Incomplete Markets: the Concept of Generalized Equilibrium and its Existence Theorem*

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Abstract

The notion of generalized equilibrium with compensating assets is introduced for incomplete markets theory. The extended theorem of existence is proved under the assumptions, which are similar to those used in economies of Arrow–Debreu type. These assumptions are weaker than the ordinary ones applied to prove pseudo-equilibria existence. It is shown, that generalized equilibria coincide with all limits of Radner's equilibria as soon as the short sale constraint is relaxed to ∞. The main result is formulated in standard mathematical terms, but it is obtained via the nonstandard analysis methods.

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1 Introduction

In publications devoted to financial markets (see bibliography in special issue of Journal of Mathematical Economics 19) considerable attention is attached to the “incompleteness” of these markets. The incompleteness appears here in two senses. First, it means that the number of permissible types of financial tools (assets) is smaller than the number of possible states of nature (or future markets). Second source of incompleteness is the degeneration of the operator (matrix), describing financial assets that can appear for some permissible prices. The incomplete markets can be considered as a special kind of Arrow–Debreu model of economy whose difference from traditional ones is concentrated in agent’s budget constrains. The constrains have the vector form of inequalities where in the right sides asset operator is applied. For this model the naturally defined *GEI-equilibrium* (general incomplete equilibrium) is commonly considered in literature and the incompleteness (in second sense) can imply the nonexistence of it (the reader is referred to [1] for definitions and examples), that entail principal conceptual difficulties. Two approaches have been designed to cope with the puzzle and to suggest such a notion of equilibrium that would exist in the model with rather generally structure of assets. One of them is *Radner’s equilibrium* concept (see [2]), which impose exogenous restrictions (“short sales constrains”) on trade with assets. It gives the existence property because of the compactness of agents strategy sets. However, arbitrariness of the short sales constrains yields arbitrariness of equilibria and it does not look satisfactory. Another approach is based on the notion of *pseudo-equilibrium.* It is the modification of GEI-equilibrium which coincides generically with it [1]. As a matter of fact this notion means the extension of the net of permitted assets in degenerated cases by some new assets, ”compensating” this degeneration. In our opinion the pseudo-equilibrium concept has two defects. First, it is the strictness of assumptions necessary to guarantee its existence. These assumptions (on assets and preferences) are almost equivalent to the conditions guaranteeing generic coincidence of pseudo-equilibria and GEI-equilibria. Second, this concept appears in economic theory as a pure auxiliary mathematical technique, which has no economic interpretation.

The goal of this paper is to combine two approaches avoiding their shortcomings. We introduce the concept of *generalized equilibrium*, give its interpretation in economic terms and prove the existence theorem under the conditions similar to those applied in Arrow–Debreu type economies.

The main difference between pseudo and generalized equilibria is concentrated in the right side of the budget constrains. Roughly speaking, for the pseudo-equilibrium they are determined by k-dimensional subspaces, containing the subspace of assets (taken together with prices) and for generalized equilibrium they are determined by k-dimensional cylinders, which are constructed as a polyhedron extended along the subspace, containing the subspace of assets (here k is the number of assets). It explains why pseudo-equilibria are the particular type of generalized equilibria, having fewer chances to exist. In this context the reasonable conjecture is appeared that the generic coincidence of GEI and generalized equilibria takes place — the problem, which is still open.

The main result of the paper, based on the original previous author's result [3], is the existence theorem of generalized equilibrium, which is obtained applying the nonstandard analysis methods via the limit transition of Radner's equilibria. The short version of this paper written in terms of standard mathematics one can find in [4]. Thus, due to applied method, the paper gives also the answer to the problem of description of all limits of R-equilibria as the short sales constraint is relaxed to infinity. This interesting question was raised in Geanakoplos [1] and may be reformulated in an equivalent form: on what kind of asset trading are based the financial transfers which support these limit allocation and prices. During the proving the main theorem we also prove the extended existence theorem of Radner's equilibria that can be interesting as well.

In conclusion, speaking of technique, we should like to mention Keiding's method [5] to prove the existence of pseudo-equilibria. This method is based on standard tools of existence theory, in contrast with previous papers, which searched for fixed points on non-convex domains. He has applied a suitable parameterization of the family of subspaces with the given dimension (obtained by the exploitation of real projective space). It seems possible that this approach might lead to an existence theorem for pseudo-equilibria of the same generality as one obtained in this paper for generalized equilibria.

2 The model and main result

2.1 *Notation*

2^A	denotes the set of all subsets of the set A;
$\mathbf{R}$	denotes the set of real numbers;
$\mathbf{R}^l$	denotes the $l-$fold product of $\mathbf{R}$;
$\langle x, y\rangle$	denotes the inner product of $x, y, \in \mathbf{R}^l$;
$conv\ A$	denotes the convex hull of the set A;
$\mathcal{L}(A)$	denotes the linear hull of the set A;
$int(A)$	denotes the interior of the set A;
$cl(A)$	denotes the closure of the set A;
$X \times Y$	and $\prod_{i\in N} X_i$ denotes Cart′s product.

If $\Psi : X \to 2^Y$ is a point-to-set mapping then

$$Gr\Psi := \{(x, y) \in X \times Y \ \mid \ y \in \Psi(x)\}.$$

${}^*\mathbf{R}$	denotes the set of all nonstandard numbers;
$\mu(x)$	denotes the monad of the point $x \in \mathbf{R}^l$;
$st(x)$	denotes the standard part of the nonstandard vector $x \in {}^*\mathbf{R}^l$;
$st(A)$	denotes the set of all vectors $st(a),\ \ a \in A$;
$si(A)$	denotes the set $\{a \in X \ \mid\ \mu(a) \subset A\},\ A \subset {}^*X$.

2.2 *The economy and definitions*

The investigated exchange economy with financial structure is

$$\mathcal{E} = \langle N, \{(X_i^t, \alpha_i^t(\cdot))_{t\in S},\ \mathcal{P}_i(\cdot)\}_{i\in N},\ A(\cdot),\ (Q^t)_{t\in S},\ Q^{-1}, w\rangle.$$

The parameters in brackets have the following meanings.

$N = \{1, ..., n\}$	is the set of numbers of economic agents;
$S = \{0, ..., s\}$	is the set of states of the world (nature), where 0 corresponds to a present spot market and $t = 1, ..., s$ correspond to future spot markets;
l_t	is the number of products in a state $t \in S$;
$X_i^t \subset \mathbf{R}^{l_t}$	is the set of consumption bundles, which are permissible for $i \in N$ in a state $t \in S$.

$$X_i = \prod_S X_i^t, \quad X = \prod_N X_i,$$

$\mathcal{P}_i : X \to 2^{X_i}$	is the i-th agent preferences, where
$\mathcal{P}_i(x) \subset X_i$	is the set of all commodity bundles strictly preferred to x_i by i-th agent with respect to $x = (x_1, x_2, ..., x_n)$;
p_t	denotes the prices of commodities and
$Q^t \subset \mathbf{R}^{l_t}$	is the set of all permissible prices in a state $t \in S$;
$K = \{1, ..., k\}$	is the set of asset numbers;
$q = (q_1, ..., q_k)$	denotes the prices of assets and
$Q^{-1} \subset \mathbf{R}^k$	is the set all permissible assets prices

$$Q = \prod_{t=-1}^{s} Q^t \subset \mathbf{R}^{k+l}, \quad l = \sum_S l_t, \quad \pi = (q, p_0, ..., p_s,) \in Q;$$

$\alpha_i^t : X \times Q \to \mathbf{R}$	is the profit function for i-th agent in a state $t \in S$, and
$\alpha_i(\cdot) = (\alpha_i^0(\cdot), ..., \alpha_i^s(\cdot))$	is the vector-functions of profits;
$A = [a_j(\cdot)]_{j\in K}$	is the matrix of assets, where column-function
$a_j(\cdot) : X \times Q \to \mathbf{R}^s$	denotes the promised monetary-valued payoff in all future states of nature, associated with buying an unit of j-th asset;
$\omega \in \mathbf{R}^l$	is the total initial endowments bundle, $\omega = (\omega^0, ..., \omega^s)$.

Considered incomplete market model has rather general form and presumes that there are two periods of economy life — present and future, to which correspond $s+1$ event-date (states of the world), where "0" is associated with the present and $1, 2, ..., s$ — with the future events. At each event the proper spot market of commodities is functioning and there is the *bounded freedom* of value transferring across different events.The value transfers are realized as a result of the trade at event "0" with the special financial tools of different kinds, called as assets (usually they are interpreted as securities, modelling the functioning of the insurance business). Price for j-s asset is represented by the value q_j and $q = (q_1, q_2, ..., q_k) \in Q^{-1}$ is the price-vector for assets. Now, if we denote

$$\lambda_j(x, \pi) = (-q_j, a_j(x, p)), \quad \Lambda = [\lambda_j]_{j \in K},$$

then the total transfer of wealth across different states of the world, which some agent can obtain from the market of assets with respect to his/her portfolio $z = (z_1, ..., z_k)$ (trade program for assets), is described by the vector

$$\Lambda \cdot z = z_1 \begin{bmatrix} -q_1 \\ a_1 \end{bmatrix} + \cdots + z_k \begin{bmatrix} -q_k \\ a_k \end{bmatrix}.$$

Therefore, if in spot markets the price-vectors for commodities $(p_t)_{t \in S}$ are defined, then each i-th consumer can choose the net of his/her consumption bundles under the following budget constrains, having vector-inequality form:

$$Px_i \leq \alpha_i(\overline{x}, \pi) + \Lambda z^i, \quad x_i \in X_i, z^i \in \mathbf{R}^K,$$

where the matrix

$$P = \begin{bmatrix} p_0 & & 0 \\ & \ddots & \\ 0 & & p_s \end{bmatrix}$$

defines the consumption cost operator and π denotes the vector of prices of all kinds (p, q). Note that commonly used in incomplete market theory "squared product", defined as $p \,\square\, x' = (p_t x'_t)_{t \in S}, \quad x' \in \mathbf{R}^l$, coincides with the ordinary matrix-vector product Px', $x' \in \mathbf{R}^l$. Also note that the model admits that agent's profit function $\alpha_i^t(.,.)$ may depends on the current vector of agents' consumption bundles $\overline{x} = (\overline{x}_1, ..., \overline{x}_n)$. We shall not discuss the economic sense of the model and its parameters more detailed; for all necessary comments the reader is referred to the review in Journal of Mathematical Economics, v.19, special issue.

The definition of the agents' budget sets plays the key role in equilibrium notions. Under the traditional GEI-approach the following budget sets are used

$$B_i(x, \pi) = \{x'_i \in X_i \mid \exists\, z \in \mathbf{R}^k : Px'_i \leq \alpha_i(x, \pi) + \Lambda(x, \pi) z\}.$$

One can see, they are exactly the sets of all possible budget feasible consumption bundles.

Definition 1. *A triplet* (x, π, z), $x = (x_1, ..., x_n) \in X$, $\pi \in Q$, $z = (z^1, ..., z^n) \in \mathbf{R}^{kn}$ *is called GEI-equilibrium if following conditions hold:*
(i) *attainability*

$$Px_i \leq \alpha_i(x, \pi) + \Lambda(x, \pi)z^i, \quad i \in N,$$

(ii) *individual rationality*

$$\mathcal{P}_i(x) \bigcap B_i(x, \pi) = \emptyset, \quad i \in N,$$

(iii) and the balance of consumption bundles and asset portfolios

$$\sum_N x_i = \omega, \quad \sum_N z^i = 0.$$

Classically (i), (ii) mean that each (x_i, z^i) is an optimal budget feasible plan for agent i, given (x, π). The condition (iii) is a couple of market clearing requirements under the assumption that no production or intertemporal storage is possible and assets are in zero net supply.

Previous results show, that GEI-equilibria by definition 1 are not far existing, though they exist "almost everywhere" (for generic real asset structures and initial endowments). Ordinary method to demonstrate it is to prove the existence of so-called pseudo-equilibria (see below on this definition) and then state, (under rather strong assumptions), that these notions "almost everywhere" realize the same equilibrium pair of agents' actions and prices (see [1] for more on this subject). The possible nonexistence of GEI-equilibria motivates the search of economically reasonable transformation of this concept to achieve the satisfactory existence theorem.

There are two potential possibilities of reasonable transformation of the GEI-notion to extend the field of equilibrium existence. The first of them amounts to taking the additional short sales constrains, i.e. instead of $z^i \in \mathbf{R}^K$ we have to require

$$z^i \in U_i \subset \mathbf{R}^K, \tag{1}$$

where the sets U_i can have rather general form. This approach was suggested by Radner [2], who was the first to remark the existence of equilibria with the limitations on the volumes of sales of assets. Below we shall speak about Radner's equilibria in the case of convex, closed and bounded from below $U_i, \quad i \in N$. This approach involves the budget sets definition as

$$B_i^R(x, \pi) := \{x_i' \in X_i \mid \exists\, z \in U_i : Px_i' \leq \alpha_i(x, \pi) + \Lambda(x, \pi)z\}.$$

So, in the first case we take the additional restrictions on the domain of the operator $\Lambda(x, \pi)$. The second possibility consists in changing its image. Here condition (i) is substituted by

$$Px \leq \alpha_i(x, \pi) + u_i, \quad u_i \in L_\pi + \mathbf{C}, \quad L_\pi = \Lambda(x, \pi)[\mathbf{R}^K], \tag{2}$$

where $\mathbf{C} \subset \mathbf{R}^S$ is a closed convex set, chosen from some class of subsets. The budgets sets in this case are transformed as follow:

$$B_i^{\mathbf{C}}(x,p) := \{x'_i \in X_i \mid \ \exists\, u'_i \in L_\pi + \mathbf{C} : Px'_i \leq \alpha_i(x,\pi) + u'_i\},$$

and it involves also the changes of the balance condition *(iii)*, which turns now into

$$\sum_N x_i = \omega, \quad \sum_N u_i = 0.$$

Commenting this approach, first of all we would like to note, that variant (2) seems more appropriate than (1) for describing of financial markets. Really, the short sales constraint (1) means that there are some exogenous physical or institutional restrictions on the volume of trade with assets, which does not seem realistic. On the contrary, variant (2) introduces additional forms of possible contracts, where $\mathbf{C}$ regulates the trade with new assets (indirectly it influence initial ones also). Of course, (2) is also some intervention in business activities but having another sign: enhancing of agent's possibilities (though, not violating the balance *(iii)*). However, an excessive extension of these possibilities would look like an arbitrary change of the model studied. It is the reason for the following "minimality" restriction on the class of permitted sets $\mathbf{C}$. We call convex $\mathbf{C} \subset \mathbf{R}^S$ an *effective set* with respect to the pair (x,π) if

$$dim(L_\pi + \mathbf{C}) \leq k.$$

Definition 2. *A permissible quadruple $(x, z, \pi, \mathbf{C})$ is called the $\mathbf{C}$-equilibrium if $\mathbf{C}$ is an effective set and there are $d_i \in \mathbf{C}$, $i \in N$, such that following conditions hold:*
(i) attainability

$$Px_i \leq \alpha_i(x,\pi) + \Lambda(x,\pi)z^i + d_i, \quad i \in N,$$

(ii) individual rationality

$$\mathcal{P}_i(x) \bigcap B_i^{\mathbf{C}}(x,\pi) = \emptyset, \quad i \in N,$$

(iii) and balance

$$\sum_N x_i = \omega, \quad \sum_N z^i = 0, \quad \sum_N d_i = 0.$$

Commenting this definition, first we would like to compare $\mathbf{C}$-equilibria and pseudo-equilibria. In fact, in accordance with its definition being considered in our terms, pseudo-equilibrium is defined as a triplet (x,π,L), where $L \subset \mathbf{R}^S$ is a subspace, satisfying the additional condition $dimL = k$ and $L_\pi \subset L$ and so that in the right hand side of i-th agent's budget constrains instead of $\Lambda(x,\pi)z^i + d_i$ the elements of L are applied, all constrains are realized in the form of equalities, and the allocation $x \in X$ is balanced. Clear that under assumptions when pseudo-equilibria are applied (strict monotonicity and differentiability of utility functions and the fact that consumption sets coincide with the positive orthant, see [1]), pseudo-equilibria are exactly $\mathbf{C}$-equilibria where $\mathbf{C}$ is a subspace of $\mathbf{R}^S$, such that $dim(\mathbf{C}+L_\pi)=k$. Therefore, pseudo-equilibria may be considered as a particular case of $\mathbf{C}$-equilibria.

Of special interest is to characterize the classes of the permissible effective sets, which guarantee the existence of $\mathbf{C}$-equilibria. Generally speaking, we can take for such class the set of all convex closed subsets of $\mathbf{R^S}$. But it is more interesting to consider narrower classes as it is done for the pseudo-equilibrium case, where all subspaces of $\mathbf{R^S}$ can be taken. For the generalized equilibrium we use the class of all polyhedrons of $\mathbf{R^S}$. It involves the effective with respect to (x,π) set $\mathbf{C}$ which can be described as

$$\mathbf{C}=\{\sum_{j=1}^{\nu} y_j c_j \ \mid \ \sum_{j=1}^{\nu} y_j b_j \geq \beta, \ y_j \in \mathbf{R}, \ j=1,2,...,\nu\}, \tag{3}$$

where

$$c_j=(c_j^0,c_j^1,...,c_j^s)\in \mathbf{R^S}, \quad \beta\in\mathbf{R}^K, \quad b_j\in\mathbf{R}^K, \quad j=1,2,...,\nu,$$

and

$$\nu \leq k - dim\ L_\pi(x).$$

The vectors c_j can be interpreted as the *compensating assets*, analogous to initial ones, where the payoff vectors can be represented as $a_{k+j}^c := (c_j^1,...,c_j^s)$, and prices of assets as $q_{k+j}^c := -c_j^0, \quad j=1,...\nu$. Note, that the constrains on sales y_j of these new assets have the linear form

$$\sum_{j=1}^{\nu} y_j b_j \geq \beta, \quad \beta=(\beta_1,...,\beta_k).$$

As a result we obtain the following

Definition 3. *A permissible triplet (x,z,π) is called the generalized equilibrium with compensating assets if there is such $\mathbf{C}$ defined by formula (3), that $(x,z,\pi,\mathbf{C})$ is $\mathbf{C}$-equilibrium, and there exist portfolios $y^i=(y_1^i,...,y_\nu^i)$ such that $d_i=\sum_{j=1}^{j=\nu} y_j^i c_j, \ i\in N$ & $\sum_N y^i=0$.*

Let us give a more explicit formulation of the budget sets used in this definition:

$$B_i^g(x,\pi):=\{x_i'\in X_i \mid \ \exists\ z'\in\mathbf{R}^K, \exists\ z''\in\mathbf{R}^\nu : Bz''\geq\beta \quad \&$$
$$Px_i' \leq \alpha_i(x,\pi)+\Lambda(x,\pi)z'+\Lambda^c z''\}, \tag{4}$$

where $\Lambda^c := [c_1,...,c_\nu], \ B:=[b_1,...,b_\nu]$.

Though the last weak equilibrium concept was produced from pure mathematical arguments, it has one important advantage in comparison with the pseudo-equilibrium notion. Namely, in difference with pseudo-equilibrium, the generalized equilibrium has economically reasonable interpretation. Really, the generalized equilibria turn out to be exactly the limits of Radner's equilibria for constrains having the form

$$z_j \geq \varepsilon_j^{(r)}, \quad j = 1, 2, ..., k,$$

where

$$\varepsilon_j^{(r)} \to -\infty, \quad r \to \infty.$$

This fact can be observed during the proving of their existence theorem — the main result of this paper. Therefore the concept of generalized equilibrium may be considered to be the concept of limit equilibrium and one can interpret the limiting process, which realize generalized equilibrium, as a way of government proceeding by trail and error on (Radner) equilibrium under decreasing institutional lower bounds on portfolios before allowing a limit equilibrium to exist when these bounds are relaxed to infinity.

2.3 *The main theorem*

The main result of the paper is the existence theorem of generalized equilibria proved under weaker assumptions in comparison with the existence theorem of pseudo-equilibria (the most restrictive of them is the differentiability of utility functions and assets, see [1]). Let us formulate the sufficient conditions for existence of equilibrium in economy $\mathcal{E}$.

A1. For each $i \in N$ the set X_i is convex, closed and bounded from below.

A2 (continuity of preferences). The mappings $\mathcal{P}_i : X \to 2^{X_i}$ have open graphs in $X \times X_i,\ i \in N$.

A3 (convexity & irreflexivity). For each $x = (x_1, ..., x_n) \in X$

$$x_i \notin conv\mathcal{P}_i(x),\ i \in N.$$

A4 (continuity of profits). The functions $\alpha_i : X \times Q \to \mathbf{R}^{\mathbf{S}},\ i \in N$ and the map $A : X \times Q \to \mathbf{R}^{ks}$ are continuous.

A5 (Walras' law). $\sum_N \alpha_i(x, p) = P\omega$ for each $x \in X,\ p \in Q$.

A6 (Slater's conditions). For each $\pi = (q, p^0, ..., p^s) \in Q,\ x \in X$ and $i \in N$, if $p_t \neq 0$, then for some compact $M_t \subset \mathbf{R}^{l_t}$

$$\inf_{x' \in X_i^t \cap M_t} \langle p_t, x' \rangle < \alpha_i^t(x, p).$$

Further, let us introduce the set of all feasible balanced allocations of $\mathcal{E}$:

$$E = E(X) = \{x = (x_1, ..., x_n) \in X \mid \sum_N x_i = \omega\}$$

and denote

$$(x_i \mid \tilde{x}_i^t) = (x_i^0, ..., x_i^{t-1}, \tilde{x}_i^t, x_i^{t+1}, ..., x_i^s).$$

A7 (local nonsatiation). For each $x \in E(X)$, $i \in N$, $t \in S$ and $\varepsilon > 0$ there is $\tilde{x}_i^t \in X_i^t$ such that $(x_i \mid \tilde{x}_i^t) \in \mathcal{P}_i(x)$ & $||\tilde{x}_i^t - x_i^t|| \leq \varepsilon$.

The assumptions **A1**–**A4** are rather traditional and have ordinary mathematical sense. The Walras' law **A5** differs from the usual one only in the vector form of identity, which gives us the financial balances for each (present and future) spot markets. The same can be said about Slater's condition **A6**. Generally speaking A6 could be further weaken and reduced to the form with non-strict inequalities, that is more realistic. This seemingly can be achieved by exploiting the nonstandard prices technique, involved not only in the proof but in equilibrium definition as well (this approach had been realized for usual markets in [6, 7]). Finally, we have assumed the nonsatiation of agents' preferences for all markets, that is necessary for existence of equilibria in traditional models of the Arrow–Debreu type as well. This assumption can be omitted by the application of transfer costs (or transferable values), the method (see [8] for example) using the following construction.

Any non-negative vector

$$\gamma = (\gamma_0, \gamma_1, ..., \gamma_s) \geq 0$$

is called the vector of transferable values. These value are added to the right side of the budget constrains in (4):

$$Px_i' \leq \gamma + \alpha_i(x, \pi) + \Lambda(x, \pi)z' + \Lambda^c z''$$

and the sets $B_i^g(x, \pi, \gamma)$ determined by this inequality are called the budget sets with transferable values, which imply the corresponding notion of equilibrium.

Theorem 1. *If $\mathcal{E}$ satisfies* **A1**–**A6** *and $0 \in int\, Q$, then generalized equilibria with transferable values exist. If* **A7** *holds additionally, then there exist generalized equilibria.*

In the proof of *Theorem 1* the auxiliary theorem on the existence of Radner's equilibria is applied. Here we are exploiting a special version of this theorem which is formulated in our terms and uses the budget sets with transferable values. These sets are formed by

$$B_i^R(x, \pi, \gamma) = \{x_i' \in X_i \mid \exists\, z' \in U_i : Px_i' \leq \gamma + \alpha_i(x, \pi) + \Lambda(x, \pi)z\}, \quad i \in N.$$

Theorem 2 *(auxiliary). Let $\mathcal{E}$ satisfy* **A1**–**A6** *and U_i be convex, closed, bounded from below and $0 \in int(U_i)$ for each $i \in N$. Let also every Q^t, $t = -1, 0, ..., s$ is the ball, centered in the origin with the radius 1. Then there exists Radner's equilibrium with transferable values $\gamma = (\gamma_0, ..., \gamma_s) \geq 0$, equilibrium prices for commodities in spot markets $p_t \in Q^t$, $t \in S$ and prices for assets $q \in Q^{-1}$, such that $\gamma_t = 1 - ||p_t||$ for $t = 1, 2, ..., s$ and $\gamma_0 = 1 - min(||p_0|| + ||q||)$ for $t = 0$.*

3 The proofs of theorems

At first we give the proof of *Theorem 1* which uses *Theorem 2*. After that we describe the scheme of *Theorem 2* proof.

The idea of the *Theorem 1* proof is reduced to the investigation of the Radner's equilibrium obtained from nonstandard extension of our model, where the short sales constraint has an actually infinite form.

3.1 *Proof of Theorem 1* is achieved in three steps.
During the proving main theorem we will apply the following facts from nonstandard analysis theory. For any internal subset $A \subset {}^*\mathcal{X}$ of $*$-image of some topological space $\mathcal{X}$ define the sets

$$st(A) = \{x \in \mathcal{X} \mid \mu(x) \cap A \neq \emptyset\} \ \ \& \ \ si(A) = \{x \in \mathcal{X} \mid \mu(x) \subset A\},$$

where symbol $\mu(x)$ means the monad of a point x, defined by $\mu(x) = \{\cap^* G \mid G \in \mathcal{G}_x\}$, where $\mathcal{G}_x$ is the set of all open neighborhoods of the point x.

Fact 1. *Let A, B be any internal subsets of $*$-image of topological space $\mathcal{X}$, which satisfy the condition $A \cap B = \emptyset$. Then*

$$st(A) \cap si(B) = \emptyset.$$

Proof. Assuming contrary take $x \in st(A) \cap si(B)$. Then by definition we have

$$\mu(x) \cap A \neq \emptyset \ \& \ \mu(x) \subset B \Rightarrow A \cap B \neq \emptyset$$

that contradicts imposed condition. **Q.E.D.**

Fact 2. *Let $\mathcal{X}$ be a topological space and $P : \mathcal{X} \to 2^{\mathcal{X}}$ be a point-to-set mapping, having open graph $Gr(P)$ in $\mathcal{X} \times \mathcal{X}$. Then*

$$P(st(x)) \subset si({}^*P(x)).$$

Proof. By transfer principle we have $Gr({}^*P(.)) = {}^*Gr(P(.))$. Let us take $y \in P(st(x))$. Applying the nonstandard characterization of open sets — A is open iff $\mu(x) \subset {}^*A$ for each $x \in A$ — (see [9], ch.3, theorem 1.3 and [4,5]), one can conclude that

$$\mu[st(x)] \times \mu(y) \subset {}^*Gr(P(.)) \Rightarrow \{x\} \times \mu(y) \subset {}^*Gr(P(.)) \Rightarrow$$

$$\mu(y) \subset {}^*P(x) \Rightarrow y \in si({}^*P(x)).$$

Q.E.D.

STEP 1. Let us consider the nonstandard model ${}^*\mathcal{E}$ obtained according to $*$-images of all parameters of $\mathcal{E}$ and take for investigation its R-equilibria.

We are interested in the special case when constrains on assets sales have the form

$$\varepsilon_j z_j \geq -1, \quad \varepsilon_j > 0, \quad \varepsilon_j \approx 0, \quad j = 1, 2, ..., k.$$

In view of *Theorem 2* and the transfer principle such equilibria do exist. If (x, z, π) is one of them then

$$e^t \leq x_i^t = \omega^t - \sum_{j \neq i} x_j^t \leq \omega^t - n \cdot e^t$$

where $e^t \leq 0$ is the common low boundary of agents' consumption sets. By **A1** the left and the right sides are near-standard that implies the near-standardness of $x_i, \ i \in N$. In view of *Theorem 2* vector $\pi = (p, q)$ can be also standardized. Let us denote

$$\overline{x}_i = st(x_i), \quad i \in N, \quad \overline{p} = st(p), \quad \overline{q} = st(q), \quad \overline{\pi} = (\overline{p}, \overline{q}).$$

Our aim is to show that $(\overline{x}, \overline{\pi})$ can be represented as a generalized equilibrium. By individual rationality *(ii)* of R-equilibria we have

$$^*\mathcal{P}_i(x) \bigcap {}^*B_i^R(x, \pi, \gamma) = \emptyset, \quad i \in N.$$

In view of **Fact 1** and **Fact 2** being applied to this case, as soon as the graph of $\mathcal{P}_i$ is open (**A2**), one can conclude that

$$\mathcal{P}_i(\overline{x}) \bigcap st({}^*B_i^R(x, \pi, \gamma)) = \emptyset, \quad i \in N \tag{5}$$

The agents' budget sets ${}^*B_i^R(x, \pi, \gamma)$ can be described in the following way: they are the projection of

$$Bud_i(x, \pi, \gamma) = \{(x', z') \ \mid \ x' \in {}^*X_i, \quad z' \in {}^*\mathbf{R}^K, \quad z_j' \cdot \varepsilon_j \geq -1, \quad j \in K,$$
$$Px' \leq \gamma + \alpha_i(x, \pi) + \Lambda(x, \pi) \cdot z'\}$$

onto ${}^*X_i$. Now let us reduce these sets for the nonstandard case by taking the additional requirement that the standard parts of the values $z_j'\varepsilon_j, \ j \in K$ and of $\Lambda(x, \pi)z'$ do exist. Denote

$$Bud_i^{st}(x, \pi, \gamma) := \{(x', z') \ \mid \ x' \in {}^*X_i, \quad z' \in {}^*\mathbf{R}^K : \ \exists \ st\,(z_j'\varepsilon_j), \ \exists \ st(\Lambda z') \ \&$$
$$Px' \leq \gamma + \alpha_i(x, \pi) + \Lambda z', \qquad z_j'\varepsilon_j \geq -1, \quad j \in K\}. \tag{6}$$

If we denote

$$\overline{B}_i(x, \pi, \gamma) := \{st(x') \ \mid \ \exists \ z' \in {}^*\mathbf{R}^K \ : \ (x', z') \in Bud_i^{st}\}, \tag{7}$$

the standard part of the projection onto ${}^*X_i$ of the set defined by (6), then by (5) we will have

$$\mathcal{P}_i(\overline{x}) \bigcap \overline{B}_i(x, \pi, \gamma) = \emptyset, \quad i \in N. \tag{8}$$

At the next step we construct the compensating assets and agents' portfolios corresponding to the bundle $(\overline{x}, \overline{\pi})$.

STEP 2. Let us denote

$$E^{\varepsilon} = \begin{bmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_k \end{bmatrix}$$

and consider the nonstandard operator

$$Gz = y, \quad z \in {}^*\mathbf{R}^K, \tag{9}$$

where the matrix

$$G = \left[\frac{E^{\varepsilon}}{\Lambda}\right], \quad G = [g_1, ..., g_k]$$

consists of the matrix of portfolio constrains and of the matrix of returns from assets. The properties of the operator G seem to be the key to revealing the structure of budget sets given by (7) and to defining the compensating assets structure.

In fact, by *(iii)* the equilibrium bundle of agents' portfolios satisfies

$$b^i \leq Gz^i = -\sum_{j \neq i} Gz^j \leq -\sum_{j \neq i} b^j$$

where the vectors b^i are determined by the right sides of asset sales constrains and of agents' budget constrains

$$b^i_r = \begin{cases} -1, & r = 1, ..., k, \\ \langle p_t, x^t_i \rangle - \alpha_i(x, \pi) - \gamma^t, & r = k + t + 1, \ t = 0, 1, ..., s. \end{cases}$$

Hence to define the equilibrium portfolios we deal with the solutions of (9) for some near-standard y. Now let us consider $st(G)$ and take the maximal linear independent collection of columns of the submatrix $st(\Lambda)$. Without loss of generality they can be considered to be the first m columns.

Now define the new matrix $\mathcal{F} = [f_1, ..., f_k]$ and $\tilde{z} = (\tilde{z}_1, ..., \tilde{z}_k)$ by induction. Let

$$f_r = g_r, \quad z^{(r)} = z, \quad r \leq m.$$

For $r > m$ determine

$$h_r = g_r - \sum_{j=1}^{r-1} \mu_{jr} f_j, \quad f_r = h_r / ||h_r||, \quad \mu_{rr} = ||h_r||,$$

$$z^{(r)}_j = \begin{cases} z^{(r-1)}_j + \mu_{jr} z^{(r-1)}_j, & j < r, \\ ||h_r|| z^{(r-1)}_j, & j = r, \\ z^{(r-1)}_j, & j > r. \end{cases}$$

Here scalars μ_{jr} are determined from the conditions

$$\langle h_r, f_s \rangle = 0, \quad s < r,$$

i.e. it must be

$$\sum_{j=1}^{r-1} \mu_{jr} \langle f_j, f_s \rangle = \langle g_r, f_s \rangle, \quad s = 1, 2, ..., r-1. \tag{10}$$

In other words the procedure is reduced to the following. At the first nontrivial step the vector of normal h_{m+1} is dropped from g_{m+1} onto the subspace determined as a linear hull of the first m columns. Here f_{m+1} is the normalized vector h_{m+1}. Further G is transformed and new variables are defined to save the equality (9). At the next step the described procedure is repeated but it is applied to the matrix

$$G^{(m+1)} = [f_1, ..., f_{m+1}, g_{m+2}, ..., g_k]$$

and so on. Since by construction of G always $||h_r|| \neq 0$ hold for $r > m$, then the system (9) has unique solution and the matrix $[\langle f_j, f_s \rangle]_{r,s}$ is standard non-singular. It implies that all coefficients μ_{jr} are near-standard. So, the transformation of G is reduced to the sequence of elementary transformations of columns and variables. As a result we obtain the system

$$\mathcal{F}\tilde{z} = y, \quad \mathcal{F} = [f_1, ..., f_k]$$

where

$$\mathcal{F} = G^{(k)} = G\Theta^{-1}, \quad \tilde{z} = \Theta z$$

and the transition matrix Θ have the form

$$\Theta = \begin{bmatrix} 1 & 0 & \dots & 0 & \mu_{1,m+1} & \dots & \mu_{1k} \\ 0 & 1 & \dots & 0 & \mu_{2,m+1} & \dots & \mu_{2k} \\ \vdots & & \ddots & & \vdots & & \vdots \\ & & & 1 & \mu_{m,m+1} & \dots & \mu_{mk} \\ & & & & \mu_{m+1,m+1} & \dots & \mu_{m+1,k} \\ & & & & & \ddots & \\ 0 & & \dots & & 0 & & \mu_{kk} \end{bmatrix},$$

while $\mathcal{F}$ have the form

$$\mathcal{F} = \left[\frac{E_m^\varepsilon W}{\Lambda_m V} \right],$$

where W & V are some submatrices, $\Lambda_m = [\lambda_1, ..., \lambda_m]$, and

$$E_m^\varepsilon = \begin{bmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_m \\ 0 & \dots & 0 \end{bmatrix}.$$

Here $st(E_m^\varepsilon) = 0$ by the choice $\varepsilon_j \approx 0$, that implies

$$\overline{\mathcal{F}} = st(\mathcal{F}) = \left[\frac{0, \quad \overline{W}}{\overline{\lambda}_1, .., \overline{\lambda}_m, \overline{V}} \right],$$

where the bar means the operation $st(.)$. The columns of $\overline{\Lambda}_m = [\overline{\lambda}_1, ..., \overline{\lambda}_m]$ are linear independent by construction and linear hulls satisfy

$$\mathcal{L}(\overline{\lambda}_1, ..., \overline{\lambda}_m) = \mathcal{L}(\overline{\lambda}_1, ..., \overline{\lambda}_k). \tag{11}$$

In addition the system (9) has a solution with respect to z for near-standard y iff the system $\mathcal{F}\tilde{z} = y$ has unique near-standard solution.

Now let us take the columns of

$$\overline{V} = [v_{m+1}, ..., v_k] = \Lambda^c$$

as the compensating assets. An agent can use initial assets $\overline{\lambda}_1, ..., \overline{\lambda}_m$ and compensating assets $v_{m+1}, ..., v_k$. In so doing his portfolio $z = (z', z'')$, $\quad z' = (z_1, ..., z_m)$, $\quad z'' = (z_{m+1}, ..., z_k)$ must satisfy constrains for trade with compensating assets, these constrains are defined by the matrix $B := \overline{W}$ by vector inequality:

$$\overline{W} z'' \geq (-1, ..., -1).$$

Now we can determine some new equilibrium portfolios according to the initial nonstandard ones by

$$\overline{z}^i = st\ \Theta z^i, \quad i \in N.$$

It is easy to see that the bundle $\overline{z} = (\overline{z}^i)_N$ is balanced and the triplet $(\overline{x}, \overline{z}, \overline{\pi})$ satisfies the requirement *(i)* of equilibrium definition. Hence now the problem amounts to stating the property of individual rationality *(ii)*.

STEP 3. In accordance with the relations (8) and (11) the proof will be completed if we show that for

$$B_i^g(\overline{x}, \overline{\pi}, \overline{\gamma}) = \{y \in X_i \mid \ \exists\ z' \in \mathbf{R}^m,\ \exists\ z'' \in \mathbf{R}^{k-m} : (-1, ..., -1) \leq \overline{W} z'',$$
$$\overline{P} y \leq \overline{\gamma} + \alpha_i(\overline{x}, \overline{\pi}) + \overline{\Lambda}_m z' + \overline{V} z''\}. \tag{12}$$

the equality

$$\overline{B}_i(x, \pi, \gamma) = B_i^g(\overline{x}, \overline{\pi}, \overline{\gamma}) \tag{13}$$

holds for each fixed $i \in N$. Let us do it. The inclusion $\subset$ in (13) follows from the construction of the considered above transformation and the standardization of inequalities. It is necessary to show the inverse inclusion. Let y be chosen from the right side of (12) and $z = (z', z'')$ corresponds to y. It is sufficient to find such $\vartriangle y \approx 0$, $\vartriangle z \approx 0$ that

$$(y + \vartriangle y,\ z + \vartriangle z) \in Bud_i^{st}.$$

By definition the last inclusion means that

$$(-1,..,-1) \leq E_m^{\varepsilon}(z' + \vartriangle z') + W(z'' + \vartriangle z''),$$

$$P(y + \vartriangle y) \leq \alpha_i(x, \pi) + \Lambda_m(z' + \vartriangle z') + V(z'' + \vartriangle z'') + \gamma.$$

Extracting the main members and performing some routine transformations these inequalities can be reduced to

$$(-1, ..., -1) \leq \overline{W}(z'' + \vartriangle z'') + \rho,$$

$$\overline{P}(y + \vartriangle y) - \alpha_i(\overline{x}, \overline{\pi}) - \overline{\gamma} \leq \overline{\Lambda}_m(z' + \vartriangle z') + \overline{V}(z'' + \vartriangle z'') + \mu, \qquad (14)$$

where ρ, μ are some infinitesimal vectors (remember that bar means the standard part). Now the problem is to find $\vartriangle y,\ \vartriangle z$ such that (14) holds for some fixed $\rho \approx 0,\ \mu \approx 0$. Let us do it.

One can assume, without loss of generality, that all inequalities in the right side of (12) turn into equalities for bundle (y, z) (since if some of given inequalities is strict for standard values, it will be true for every elements taken from the monads of standard variables and therefore this inequality may be excluded from below considerations). Next using *Theorem 2* result, we shall distinguish two cases. For the first case $\overline{p}_0 \neq 0$ *or* $\overline{\gamma}_0 > 0$ (for $\overline{p}_0 = 0$) holds. In the second case $\overline{p}_0 = 0$ and $\overline{\gamma}_0 = 0$.

First case. Let us extract all states $t \in S$, where

$$b_t = \langle \overline{p}_t, y^t \rangle - \alpha_i^t(\overline{x}, \overline{\pi}) - \overline{\gamma}^t < 0 \iff t \in S_-$$

and find $\tau < 0,\ \ \tau \approx 0$ such that

$$\tau \cdot \overline{W} z'' \geq \rho \approx 0,$$

$$\tau(\overline{\Lambda}_m^t z' + \overline{V}^t z'') \geq \mu^t \approx 0, \quad t \in S_-.$$

Since ρ & μ are infinitesimal, by definition of S_- such choice of τ is possible. Take

$$\vartriangle z = \tau z, \quad \vartriangle y^t = 0, \quad t \in S_-.$$

Further, if $t \in S \backslash S_-$ then

$$\langle \overline{p}_t, y^t \rangle \geq \alpha_i^t(\overline{x}, \overline{\pi})$$

because $\gamma^t \geq 0$. Note that for each $t \notin S_-$ we have $\overline{p}_t \neq 0$ and therefore by **A6** there is such $\tilde{x}_i^t \in X_i^t$ that

$$\langle \overline{p}_t, \tilde{x}_i^t \rangle < \alpha_i^t(\overline{x}, \overline{\pi})$$

(due to *Theorem 2*, $\overline{p}_t = 0$ implies $\overline{\gamma}^t > 0$ for all $t \neq 0$ and by the condition of chosen case the same takes place for $t = 0$, that due to $\alpha_i^t(\overline{x}, \overline{\pi}) = 0$ for

$\overline{p}_t = 0$[1] in both cases means $t \in S_-$ if $\overline{p}_t = 0$). Now choose such $\sigma > 0, \;\; \sigma \approx 0$ that

$$\sigma \cdot \overline{p}_t(\tilde{x}_i^t - y^t) \leq \overline{\Lambda}_m^t \vartriangle z' + \overline{V}^t \vartriangle z'' + \mu^t \approx 0, \quad t \in S \setminus S_-$$

holds and define

$$\vartriangle y^t = \sigma(\tilde{x}_i^t - y^t), \quad t \in S \setminus S_-.$$

It is easy to see that all inequalities of (14) hold for such $\vartriangle y, \vartriangle z$.

Second case. Due to *Theorem 2* we have $\gamma_0 = 1 - min(1, ||p_0|| + ||q||) \approx 0$ in this case, that for standard parts implies $||\overline{q}|| = 1$. Moreover, since we assumed that all constrains in the right part of (12) are realized in the form of equality for given $\overline{x}, \overline{\pi}, \overline{\gamma}$ and y, and because of $\alpha_i^0(\overline{x}, \tilde{\pi}) = 0$ for every permissible $\tilde{\pi}$, such that $\tilde{p}_0 = 0$ (see footnote above), one can see that

$$(\overline{\Lambda}_m z' + \overline{V} z'')_0 = 0. \tag{15}$$

Further, since $||\overline{q}|| \neq 0$, in the matrix $\overline{\Lambda}_m$ there is the column-vector, in which first component (corresponding the state of nature $t = 0$) is not zero (otherwise it contradicts the choice of matrix $\overline{\Lambda}_m$). We may think that this is the first column-vector. So, we have $\overline{\lambda}_{01} = -\overline{q}_1 \neq 0$. Now take $\delta = \mu_0 / \overline{q}_1$ and check that for $\tilde{z}_1' = z_1' + \delta$, $\tilde{z}_j' = z_j' \;\; j = 2, ..., m$

$$\overline{p}_0 y_0 = (\overline{\gamma} + \alpha_i(\overline{x}, \overline{\pi}) + \overline{\Lambda}_m \tilde{z}' + \overline{V} z'' + \mu)_0 = 0$$

holds. Now consider the system of linear inequalities with respect to given nonstandard $\tilde{z}'$ and unknown infinitesimal $\vartriangle y$ and $\vartriangle z$, which is equivalent the system (14) but has the new "second part" of inequalities:

$$\overline{P}(y + \vartriangle y) - \alpha_i(\overline{x}, \overline{\pi}) - \overline{\gamma} \leq \overline{\Lambda}_m(\tilde{z}' + \vartriangle z') + \overline{V}(z'' + \vartriangle z'') + \tilde{\mu},$$

where $\tilde{z}'$ was defined above and $\tilde{\mu} = \mu + \delta \overline{\lambda}_1$. Further one can realize the method used in *first case* and find $\tilde{\tau} < 0, \;\; \tilde{\tau} \approx 0$ such that

$$\tilde{\tau} \cdot \overline{W} z'' \geq \rho \approx 0,$$

$$\tilde{\tau}(\overline{\Lambda}_m^t z' + \overline{V}^t z'') \geq \tilde{\mu}^t \approx 0, \quad t \in S_-.$$

Again, since ρ & $\tilde{\mu}$ are infinitesimal, by definition of S_- such choice of $\tilde{\tau}$ is possible. Put

$$\vartriangle z = \tilde{\tau} z, \quad \vartriangle y^t = 0, \quad t \in S_-$$

and find $\vartriangle y_t$, for $t \notin S_-, \;\; t \neq 0$ exactly how it was down in the *first case* but with respect to vector $\tilde{\mu}$, taken instead of μ. Since $\vartriangle z = \tilde{\tau} z$, in view of (15), the equality corresponding $t = 0$ will be still true and we have the desired result with respect to $y + \vartriangle y$, where $(\vartriangle y)_0 = 0$, and nonstandard portfolio $\tilde{z} + \vartriangle z$. This completes the verification of (13).

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[1] Apply $\mathbf{A7^s}$ for $p_\xi \neq 0$, $p_\xi \to 0$ and pass to limits proving $\alpha_i^\sigma(\bar{x}, \bar{\pi}) \geq 0 \; \forall i$, that by Walras' law is possible only if $\alpha_i^\sigma(\bar{x}, \bar{\pi}) = 0$ for all i.

To finish the proof note, that if assumption **A7** holds for $\mathcal{E}$, then every nonstandard R-equilibrium with the transferable values γ can be realized in ${}^*\mathcal{E}$ if and only if $\gamma = 0$. Really, **A7** implies that for all agents all budget inequalities turn into equalities in equilibrium and after their summing Walras' law **A5** gives $\gamma = 0$. **Q.E.D.**

3.2 *Proof of Theorem 2*

We are going to describe the main stages of the proof only and to omit the details, because the ideas are well-known and commonly used in Arrow–Debreu type models.

The proof is reduced to the construction of an appropriate point-to-set mapping and studying its fixed points. In the conditions of Theorem 2 we assumed that $Q^t,\ \ t = -1, ..., s$ are balls with the unit radius centered in the origin. Now define

$$\gamma^t(p_t) = 1 - ||p_t||, \quad p_t \in Q^t, \quad t = 1, ..., s,$$

$$\gamma^0(p_0, q) = 1 - min\,(1, ||p_0|| + ||q||), \quad \gamma = (\gamma^0, ..., \gamma^s).$$

Determine the budget sets by

$$Bud_i(x, \pi, \gamma) = \{(\tilde{x}_i, \tilde{z}^i) \in X_i \times U_i \mid \ P\tilde{x}_i \le \gamma(\pi) + \alpha_i(x, \pi) + \Lambda\tilde{z}^i\}.$$

Note that by **A6** and $0 \in int(U_i)$ they are non-empty for every $(p', q') \in Q$ and $x \in X$. Moreover, it is easy to see, that for every permissible (p, q), x there is permissible x'_i and z'^i such that

$$Px'_i \ll \gamma(p, q) + \alpha_i(x, p, q) + \Lambda z'^i$$

holds. Latter one allows to show that $Bud_i(.)$ is lower semi-continuous and, since it is obviously upper semi-continuous, the map $(x, \pi) \to Bud_i(x, \pi, \gamma)$ is continuous and has non-empty, convex, closed images for every $(x, \pi) \in X \times Q$.

Further let us determine individual reactions of agents by

$$(x, z, \pi) \overset{\Psi_i}{\to} (x'_i, z'^i), \quad i \in N,$$

$$\Psi_i(x, z, \pi) = conv(arg \max_{(\tilde{x}_i, \tilde{z}^i) \in Bud_i(x, \pi, \gamma)} dist\,(x, \tilde{x}_i)),$$

where $dist(x, \tilde{x}_i)$ is the Euclidean distance between the point $(x, \tilde{x}_i)$ and the set $X \times X_i \setminus Gr\,\mathcal{P}_i$ where $Gr\,\mathcal{P}_i$ is the "graph" of the i-th preference. The reactions of the price-setting body are defined by

$$(x, z) \quad \overset{\Upsilon}{\to} \quad \pi = (p, q)$$

$$\Upsilon(x, z) = arg \max_{\pi \in Q} \langle \pi, (\sum_N x_i - \omega, \sum_N z^i) \rangle.$$

The resulting mapping is

$$\Phi : \mathcal{X} \to \mathcal{X}, \quad \mathcal{X} = X \times \prod_N U_i \times Q,$$

which is defined as a product of mappings

$$\Phi = \prod_N \Psi_i \times \Upsilon.$$

Without loss of generality we can assume that X_i and U_i are convex compacts, $i \in N$. Then one can see that by **A1**, **A2**, **A4**, **A6** the mapping Φ and the set $\mathcal{X}$ satisfies all conditions of Kakutani's fixed point theorem and therefore there exists a fixed point

$$(\overline{x}, \overline{z}, \overline{\pi}) \in \Phi(\overline{x}, \overline{z}, \overline{\pi})$$

which realizes equilibrium. Let us check conditions *(ii)*, *(iii)* of equilibrium definition.

In fact, if $(\overline{x}, \overline{z})$ is not balanced then by $0 \in int\, Q$ and fixed point properties we have

$$\langle \overline{p}, (\sum \overline{x}_i - \omega, \sum \overline{z}^i) \rangle > 0.$$

If $\sum \overline{z}^i \neq 0$ then $||\overline{q}|| = 1$ and $\gamma^0(\overline{p}_0, \overline{q}) = 0$ that implies

$$\langle \overline{p}_0, \sum \overline{x}_i^0 - \omega^0 \rangle + \langle \overline{q}, \sum \overline{z}^i \rangle > 0.$$

On the other hand after summing of budget constrains we shall obtain the inequality

$$\langle \overline{p}_0, \sum \overline{x}_i^0 \rangle \leq \sum \alpha_i^0(\overline{x}, \overline{\pi}) - \langle \overline{q}, \sum \overline{z}^i \rangle$$

that by **A5** contradicts the previous one. If $\sum \overline{z}^i = 0$ and $\sum \overline{x}_i^t \neq \omega^t$ for some $t \in S$ then $\gamma^t(\overline{p}_t) = 0$ and

$$\langle \overline{p}_t, \sum \overline{x}_i^t - \omega^t \rangle > 0.$$

Again the budget constrains of the t-th market give

$$\langle \overline{p}_t, \sum \overline{x}_i^t \rangle \leq \sum \alpha_i^t(\overline{x}, \overline{\pi}) = \langle \overline{p}_t, \omega^t \rangle$$

that proves *(iii)*.

If *(ii)* is false for some $i \in N$ then

$$\max_{(\tilde{x}_i, \tilde{z}^i) \in Bud_i} dist(\overline{x}, \tilde{x}_i) > 0 \;\Rightarrow\; arg \max_{(\tilde{x}_i, \tilde{z}^i) \in Bud_i} dist(\overline{x}, \tilde{x}_i) \subset \mathcal{P}_i(\overline{x})$$

that involves

$$\overline{x}_i \in \; conv \mathcal{P}_i(\overline{x} \mid \overline{x}_i).$$

But this contradicts **A3**. The proof is completed. **Q.E.D.**

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