saving separation properties of sets. This is possible due to nontrivial properness assumption, since

[6] Because $(A + B) \cap C = \emptyset \iff A \cap (C - B) = \emptyset$ for any sets $A, B, C \subseteq L$.

$V \cap [L(u^v)]^{\mathcal{I}} \neq \emptyset$, $[L(u^v)]^{\mathcal{I}} \cap \big[Z \cap [\llbracket 0, x-\omega \rrbracket + \mathcal{A}(L^{\mathcal{I}})]\big] \neq \emptyset$. In a final step applying extended functional we construct quasiequilibrium prices for x. These prices are obtained as the supremum of components in extended functional, which is also continuous due to **SA** (iii). The described program is realized in subsequent lemma and theorem.

Lemma 3.1 *Let $\mathcal{E}$ be a nontrivial E-proper economy and $x \in \mathcal{C}^f(\mathcal{E})$. Then there exist τ-continuous linear functionals q_i on L such that*

$$q_i \cdot V_x^i \geq q_i \cdot x_i, \; i \in \mathcal{I}, \tag{3.11}$$

for $\bar{q} = \vee_{\mathcal{I}} q_i$, $\bar{q} x_i = \bar{q} \omega_i$, $i \in \mathcal{I}$ is true, moreover, $\bar{q} v_i > 0$ for some $i \in \mathcal{I}$ relative to his properness vector v_i, and

$$\sum_{i \in \mathcal{I}} q_i t_i \leq 0, \quad \forall\, t = (t_1, \ldots, t_n) \; : \; \sum_{i \in \mathcal{I}} t_i = 0, \; x_i + t_i \in Z_x^i \cap L(u^v), \; \forall\, i \in \mathcal{I} \tag{3.12}$$

holds.

Proof. The statement of this lemma is, in fact, a corollary of Propositions 3.1, 2.1, 2.2 from [14]. However we shortly recall the argumentation. Really, let $K = L(u^v)$. In view of fuzzy core elements property (2.3) and from F-properness we obtain

$$0 \notin \mathrm{co}[\bigcup_{\mathcal{I}} (V_x^i \cap Z_x^i - \omega_i)].$$

The set $K \cap \mathrm{co}[\cup_{\mathcal{I}} (V_x^i \cap Z_x^i - \omega_i)]$ has non-empty interior in K in Riesz's norm, determined by vector u^v, therefore this set can be separated on K from zero by a linear functional $p \neq 0$. Applying (3.7), one can standardly state, that (x, p) is a quasiequilibrium of $\mathcal{E}_{|_K}$, that in particular implies $p x_i = p \omega_i$, $i \in \mathcal{I}$.

Further, it is easy to see that $\mathcal{E}$ is F-proper relative to K by Definition 2.1 from [14] (instead of Z_x^i in (3.6) one needs to use $Z_x^i \cap K$). Thus Proposition 2.1 from [14] can be applied, that implies the existence of functionals $q_i \in (L, \tau)'$, $i \in \mathcal{I}$, such that $q_{i|_K} \leq p$ and

$$q_i \cdot V_x^i \geq q_i \cdot x_i, \quad i \in \mathcal{I}.$$

Moreover, for $\bar{q} = \vee_{\mathcal{I}} q_t$

$$q_i(x_i - z) = \bar{q}(x_i - z) = p(x_i - z), \; \forall\, z \in Z_x^i \cap K, \; z \leq x_i, \; \forall\, i \in \mathcal{I} \tag{3.13}$$

and

$$\bar{q}(\omega' - z) = p(\omega' - z), \; \forall\, z \in \sum_{i \in \mathcal{I}} Z_i \cap K, \; z \leq \omega' \tag{3.14}$$

holds.[7] Finally, in view of non-trivial E-properness of $\mathcal{E}$, we are now under conditions of Proposition 2.2 from [14], that yields $\bar{q}_{|K} = p$ and $\bar{q}_{|K} v_i = pv_i > 0$ for some $i \in \mathcal{I}$. For the presentation of proof to be complete let us briefly describe the last argument.

To show $\bar{q}_{|K} = p$ take an arbitrary $y \in K$. Since K is an ideal, $y^+, y^- \in K$.[8] Due to non-trivial properness $Z' = \sum_{\mathcal{I}}[Z_x^i \cap K]$ is radial at $\omega' = \sum_{i\in\mathcal{I}} \omega_i$ when $K = L(u^v)$. Hence $\omega' - \lambda y^+, \omega' - \lambda y^- \in Z'$ for some real $\lambda > 0$. Now (3.14) yields

$$\bar{q}(\omega' - (\omega' - \lambda y^+)) = p(\omega' - (\omega' - \lambda y^+)) \;\Rightarrow\; \lambda \bar{q} y^+ = \lambda p y^+ \;\Rightarrow\; \bar{q} y^+ = p y^+.$$

Similarly we obtain $\bar{q} y^- = p y^-$, that because of $y = y^+ - y^-$ gives $\bar{q} y = py$.

To check $\bar{q}_{|K} v_i = pv_i > 0$ for some $i \in \mathcal{I}$ notice that from (3.11) and $x_i + v_i \in V_i$ we obtain $q_i v_i \geq 0$. Moreover, if $q_i v_i = 0$, then $q_i = 0$. However $q_i = 0$ for all i contradicts $\bar{q}_{|K} = p \neq 0$. Further let $q_{i_0} v_{i_0} > 0$ for given i_0. Define $z_{i_0} = (x_{i_0} + v_{i_0}) \wedge x_{i_0} \in Z_x^{i_0}$. From $\bar{q}$ specification and (3.13) conclude

$$pv_{i_0} = p(x_{i_0} + v_{i_0} - z_{i_0}) - p(x_{i_0} - z_{i_0}) \geq q_{i_0}(x_{i_0} + v_{i_0} - z_{i_0}) - q_{i_0}(x_{i_0} - z_{i_0}) = q_{i_0} v_{i_0} > 0,$$

that implies $pv_{i_0} > 0$. Since $\bar{q}_{|K} = p$ and $v_{i_0} \in K$, then $pv_{i_0} = \bar{q} v_{i_0}$. This finishes the first part of lemma proving.

Further let us check (3.12). Let $t = (t_1, \ldots, t_n)$ satisfy the right hand side of (3.12). Define $z_i = x_i \wedge (x_i + t_i)$. Since $x_i \in Z_x^i \cap K$, $x_i + t_i \in Z_x^i \cap K$, and Z_x^i is a lattice, K is an ideal, we obtain $z_i \in Z_x^i \cap K$. This in view of (3.13) and $p \geq q_{i_{|K}}$, $x_i - z_i + t_i \geq 0$, yields

$$p(x_i - z_i) + p(t_i) = p(x_i - z_i + t_i) \geq q_i(x_i - z_i + t_i) = q_i(x_i - z_i) + q_i(t_i) = p(x_i - z_i) + q_i(t_i),$$

that implies $p(t_i) \geq q_i(t_i)$, $\forall\, i \in \mathcal{I}$. Summing the last equalities find

$$0 = p(\sum_{i\in\mathcal{I}} t_i) \geq \sum_{i\in\mathcal{I}} q_i t_i,$$

that completes the proof of Lemma 3.1. □

The next theorem is the main result of this paper.

Theorem 3.1 *Let $\mathcal{E}$ be a nontrivial E-proper exchange economy and $x \in \mathcal{C}^f(\mathcal{E})$. Then there exists a τ-continuous linear functional $\bar{\pi} \in (L, \tau)'$, such that $(x, \bar{\pi})$ is a nontrivial quasi-equilibrium.*

[7] Notice that assumption $\omega_i \in Z_x^i$, $i \in \mathcal{I}$ being postulated in Proposition 2.1 from [14] was not applied in this part.

[8] Here, by tradition, $y^+ = y \vee 0$ and $y^- = (-y) \vee 0$.

Of course, this theorem still is not a theorem on existence of quasiequilibria. However the conditions under which fuzzy core is nonempty are well known in literature (e.g., see [14], [13]) so taking them and adding the requirement of nontrivial E-properness we obtain an existence theorem of nontrivial quasiequilibria. We would like to recall further important Riesz–Kantorovich formula on the presentation of supremum of linear functionals in linear vector lattice.

It is well known that the space $(L, \geq)^{\sim}$ of all order bounded linear functionals over $(L, \geq)$, being ordered as

$$f \geq g \iff f(x) \geq g(x), \quad \forall\, x \in L^+,$$

for $f, g \in (L, \geq)^{\sim}$, is itself (complete) Riesz space when L is a vector lattice,[9] and is called *order dual*. In doing so for every $x \geq 0$, $x \in L$ and any $f, g \in (L, \tau)^{\sim}$ the following *Riesz–Kantorovich* formula takes place:

$$(f \vee g)(x) = \sup\{f(y) + g(z) \mid y + z = x,\ y \geq 0,\ z \geq 0,\ y, z \in L\}.$$

Proof. Consider a functional $q = (q_1, \ldots, q_n) \in (L')^{\mathcal{I}}$, existing via Lemma 3.1. Due to (3.11), (3.12) this functional separates the set $V = \prod_{\mathcal{I}} V_x^i$ and

$$M_x = x + \{(t_1, \ldots, t_n) \in L^{\mathcal{I}} \mid \sum_{i \in \mathcal{I}} t_i = 0,\ x_i + t_i \in Z_x^i\}$$

on subspace $\mathcal{K} = L(u^v)^{\mathcal{I}}$. Since $M_x \subset \left[Z \cap [\llbracket 0, x - \omega \rrbracket + \mathcal{A}(L^{\mathcal{I}})]\right]$, (3.10) yields $M_x \cap V = \emptyset$. Moreover we have $\mathcal{K} \cap V \neq \emptyset$, where V is a convex, τ-open set. Therefore the functional $q_{|\mathcal{K}}$ can be extended onto whole space $L^{\mathcal{I}}$, saving separating property. Denote this extension $\pi = (\pi_1, \ldots, \pi_n)$ and consider $\bar{\pi} = \vee_{\mathcal{I}} \pi_i$. By construction $\pi_{|\mathcal{K}} = q_{|\mathcal{K}}$, that in view of Riesz-Kantorovich formula and by the properties of ideal implies $\bar{\pi}_{|_{L(u^v)}} = \bar{q}_{|_{L(u^v)}}$, where $\bar{q} = \vee_{\mathcal{I}} q_i$ is the functional from Lemma 3.1. Thus $\bar{\pi}(x_i) = \bar{\pi}(\omega_i)$, $\forall\, i \in \mathcal{I}$ and $\bar{\pi}(v_i) > 0$ for some i, where v_i is a properness vector by Definition 3.2. Further let us apply E-properness (3.7) and show

$$\langle \bar{\pi}, V_x^i \cap Z_x^i \rangle \geq \bar{\pi} x_i. \tag{3.15}$$

With this in mind we first prove

$$\pi_i(x_i - z_i) = \bar{\pi}(x_i - z_i), \quad \forall\, z_i \leq x_i,\ z_i \in Z_x^i. \tag{3.16}$$

To do it fix some i_0 and z_{i_0}. Since $x_{i_0} - z_{i_0} \geq 0$, then by Riesz-Kantorovich formula it is sufficient to check inequality

$$\sum_{\xi=1}^{n} \pi_\xi s_\xi \leq \pi_{i_0}(x_{i_0} - z_{i_0})$$

[9] This requirement to L be a lattice can be relaxed, but it is not important for our aims.

for an arbitrary $s \in L^n$, $s = (s_1, \dots, s_n) \geq 0$, $\sum s_\xi = x_{i_0} - z_{i_0}$. In fact, by definition of F-properness we have

$$y = (x_1 + s_1, \dots, z_{i_0} + s_{i_0}, \dots, x_n + s_n) \in Z.$$

By the choice of s and due to $x \in \mathcal{A}(\mathcal{X})$ we also have

$$\sum_{\mathcal{I}} y_i = \sum_{i \neq i_0} x_i + z_{i_0} + \sum_{\xi=1}^{n} s_\xi = \sum_{\mathcal{I}} \omega_i.$$

Thus the element $y \in M_x$, to the set, which the functional π separates from V. Therefore, since $x \in \overline{V}$, then $\langle \pi, y \rangle \leq \langle \pi, x \rangle$, that can be written in the form

$$\sum_{\mathcal{I}} \pi_i y_i = \sum_{i \neq i_0} \pi_i x_i + \pi_{i_0} z_{i_0} + \sum_{\xi=1}^{n} \pi_\xi s_\xi \leq \sum_{i \neq i_0} \pi_i x_i + \pi_{i_0} x_{i_0}.$$

Omitting common members, we obtain the inequality that proves (3.16).

Further let us state (3.15). Let $y_i \in V_x^i \cap Z_x^i$. Define $z_i = y_i \wedge x_i$. By lattice properties we obtain $z_i \in Z_x^i$. Now from $y_i - z_i \geq 0$, $\bar{\pi} \geq \pi_i$, $\pi_i y_i \geq \pi_i x_i$ and due to (3.16) conclude

$$\bar{\pi}(y_i - x_i) + \bar{\pi}(x_i - z_i) = \bar{\pi}(y_i - z_i) \geq \pi_i(y_i - z_i) \geq \pi_i(x_i - z_i) = \bar{\pi}(x_i - z_i) \ \Rightarrow$$

$$\bar{\pi}(y_i - x_i) \geq 0, \ \forall\, y_i \in V_x^i \cap Z_x^i,$$

that was required to prove. Now since $\bar{\pi}$ is continuous via (3.7) we conclude $\langle \bar{\pi}, \mathcal{P}_i(x) \rangle \geq \bar{\pi} x_i$.

Finally, the non-triviality of found quasiequilibrium follows from Definition 3.2. In fact, take i_0, satisfying $\bar{\pi} v_{i_0} > 0$. In view of (3.8) and $x \in \mathcal{A}(\mathcal{X})$ there are $x_i' \in X_i$, $i \in \mathcal{I}$ such that

$$\sum x_i - v_{i_0} = \sum x_i' \implies \sum \bar{\pi}(x_i - x_i') = \bar{\pi} v_{i_0} > 0.$$

Hence $\bar{\pi}(x_i - x_i') > 0 \Rightarrow \bar{\pi} x_i > \bar{\pi} x_i'$ for some $i \in \mathcal{I}$, $x_i' \in X_i$, and we hit the target. Theorem 3.1 has proven. □

4 Conclusion remarks

In a conclusion to this paper we would like to compare more detailed our main result with close to our Tourky's result from [27]. Tourky imposes the following preferences properness assumption, calling it M-properness, because of the similarity with properness assumed by Mas-Colell [20] for *production sets*. Below we use notations close to applied above, they differ with original Tourky's ones.

A preference relation $\mathcal{P}_i : X_i \Rightarrow X_i$ is said to be M-proper at a point $x_i \in X_i$, if there exists a convex set $V_i \subset E$ and a convex lattice $Z_i \subset L$, such that $Z_i + L^+ \subset Z_i$,[10] satisfying $x_i \in \overline{V}_i$ and

(i) $V_i \cap Z_i = \mathcal{P}_i(x)$;

(ii) $x_i + \omega' \in \text{int}V_i$;

(iii) $x_i,\ 0,\ \omega_i \in Z_i$;

(iv) $(1 + \alpha_i)x_i \in Z_i$ for some real $\alpha_i > 0$.

Assuming in addition preferences are monotonic: $x_i + L^+ \subset \mathcal{P}_i(x_i) \cup \{x_i\}$, $\forall\, x_i \in X_i$, $\forall\, i \in \mathcal{I}$, and also $\omega' > 0$, Tourky proves that every element of fuzzy core is a quasiequilibrium relative to a continuous prices $\bar{\pi} \in L'$ such that $\bar{\pi}\omega' > 0$.

The analysis of definitions shows that if a linear vector lattice economy is M-proper at x in Tourky's sense then this economy is also E-proper at x by Definition 3.1. Moreover, in our context assumptions (ii), $\omega_i, 0 \in Z_i$ and $\omega' > 0$ may be applied only to show that obtained quasiequilibrium is nontrivial, and for us (iv) is simply redundant condition[11]. However the crucial point is the requirements of quasiequilibrium to be nontrivial, here our conditions are different with Tourky's ones. In fact, we use the nontriviality in the form: there exists $i \in \mathcal{I}$, such that $\inf\langle \bar{\pi}, X_i \rangle < \bar{\pi}x_i$. It easy to see that in general case condition $\bar{\pi}\omega' > 0$, applied by Tourky, *does not implies* the nontriviality in our sense. Moreover, Tourky's nontriviality is not sufficient to characterize quasiequilibrium since, even when economy is irreducible (or under another kind of survival assumption), this nontriviality *does not imply* the existence of *equilibrium* with continuous prices — the main goal in this part of economic theory, because there is no property $\inf\langle \bar{\pi}, X_i \rangle < \bar{\pi}x_i \ \forall\, i \in \mathcal{I}$, exactly which allows us to conclude that every quasiequilibrium is an equilibrium in fact.

In Definition 3.2 we require (3.8) and that a set from (3.9) to be radial. There is no a direct analog of these conditions in Tourky's paper, however as a whole Tourky assumptions are stronger then ours. In fact, as it was mentioned in Section 3, Tourky's assumptions guarantee the set (3.9) is radial at ω' in $L(u)$. They are assumption $\omega' > 0$, requirements $\omega_i, 0 \in Z_x^i$, $\forall\, i \in \mathcal{I}$ and the fact that $\omega' = \sum \omega_i$ is chosen as a common properness vector (see also Florenzano-Marakulin [14]). At the same time condition (3.8) is additional with respect to Tourky's result, but we applied it *only* to prove, that quasiequilibrium is nontrivial in our sense (see the end of main theorem proof). Moreover, for the case when ω' is common properness

[10]Notice, that this implies the convexity of Z_i without special assuming.

[11]Tourky uses it to state $\bar{\pi}_i x_i = \bar{\pi}\omega_i$, that is applied to prove the "equilibrium properties" of $(x, \bar{\pi})$.

vector, in our conditions and without (3.8) it was proved, in fact, that price functional $\bar{\pi}$ satisfies $\bar{\pi}\omega' > 0$ (see Lemma 3.1 when $v_i = \omega'$, $\forall\, i$). However, we would like to repeat that it is not sufficient to a quasiequilibrium be really nontrivial. In our opinion Tourky's result for exchange economy has to be supplemented by, for example, requirement $\lambda\omega' \in \sum_{\mathcal{I}} X_i$ for some real $\lambda < 1$. However this is exactly our case when condition (3.8) is true.

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