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REVIEW ARTICLE

Contracts and domination in incomplete markets: what is a true core?

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Abstract The goal of the paper is to propose and study a concept of domination by coalitions for incomplete markets. Previously elaborated in the context of contractual approach, now it is presented in more or less standard terms and style. This concept is described as a set of allocations implemented by the net trades (webs of contracts) that are characterized by a special kind of stability in future markets: (i) for every state of the world inducted allocation has to be Pareto optimal and individually rational and (ii) there is no coalition which is able to dominate the allocation via financially feasible trades in future spot markets using real assets and relative to prices specified by (i) (partial Pareto prices). This core converts into a classical one when the market turns complete. Under perfect competition conditions core allocations are GEI-equilibria. These properties prove the validity of suggested core.

Keywords Incomplete markets · Core · Net trades · GEI-equilibrium

JEL Classification C62 · C71 · D51 · D52

1 Introduction

In the world of real economy, individuals are forced to make decisions under uncertainty arising from incomplete information and the objective ambiguity of future

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events. As a result, in a modern economy one can observe not only ordinary commodity markets, but also a rich array of markets of specific financial tools, so-called assets. These markets are directly aimed at solving problems deeply related to the uncertainty of the future; they are: insurance business, the markets of futures contracts, trade with options, etc. This subject has received an attention in the early 1980s when an extension of Arrow–Debreu–McKenzie model (see Mas-Colell et al. 1995) known as incomplete market (e.g., see Geanakoplos 1990; Magill and Shafer 1991) was introduced in theoretical economics; it formalizes and studies the trade with assets. The term incomplete refers to the fact that the potentially infinite set of possible realizations of the future is surely wider than those created by people, "insurance variants" of financial assets. Thus the incomplete market theory models economic environments in which economic agents live and function under the constraints related to the possible differences in times and ambiguity.

However, the modern version of incomplete market theory has one essential gap—there is no satisfactory concept of domination by coalitions (of allocations) and consequently an appropriate core notion is lacking. In fact, in a classical setting, the competitive equilibrium concept, having descriptive power, is also supported by the fact that there is no group of agents (coalition) that has incentives to form an autonomous subeconomy (it is said that the equilibrium belongs to the core—i.e., this is not dominated by any coalition). Moreover, in the conditions of perfect competition, every allocation from the core allows price decentralization, i.e., it is an equilibrium relative to some prices—this is Edgeworth's well known conjecture. So in the ideal world of an Arrow–Debreu economy, competitive equilibrium, primarily defined in a purely descriptive way, obtains the normative foundation as an ideally stable (in a given sense) allocation. This is why it seems natural to raise the question about the core definition in an incomplete market environment and to clarify its relations with the financial equilibrium. Related literature on this subject can be found in Grossman (1977), Radner (1982), Koutsougeras (1998).

The goal of this paper is to suggest and to investigate a core concept in incomplete (financial) markets—a difficult quest for economic theory, which still has not found a satisfactory solution. In the author's strong opinion, a "correct" core concept has to embody the main properties of classical markets and has to satisfy the following two requirements:

- Let the economy be described as an incomplete market but in fact be complete, i.e., it is mathematically equivalent to a standard pure exchange model, in which equilibria correspond to financial equilibria. Formally, this means that the rank of the matrix of value returns from assets is equal to the number of future events. Then the classical concept of the core and a new concept can be applied simultaneously. In such a case, the set of allocations for the core of an incomplete market should coincide with the set of standard core allocations.
- Under perfect competition conditions, the core and equilibria have to coincide—for a standard exchange economy this is the coincidence of Edgeworth's equilibria (the allocations that belong to the core of each replicated economy) with competitive equilibria.

So I take these properties as the main criterion for a correct definition of core and coalition domination in financial markets.

Initially the notion of core for incomplete markets was elaborated in the context of contractual approach—it is a new theory which I constructed and investigated during the last 15 years. Moreover, the problem of core for incomplete market stimulated me and was a starting point in this study. This contractual approach presents a new and simplest model of perfect competition. It seemingly goes back to Edgeworth approach; an interested reader can find modern studies in Marakulin (2003, 2011, 2013). In contractual approach equilibria can be described in pure game-theoretical terms without addressing to any kind of value parameters. The mathematical nature of this phenomenon is similar to the coincidence of equilibrium allocations with fuzzy core elements (or Edgeworth's equilibria), which is a possible way to model perfect competition conditions and not only for usual commodities but also at least for public goods. Now contractual approach is rather developed and voluminous theory that is not easy to understand for unfamiliar reader. That is why in this paper I am presenting a simplified version of incomplete market core based on the possibility to consider, for every future state and every Pareto optimal for this state allocation, supporting prices. As a result core allocations are described as having two kinds of properties: (i) for every state of the world inducted allocation has to be Pareto optimal and individually rational and (ii) there is no coalition which is able to dominate the allocation via financially feasible trades in future spot markets using real assets exchanges and relative to prices specified by (i) (partial Pareto prices). It is proved that this core converts into a classical one when the market turns complete. Under perfect competition conditions core allocations are GEI-equilibria. These properties prove the validity of suggested core.

The paper is organized as follows. Section 2 briefly presents an exchange economy, describes the model of incomplete market and specifies GEI-equilibrium. Some preliminary results including an analog of First and Second Welfare Theorem for incomplete market are formed as Sect. 3. Thereupon formal definition of core, its properties and relations with equilibrium allocations and discussion of obtained results form Sect. 4. An analysis of perfect competition and the proof of asymptotic theorem on coincidence of core and GEI-equilibria is presented in Sect. 5. Finally, introduced core is illustrated via Hart's example and these results are presented in Sect. 6. Conclusion finishes the main part of the paper. The most difficult and long proofs are presented in the Appendix.

2 Equilibrium and core in incomplete markets

2.1 Incomplete market model

Let us consider a typical exchange economy of Arrow–Debreu type. For this economy let $E = \mathbb{R}^l$ denote the commodity space (l is a number of commodities) and $\mathcal{I} = \{1, \ldots, n\}$ is the set of agents. A consumer $i \in \mathcal{I}$ is characterized by a consumption set $X_i \subset E$, an initial endowment $\mathbf{e}_i \in X_i$, and a preference relation described by a point-to-set mapping $\mathcal{P}_i : X_i \Rightarrow X_i$ where $\mathcal{P}_i(x_i)$ denotes the set of all consumption bundles which are strictly preferred to the bundle x_i; notation $y_i \succ_i x_i$ is equivalent to $y_i \in \mathcal{P}_i(x_i)$. Let us denote by $\mathbf{e} = (\mathbf{e}_i)_{i \in \mathcal{I}}$ the vector of initial endowments of all traders of the economy. Denote $X = \prod_{i \in \mathcal{I}} X_i$ and let

$$\mathcal{A}(X) = \left\{ x \in X \mid \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i \right\}$$

be the set of all feasible allocations.

In the general framework of a pure exchange economy let us consider a model with two periods $t = 0, 1$, in which there are l kinds of physically different (potentially) commodities available either today (with certainty) or tomorrow (contingent on each of a finite number s of possible future states of nature). So for this (market) economy, the total space of commodities E is associated with the space $\mathbb{R}^{l(s+1)}$. For convenience, we denote by $\sigma = 0$ the state of nature today. At each state $\sigma = 0, 1, \ldots, s$, there is a spot market for each of the l commodities, whose price-vector is $p_\sigma \in \mathbb{R}^l$; at time 0, there exists also a financial market for k assets that deliver a random return across the states at $t = 1$. Let q_j be the price for j-s asset and $q = (q_1, q_2, \ldots, q_k)$. Let $\pi = (p, q) \in \mathbb{R}^{l(s+1)} \times \mathbb{R}^k$ denote the total price-vector.

We are studying a particular case of real assets—the vectors: $a^1, a^2, \ldots, a^k \in \mathbb{R}^{ls}$, which present commodity returns in all future states of nature associated with buying a unit of jth asset; as vector-columns they form the $(sl \times k)$-matrix $\mathrm{A} = [a^j]_{j=1}^{j=k}$, i.e.,

$$\mathrm{A} = \begin{bmatrix} a_1^1 & a_1^2 & \ldots & a_1^k \\ \vdots & \vdots & \ddots & \vdots \\ a_s^1 & a_s^2 & \ldots & a_s^k \end{bmatrix}$$

which defines the matrix of financial returns across the future states of the world by formula

$$\mathcal{A}(p, q) = (p_\sigma \cdot a_\sigma^j)_{\sigma=1,\ldots,s \cdot j=1,\ldots,k}.$$

Now, if we denote

$$\lambda_j(x, \pi) = (-q_j, a_j(x, \pi)), \quad \Lambda = \begin{pmatrix} -q \\ \mathcal{A}(\pi) \end{pmatrix},$$

then the total transfer of wealth across different states of the world, which some agent can obtain from the market of assets with respect to his/her portfolio $z = (z^1, \ldots, z^k)$ (trade program for assets), is described by the vector

$$\Lambda \cdot z = z^1 \begin{bmatrix} -q_1 \\ a_1(\pi) \end{bmatrix} + \cdots + z^k \begin{bmatrix} -q_k \\ a_k(\pi) \end{bmatrix}.$$

In an incomplete market setting, consumer i is ordinary described by a consumption set $X_i \subset E$ and a preference correspondence $\mathcal{P}_i : X_i \rightrightarrows X_i$. The agents are also equipped with the vectors of individualized consumers' initial endowments $\mathbf{e}_i \in X_i$, $i \in \mathcal{I}$ for which I put $\bar{\mathbf{e}} = \sum_{i \in \mathcal{I}} \mathbf{e}_i$. Now the agents' profit functions are specified as the value of individualized vectors of initial endowments $\mathbf{e}_i^\sigma$, as $p_\sigma \mathbf{e}_i^\sigma$, relative to prices $p_\sigma, \sigma = 1, \dots, s$.

Note that in this case the ith consumer budget constraints have the vector-inequality form

$$\mathrm{P}x_i \le \mathrm{P}\mathbf{e}_i + \begin{pmatrix} -\mathrm{q} \\ \mathrm{P}_1 \mathrm{A} \end{pmatrix} z_i, \quad \mathbf{e}_i = (\mathbf{e}_i^0, \dots, \mathbf{e}_i^s), \quad x_i \in X_i, \quad z_i \in \mathbb{R}^k, \tag{1}$$

where the matrixes

$$\mathrm{P} = \begin{bmatrix} p_0 & & 0 \\ & \ddots & \\ 0 & & p_s \end{bmatrix} = \begin{bmatrix} p_0 & \dots 0 \\ \vdots & \mathrm{P}_1 \\ 0 & \end{bmatrix}, \quad \mathrm{P}_1 = \begin{bmatrix} p_1 & & 0 \\ & \ddots & \\ 0 & & p_s \end{bmatrix}$$

define the consumption cost operators for the present and future events ($t = 1$). Here P_1 is the submatrix of P, which is formed by the rows $\sigma = 1, \dots, s$ and omitting the first l zero columns. Clearly in this case, we have $\mathcal{A}(p, q) = \mathrm{P}_1\mathrm{A}$. Note that "squared product" (the standard notation $p \Box x$ denoting the vector $(p_\sigma \cdot x_\sigma)_{\sigma=0}^{\sigma=s}$, $x \in \mathbb{R}^{l(s+1)}$), commonly used in incomplete market theory, coincides with the ordinary matrix–vector product $\mathrm{P}x$, $x \in \mathbb{R}^{l(s+1)}$ applied above.

As a result this model under study may be written in the following short form:

$$\mathcal{E}^{\mathrm{in}} = \langle \mathcal{I}, E, (X_i, \mathcal{P}_i, \mathbf{e}_i)_{i \in \mathcal{I}}, A \rangle.$$

Now let us recall the concept of equilibrium. So, taking as given a market system of commodity and asset prices, the budget set of ith consumer is

$$B_i(p, q) = \{x_i \in X_i \mid \exists z_i \in \mathbb{R}^k : \; p \Box x_i \le p \Box \mathbf{e}_i + \Lambda(p, q) z_i\}.$$

Definition 1 A couple of actions $(\bar{x}_i, \bar{z}_i)_{i \in \mathcal{I}} \in X \times \mathbb{R}^{kn}$ and admissible prices $(\bar{p}, \bar{q}) \in \mathbb{R}^{l(s+1)} \times \mathbb{R}^k$ is said to be GEI-equilibrium if it obeys

(i) for each $i \in \mathcal{I}$: $\bar{p} \Box \bar{x}_i = \bar{p} \Box \mathbf{e}_i + \Lambda(\bar{p}, \bar{q}) \bar{z}_i$ and $\mathcal{P}_i(\bar{x}_i) \bigcap B_i(\bar{p}, \bar{q},) = \emptyset$,
(ii) $\sum_{i \in \mathcal{I}} \bar{x}_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i$ and $\sum_{i \in \mathcal{I}} \bar{z}_i = 0$.

Classically, (i) means that each $(\bar{x}_i, \bar{z}_i)$ is an optimal feasible budget plan for agent i, given $(\bar{p}, \bar{q})$; condition (ii) is a couple of market clearing conditions.

The reader can find more on incomplete market theory in Geanakoplos (1990), Magill and Shafer (1991), Magill and Quinzii (2002).

2.2 Assumptions

First let us consider assumptions and recall some notions. Let us assume that economy $\mathcal{E}^{\mathrm{in}}$ is convex, i.e.,

- the sets X_i are convex, closed, $\mathbf{e}_i \in X_i$ and for every $x_i \in X_i$ there exists an open convex $G_i \subset E$ such that $\mathcal{P}_i(x_i) = G_i \cap X_i$ for all $i \in \mathcal{I}$. In addition, we will postulate that
- Every X_i is "rectangular" over the states of the world, i.e., $X_i = \prod_{\sigma=0}^{s} X_i^\sigma$ and $\text{int}X_i^\sigma \neq \emptyset$, $\sigma = 0, \ldots, s$.

An incomplete market economy is called locally non-satiated if the following assumption is fulfilled.

(S) *Preferences are locally non-satiated in each spot market,i.e., for every σ and each $i \in \mathcal{I}$*

$$x_i^\sigma \in \overline{\mathcal{P}_i(x_i^\sigma, x_i^{-\sigma}) \cap X_i^\sigma}, \quad \forall x_i = (x_i^\sigma, x_i^{-\sigma}) \in X_i,$$

holds, where $x_i^{-\sigma} = (x_i^0, \ldots, x_i^{\sigma-1}, x_i^{\sigma+1}, \ldots, x_i^s)$ is a fragment-vector of x_i, complementing x_i^σ to x_i.

In addition, we sometimes will assume the (strong) monotonicity of preferences.

(M) *For some $i \in \mathcal{I}$ and every $x \in \mathcal{A}(X)$*

$$(\{x_i\} + E^+) \setminus \{x_i\} \subset \mathcal{P}_i(x_i).$$

For the convenience of below presentation we introduce the following term. The economy is called smooth if for every $i \in \mathcal{I}$,

$$\mathcal{P}_i(x_i) = \{y \in X_i \mid u_i(y) > u_i(x_i)\}, \quad \forall x_i \in X_i$$

for some differentiable function u_i defined on an open neighborhood of X_i, such that $\text{grad}_{|_{x_i^\sigma}} u_i(x_i) \neq 0$, $\forall x_i \in \mathcal{A}(X)$, $\sigma \geq 0$. Here $\text{grad}_{|_{x_i^\sigma}} u_i(x_i)$ denotes the subvector of the gradient of utility function, calculated at the point x_i, and corresponding to the state σ.

3 Preliminary results and GEI-equilibrium characterization

Let us introduce notations and consider some auxiliary notions which will be applied in our analysis.

Denote the total vector-price in future markets by

$$p^{\mathbf{1}} = (p_\sigma)_{\sigma=1}^{\sigma=s}, \quad p_\sigma \in \mathbb{R}^l, \quad \sigma \geq 1.$$

Define

$$H = H(p^{\mathbf{1}}) = \{x \in \mathbb{R}^{nl(s+1)} \mid \exists z \in \mathbb{R}^{nk} \ : \ \sum_{i \in \mathcal{I}} z_i = 0 \quad \& \\ p_\sigma x_i^\sigma - p_\sigma \mathbf{e}_i^\sigma = p_\sigma A_\sigma z_i, \quad \forall \sigma = 1, \ldots, s, \quad \forall i \in \mathcal{I}\}.$$

By construction for equilibrium allocation $\bar{x} = (\bar{x}_i)_{\mathcal{I}} \in X$ we have $\bar{x} \in H$. Now put

$$\mathcal{H}_i = \mathcal{H}_i(p^{\mathbf{1}}) = \{x_i \in \mathbb{R}^{l(s+1)} \mid \exists z_i \in \mathbb{R}^k \ : \ p_\sigma x_i^\sigma - p_\sigma \mathbf{e}_i^\sigma = p_\sigma A_\sigma z_i, \ \sigma = 1, \ldots, s\};$$

this is (in fact) the projection of subspace H onto a subspace corresponding to agent i's consumption bundles. Clearly, for all $i \in \mathcal{I}$

$$\mathcal{H}_i = \mathcal{H} + \{\mathbf{e}_i\}, \quad \mathcal{H} = \{y \in \mathbb{R}^{l(s+1)} \mid \exists z \in \mathbb{R}^k \ : \ p_\sigma y = p_\sigma A_\sigma z, \ \forall \sigma \geq 1\}. \quad (2)$$

The useful properties of equilibrium allocations in an incomplete market are stated in the following lemma and will be applied in below analysis. This lemma can be considered as an analog of First Welfare Theorem for incomplete market.

Lemma 1 *Let* $\left(\bar{x}_i, \bar{z}_i\right)_{i \in \mathcal{I}} \in X \times \mathbb{R}^{kn}$ *and admissible prices* $\left(\bar{p}, \bar{q}\right)\right) \in \mathbb{R}^{l(s+1)} \times \mathbb{R}^k$ *be a GEI-equilibrium couple of locally non-satiated incomplete market* $\mathcal{E}^{\text{in}}$*. Let* $\bar{p} = (\bar{p}^0, \bar{p}^{\mathbf{1}})$ *where* $\bar{p}^{\mathbf{1}} = (\bar{p}_\sigma)_{\sigma=1}^{\sigma=s}$*. Then allocation* $\bar{x} = (\bar{x}_i)_{i \in \mathcal{I}}$ *obeys*

(i) $\bar{x} \in H = H(\bar{p}^{\mathbf{1}})$*,*
(ii) $\bar{x}$ *is not Pareto-dominated via an allocation from* $H(\bar{p}^{\mathbf{1}}) \cap \mathcal{A}(X)$*,*
(iii) $\bar{x}$ *is not Pareto-dominated via a feasible allocation from the space* $E^\sigma_{\bar{x}} = \{y = (y_i)_\mathcal{I} \in E^\mathcal{I} \mid y_i^{-\sigma} = \bar{x}_i^{-\sigma}, \ \forall i \in \mathcal{I}\}, \ \forall \sigma \geq 0.$

Proof of Lemma 1 Item (i) is due to Definition 1 (i), first part. To prove (ii) assume contrary and let $y = (y_i)_\mathcal{I} \in H(\bar{p}^{\mathbf{1}}) \cap \mathcal{A}(X)$ dominates $\bar{x}$, i.e., $y_i \succ_i \bar{x}_i \ \forall i \in \mathcal{I}$. Applying second part of Definition 1 (i) and budget set specification one concludes $y_i \notin B_i(\bar{p}, \bar{q})$. Since $y \in H(\bar{p}^{\mathbf{1}})$ then there are exist $z_i \in \mathbb{R}^k$, $i \in \mathcal{I}$, $\sum_\mathcal{I} z_i = 0$ such that $\bar{p}^\sigma y_i^\sigma = \bar{p}^\sigma \mathbf{e}_i^\sigma + \bar{p}^\sigma A_\sigma z_i$. Now the former one implies $\bar{p}^0 y_i^0 > \bar{p}^0 \mathbf{e}_i^0 - \bar{q} z_i$. Summing these inequalities one obtains $\bar{p}^0 \sum_\mathcal{I} y_i^0 > \bar{p}^0 \sum_\mathcal{I} \mathbf{e}_i^0$ that contradicts the choice of $y \in \mathcal{A}(X)$. Item (iii) is stated similarly. □

The items (ii) and (iii) of this lemma and the above considerations induce the following terminology.

An allocation $x \in \mathcal{A}(X)$ is called σ*-Pareto optimal*, $\sigma = 0, \ldots, s$, if it is not Pareto dominated via an allocation $y \in \mathcal{A}(X)$ from the space

$$E^\sigma_{\bar{x}} = \{y = (y_i)_\mathcal{I} \in E^\mathcal{I} \mid y_i^{-\sigma} = \bar{x}_i^{-\sigma}, \quad \forall i \in \mathcal{I}\}.$$

An allocation, which is σ-Pareto optimal for every $\sigma \geq 0$, is called partially Pareto optimal.

Let $x = (x_\sigma)_{\sigma=0}^{\sigma=s} \in \mathcal{A}(X)$ be a σ-Pareto optimal allocation. The nonzero vector (functional) $p_\sigma \in \mathbb{R}^l$ is called σ-Pareto prices if

$$p_\sigma y_i^\sigma \geq p_\sigma \bar{x}_i^\sigma, \quad \forall (y_i^\sigma, \bar{x}_i^{-\sigma}) \in \mathcal{P}_i(\bar{x}_i), \quad \forall i \in \mathcal{I} \quad (3)$$

and there is $i \in \mathcal{I}$ and y_i^σ such that

$$p_\sigma y_i^\sigma > p_\sigma \bar{x}_i^\sigma, \quad (y_i^\sigma, \bar{x}_i^{-\sigma}) \in \mathcal{P}_i(\bar{x}_i).$$

Notice that for smooth preferences and if $x \in \text{int}\, X$, relation (3) is equivalent to the existence of $\gamma_i^\sigma > 0$, satisfying

$$\text{grad}_{|_{x_i^\sigma}} u_i(x_i) = \gamma_i^\sigma p_\sigma, \quad \forall i \in \mathcal{I}. \quad (4)$$

A collection of vectors $(p_\sigma)_{\sigma=0}^{\sigma=s}$, $p_\sigma \in \mathbb{R}^l$ is called (partial) Pareto prices if (3) is true for all $\sigma = 0, \dots, s$.

An allocation from $H \cap \mathcal{A}(X)$ is called Pareto H-optimal if it cannot be Pareto-dominated via a feasible allocation from $H = H(p^{\mathbf{1}})$.[1] Using (2) one can see that an allocation is Pareto H-optimal if and only if it cannot be Pareto-dominated via an allocation $y \in \mathcal{A}(X)$, for which $y - \mathbf{e} \in \mathcal{H}^{\mathcal{I}}$, and this is the specific form of second best optimality.

Remark 1 Introduced terminology does not repeat or change terms known in literature. For example "social Nash optimum" from Grossman (1977) or "weakly constraint efficient" (allocation) from Magill and Shafer (1991): this notion looks almost the same as "partial Pareto optimum" but according to SNO for domination at date $t = 0$ barter exchanges via assets can be applied, and it is not so for PPO. □

The next lemma, states the key properties of H-optimal allocations. This lemma and its Corollaries 1, 2 can be viewed as an analog of Second Welfare Theorem for incomplete markets; the proof of Lemma 2 and Corollary 1 is presented in Appendix.

Lemma 2 *Let $\mathcal{E}^{\text{in}}$ be an incomplete market, $\bar{x} \in \text{int}\, X$ and $p^{\mathbf{1}} = (p_\sigma)_{\sigma=1}^{\sigma=s}$ be prices for future states of the world. Let $i_0 \in \mathcal{I}$ be an arbitrarily chosen and fixed agent. Then $\bar{x} = (\bar{x}_i)_{\mathcal{I}} \in H(p^{\mathbf{1}}) \cap \mathcal{A}(X)$ is Pareto $H(p^{\mathbf{1}})$-optimal if and only if the following property is true.*

There exists $\bar{p} = (\bar{p}_0, \bar{p}_1, \dots, \bar{p}_s)$ such that $\bar{p}_\sigma \neq 0$ for all $\sigma = 0, \dots, s$ and

$$\bar{p} y_i > \bar{p} \bar{x}_i \;\; \forall y_i \in \mathcal{P}_i(\bar{x}_i) \;\mid\; \exists z_i \in \mathbb{R}^k \;:\; p_\sigma (y_i^\sigma - \mathbf{e}_i^\sigma) = p_\sigma A_\sigma z_i, \quad \forall \sigma \geq 1 \quad (5)$$

is true for all $i \in \mathcal{I}$, $i \neq i_0$. For i_0, a stronger property is true:

$$\bar{p} y_{i_0} > \bar{p} \bar{x}_{i_0}, \quad \forall y_{i_0} \in \mathcal{P}_{i_0}(\bar{x}_{i_0}). \quad (6)$$

Note that requirement $\bar{x} \in \text{int}X$ in the statement of the lemma is applied only to obtain the strict inequality in relations (5), (6) and to avoid additional cumbersome considerations (irreducibility or similar ones). The next corollary continues our analysis and gives us a convenient reformulation of Lemma 2 for a smooth case.

Corollary 1 *In Lemma 2 conditions, let us assume that $\mathcal{E}^{\text{in}}$ is a smooth market and let $u_i(.)$ be a utility function for $i \in \mathcal{I}$. Then for an allocation $\bar{x} \in \mathcal{A}(X) \cap \text{int}X$ to be Pareto $H(p^{\mathbf{1}})$-optimal, the following property is necessary and sufficient. Let*

$$\bar{p} = \text{grad}\, u_{i_0}(\bar{x}_{i_0})$$

for some i_0. Then for all $i \neq i_0$ and each $\sigma \geq 1$, there exist real $\alpha_i > 0$ and some real λ_i^σ, such that

$$\text{grad}_{|_{x_i^0}} u_i(\bar{x}_i) = \alpha_i \bar{p}_0 \;\;\&\;\; \text{grad}_{|_{x_i^\sigma}} u_i(\bar{x}_i) = \alpha_i \bar{p}_\sigma + \lambda_i^\sigma p_\sigma, \quad \forall \sigma \geq 1$$

hold and, moreover, $\sum_{\sigma=1}^{\sigma=s} \lambda_i^\sigma p_\sigma A_\sigma = 0$ is fulfilled.

[1] Notice that now $p^{\mathbf{1}}$ may not be partially Pareto prices.

Now let us consider a case of a smooth economy when allocation $\bar{x} \in \text{int}X$ is also Pareto optimal in each future market and (nonzero) spot prices satisfy $p_\sigma = \gamma_i^\sigma \text{grad}_{|_{x_i^\sigma}} u_i(\bar{x}_i)$, $\gamma_i^\sigma > 0$, i.e., prices are (uniquely) derived from necessary optimal conditions (hence they are Pareto prices). Now applying Lemma 2 and its Corollary 1 one can immediately conclude

Corollary 2 *Let $\mathcal{E}^{\text{in}}$ be a smooth incomplete market and $\bar{x} \in \text{int}X$. Suppose $\bar{x}$ is partially Pareto optimal and let $p^{\mathbf{1}} = (p_\sigma)_{\sigma=1}^{\sigma=s}$ be a bundle of σ-Pareto prices (i.e., (3) is true for $\sigma \geq 1$). Presume also that $\bar{x} = (\bar{x}_i)_{\mathcal{I}} \in H(p^{\mathbf{1}})$ and is Pareto $H(p^{\mathbf{1}})$-optimal. Then there exists a vector $\bar{p} = (\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_s)$ such that $\bar{p}_\sigma = \beta_\sigma p_\sigma$ for some $\beta_\sigma > 0$ and all $\sigma \geq 1$, so that for $\bar{q} = \sum_{\sigma=1}^{\sigma=s} \bar{p}_\sigma A_\sigma$ and every $i \in \mathcal{I}$*

$$\bar{p}_0 y_i^0 - \bar{p}_0 \mathbf{e}_i^0 + \bar{q} z_i > \bar{p}_0 \bar{x}_i^0 - \bar{p}_0 \mathbf{e}_i^0 + \bar{q} \bar{z}_i, \quad \forall y_i \in \mathcal{P}_i(\bar{x}_i) \tag{7}$$

is true for all $z_i, \bar{z}_i \in \mathbb{R}^k$, which satisfy

$$\bar{p}_\sigma (y_i^\sigma - \mathbf{e}_i^\sigma) = \bar{p}_\sigma A_\sigma z_i \;\;\&\;\; \bar{p}_\sigma (\bar{x}_i^\sigma - \mathbf{e}_i^\sigma) = \bar{p}_\sigma A_\sigma \bar{z}_i, \quad \sigma = 1, \ldots, s.$$

Proof of Corollary 2 To verify this corollary, first note that since $(p_\sigma)_{\sigma=1}^{\sigma=s}$ is a bundle of σ-Pareto prices, in the corollary conditions (4) is true. Now let us take vector $\bar{p} = \text{grad}\, u_{i_0}(x_{i_0})$ for $i_0 \in \mathcal{I}$ from the statement of Lemma 2 and via (4) put $\beta_\sigma = \gamma_{i_0}^\sigma > 0$. It is easy to see that we have $\mathcal{H}_i(p^{\mathbf{1}}) = \mathcal{H}_i(\bar{p}^{\mathbf{1}})$, i.e., in the right-hand side of (5) one can equivalently change vector p_σ by $\bar{p}_\sigma$ for all $\sigma \geq 1$. Now rewrite the inequality from the left-hand side of (5) in the form $\bar{p} y_i - \bar{p} \mathbf{e}_i > \bar{p} \bar{x}_i - \bar{p} \mathbf{e}_i$ and substitute the following representations:

$$\sum_{\sigma=1}^{s} (\bar{p}_\sigma y_i^\sigma - \bar{p}_\sigma \mathbf{e}_i^\sigma) = \sum_{\sigma=1}^{s} \bar{p}_\sigma A_\sigma z_i = \bar{q} z_i \;\;\&\;\; \sum_{\sigma=1}^{s} (\bar{p}_\sigma \bar{x}_i^\sigma - \bar{p}_\sigma \mathbf{e}_i^\sigma) = \sum_{\sigma=1}^{s} \bar{p}_\sigma A_\sigma \bar{z}_i = \bar{q} \bar{z}_i.$$

This proves the result. □

Now let us pass to the analysis of the core concept for incomplete markets.

4 Incomplete market core and its analysis

Consider $p^{\mathbf{1}} = (p_\sigma)_{\sigma=1}^{\sigma=s}$, $p_\sigma \in \mathbb{R}^l, \sigma = 1, \ldots, s$ as a fixed vector of prices in the spot markets of the future states of the world.

Definition 2 An allocation $x = (x_1, \ldots, x_n) \in \mathcal{A}(X)$ is called $p^{\mathbf{1}}$- feasible if there are portfolios $z = (z_1, \ldots, z_n)$, $z_i \in \mathbb{R}^k$, $\sum_{i \in \mathcal{I}} z_i = 0$ such that equalities

$$p_\sigma x_i^\sigma = p_\sigma \mathbf{e}_i^\sigma + p_\sigma A^\sigma z_i, \quad \forall i \in \mathcal{I}, \quad \forall \sigma = 1, \ldots, s$$

hold.

Analogously one can define the notion of a $p^{\mathbf{1}}$-feasible allocation for an arbitrary (nonempty) coalition $S \subset \mathcal{I}$, substituting in Definition 2 the set $\mathcal{I}$ by means of S.

Let $\mathcal{A}_{p^{\mathbf{1}}}(S)$ (or $\mathcal{A}_p(S)$) denote the set of all $p^{\mathbf{1}}$-feasible via coalition S allocations. Note that the set $\mathcal{A}_{p^{\mathbf{1}}}(S) \neq \emptyset$ for every $S \subset \mathcal{I}$ since the vector of initial endowments $\mathbf{e}^S = (\mathbf{e}_i)_{i \in S}$ always belongs to $\mathcal{A}_{p^{\mathbf{1}}}(S)$. Moreover, it has to be clear from the above definitions that $\mathcal{A}_{p^{\mathbf{1}}}(\mathcal{I}) = H(p^{\mathbf{1}}) \cap \mathcal{A}(X)$.

The following notion plays an important auxiliary role in our analysis below.

Definition 3 p-core is the set $C_{\mathbf{p}}(\mathcal{E}^{\text{in}})$ of all $p^{\mathbf{1}}$-feasible allocations which cannot be dominated via coalitions, i.e.,

$$x \in C_{\mathbf{p}}(\mathcal{E}^{\text{in}}) \iff x \in \mathcal{A}_p(\mathcal{I}) \ \& \ \nexists S \subset \mathcal{I} : \exists\, y \in \mathcal{A}_p(S) \mid y_i \succ_i x_i \ \forall i \in S.$$

Let agents' preferences be defined via utility functions (which are presumed to be concave and continuous) and let a price-vector $p^{\mathbf{1}}$ for future markets be fixed. Then for an incomplete market one can put into correspondence some cooperative game with non-transferable utility (to be short, the NTU-game). Recall that formally NTU-game (e.g., see Moulin 1988, for details) is a couple $(\mathcal{I}, (V(S))_{S \subseteq \mathcal{I}})$, described by the set of players $\mathcal{I} = \{1, \ldots, n\}$ and the sets of permissible vector-payoffs $V(S) \subseteq \mathbb{R}^S$ for every (nonempty) coalition $S \subset \mathcal{I}$. In our case the set of all permissible vector-payoffs for coalition S is determined by formula

$$V_p(S) = \bigcup_{x \in \mathcal{A}_p(S)} V_p^x(S),$$

where

$$V_p^x(S) = \{(v_i)_{i \in S} \leq (u_i(x_i))_{i \in S} \mid (x_i)_{i \in S} \in \mathcal{A}_p(S)\}.$$

The sets $V_p(S)$ satisfy all additional necessary conditions that can be easily checked.

The famous Scarf's theorem states that the core of a balanced game $(\mathcal{I}, (V(S))_{S \subseteq \mathcal{I}})$ is nonempty. Applying this theorem and using standard arguments, one can prove the following.

Proposition 1 *Let the set of all individually rational feasible allocations of $\mathcal{E}^{\text{in}}$ be bounded and let agents' preferences be defined via concave continuous utility functions. Then $C_{\mathbf{p}}(\mathcal{E}^{\text{in}}) \neq \emptyset$.*

In this paper I omit the proof of this proposition.

Now I pass to introduction of the main concept of the paper, the notion of incomplete market core $\mathcal{C}(\mathcal{E}^{\text{in}})$.

Definition 4 An allocation $x \in \mathcal{C}(\mathcal{E}^{\text{in}})$ if and only if

(i) x is a partially Pareto optimal allocation and
(ii) $x \in C_{\mathbf{p}}(\mathcal{E}^{\text{in}})$ for a bundle of σ-Pareto prices $p^{\mathbf{1}} = (p_\sigma)_{\sigma=1}^{\sigma=s}$ that corresponds to item (i).

The first important fact is that similarly to standard exchange economy, for incomplete market equilibria always belong to the core: it immediately follows from GEI-equilibrium definition and incomplete market core specification.

Theorem 1 *Every GEI-equilibrium belongs to incomplete market core.*

Proof of Theorem 1. Let $\left(x_i, z_i\right)_{i\in\mathcal{I}} \in X \times \mathbb{R}^{kn}$ and prices $\left(p, q\right) \in \mathbb{R}^{l(s+1)} \times \mathbb{R}^k$ be GEI-equilibrium. Now Lemma 1 (iii) implies that $p = (p_\sigma)_{\sigma=0}^{\sigma=s}$ are partial Pareto prices. Let us show that $x \in C_{\mathbf{p}}(\mathcal{E}^{\text{in}})$ for these prices. Assume contrary and find $S \subset \mathcal{I}$ and $y = (y_i)_{i\in S} \in \mathcal{A}_p(S)$ such that $y_i \succ_i x_i, i \in S$. Due to $(y_i)_{i\in S} \in \mathcal{A}_p(S)$ we have: $\exists z_S = (z'_i)_{i\in S}, z'_i \in \mathbb{R}^k, \sum_{i\in S} z'_i = 0$ such that equalities

$$p_\sigma y_i^\sigma = p_\sigma \mathbf{e}_i^\sigma + p_\sigma A^\sigma z'_i, \quad \forall i \in S, \quad \forall \sigma = 1, \ldots, s$$

hold. Now the fact that $y_i \in B_i(p, q)$ for some $i \in S$ contradicts Definition 1 (i) that due to budget set specification implies $p_0 y_i^0 > p_0 \mathbf{e}_i^0 - q z'_i, \forall i \in S$. Summing the inequalities we obtain $p_0 \sum_S y_i^0 > p_0 \sum_S \mathbf{e}_i^0$, that contradicts $\sum_S y_i^0 = \sum_S \mathbf{e}_i^0$. □

Below the main properties of $\mathcal{C}(\mathcal{E}^{\text{in}})$ are investigated. It is shown in particular that the set $\mathcal{C}(\mathcal{E}^{\text{in}})$ fits with the ordinary notion of core as soon as the market becomes complete. Recall that an allocation is called individually rational if $\mathbf{e}_i \not\succ_i x_i, \forall i \in \mathcal{I}$.

Let us call a market complete relative to prices $p^{\mathbf{1}} = (p_\sigma)_{\sigma=1}^{\sigma=s}$ if the rank of matrix $\mathrm{P_l A}$ is equal to s, the total number of possible future states of the world.

A market is complete if for every individually rational σ-Pareto optimal allocation market is complete relative to every bundle of its σ-Pareto prices $p_1, \ldots, p_s \in \mathbb{R}^l$.

Clearly, this completeness is a kind of restriction for matrix A, more exactly for financial markets of assets, the number of which under this hypothesis has to be not less then s.[2] For monotonic preferences satisfying (**M**) one can consider a stronger assumption: the rank of matrix $\mathrm{P_l A}$ is equal to s uniformly relative to $p^{\mathbf{1}} \gg 0$. An example of an incomplete market, satisfying this completeness assumption, is the market of numeraire assets $a_\sigma^j = r_\sigma^j e_\sigma,\ r_\sigma^j \in \mathbb{R}\ j = 1, \ldots, k$ for $e_\sigma > 0$,[3] $\sigma \geq 1$, in which the matrix $\mathrm{R} = (r_\sigma^j)_{\sigma=1,\ldots,s \cdot j=1,\ldots,k}$ has rank equal to s.

The incomplete market core introduced above can be described in familiar terms when the model is compete. This important fact is stated below applying Definition 4 and the following.

Proposition 2 *Let $\mathcal{E}^{\text{in}}$ be a complete model of incomplete market economy. Then*

$$\mathcal{C}(\mathcal{E}^{\text{in}}) = \bigcup_{p^{\mathbf{1}}} \{C_{\mathbf{p}}(\mathcal{E}^{\text{in}}) \mid p = (p^0, p^{\mathbf{1}}) \text{ is partial Pareto for some } x \in \mathcal{A}(X)\}.$$

[2] Of course this condition cannot be sufficient.

[3] This is a consumption bundle $e_\sigma \in \mathbb{R}^l$ chosen as a unit of "numeraire" for assets and for future spot market σ; $e_\sigma > 0 \iff e_\sigma \geq 0 \;\&\; e_\sigma \neq 0$.

For the strongest form of assumption on market completeness, specified for monotonic preferences one can postulate

$$\mathcal{C}(\mathcal{E}^{\text{in}}) = \bigcup_{p^{\mathbf{1}} \gg 0} C_{\mathbf{p}}(\mathcal{E}^{\text{in}}).$$

Proof of Proposition 2 Let us prove $\subseteq$. Due to Definition 4 (i) every $x \in \mathcal{C}(\mathcal{E}^{\text{in}})$ is partially Pareto optimal feasible allocation. Therefore there exist σ-Pareto prices $\bar{p}^{\mathbf{1}} = (\bar{p}_\sigma)_{\sigma=1}^{\sigma=s}$. Now applying (ii) of Definition 4, we obtain

$$x \in C_{\bar{\boldsymbol{p}}}(\mathcal{E}^{\text{in}}) \subset \bigcup_{p^{\mathbf{1}}} \{C_{\boldsymbol{p}}(\mathcal{E}^{\text{in}}) \mid p = (p^0, p^{\mathbf{1}}) \text{ is partial Pareto for some } x \in \mathcal{A}(X)\}.$$

Let us prove the inverse inclusion for complete model. Chose any $x \in C_{\hat{\boldsymbol{p}}}(\mathcal{E}^{\text{in}})$ for fixed σ-Pareto prices $\hat{p}$. Now due to the completeness of $\mathcal{E}^{\text{in}}$ one can conclude that under these prices the matrix $\mathrm{P_1 A}$ has full rank and, therefore, the system of linear equations

$$\mathrm{P_1} \hat{x}_i^{\mathbf{1}} = \mathrm{P_1} \mathbf{e}_i^{\mathbf{1}} + \mathrm{P_1 A} \hat{z}_i \tag{8}$$

has a solution relative to $\hat{z}_i$ and arbitrary chosen other parameters. Note that one can think that these solutions satisfy $\sum_{i \in \mathcal{I}} \hat{z}_i = 0$ when $(\hat{x}_i)_{i \in \mathcal{I}}$ is feasible; in (8) instead of $\mathrm{P_1 A}$, one can take any square non-singular submatrix whose dimension is $s \times s$. Therefore every feasible $(\hat{x}_i)_{\mathcal{I}}$ is $p^{\mathbf{1}}$-feasible for a partially Pareto prices $p^{\mathbf{1}}$. So the condition $x \in C_{\hat{\boldsymbol{p}}}(\mathcal{E}^{\text{in}})$ implies that x is individually rational and Pareto optimal, which entails its partial Pareto optimality. From this, applying Second Welfare Theorem one can conclude the existence of σ-Pareto prices $\bar{p}^{\mathbf{1}} = (\bar{p}_\sigma)_{\sigma=1}^{\sigma=s}$. Now we have to show only that $x \in C_{\bar{\boldsymbol{p}}}(\mathcal{E}^{\text{in}})$. Let $y \in \mathcal{A}_{\bar{p}}(S)$ for $S \subset \mathcal{I}$. Once again, using the completeness of the market, we can conclude that system (8) may be solved with respect to $\hat{z}_i$ for all $i \in S$ if one substitutes y_i for $\hat{x}_i$ and $\bar{p}^{\mathbf{1}}$ by $\hat{p}^{\mathbf{1}}$. Thus we obtain $\mathcal{A}_{\bar{p}}(S) \subseteq \mathcal{A}_{\hat{p}}(S)$ and, using $x \in C_{\hat{\boldsymbol{p}}}(\mathcal{E}^{\text{in}})$, may conclude that coalition S cannot dominate allocation x under prices $\bar{p}$. □

The characterization of an incomplete market core for complete exchange economies gives the following

Theorem 2 *If $\mathcal{E}^{\text{in}}$ is complete, then $\mathcal{C}(\mathcal{E}^{\text{in}}) = \mathcal{C}(\mathcal{E})$.*

Proof of Theorem 2 It immediately follows from the fact that for every partial Pareto prices $p = (p_\sigma)_{\sigma=0}^{\sigma=s}$ the set $C_{\mathbf{p}}(\mathcal{E}^{\text{in}})$ coincides with the standard core of exchange economy. It is so because every allocation is $p^{\mathbf{1}}$-feasible: the system (8) has solution relative to $\hat{z}_i = (\hat{z}_i^1, \ldots, \hat{z}_i^k)$ for every possible $\hat{x}_i \in X_i$ and therefore $\mathcal{A}_p(S) = \mathcal{A}(S)$ $\forall S \subseteq \mathcal{I}$, $S \neq \emptyset$. □

5 Incomplete market core and perfect competition

In this section, we are going to study introduced incomplete market core under perfect competition conditions. From the different ways to model perfect competition we chose the method expressed in the form of a replicated model that corresponds to the classical representation of perfect competition conditions. The proof is based on the reducing of the domination in replicas to the study of domination via fuzzy coalitions with the succeeding fuzzy core consideration. Of course the concept of fuzzy core has to be adopted to the case of incomplete markets in a proper way. In what follows, the mathematical problem is reduced to the separation theorem being applied to separate some convex set from zero (zero cannot belong the set due to the fuzzy core property). Certainly our analysis is essentially based on the characteristic Definition 4 and on the fact that rational numbers are dense in the set of all real ones.

An incomplete market replica of volume $r \in \mathbb{N}$ is called the economy $\mathcal{E}_r^{\text{in}}$, in which r exact copies of each consumer from initial model $\mathcal{E}^{\text{in}}$ is put into correspondence in $\mathcal{E}_r^{\text{in}}$. The agents from $\mathcal{E}_r^{\text{in}}$ are numbered by double indices (i, m), $i \in \mathcal{I}$, $m = 1, \dots, r$, and it is put $X_{im} = X_i$, $\mathbf{e}_{im} = \mathbf{e}_i$. Agents' preferences are defined and take values in X_{im} due to identification $\mathcal{P}_{im} = \mathcal{P}_i$. An assets structure for a replica exactly repeats the structure of the initial model. To an initial economy $\mathcal{E}^{\text{in}}$ allocation $x = (x_i)_{\mathcal{I}}$, we can put into correspondence the replicated economy allocation $x^r = (x^r_{im})$ by the rule $x_{im} = x_i$, $\forall i, m$.

Definition 5 An allocation x is called incomplete market Edgeworth equilibrium if $x^r \in \mathcal{C}(\mathcal{E}_r^{\text{in}})$ for every natural $r = 1, 2, \dots$ and $\mathcal{C}^e(\mathcal{E}^{\text{in}})$ denotes the set of all Edgeworth equilibria for the model $\mathcal{E}^{\text{in}}$.

Now let us consider the most characteristic properties of the Edgeworth equilibria. By Definition 4, the property $x \in \mathcal{C}^e(\mathcal{E}^{\text{in}})$ is equivalent to the facts that allocation x is partially Pareto optimal and for partial Pareto prices $p^{\mathbf{1}} = (p_\sigma)_{\sigma \geq 1}$ the allocation belongs to the p-core of $\mathcal{E}_r^{\text{in}}$ for every natural r. Now consider the last requirement in more detail. It is very important that a domination is admitted via any coalitions and via any inter-coalition allocation.

Presume that for some r a coalition $S \subseteq \mathcal{I} \times \{1, \dots, r\}$ dominates the allocation x^r. Let $\mathcal{I}(S) \subseteq \mathcal{I}$ be the set of all agent types non-trivially presented in the coalition S. Due to p-core specification, this domination means that for every $(i, m) \in S$ there is $y_{im} \in \mathcal{P}_i(x_i)$ such that for some $z_{im} \in \mathbb{R}^k$, $i \in \mathcal{I}(S)$

$$\mathrm{P}_1 y^{\mathbf{1}}_{im} = \mathrm{P}_1 \mathbf{e}^{\mathbf{1}}_i + \mathrm{P}_1 \mathrm{A} z_{im}, \quad \forall m \ : \ (i, m) \in S$$

holds and in addition

$$\sum_{(i,m)\in S} y_{im} = \sum_{(i,m)\in S} \mathbf{e}_{im}$$

takes place. Now if we "average out" the dominating consumption bundles and portfolios for each given type of agents, i.e., if we put

$$y_i = \left(\sum_{m|(i,m)\in S} y_{im} \right) / s_i \quad \& \quad z_i = \left(\sum_{m|(i,m)\in S} z_{im} \right) / s_i \quad \forall i \in \mathcal{I}(S),$$

where s_i is the number of elements (capacity) in the set $S^i = \{m \mid (i,m) \in S\}$ (we have $i \in \mathcal{I}(S) \iff S^i \neq \emptyset$), then former equalities yield

$$\mathrm{P}_1 y_i^{\mathbf{1}} = \mathrm{P}_1 \mathbf{e}_i^{\mathbf{1}} + \mathrm{P}_1 \mathrm{A} z_i \ \forall i \in \mathcal{I}(S) \quad \& \quad \sum_{\mathcal{I}(S)} s_i y_i = \sum_{\mathcal{I}(S)} s_i \mathbf{e}_i.$$

Since $\mathcal{P}_i(x_i)$ is a convex set, we also obtain $y_i \in \mathcal{P}_i(x_i)$ for all $i \in \mathcal{I}(S)$. Next define a vector $t = (t_1, \ldots, t_n)$ by putting

$$t_i = s_i/r, \ i \in \mathcal{I}(S) \ \& \ t_i = 0, \quad i \in \mathcal{I} \setminus \mathcal{I}(S).$$

Clearly, that in the previous equality natural numbers s_i can be equivalently substituted by rational t_i. Moreover, under imposed assumptions the described logical chain can be inverted, i.e., one can show the sufficiency of described properties for a partially Pareto optimal allocation to be dominated via a coalition in a replica. Resuming the described arguments we are going to a fuzzy core concept for incomplete markets.

Recall that any n-dimension vector $t = (t_1, \ldots, t_n) \neq 0, \ \ 0 \leq t_i \leq 1 \ \ \forall i \in \mathcal{I}$ is said to be a fuzzy coalition. Let $p^{\mathbf{1}} = (p_\sigma)_{\sigma \geq 1}$ be some fixed bundle of spot prices for future states of the world. Introduce now the notion of fuzzy p-domination.

A fuzzy coalition t is called p-dominating $p^{\mathbf{1}}$-feasible allocation $x \in \mathcal{A}_p(\mathcal{I})$, if there is $y^t \in \prod_{i\in\mathcal{I}} X_i$ such that

$$\sum_{i\in\mathcal{I}} t_i y_i^t = \sum_{i\in\mathcal{I}} t_i \mathbf{e}_i \tag{9}$$

and it is fulfilled

$$y_i^t \succ_i x_i \ \& \ \exists\, z_i \in \mathbb{R}^k : \ \mathrm{P}_1 y_i^{t\,\mathbf{1}} = \mathrm{P}_1 \mathbf{e}_i^{\mathbf{1}} + \mathrm{P}_1 \mathrm{A} z_i \ \forall i \in \mathrm{supp}(t) = \{i \in \mathcal{I} \mid t_i > 0\}. \tag{10}$$

Notice that if $\mathcal{E}^{\mathrm{in}}$ is a smooth economy and if allocation $x \in \mathrm{int}\, X$, then the fact $x \notin \mathcal{C}^e(\mathcal{E}^{\mathrm{in}})$ is equivalent to the ability of its p-domination via a fuzzy coalition with rational components relative to a partial Pareto prices, corresponding to this allocation.

Definition 6 The set $C_{\mathbf{p}}^f(\mathcal{E}^{\mathrm{in}})$ of all $p^{\mathbf{1}}$-feasible allocations $x \in \mathcal{A}_p(\mathcal{I})$, for which there is no p-dominating fuzzy coalition, is called a fuzzy p-core.

In accordance with this definition the concept of fuzzy p-core differs from ordinary requirements only in the right-hand side of (10), where the potential financial marketability of consumption bundles relative to the given prices is additionally required. If, moreover, one requires these prices to be partial Pareto, then one achieves the notion of incomplete market fuzzy core.

Definition 7 The fuzzy core is the set $\mathcal{C}^f(\mathcal{E}^{\text{in}})$ of all feasible allocations satisfying the following properties.

(i) x is partial Pareto optimal, i.e., for every $\sigma \geq 0$ it cannot be Pareto dominated via a feasible allocation from subspace $E_x^\sigma = \{y = (y_i)_\mathcal{I} \in E^\mathcal{I} \mid y_i^{-\sigma} = x_i^{-\sigma}, \forall i \in \mathcal{I}\}$,

(ii) $x \in \mathcal{A}_p(\mathcal{I})$, i.e., it is $p^\mathbf{1}$-feasible, where $p^\mathbf{1} = (p_\sigma)_{\sigma=1}^{\sigma=s}$ is a bundle of σ-Pareto prices, existing due to item (i),

(iii) $x \in C_\mathbf{p}^f(\mathcal{E}^{\text{in}})$, i.e., it belongs to the fuzzy p-core of an incomplete market.

The following lemma states the key properties of a fuzzy p-core.

Lemma 3 *Let $p^\mathbf{1} = (p_\sigma)_{\sigma \geq 1}$ be a bundle of spot prices for future states and let x be a $p^\mathbf{1}$-feasible allocation. Let $x \in C_\mathbf{p}^f(\mathcal{E}^{\text{in}})$ and $x_{i_0} \in \text{int}\, X_{i_0}$ for some i_0. Then there is a vector $\bar{p} = (\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_s)$, such that $\bar{p}_\sigma \neq 0$ for all $\sigma \geq 0$ and*

$$\bar{p} y_i \geq \bar{p}\mathbf{e}_i, \quad \forall y_i \in \mathcal{P}_i(\bar{x}_i) \mid \exists z_i \in \mathbb{R}^k \,:\, p_\sigma(y_i^\sigma - \mathbf{e}_i^\sigma) = p_\sigma A_\sigma z_i, \quad \forall \sigma \geq 1 \quad (11)$$

is true for all $i \in \mathcal{I}$. Moreover, stronger property

$$\bar{p} y_{i_0} > \bar{p}\mathbf{e}_{i_0}, \quad \forall y_{i_0} \in \mathcal{P}_{i_0}(\bar{x}_{i_0}) \quad (12)$$

is true for agent i_0.

It is useful to compare the statement of this lemma with the statement of Lemma 2. The first difference is that the inequalities in Lemma 3 are non-strict. The second one is that in the right-hand side of the inequalities, the value of initial endowments is applied. One can find similarities between these facts with classical market case, when the property of an allocation be Pareto optimal is compared with its quasiequilibrium properties. Also notice that via (**S**) and passing to limits in the left-hand side of inequalities (11), one can state $\bar{p}x_i \geq \bar{p}\mathbf{e}_i \;\; \forall i \in \mathcal{I}$, that due to $\sum_{i\in\mathcal{I}} x_i = \sum_{i\in\mathcal{I}} \mathbf{e}_i$ eventually yields

$$\bar{p}x_i = \bar{p}\mathbf{e}_i \;\; \forall i \in \mathcal{I}.$$

Finally, let $x \in \text{int}\, X$, the economy be smooth and prices $p^\mathbf{1} = (p_\sigma)_{\sigma \geq 1}$ be partially Pareto optimal. Then the first, non-strict inequalities from the left side of (11) are turned into strict ones. Therefore now the conditions of Lemma 2 and its Corollary 2 are true. This is why, due to similar arguments applied in Corollary 2 proof, one can state the following

Corollary 3 *Let $\mathcal{E}^{\text{in}}$ be a smooth incomplete market and $\bar{x} \in \text{int}X \cap \mathcal{C}^f(\mathcal{E}^{\text{in}})$. Then there is a vector $\bar{p} = (\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_s)$ such that $\bar{p}_\sigma \neq 0$ for all $\sigma \geq 0$, and for $\bar{q} = \sum_{\sigma=1}^{\sigma=s} \bar{p}_\sigma A_\sigma$ and each $i \in \mathcal{I}$*

$$\bar{p}_0 y_i^0 > \bar{p}_0\mathbf{e}_i^0 - \bar{q}z_i \; \forall y_i \in \mathcal{P}_i(\bar{x}_i) \mid \exists z_i \in \mathbb{R}^k \,:\, \bar{p}_\sigma y_i^\sigma = \bar{p}_\sigma\mathbf{e}_i^\sigma + \bar{p}_\sigma A_\sigma z_i \; \forall \sigma \geq 1 \quad (13)$$

is true.

Applying this corollary one can easily prove the asymptotic theorem, which asserts that for every smooth incomplete market Edgeworth equilibria coincide with GEI-equilibria. Let $W(\mathcal{E}^{\text{in}})$ denote the set of all GEI-equilibria of the model $\mathcal{E}^{\text{in}}$.

Theorem 3 *Let $\mathcal{E}^{\text{in}}$ be a smooth incomplete market. Then*

$$\mathcal{C}^f(\mathcal{E}^{\text{in}}) = \mathcal{C}^e(\mathcal{E}^{\text{in}}) \quad \& \quad \text{int}X \cap \mathcal{C}^e(\mathcal{E}^{\text{in}}) = W(\mathcal{E}^{\text{in}}) \cap \text{int}X.$$

Proof of Theorem 3. First let us show $\mathcal{C}^e(\mathcal{E}^{\text{in}}) = \mathcal{C}^f(\mathcal{E}^{\text{in}})$. To do it we need to state $\mathcal{C}^e(\mathcal{E}^{\text{in}}) \subseteq \mathcal{C}^f(\mathcal{E}^{\text{in}})$. Assuming contrary find an allocation $x \in \mathcal{C}^e(\mathcal{E}^{\text{in}})$, which is dominated via a fuzzy coalition $t \neq 0$. By definition this means the existence of $y^t \in \prod_{\mathcal{I}} X_i$, satisfying (9) and (10). Now show that the allocation x is dominated via a fuzzy coalition $g = (g_1, \ldots, g_n)$ with rational components g_i, $i \in \mathcal{I}$. With this in mind for $t_i > 0$, put

$$x_i' = (t_i/g_i)y_i^t + (1 - t_i/g_i)\mathbf{e}_i \implies g_i(x_i' - \mathbf{e}_i) = t_i(y_i^t - \mathbf{e}_i),$$

where rational g_i satisfies the condition $0 < t_i \leq g_i \leq 1$, and for $t_i = 0$ define $g_i = 0$ and $x_i' = y_i^t$. Since $\mathbf{e}_i \in X_i$, then $x' = (x_i')_{\mathcal{I}} \in \prod_{\mathcal{I}} X_i$ and

$$\sum_{i \in \mathcal{I}} g_i(x_i' - \mathbf{e}_i) = 0.$$

However, each $\mathcal{P}_i(x_i)$ is an open subset in X_i that implies the scalars g_i can be chosen in such a way that $x_i' \in \mathcal{P}_i(x_i)$ is true for all i, satisfying $g_i > 0$. Moreover, for these i

$$\exists z_i' \in \mathbb{R}^k \; : \; \text{P}_1 x_i'^{\mathbf{1}} = \text{P}_1 \mathbf{e}_i^{\mathbf{1}} + \text{P}_1 \text{A} z_i'$$

holds relative to σ-Pareto prices, corresponding to x, as soon as similar relations are true for y^t (put $z_i' = \frac{t_i}{g_i} z_i$). We obtain a contradiction with the choice of $x \in \mathcal{C}^e(\mathcal{E}^{\text{in}})$.

So, the coincidence of a fuzzy core with the set of Edgeworth equilibria is proved. Now we need to prove $\text{int}X \cap \mathcal{C}^f(\mathcal{E}^{\text{in}}) \subseteq W(\mathcal{E}^{\text{in}}) \cap \text{int}X$. To do it one can apply Corollary 3 and conclude that for each i the vector $\bar{x}_i$ is the maximal element of $\succ_i$ on the set $\mathcal{B}_i(\bar{p}, \bar{q})$ of all $x_i \in X_i$, satisfying the conditions

$$\exists\, z_i \in \mathbb{R}^k \; : \; \bar{p}_0 x_i^0 = \bar{p}\mathbf{e}_i^0 - \bar{q} z_i \;\; \& \;\; \bar{p}_\sigma x_i^\sigma = \bar{p}\mathbf{e}_i^\sigma + \bar{p}_\sigma A_\sigma z_i \;\; \forall \sigma \geq 1.$$

However, using (**S**), local-nonsatiation in each of the spot markets, and following along standard line of augmentation, one can state that if $\succ_i$ attains a maximal point

$$\exists\, z_i \in \mathbb{R}^k \; : \; \bar{p}_0 x_i^0 \leq \bar{p}\mathbf{e}_i^0 - \bar{q} z_i \;\; \& \;\; \bar{p}_\sigma x_i^\sigma \leq \bar{p}\mathbf{e}_i^\sigma + \bar{p}_\sigma A_\sigma z_i \;\; \forall \sigma \geq 1,$$

then this point undoubtedly has to belong to the set $\mathcal{B}_i(\bar{p}, \bar{q})$ (i.e., for this point all inequalities are realized in the form of an equality). So we state $\text{int}X \cap \mathcal{C}^f(\mathcal{E}^{\text{in}}) = W(\mathcal{E}^{\text{in}}) \cap \text{int}X$. Theorem 3 is proved. □

6 Contractual incomplete core in examples

In this section we are going to illustrate the obtained results and study an incomplete market example with two future states, two physical commodities, two real assets and Cobb–Douglas utilities. The particular case of this market is known in literature as Hart's example, in which GEI-equilibrium may not exist. Notice conditions quarantining existence of equilibrium Anderson and Raimondo (2007). First we analyze it in more or less general framework and then pass to Hart's example.

Example 1 Let us consider an economic model with two consumers, two states of the world in the future and no present.[4]

Let there be two commodities in states $\sigma = 1, 2$, and let $x = (x^{\sigma=1}, x^{\sigma=2})$ correspond to the consumption of the 1st agent, but $y = (y^{\sigma=1}, y^{\sigma=2})$ be the consumption program for the 2nd one. Let $X_i = \mathbb{R}^4_+$, $i = 1, 2$, a total vector of initial endowments $\mathbf{e} = (\mathbf{e}_i)_{i=1,2} \in \mathbb{R}^8_+$ satisfy $\mathbf{e}_i \gg 0$ $i = 1, 2$, and let utilities be described by functions

$$u_1(x) = \rho^1_{\sigma=1} U_1^{\sigma=1}(x^{\sigma=1}) + \rho^1_{\sigma=2} U_1^{\sigma=2}(x^{\sigma=2}),$$
$$u_2(y) = \rho^2_{\sigma=1} U_2^{\sigma=1}(y^{\sigma=1}) + \rho^2_{\sigma=2} U_2^{\sigma=2}(y^{\sigma=2}),$$

where for $i, \sigma = 1, 2$ real $\rho^i_\sigma > 0$, and U_i^σ are (logarithmed) Cobb–Douglas functions:

$$U_1^\sigma(z) = \alpha_\sigma \ln(z_1) + (1 - \alpha_\sigma)\ln(z_2), \quad 0 < \alpha_\sigma < 1,$$
$$U_2^\sigma(z) = \beta_\sigma \ln(z_1) + (1 - \beta_\sigma)\ln(z_2), \quad 0 < \beta_\sigma < 1.$$

Let us also consider as incorporated into the model a financial market with two assets having the following structure:

$$a^1 = (a^1_{\sigma=1}, a^1_{\sigma=2}), \quad a^1_{\sigma=1} = a^1_{\sigma=2} = (1, 0)$$
$$a^2 = (a^2_{\sigma=1}, a^2_{\sigma=2}), \quad a^2_{\sigma=1} = a^2_{\sigma=2} = (0, 1).$$

Now as a whole matrix A of real returns has the form

$$\mathrm{A} = \left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] \begin{array}{l} = \mathrm{A}_{\sigma=1} \\ \\ = \mathrm{A}_{\sigma=2} \end{array}$$

From this one can find the matrix of financial returns $\mathrm{P}_1\mathrm{A}$ for the trade portfolios of the financial sector relative to given prices $p^{\mathbf{1}}$ for spot markets:

[4] Note, that formally the present can always be added to the model description without the loss of important properties.

$$\mathrm{P_l A} = \left[\begin{array}{c|c} p_{\sigma=1}\begin{pmatrix}1\\0\end{pmatrix} & p_{\sigma=1}\begin{pmatrix}0\\1\end{pmatrix} \\ \hline p_{\sigma=2}\begin{pmatrix}1\\0\end{pmatrix} & p_{\sigma=2}\begin{pmatrix}0\\1\end{pmatrix}\end{array}\right] = \left[\begin{array}{c|c} p^1_{\sigma=1} & p^2_{\sigma=1} \\ \hline p^1_{\sigma=2} & p^2_{\sigma=2}\end{array}\right] = \left[\begin{array}{c} p_{\sigma=1} \\ \hline p_{\sigma=2}\end{array}\right].$$

The analysis of the core is based on Definition 4. In order to do it we need to find partial Pareto boundary which for separable utilities is presented as a Cart's product of boundaries defined for each state of the world. A standard analysis for 2×2 economy gives

$$x + y = \bar{\mathbf{e}} = (\bar{\mathbf{e}}^1, \bar{\mathbf{e}}^2) : \ \ x = \left(\frac{\alpha}{p_1}, \frac{1-\alpha}{p_2}\right) \ \ \& \ \ y = \lambda\left(\frac{\beta}{p_1}, \frac{1-\beta}{p_2}\right),$$

where

$$p = \left(\frac{\alpha}{\bar{\mathbf{e}}^1}, \frac{1-\alpha}{\bar{\mathbf{e}}^2}\right) + \lambda\left(\frac{\beta}{\bar{\mathbf{e}}^1}, \frac{1-\beta}{\bar{\mathbf{e}}^2}\right) \ \overset{\bar{\mathbf{e}}=(1,1)}{\Longleftrightarrow} \ p = (\alpha + \lambda\beta, 1 - \alpha + \lambda(1-\beta)) \tag{14}$$

are the current Pareto prices depending on a parameter $\lambda > 0$ which unambiguously define every Pareto optimal point. Here parameter $\alpha \in (0, 1)$ corresponds to the first agent utility data and $\beta \in (0, 1)$ to the second one.

Due to item (i) of Definition 4, a partial Pareto prices $p^{\mathbf{1}} = (p_\sigma)_{\sigma=1,2}$ may be put into correspondence to every core allocation (unambiguously); these prices were defined in (14). So for $\bar{\mathbf{e}}_{\sigma=1} = \bar{\mathbf{e}}_{\sigma=2} = (1, 1)$ (without loss of generality), we obtain

$$\mathrm{P_l A} = \left[\begin{array}{c|c} \alpha_1 + \lambda\beta_1 & 1 - \alpha_1 + \lambda(1-\beta_1) \\ \hline \alpha_2 + \gamma\beta_2 & 1 - \alpha_2 + \gamma(1-\beta_2)\end{array}\right],$$

where $\lambda > 0$ and $\gamma > 0$ are some real parameters, which unambiguously determine the partial Pareto boundary.

Due to item (ii) of Definition 4 in order to current partially Pareto optimal allocation to be an element of the core it is also necessary (and sufficient) that the allocation be $p^{\mathbf{1}}$-feasible and be an element of p-core for its partial Pareto prices $p^{\mathbf{1}}$.

The condition of $p^{\mathbf{1}}$-feasibility says that there is such vector $z = (z_1, z_2)$, that $\mathrm{P_l}x = \mathrm{P_l}\mathbf{e}_1 + \mathrm{P_l A}z$. This, for the chosen normalization of prices (it implies $p_{\sigma=1}x_{\sigma=1} = p_{\sigma=2}x_{\sigma=2} = 1$), is equivalent to the fact that the system of linear equations

$$\mathrm{P_l A}z = \begin{pmatrix}1\\1\end{pmatrix} - \mathrm{P_l}\mathbf{e}_1, \tag{15}$$

has a solution relative to z. Clearly this can be a constraint only when the matrix $\mathrm{P_l A}$ is degenerated one, and this is similar to one asset case. If it is so then we have $\det(\mathrm{P_l A}) = 0 \iff$

$$(\alpha_1 + \lambda\beta_1)(1 - \alpha_2 + \gamma(1-\beta_2)) = (\alpha_2 + \gamma\beta_2)(1 - \alpha_1 + \lambda(1-\beta_1)). \tag{16}$$

Furthermore, (15) can be solved only if the following system of linear equations is solvable: the system with the only unknown variable in the left-hand side of (15) multiplied by a column vector (the first for specificity) and the same right-hand side, i.e., if for some real z

$$1 = \langle p_{\sigma=1}, \mathbf{e}_1^{\sigma=1}\rangle + p^1_{\sigma=1} z, \quad 1 = \langle p_{\sigma=2}, \mathbf{e}_1^{\sigma=2}\rangle + p^1_{\sigma=2} z,$$

which is equivalent to

$$p^1_{\sigma=2} - p^1_{\sigma=1} = p^1_{\sigma=2}\langle p_{\sigma=1}, \mathbf{e}_1^{\sigma=1}\rangle - p^1_{\sigma=1}\langle p_{\sigma=2}, \mathbf{e}_1^{\sigma=2}\rangle.$$

Finally, due to the fact that the allocation has to belong to the p-core, it is necessary to require the following relations to be true:

(i) $u_1(x) \geq u_1(\mathbf{e}_1)$ & $u_2(y) \geq u_2(\mathbf{e}_2)$,
(ii) allocation (x, y) is Pareto $H(p^{\mathbf{1}})$-optimal for partial Pareto prices $p^{\mathbf{1}}$.

For the analysis of (ii) one can apply Corollary 1 which taking $i_0 = 1$ and $p = \text{grad}u_1(y)$, now states that an allocation is $H(p^{\mathbf{1}})$-optimal iff there is $\mu > 0$ such that (here $\bar{p} = p$ is a vector of partial Pareto prices) $\tilde{p} = \text{grad}u_2(y) - \mu p$ satisfies

$$\tilde{p}_{\sigma=1} a^1_{\sigma=1} + \tilde{p}_{\sigma=2} a^2_{\sigma=2} = 0 \implies \tilde{p}^1_{\sigma=1} + \tilde{p}^1_{\sigma=2} = 0.$$

Now from partial Pareto optimality we have $\text{grad}u_2(y) = (\lambda p_{\sigma=1}, \gamma p_{\sigma=2})$ for some $\lambda > 0$, $\gamma > 0$, that gives

$$\left(\frac{1}{\lambda} - \mu\right) p^1_{\sigma=1} + \left(\frac{1}{\gamma} - \mu\right) p^1_{\sigma=2} = 0 \iff \mu(p^1_{\sigma=2} + p^1_{\sigma=1}) = \frac{1}{\gamma} p^1_{\sigma=2} + \frac{1}{\lambda} p^1_{\sigma=1}.$$

Clearly, for positive prices and other parameters $\mu > 0$ that we need do exist. It means that (ii) is always true and only $p^{\mathbf{1}}$-feasibility and individual rationality (i) are essential ones.

Now let (16) be false, i.e., the matrix of financial returns for partial Pareto prices is non-degenerated. In this case an allocation belongs to the incomplete core only if it belongs to the classical core, i.e., requirements (i) and (ii) are true, where (ii) is transformed into ordinary Pareto optimality. Now according to the chosen partial Pareto prices normalization, we have

$$\text{grad}u_1(x) = (\rho^1_{\sigma=1} p_{\sigma=1}, \rho^1_{\sigma=2} p_{\sigma=2}) \ \& \ \text{grad}u_2(y) = \left(\tfrac{\rho^2_{\sigma=1}}{\lambda} p_{\sigma=1}, \tfrac{\rho^2_{\sigma=2}}{\gamma} p_{\sigma=2}\right)$$

that via the collinearity of vectors yields

$$\lambda \rho^1_{\sigma=1}/\rho^2_{\sigma=1} = \gamma \rho^1_{\sigma=2}/\rho^2_{\sigma=2} \iff \lambda = \gamma \frac{\rho^2_{\sigma=1}\rho^1_{\sigma=2}}{\rho^1_{\sigma=1}\rho^2_{\sigma=2}}. \tag{17}$$

Thus if parameters $\lambda > 0$ and $\gamma > 0$ satisfy (17) and simultaneously do not satisfy (16), then the generated individually rational allocation belongs to the incomplete core. □

Example 2 (Hart's example) Let us consider properly Hart's example, which corresponds to the economy with two assets from Example 1 under an additional condition:

$$\rho_\sigma^1 = \rho_\sigma^2 = \rho_\sigma, \ \sigma = 1, 2 \quad \& \quad \alpha_{\sigma=1} = \alpha_{\sigma=2} = \alpha, \ \beta_{\sigma=1} = \beta_{\sigma=2} = \beta.$$

Now one can note that the first part of this requirement and (17) imply $\lambda = \gamma$, i.e., an allocation is Pareto optimal iff $\lambda = \gamma$. Initial endowments for Hart's example are determined as

$$\mathbf{e}_1^{\sigma=1} = (1-\varepsilon, 1-\varepsilon), \ \ \mathbf{e}_1^{\sigma=2} = (\varepsilon, \varepsilon), \ \ \mathbf{e}_1^\sigma + \mathbf{e}_2^\sigma = (1, 1), \ \sigma = 1, 2,$$

where real $0 < \varepsilon < 1$.

Let us show that for Hart's example the set of allocations from the incomplete core, which corresponds to the non-degenerated matrix of financial returns, forms the empty set. In fact if $(x, y) \in \mathcal{C}(\mathcal{E}^{\text{in}})$ and $\det(\mathrm{P_I A}) \neq 0$, then (x, y) is Pareto optimal and $\lambda = \gamma$. However, then from $\alpha_\sigma = \alpha$, $\beta_\sigma = \beta$, $\sigma = 1, 2$ one can conclude the coincidence of matrix $\mathrm{P_I A}$ rows and therefore $\det(\mathrm{P_I A}) = 0$—contradiction.

Further consider the second possibility: $(x, y) \in \mathcal{C}(\mathcal{E}^{\text{in}})$ and $\det(\mathrm{P_I A}) = 0$. It may be realized only if system (15) is solvable. The latter one is equivalent to the solvability of

$$\begin{pmatrix} \alpha + \lambda\beta \\ \alpha + \gamma\beta \end{pmatrix} z = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \langle(\alpha + \lambda\beta, 1 - \alpha + \lambda(1-\beta)), \mathbf{e}_1^{\sigma=1}\rangle \\ \langle(\alpha + \gamma\beta, 1 - \alpha + \gamma(1-\beta)), \mathbf{e}_1^{\sigma=2}\rangle \end{pmatrix};$$

substituting for the value of initial endowments, and realizing some elementary transformations we find

$$\begin{pmatrix} \alpha + \lambda\beta \\ \alpha + \gamma\beta \end{pmatrix} z = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} (1-\varepsilon)(1+\lambda) \\ \varepsilon(1+\gamma) \end{pmatrix}. \tag{18}$$

Next, substituting $\alpha_\sigma = \alpha$, $\beta_\sigma = \beta$, $\sigma = 1, 2$ in (16) and doing transformations, we obtain

$$\det(\mathrm{P_I A}) = 0 \quad \Longleftrightarrow \quad (\gamma - \lambda)(\alpha - \beta) = 0.$$

Thus the matrix of financial returns is degenerated only either $\gamma = \lambda$ (as seen above) or $\alpha = \beta$. In the first case (18) may be fulfilled if and only if $\varepsilon = 1/2$. In the second case the further analysis shows that if $\alpha = \beta$ then core is a nonempty one-dimensional manifold, which points for $\varepsilon \neq 1/2$ are not Pareto optimal ones.

Let us resume our analysis. For Hart's example if $\alpha \neq \beta$ then the core of an incomplete market is a nonempty set only for $\varepsilon = 1/2$, and for this case the incomplete core coincides with the classical market core (since then the solvability of (18) is equivalent to an allocation be Pareto optimal). For $\varepsilon \neq 1/2$, the core is empty, which can be explained via the specific features of given model parameters: preferences and real assets. This peculiarity is such that contracting each other in every state of nature and applying real assets in the present, the agents are not able to arrive at Pareto optimal allocation, regardless of the fact that there are potentially enough assets (so much as there are many future states of the world). In other words any feasible net trades are unstable in the sense that coalition $\{1, 2\}$ of all market operators is able to find a new beneficial barter contract, taking into account the possibility to break some of the given contracts.

Notice that one can infinitesimally closely approach the Pareto optimal points via allocations which are partially Pareto optimal and $p^{\mathbf{1}}$-feasible relative to partial Pareto prices. The crucial point is that the p-feasibility of allocation may conflicts with its optimality, it is so in Hart's example, there every Pareto prices for complete case (non-degenerated matrix of assets value returns) produce degenerated matrix $\mathrm{P}_{\mathrm{I}} \mathrm{A}$. In this economy Pareto optimal allocations may be attained only in a limit and via a sequence of contracts which is an unbounded one (for the exchange of assets in view of the $p^{\mathbf{1}}$-feasibility condition). So, in terms of contracts, one may see that during the contracting and recontracting process, market operators may realize "a race to infinity," i.e., the total volume of contracts for one of the agents may rise with no limit. The problem can be solved if similarly to how it was analyzed in Marakulin (1999), one imposes constraints on the total volume of contracts from the asset market for each agent. Imposing sufficiently large constraints, one can realize allocations (or utilities) neighboring with classical core. □

7 Conclusion

The paper presents a domination via coalitions and subsequently a core concept for incomplete markets. I suggest that the notion of correct core has to satisfy the following (two) requirements:

- If an economy is described as an incomplete market, but is mathematically equivalent to a classical pure exchange model, then the core in the context of an incomplete market coincides with the classical core of a pure exchange economy.
- In conditions of perfect competition the core of an incomplete market coincides with the set of equilibrium allocations.

Exactly these two properties allow us to assert that the concept of core presents a generalization of the classical approach and preserves its most meaningful properties. According to introduced notion, core allocations are described as possessing two kinds of properties: (i) for every state of the world inducted allocation has to be Pareto optimal and (ii) there is no coalition which is able to dominate the allocation via financially feasible trades in future spot markets using real assets exchanges and relative to prices specified by (i) (partial Pareto prices).

The goal was achieved and the paper presents the following results:

(i) The analog of First and Second Welfare Theorems stated for incomplete market economy; see Lemmas 1, 2.
(ii) Theorem 1 states that every GEI-equilibrium belongs to incomplete market core.
(iii) If incomplete market turns to complete then incomplete market core coincides with classical core; see Theorem 2.
(iv) Under perfect competition core and equilibria coincide; see Theorem 3. This analysis follows the classical modeling tradition of perfect competition

conditions—being replicated, an allocation can be considered as an allocation of a replicated economy and must belong to the core of the replicated model. Then such an allocation may be decentralized.

(v) The core in the simplest examples ($2\times2\times2$—economy) including as a particular case Hart's example is analyzed. It is shown that core may be empty and in general presents nonclosed set.

More about the last point. Hart's example is known as an economy in which GEI-equilibrium may not exist for specific model parameters. It was clarified, that the core in the described sense may also not exist or may be presented as a nonclosed set. The cause of this are the specific properties of financial market. Namely, if the number of assets is limited, the situation may occur when market operators tend to raise contract volumes for assets with no limit. It seems to be a degenerate case, which may happen only under a specific relationship between preferences and assets (it is known that financial equilibria, which always are in the core, generically do exist). The example also shows that an incomplete market core may be a nonclosed set.

As a subject for a further investigations: the problem of core emptiness can be solved if one imposes some institutional constraints for the trade in financial markets. It is known that when some constrains of this kind are imposed, then equilibria exist under rather weak model assumptions. Moreover, relaxing constraints to infinity, one can also consider limit equilibrium allocations (e.g., see Marakulin 1999). These arguments may be incorporated to the core concept also, and in such a case one can pass to the consideration of approximating and marginal dominating variants.

Appendix: Proofs

Proof of Lemma 2 Let us write in matrix form the conditions, which define the allocations from $H = H(p^{\mathbf{1}})$. In fact for $x \in X$ being $x \in H \cap \mathcal{A}(X) \iff$ there are $z_i \in \mathbb{R}^k,\ i \in \mathcal{I}$ such that

$$\begin{cases} \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \mathbf{e}_i; \\ \sum_{i=1}^{n} z_i = 0; \\ p_\sigma x_i^\sigma - p_\sigma A_\sigma z_i = p_\sigma \mathbf{e}_i^\sigma,\ \sigma = 1, \ldots, s,\ i \in \mathcal{I} \end{cases}$$

holds. Notice that if balance relations and budget constraints $p_\sigma x_i^\sigma - p_\sigma A_\sigma z_i = p_\sigma \mathbf{e}_i^\sigma$ are satisfied for some fixed $\sigma \geq 1$ and all $i \in \mathcal{I} \setminus \{i_0\}$, then the last budget constraint is also true automatically. This is why all agent i_0's budget constraints, being linear dependent, may be removed from the system of linear equations defining the space H. One may think without loss of generality that $i_0 = n$. Denote by B the following $(l + sl + k + n - 1) \times n(l + sl + k)$ matrix

$$B = \begin{pmatrix}
E_l & 0 & \dots & 0 & . & E_l & 0 & \dots & 0 & E_l & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\
0 & E_l & . & 0 & \cdots & 0 & E_l & . & 0 & 0 & E_l & . & 0 & 0 & \dots & 0 & 0 \\
\vdots & . & . & . & . & \vdots & . & . & . & . & . & . & . & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & . & E_l & . & 0 & 0 & . & E_l & 0 & 0 & . & E_l & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & E_k & \dots & E_k & E_k \\
0 & p_1 & . & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -p_1 A_1 & . & 0 & 0 \\
\vdots & . & . & . & . & \vdots & \vdots & . & \vdots & \vdots & \vdots & . & \vdots & \vdots & . & \vdots & \vdots \\
0 & 0 & . & p_s & . & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -p_s A_s & . & 0 & 0 \\
\vdots & \vdots & . & . & . & \vdots & . & . & \vdots & \vdots & \vdots & . & \vdots & . & . & . & \vdots \\
0 & 0 & \dots & 0 & \dots & 0 & p_1 & . & 0 & 0 & 0 & \dots & 0 & 0 & . & -p_1 A_1 & 0 \\
\vdots & \vdots & . & \vdots & . & \vdots & . & . & . & \vdots & \vdots & . & \vdots & \vdots & . & \vdots & \vdots \\
0 & 0 & \dots & 0 & \dots & 0 & 0 & . & p_s & 0 & 0 & \dots & 0 & 0 & . & -p_s A_s & 0
\end{pmatrix}.$$

Here in the standard manner E_l and E_k denote the unit matrices of an appropriate size and p_1, p_s, $p_1 A_1$ and $p_s A_s$ are row-vectors. Clearly for $x \in X$ we have the following equivalence: $x \in H$, $\sum_{\mathcal{I}} x_i = \sum_{\mathcal{I}} \mathbf{e}_i \iff$ there exists $z = (z_1, \dots, z_n)$ such that

$$\mathrm{B} \quad * \quad \begin{pmatrix} x_1^0 \\ \vdots \\ x_1^s \\ \vdots \\ x_n^0 \\ \vdots \\ x_n^s \\ z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{e}}^0 \\ \vdots \\ \bar{\mathbf{e}}^s \\ \mathbf{0}_k \\ p_1 \mathbf{e}_1^1 \\ \vdots \\ p_s \mathbf{e}_1^s \\ \vdots \\ p_1 \mathbf{e}_{n-1}^1 \\ \vdots \\ p_s \mathbf{e}_{n-1}^s \end{pmatrix}.$$

Consider the subspace

$$H^Z = \{((x_1, z_1), ., (x_n, z_n)) \in (\mathbb{R}^{l(s+1)} \times \mathbb{R}^k)^{\mathcal{I}} \mid B(x_1, \dots, x_n, z_1, \dots, z_n)^t = 0\};$$

this is the kernel of operator $\mathrm{B}(\cdot)$, in which the order of components is changed for the convenience of the below considerations.

Now let an allocation $\bar{x} \in H$ and be Pareto-optimal relative to H. Therefore

$$\prod_{\mathcal{I}} \mathcal{P}_i(\bar{x}_i) \cap H(p^{\mathbf{1}}) = \emptyset \iff \prod_{\mathcal{I}} [(\mathcal{P}_i(\bar{x}_i) - \bar{x}_i) \times \mathbb{R}^k] \cap H^Z = \emptyset$$

takes place (note that via (**S**) each of these sets is nonempty). Now by the separation theorem we may find a linear functional $f = (f_1, \dots, f_n) \neq 0$, $f_i = (f_i^x, f_i^z)$, $f_i^x \in \mathbb{R}^{l(s+1)}$ and $f_i^z \in \mathbb{R}^k$, $i \in \mathcal{I}$ such that

$$\left\langle f, \prod_{\mathcal{I}}[(\mathcal{P}_i(\bar{x}_i) - \bar{x}_i) \times \mathbb{R}^k] \right\rangle \geq \langle f, H^Z \rangle$$

holds. Notice the functional f is constant (and hence is equal to zero) onto subspace H^Z, since the right-hand side of the last inequality is bounded. Therefore

$$\left\langle f, \prod_{\mathcal{I}}[(\mathcal{P}_i(\bar{x}_i) - \bar{x}_i) \times \mathbb{R}^k] \right\rangle \geq 0 \tag{19}$$

is true. Let us show further that $f_i^z = 0, \ i \in \mathcal{I}$, i.e.,

$$f_i = (f_i^0, \ldots, f_i^s, \underbrace{0, \ldots, 0}_{k}), \quad \forall i \in \mathcal{I}.$$

In fact, consider fixed $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$, $\hat{x}_i \in (\mathcal{P}_i(\bar{x}_i) - \bar{x}_i)$, $i \in \mathcal{I}$, $y_j \in \mathbb{R}^k$, $j \neq i_0$ for some $i_0 \in \mathcal{I}$. In view of (19), for any $y \in \mathbb{R}^k$ we have

$$\sum_{j \neq i_0} \langle f_j, (\hat{x}_j, y_j) \rangle + \langle f_{i_0}, (\hat{x}_{i_0}, y) \rangle \geq 0,$$

which is possible only if $f_{i_0}^z = 0$ and in view of the arbitrariness of i_0, for all $i_0 \in \mathcal{I}$.

For the convenience of the below notations I will identify the functional f_i with f_i^x, i.e., by convention let us put $f_i = (f_i^x, 0) = (f_i, 0)$. Let us show now that

$$\langle f_i, (\mathcal{P}_i(\bar{x}_i) - \bar{x}_i) \rangle \geq 0, \quad i \in \mathcal{I}, \tag{20}$$

and moreover, if $f_i \neq 0$ and $y_i \in \mathcal{P}_i(\bar{x}_i) \cap \operatorname{int} X_i$, then the inequality is strict: $\langle f_i, (y_i - \bar{x}_i) \rangle > 0$. Indeed due to the preferences are locally non-satiated, we have $\bar{x}_j \in \operatorname{cl}(\mathcal{P}_j(\bar{x}_j))$; now the substitution of $\bar{x}_j$ instead of $\mathcal{P}_j(\bar{x}_j)$ in (19) for all $j \neq i$ and due to $f_j^z = 0, \ j \in \mathcal{I}$ immediately gives us the result. So, (20) is true in the non-strict form of inequalities. Now assumption $y_i \in \mathcal{P}_i(\bar{x}_i) \cap \operatorname{int} X_i$ and the fact that each $\mathcal{P}_i(\bar{x})$ is an open in X_i standardly implies the strict inequalities for $f_i \neq 0$.

Further, the fact that functional $f = (f_1, \ldots, f_n)$ is constant onto subspace H^Z implies that this functional presenting vector can be represented as a linear combination of the vector-rows of matrix B. Now using the structure of matrix B, one can conclude the existence of such real λ_i^σ, $\sigma \geq 1$, $i \in \mathcal{I}$, $i \neq n$ and such vectors $q \in \mathbb{R}^k$, $\bar{p} = (\bar{p}_0, \ldots, \bar{p}_s) \in E'$, that for all $i \neq n$ the following system of linear equations is true:

$$\begin{cases} f_i^0 = \bar{p}_0, \\ f_i^\sigma = \bar{p}_\sigma + \lambda_i^\sigma p_\sigma, \qquad \sigma \geq 1, \\ f_i^z = 0 = -q + \sum\limits_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma A_\sigma, \end{cases} \tag{21}$$

and for $i = n$ we have $f_n^z = 0 = -q + 0$ and $f_n^x = \bar{p}$. Putting $\lambda_n^\sigma = 0$ for all $\sigma \geq 1$, one may think (21) is true for all $i \in \mathcal{I}$. Moreover, system (21) implies that

$q = \sum_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma A_\sigma = 0$ for all i. Note also that assumption $\bar{p}_0 = 0$ contradicts (**S**) (we have the local non-satiation in each spot market[5]). Therefore $f_i \neq 0$ and we have strict inequality in (20) for int $\mathcal{P}_i(\bar{x})$ and all i. Moreover, as soon as $f_n = \bar{p} \neq 0$, due to the similar arguments (from the local-nonsatiation) we conclude that $\bar{p}_\sigma \neq 0$ for all σ. It is also clear that due to (20) and (**S**), for $x_i \in \text{int}\, X_i$ we have $f_i^\sigma \neq 0$ for all i and σ.

Now let us show that $\bar{p}$ satisfies the other requirements of Lemma 2. First note that by subspaces $\mathcal{H}_i$ specification for every $x_i \in \mathcal{H}_i$ we have

$$p_\sigma(x_i^\sigma - \mathbf{e}_i^\sigma) = p_\sigma A_\sigma z_i, \;\; \sigma = 1, \ldots, s$$

for some $z_i \in \mathbb{R}^k$. Now multiplying equalities by λ_i^σ and then summing them by $\sigma = 1, \ldots, s$, one obtains

$$\sum_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma(x_i^\sigma - \mathbf{e}_i^\sigma) = \sum_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma A_\sigma z_i = \left(\sum_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma A_\sigma\right) z_i = q z_i. \tag{22}$$

Next let us recall that due to (20) and the above considerations we also have

$$\langle f_i, (\mathcal{P}_i(\bar{x}_i) - \bar{x}_i)\rangle > 0 \;\Rightarrow\; \langle f_i, (y_i - \bar{x}_i)\rangle > 0 \quad \forall y_i \in \mathcal{P}_i(\bar{x}) \cap \mathcal{H}_i \neq \emptyset \tag{23}$$

for all $i \in \mathcal{I}$. Now substituting the representation of f_i from (21), we obtain

$$\bar{p}_0(y_i^0 - \bar{x}_i^0) + \sum_{\sigma=1}^{s} \langle (\bar{p}_\sigma + \lambda_i^\sigma p_\sigma), (y_i^\sigma - \bar{x}_i^\sigma)\rangle > 0 \;\; \forall y_i \in \mathcal{P}_i(\bar{x}) \cap \mathcal{H}_i.$$

Subtracting from the left and right-hand sides of the inequality the value $\sum_{\sigma=1}^{s} \lambda_i^\sigma (p_\sigma \mathbf{e}_i^\sigma)$, after transformations we obtain

$$\bar{p}_0 y_i^0 + \sum_{\sigma=1}^{s} \bar{p}_\sigma y_i^\sigma + \sum_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma(y_i^\sigma - \mathbf{e}_i^\sigma) > \bar{p}_0 \bar{x}_i^0 + \sum_{\sigma=1}^{s} \bar{p}_\sigma \bar{x}_i^\sigma + \sum_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma(\bar{x}_i^\sigma - \mathbf{e}_i^\sigma).$$

Since $\bar{x}_i, y_i \in \mathcal{H}_i$, there are $\bar{z}_i = \bar{z}_i(\bar{x}_i)$, $z_i = z_i(y_i)$ such that relations (22) are true. Now due to the previous inequality we get

$$\langle \bar{p}, y_i\rangle + q z_i > \langle \bar{p}, \bar{x}_i\rangle + q \bar{z}_i.$$

However, from the last equation of (21) we have $q = f_n^z = 0$, which gives

$$\langle \bar{p}, y_i\rangle > \langle \bar{p}, \bar{x}_i\rangle \;\; \forall y_i \in \mathcal{P}_i(\bar{x}_i) \cap \mathcal{H}_i \iff \langle \bar{p}, ((\mathcal{P}_i(\bar{x}_i) \cap \mathcal{H}_i) - \bar{x}_i)\rangle > 0. \tag{24}$$

[5] Due to $\mathcal{H}_i$ specification, if $y_i \in \mathcal{H}_i$ then for $\bar{y}_i = (\bar{y}_i^\sigma)_{\sigma=0}^{\sigma=s}$, where $\bar{y}_i^\sigma = y_i^\sigma$ for $\sigma \geq 1$, we have $\bar{y}_i \in \mathcal{H}_i$ for every $\bar{y}_i^0$, and therefore $\mathcal{P}_i(\bar{x}) \cap \mathcal{H}_i \neq \emptyset$ for all i.

The sufficiency of relations (5) and (6) for an allocation $\bar{x} \in \mathcal{A}(X) \cap H(p^{\mathbf{1}})$ to be Pareto $H(p^{\mathbf{1}})$-optimal is stated quite standardly. In fact let $y = (y_i)_{\mathcal{I}} \in \mathcal{A}(X) \cap H(p^{\mathbf{1}})$ be such that $y_i \succ_i \bar{x}_i$ is true for all i. Then as soon as the right-hand side in (5) is fulfilled for $y = (y_i)_{\mathcal{I}} \in H(p^{\mathbf{1}})$ due to $H(p^{\mathbf{1}})$ determination, via the left-hand part of (5), we can conclude $\bar{p} y_i > \bar{p} \bar{x}_i$ for all i. Now, summing inequalities over i one finds $\bar{p} \sum_{i \in \mathcal{I}} y_i > \bar{p} \sum_{i \in \mathcal{I}} \bar{x}_i$. Since $\sum_{i \in \mathcal{I}} y_i = \sum_{i \in \mathcal{I}} \bar{x}_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i$, we are coming to a contradiction. Lemma 2 is proved. □

Proof of Corollary 1 We have to consider the smooth case in the context of Lemma 2. On the necessary side, for the existence of values $\alpha_i > 0$ and λ_i^σ $\forall \sigma \geq 1$, one can state it directly from relations (5) and (6), applying separation theorem (or via the necessary conditions of a convex programming problem). However, the easiest way to see it may be found from condition $\bar{x} \in \operatorname{int} X$ and relations (20), stated in the proof of Lemma 2. From this we conclude in a standard manner the existence of $\alpha_i > 0$ such that $\operatorname{grad} u_i(\bar{x}_i) = \alpha_i f_i$ ($\alpha_i \neq 0$ due to $\operatorname{grad} u_i(\bar{x}_i) \neq 0$). Finally, one needs to apply (21).

To state the sufficiency, let us show that relations (5) and (6) are true. For some i and $y_i \in \mathcal{P}_i(\bar{x}_i)$, assume $\exists z_i \in \mathbb{R}^k \; : \; p_\sigma(y_i^\sigma - \mathbf{e}_i^\sigma) = p_\sigma A_\sigma z_i$ $\forall \sigma \geq 1$. Due to gradient's properties for interior points we have

$$\langle \operatorname{grad} u_i(\bar{x}_i), y_i \rangle > \langle \operatorname{grad} u_i(\bar{x}_i), \bar{x}_i \rangle \quad \forall i \in \mathcal{I}.$$

Now substituting the gradient presentation given in the corollary conditions, one can conclude

$$\alpha_i \bar{p} y_i + \sum_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma y_i^\sigma > \alpha_i \bar{p} \bar{x}_i + \sum_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma \bar{x}_i^\sigma.$$

However, there are $z_i, \bar{z}_i \in \mathbb{R}^k$, such that $p_\sigma y_i^\sigma = p_\sigma \mathbf{e}_i^\sigma + p_\sigma A_\sigma z_i$ & $p_\sigma \bar{x}_i^\sigma = p_\sigma \mathbf{e}_i^\sigma + p_\sigma A_\sigma \bar{z}_i$ $\forall \sigma \geq 1$. Substituting these expressions under summation in the formula, one can find

$$\alpha_i \bar{p} y_i + \sum_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma \mathbf{e}_i^\sigma + \left(\sum_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma A_\sigma \right) z_i > \alpha_i \bar{p} \bar{x}_i + \sum_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma \mathbf{e}_i^\sigma + \left(\sum_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma A_\sigma \right) \bar{z}_i,$$

that due to $\sum_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma A_\sigma = 0$ and $\alpha_i > 0$ gives the result. □

Proof of Lemma 3 The proof of lemma is reduced to the application of the separation theorem to a convex set properly constructed; this set corresponds to the ability of fuzzy coalitions to dominate an allocation.

Analogously to formula (2), let us determine the subspaces

$$\mathcal{H}_i = \mathcal{H} + \mathbf{e}_i, \;\; \mathcal{H} = \{ y \in \mathbb{R}^{l(s+1)} \mid \exists z \in \mathbb{R}^k \; : \; p_\sigma y^\sigma = p_\sigma A_\sigma z, \; \forall \sigma \geq 1 \}.$$

Next let us take any consumer $i_0 \in \mathcal{I}$, let $i_0 = 1$, and determine the following set

$$G = G(x) = \mathrm{co}\left[(\mathcal{P}_1(x_1) - \{\mathbf{e}_1\}) \bigcup \left(\bigcup_{i=2}^{n} [(\mathcal{P}_i(x_i) - \{\mathbf{e}_i\}) \cap \mathcal{H}] \right) \right].$$

Now we show that if $0 \in G$, then there is a fuzzy coalition p-dominating given allocation x in the incomplete market. In fact, $0 \in G$ implies the existence of $t = (t_1, \ldots, t_n) \geq 0$, $\sum t_i = 1$ such that for some $y_i \in \mathcal{P}_i(x_i)$

$$\sum_{i \in \mathcal{I}} t_i(y_i - \mathbf{e}_i) = 0 \iff \sum_{i \in \mathcal{I}} t_i y_i = \sum_{i \in \mathcal{I}} t_i \mathbf{e}_i \tag{25}$$

is true and, moreover, for $i = 2, \ldots, n$

$$\exists z_i \in \mathbb{R}^k \ : \ \mathrm{P}_1 y_i = \mathrm{P}_1 \mathbf{e}_i + \mathrm{P}_1 \mathrm{A} z_i$$

takes place. To check the fuzzy domination definition in part (10), it is sufficient to state the latter relation for $i = 1 \in \mathrm{supp}(t)$. To realize this, multiply (25) on matrix P_1, which after transformations due to $t_1 \neq 0$ yields

$$\mathrm{P}_1 y_1 = \mathrm{P}_1 \mathbf{e}_1 + \mathrm{P}_1 \mathrm{A} \left(- \sum_{i=2}^{n} \frac{t_i}{t_1} z_i \right).$$

Thus one can take $z_1 = -\sum_{i=2}^{n} \frac{t_i}{t_1} z_i$ as the necessary solution (portfolio) for agent 1. As a result we conclude that coalition t can p-dominate the allocation.

Therefore, for every $x \in C_{\mathbf{p}}^{f}(\mathcal{E}^{\mathrm{in}})$, it has to be that $0 \notin G$ and since $\mathrm{int}\, G \neq \emptyset$ (due to (**S**) we have $\mathrm{int}\, \mathcal{P}_1(x_1) \neq \emptyset$), then one can apply the separation theorem and find nonzero $\bar{p}$ such that

$$\langle \bar{p}, G \rangle \geq 0.$$

Since $\mathcal{P}_1(x_1) - \{\mathbf{e}_1\}$ and $(\mathcal{P}_i(x_i) - \{\mathbf{e}_i\}) \cap \mathcal{H}$, $i = 2, \ldots, n$ are the subsets of G, we conclude

$$\langle \bar{p}, \mathcal{P}_1(x_1) \rangle \geq \bar{p}\mathbf{e}_1 \ \ \& \ \langle \bar{p}, \mathcal{P}_i(x_i) \cap (\{\mathbf{e}_i\} + \mathcal{H}) \rangle \geq \bar{p}\mathbf{e}_i, \quad \forall i = 2, \ldots, n.$$

Moreover, the first of these inequalities due to $x_1 \in \mathrm{int}\, X_1$ and $\bar{p} \neq 0$ is realized in strict form that via (**S**) implies $\bar{p}_\sigma \neq 0$, $\forall \sigma$ in a standard way. The lemma is proved. □

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