Production equilibria in vector lattices*

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Received: March 9, 1999/ Revised: September 4, 2000

Abstract

The general purpose of this paper is to prove quasiequilibrium existence theorems for production economies with general consumption sets in an infinite dimensional commodity space, without assuming any monotonicity of preferences or free-disposal in production.

The commodity space is a vector lattice commodity space whose topological dual is a sublattice of its order dual. We formulate two kinds of properness concepts for agents' preferences and production sets, which reduce to more classical ones when the commodity space is locally convex and the consumption sets coincide with the positive cone. Assuming properness allows for extension theorems of quasiequilibrium prices obtained for the economy restricted to some order ideal of the commodity space. As an application, the existence of quasiequilibrium in the whole economy is proved without any assumption of monotonicity of preferences or free-disposal in production.

Keywords and Phrases: linear vector lattices, competitive production equilibrium, quasiequilibrium, E-properness, F-properness.

JEL Classification Numbers: C 62, D 51

*The research of V. Marakulin was supported by the Russian Foundation for Humanities, grant N 99-02-00141 and by the Russian Foundation of Basic Researches, grants N 98-01-00664, N 96-15-98656. The support of the cooperation agreement 98-4977 CNRS-Russian Academy of Sciences is also gratefully acknowledged. We thank P. Gourdel (who pointed an error in a previous version) and R. Tourky for valuable discussions before, during and after the 1999 Rhodos S.A.E.T. Conference.

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1 Introduction

The purpose of this paper is to prove quasiequilibrium existence theorems for production economies with general consumption sets in an infinite dimensional commodity space, without assuming any monotonicity of preferences or free-disposal in production.

As indicated by the title of this paper, we consider economies defined on (infinite dimensional) vector lattice commodity spaces, a notion introduced by Aliprantis and Brown [2]. Here, as in Mas-Colell and Richard [22], Richard [26] and many others papers ([11], [24], [1], [27], [28], [14], [19]), we assume that the commodity space is a vector lattice whose topological dual is a sublattice of its order dual. As well-known, this setting, which covers most of the important infinite dimensional models, was introduced in order to include the models of commodity differentiation in Jones [18] and of intertemporal consumption in Huang and Kreps [17], not covered before by a number of equilibrium existence results (e.g., [20], [30], [31], [25], [4], [5], [16], [29]) requiring that the commodity space be a topological vector lattice. Even if it leaves out of its scope some commodity-price dualities of economic interest (a detailed discussion on relevant commodity-price dualities can be found in [3], [23]), such a setting is also the most general one used by now in equilibrium existence proofs, if one excepts a thought provoking paper by Aliprantis et al. [8] discarding the vector lattice property of the commodity space and its dual at the cost of an alternate theory of value with non-linear prices.

Our method of proof is to get quasiequilibria by decentralizing Edgeworth equilibria whose existence is guaranteed under relatively weak assumptions. When the preferred sets have an empty interior, the decentralization arguments use properness assumptions first introduced by Mas-Colell [20], then substantially weakened. Specifically, uniform properness of preferences was replaced by an assumption of pointwise properness at some particular allocations by Araujo and Monteiro [9], Duffie and Zame [15], in the particular case where the total endowment is a strictly positive element of the commodity space of an exchange economy with the positive cone as consumption sets of the agents. This result was extended by Podczeck [24] to the nonordered case and proved without any monotonicity assumption. For the more general case where the total endowment is not a quasi-interior element of the commodity space (specially, if this one has no quasi-interior element), Podczeck introduced a properness concept, called E-properness, stronger than pointwise properness but weaker than uniform properness.

It is this concept of E-properness that we mainly address in this paper. The economy under consideration is a production economy, a case not studied by Podczeck. We also consider more general consumption sets than the positive cone of the commodity space. We were stimulated to do it by two papers of Tourky [27], [28] which perform, in a more general framework, an objective previously claimed by Back [11] and Boyd and McKenzie [13]. As a counterpart, the formulation of properness becomes then somewhat abstract. It uses, as well for preferences as for production sets, comprehensive lattices which play the same role as the pretechnology sets introduced by Mas-Colell [21]. For preferences defined on the positive cone of a locally convex Riesz space, the E-properness defined in this paper exactly corresponds to Podczeck's E-properness, while for a production set in a locally convex Riesz commodity space, E-properness is quite similar to uniform properness as defined in Mas-Colell [21]. We introduce also a weaker properness that we call F-properness. As it will be seen, this concept, so useful for proving the existence of equilibrium in an exchange economy whose consumers have the positive cone as consumption set, is of a difficult use in a production economy or in an exchange economy with more general consumption sets.

As we look for an equilibrium existence theorem without assuming any monotonicity in preferences[1] or free-disposal in production, unlike Tourky's papers, we cannot make an argument directly in the whole commodity space. We first decentralize Edgeworth equilibria of an economy restricted to some well-chosen order ideal of the commodity space, an idea originated from Aliprantis et al. [4]. The extension of equilibrium prices to the whole commodity space is done using a technique borrowed from Podczeck [24] and adapted here to the case of a production economy.

The paper is organized as follows. In the next section, we define the model, set the main assumptions, discuss the properness definitions, state and prove theorems extending equilibrium prices of a restricted economy to continuous equilibrium prices for the initial economy. These theorems have their own interest. As a by-product, they show in particular that under E-properness (relative to the whole commodity space), a feasible allocation sustainable as a nontrivial quasiequilibrium with discontinuous prices is also sustainable as a (nontrivial) quasiequilibrium with continuous prices. Such a property, proved first by Yannelis and Zame [30] in the particular case of an exchange economy defined on a topological vector lattice having the total initial endowment as a strictly positive element, was re-proved later in several contexts ([3], [24],[27], [28]). It is obtained here in our general framework. The extension theorems are applied in Section 3 to establishing quasiequilibrium existence theorems. We then compare these results with similar ones obtained by Tourky [27], [28] in the same framework but under assumptions of strict monotonicity of preferences and free-disposal in production. The main proofs are given in the last section.

2 The model and extension results

2.1 *The model*

We consider a typical production economy in which the commodity space L is a partially ordered vector space equipped with a Hausdorff, linear topology τ. Let $I = \{1, \ldots, I\}$ and $J = \{1, \ldots, J\}$ be respectively the set of consumers and the

[1] Other than the desirability assumption involved in the formulation of E(or F)-properness.

set of firms. A consumer $i \in I$ is characterized by a consumption set $X_i \subset L$, an initial endowment $\omega_i \in L$ and a preference relation described by the point-to-set mapping $P_i : X_i \to X_i$, such that $P_i(x_i)$ denotes the set of all consumption bundles strictly preferred by the i-th agent to the bundle x_i. We will also use the notation $y_i \underset{i}{\succ} x_i$ which is equivalent to $y_i \in P_i(x_i)$. A producer (a firm) j is characterized by a production set $Y_j \subset L$. For every $j \in J$, each consumer i is also endowed with a share $\theta_i^j \geq 0$ of the profit of firm j, with $\sum_{i \in I} \theta_i^j = 1$. Let us set $\theta_i = (\theta_i^1, ..., \theta_i^J)$. The model under study is a 5-tuple

$$\mathcal{E} = (I, J, (L, \tau), (X_i, P_i, \omega_i, \theta_i)_{i \in I}, (Y_j)_{j \in J}).$$

Let us denote by $\omega = \sum_{i \in I} \omega_i$ the total resources of the economy and let

$$\mathcal{A}(\mathcal{E}) = \{(x, y) \in \prod_{i \in I} X_i \times \prod_{j \in J} Y_j \mid \sum_{i \in I} x_i = \omega + \sum_{j \in J} y_j\}$$

be the set of all *feasible allocations*. In the following, $\mathcal{A}_X(\mathcal{E})$ will denote the projection of $\mathcal{A}(\mathcal{E})$ on $\prod_{i \in I} X_i$. We first recall some definitions.

A triple (x, y, p) is said to be a *quasiequilibrium* of $\mathcal{E}$ iff $(x, y) \in \mathcal{A}(\mathcal{E})$, p is a linear functional, with $p \neq 0$, and

(i) for every $i \in I$, $p \cdot x_i = p \cdot (\omega_i + \sum_{j \in J} \theta_i^j y_j)$ and $p \cdot x'_i \geq p \cdot x_i \ \forall x'_i \in P_i(x_i)$;

(ii) for every $j \in J$, $p \cdot y'_j \leq p \cdot y_j \ \forall y'_j \in Y_j$.

This quasiequilibrium is said to be *nontrivial* if for some i_0, $\inf p \cdot X_{i_0} < p \cdot x_{i_0}$. A quasiequilibrium such that $x'_i \in P_i(x_i)$ actually implies $p \cdot x'_i > p \cdot x_i$ is a *Walrasian equilibrium*. As well known, under some continuity assumptions on preferences, classical assumptions on production and some irreducibility condition on the economy, a nontrivial quasiequilibrium is easily proved to be a Walrasian equilibrium.

On the other hand, $x \in \mathcal{A}_X(\mathcal{E})$ is said to be blocked by a nonempty coalition $B \subset I$ if there exists $x'_B \in \prod_{i \in B} X_i$ such that $\sum_{i \in B}(x'_{iB} - \omega_i) \in \sum_{i \in B} \sum_{j \in J} \theta_i^j Y_j$ and $x'_{iB} \in P_i(x_i) \ \forall i \in B$. The *core* of $\mathcal{E}$ is the set of all $x \in \mathcal{A}_X(\mathcal{E})$ which are blocked by no (nonempty) coalition. Following Aliprantis et al. ([3], [4], [5]), $x \in \mathcal{A}_X(\mathcal{E})$ is said to be an *Edgeworth equilibrium* if, for every integer $r \geq 1$, the r-repetition of x belongs to the core of the r-replication of $\mathcal{E}$. Let $\mathcal{C}^e(\mathcal{E})$ denote the set of all Edgeworth equilibria of $\mathcal{E}$. As it is easily seen and proved in Florenzano [16], under convexity assumptions for consumption and production sets, the set of all Edgeworth equilibria $\mathcal{C}^e(\mathcal{E})$ contains the set $\mathcal{C}^f(\mathcal{E})$ of all $x \in \mathcal{A}_X(\mathcal{E})$ such that there exists no $t = (t_i) \in [0,1]^I$, $t \neq 0$, and no $x'_t \in \prod_{t_i > 0} X_i$ satisfying

$$\sum_{i \in I} t_i(x'_{it} - \omega_i) \in \sum_{i \in I} t_i \sum_{j \in J} \theta_i^j Y_j$$

$$x'_{it} \in \operatorname{co} P_i(x_i) \ \forall i : t_i > 0.$$

The assumptions the economy will be required to satisfy are divided into several groups.

Structural assumptions *(SA)*

(i) L is a linear vector lattice (or Riesz space) endowed with a Hausdorff linear topology τ;

(ii) L_+ is a closed cone in the τ-topology of L;

(iii) L' is a sublattice of the order dual of L.

It is worth noticing that we do not assume the topology τ to be locally solid. Note that if L were a locally solid Riesz space then the requirements *(ii), (iii)* would be automatically valid. Since we avoid the solidness hypothesis, we need to require them explicitly. For more specific explanations and references, the reader is referred to [3], [6], [22].

The three following groups of assumptions are classical for existence of equilibrium and do not require special explanations. It should only be noticed that we do not make in *(C)* any local nonsatiation assumption. Local nonsatiation at every component of well-chosen consumption allocations will be a consequence of properness assumptions to be made later.

Consumption Assumptions *(C)*

For all $i \in I$,

(i) $X_i \subset L$ is convex, τ-closed, and $\omega_i \in X_i$;

(ii) $\forall x_i \in X_i$, the set $P_i^{-1}(x_i) = \{y_i \in X_i \mid x_i \in P_i(y_i)\}$ is $\sigma(L, L')$[2]-open in X_i; and for each $x = (x_i) \in \mathcal{A}_X(\mathcal{E})$,

(iii) (convexity and irreflexivity) $P_i(x_i)$ is convex and $x_i \notin P_i(x_i)$;

(iv) $P_i(x_i)$ is τ-open in X_i.

Production Assumption *(P)*

For all $j \in J$,

$Y_j \subset L$ is convex, τ-closed and $0 \in Y_j$.

Boundedness Assumption *(B)*

$\mathcal{A}(\mathcal{E})$ is $(\sigma(L, L'))^{|I|+|J|}$-compact.

The previous definitions and assumptions can be adapted in an obvious way to the case of a pure exchange economy

$$\mathcal{E} = (I, (L, \tau), (X_i, P_i, \omega_i)_{i \in I}).$$

As proved in Florenzano [16], under *(C (i) – (iii)), (P), (B)* for an economy $\mathcal{E}$ defined on a Hausdorff topological vector space, $\mathcal{C}^e(\mathcal{E})$ is nonempty. The same is

[2] In what follows, $\sigma(L, L')$ denotes the weak topology on L; $\operatorname{co} A$ denotes the convex hull of the set A and $\overline{A}$ is its closure.

true for the economy $\mathcal{E}_{|_K}$ where K is any τ-closed vector subspace of L containing all ω_i and $\mathcal{E}_{|_K}$ is the production economy truncated to K whose consumption and production sets are respectively $X_i \cap K$ and $Y_j \cap K$. If one assumes in addition *(C (iv))*, then $\mathcal{C}^f(\mathcal{E})$ (resp. $\mathcal{C}^f(\mathcal{E}_{|_K})$ if K is τ-closed) is also nonempty.

2.2 *Properness assumptions and extension of linear functionals*

For decentralizing Edgeworth equilibria when the commodity space L is infinite dimensional, from now on, we assume that (L, τ) satisfies *(SA)* and introduce properness assumptions.

Definition 2.1 *Let K be some order ideal of L. A preference relation $P : X \to X$ is said to be F-proper relative to K at $x \in X$ if there exists a τ-open convex subset V_x of L, a lattice $Z_x \subset K$ verifying $Z_x + K_+ \subset Z_x$ and some subset A_x of L, radial*[3] *at x, such that $x \in \overline{V}_x \cap Z_x$ and*

$$\emptyset \neq V_x \cap Z_x \cap A_x \subset P(x) \tag{2.1}$$

If, moreover,

$$P(x) \cap A_x \subset \overline{V}_x \cap (Z_x + L_+) \tag{2.2}$$

then the preference relation is said to be E-proper at $x \in X$ relative to K.

Note that it automatically follows from the previous definition that $x \in K$. To understand Definition 2.1, let us assume that (L, τ) is locally convex and that v_x, such that $x + \alpha v_x \in K_+$ for some $\alpha > 0$, and U_x are respectively a *properness vector* and a τ-open convex *properness* 0-*neighborhood*. Let Γ_x be the open convex cone with vertex 0 generated by $(\{v_x\} + U_x)$. Then a convex-valued correspondence $P : L_+ \to L_+$ is F-proper relative to K at $x \in K_+$ if there exists some $A_x \subset L$, radial at x, such that

$$(\{x\} + \Gamma_x) \cap K_+ \cap A_x \ \subset P(x)$$

a definition originated from Yannelis and Zame [30] in the case $K = L$, while P is E-proper relative to K at $x \in K_+$ if there exists some $A_x \in L$, radial at x, such that

$$x + \beta v_x \in P(x) \text{ for some } \beta > 0$$

$$(P(x) + \Gamma_x) \cap K_+ \cap A_x \ \subset P(x).$$

In the first case, $V_x = \{x\} + \Gamma_x$ and $Z_x = K_+$ fit with F-properness as defined in Definition 2.1, while in the second case, $Z_x = K_+$ and $V_x = P(x) + \Gamma_x$ satisfy conditions (2.1) and (2.2) of the same definition, which corresponds to the definition of E-properness relative to K given by Podczeck in [24].

[3] If A is a subset of a vector space L, then A is called radial at a point $x \in A$ if for each $y \in L$ there exists a real number $\bar{\lambda}$, $0 < \bar{\lambda} \leq 1$ such that $(1 - \lambda)x + \lambda y \in A$ for every λ with $0 \leq \lambda \leq \bar{\lambda}$.

From the two previous examples, it should be clear that properness assumptions on a preference relation $P: X \to X$, as stated in Definition 2.1, are as well assumptions on P as assumptions on X. As it is usually observed, a correspondence P such that $(P(x) \cap A_x) \subset (Z_x + L_+)$ is E-proper at x if $P(x) \cap A_x$ can be extended (i.e. $P(x) \cap A_x = \widehat{P}(x) \cap \underline{Z_x \cap A_x}$) to a convex set $\widehat{P}(x)$ with a τ-interior point in $Z_x \cap A_x$ and such that $x \in \overline{\widehat{P}(x)} \cap Z_x$. Such an extendibility property is precisely assumed by Tourky who defines, in [27], [28], M-properness at x of a convex-valued preference correspondence such that $x \in \overline{P(x)}$ by the condition $P(x) = \widehat{P}(x) \cap Z_x$, where Z_x is a lattice containing x, such that $Z_x + L_+ = Z_x$, and $\widehat{P}(x)$ is a convex set with an interior point in Z_x ($x + \omega$, in Tourky's context). It is also worth noticing that condition (2.1) in Definition 2.1 together with $x \in \overline{V}_x \cap Z_x$ imply that x is a point of local nonsatiation for $\operatorname{co} P$ in K.

To end with the remarks on properness of a preference relation, let us notice that assumptions of E-properness at x relative to the whole space L and of E-properness at x relative to some order ideal K of L containing x are not directly comparable, even if $V_x \cap Z_x \cap K \neq \emptyset$. We will come back to this point later.

Definition 2.2 *Let K be some order ideal of L. A set $Y \subset L$ is said to be F-proper relative to K at $y \in Y$ if there exists a τ-open convex subset V_y of L, a convex lattice $Z_y \subset K$ verifying $Z_y - K_+ \subset Z_y$ and some subset A_y of L, radial at y, such that $y \in \overline{V}_y \cap Z_y$ and*

$$\emptyset \neq V_y \cap Z_y \cap A_y \subset Y \tag{2.3}$$

If, moreover,

$$Y \cap A_y \subset \overline{V}_y \cap (Z_y - L_+) \tag{2.4}$$

then the set is said to be E-proper at $y \in Y$ relative to K.

Once again, it automatically follows from the previous definition that $y \in K$. It also follows from the comprehensivity of Z_y that Z_y is convex. In order to understand the difference between F- and E-properness at $y \in Y \cap K$, let us assume once more that (L, τ) is locally convex and that $v_y \in K_+$ and U_y are respectively a *properness vector* and a τ-open convex *properness* 0-*neighborhood.* Let Γ_y be the open convex cone with vertex 0 generated by $(\{v_y\} + U_y)$. Then Y is F-proper relative to K at $y \in Y \cap K$ if there exists some $A_y \subset L$, radial at y, such that

$$(\{y\} - \Gamma_y) \cap \{z \in K \mid z^+ \leq y^+\} \cap A_y \subset Y$$

a condition similar to the one imposed by Richard [25] if $K = L$, while Y is E-proper relative to L at $y \in Y$ if, as in Mas-Colell [21], there exists some pretechnology convex lattice Z_y such that $Y \subset Z_y$, $Z_y - L_+ \subset Z_y$ and

$$(Y - \Gamma) \cap Z_y \subset Y.$$

Here $v \in L_+$ is a vector of uniform properness of Y, U is a τ-open convex 0-neighborhood and Γ is the open convex cone with vertex 0 generated by $(\{v\}+U)$.

Note that a production set $Y = \{0\}$ is neither F-proper nor E-proper. On the other hand, $Y = -L_+$ is F-proper and E-proper, relative to any order ideal K of L, at any point $y \in Y \cap K$.

Note also that E-properness relative to L at y is implied by M-properness at y, as defined in Tourky [28]. For a (production) set as for a preference relation, assumptions of properness relative to L at a point $y \in Y$ and of properness relative to some order ideal K of L containing y are not directly comparable, even if $V_y \cap Z_y \cap K \neq \emptyset$.

Definition 2.3 *Let K be some order ideal of L. A production economy $\mathcal{E}$ is said to be F-proper (resp. E-proper) relative to K at $(x, y) \in \prod_{i \in I} X_i \times \prod_{j \in J} Y_j$ if each preference relation and each production set are F-proper (resp. E-proper) relative to K at the corresponding component of (x, y), with for each i, $\omega_i \in Z^i_{x_i}$ and for each j, $0 \in Z^j_{y_j}$ (where the sets $Z^i_{x_i}$ and $Z^j_{y_j}$ are taken from the definition of properness).*

If $\mathcal{E}$ is a pure exchange economy, $\mathcal{E}$ is F-proper (resp. E-proper) relative to K at $x \in \prod_{i \in I} X_i$ if each preference relation is F-proper (resp. E-proper) relative to K at the corresponding component of x , with for each i, $\omega_i \in Z^i_{x_i}$ (with $Z^i_{x_i}$ taken from the definition of properness).

Once again, observe that K contains for every $i \in I$, x_i, ω_i, v_i such that $x_i + v_i \in V^i_{x_i} \cap Z^i_{x_i}$, and, in case of production, for every $j \in J$, y_j, v_j such that $y_j - v_j \in V^j_{y_j} \cap Z^j_{y_j}$. Such v_i, $i \in I$ and v_j, $j \in J$ are called *properness vectors* of $\mathcal{E}$ at (x, y).

The following definition reinforces the previous one. The reason for such a definition will become clear later.

Definition 2.4 *A F-proper (resp. E-proper) relative to K production economy $\mathcal{E}$ is said to be nontrivially F-proper (resp. E-proper) if, in the previous definition, the set $\sum_{i \in I}(Z^i_{x_i} \cap L(u)) - \sum_{j \in J}(Z^j_{y_j} \cap L(u))$ is a radial at ω subset of $L(u)$, where $L(u)$[4] denotes the ideal generated in K by*

$$u = \sum_{i \in I} |\omega_i| + \sum_{i \in I} |x_i| + \sum_{j \in J} |y_j|$$

and if for every $i \in I$ and for every $j \in J$, $\mathcal{E}$ has properness vectors in $L(u)$. Such v_i, v_j are called nontrivial properness vectors of $\mathcal{E}$ at (x, y).

A pure exchange economy $\mathcal{E}$, F-proper (resp. E-proper) relative to K is said to be nontrivially F-proper (resp. E-proper) if, in the previous definition, the set $\sum_{i \in I}(Z^i_{x_i} \cap L(u))$ is a radial at ω subset of $L(u)$, where $L(u)$ denotes the ideal generated in K by

$$u = \sum_{i \in I} |\omega_i| + \sum_{i \in I} |x_i|$$

and if for every $i \in I$, $\mathcal{E}$ has properness vectors in $L(u)$. Such v_i are called nontrivial properness vectors of $\mathcal{E}$ at x.

[4]Recall that for $u \in L$, $L(u) = \{z \in L \mid \exists \lambda > 0, |z| \leq \lambda u\}$. As K is an ideal and $u \in K$, $L(u) \subset K$.

From now on, in view of these definitions, the results of this paper will be written for a production economy. Their translation for an exchange economy should be obvious.

The considerations below are based on the following auxiliary result of convex analysis, the first part of which can be found in Podczeck [24].

Lemma 2.1 (Main auxiliary lemma) *Let (L, τ) be a Hausdorff topological vector space and let K be a vector subspace of L. Let Z be a convex subset of K and V be a convex τ-open subset of L such that $V \cap Z \neq \emptyset$. If p is a linear functional on K satisfying for some $z \in \overline{V} \cap Z$*

$$p \cdot z \leq p \cdot y \quad \forall y \in V \cap Z$$

then there exists a τ-continuous linear functional $\pi \in (L, \tau)'$ and a linear functional h on L such that $p = \pi_{|K} + h_{|K}$ and

$$\pi \cdot z \leq \pi \cdot y \quad \forall y \in V, \quad h \cdot z \leq h \cdot y \quad \forall y \in Z. \tag{2.5}$$

Let us assume in addition that (L, τ) is an ordered vector space and set $K_+ = L_+ \cap K$. If $Z + K_+ \subset Z$, then $h_{|K} \geq 0$, $\pi_{|K} \leq p$ and

$$p \cdot (z - y) = \pi \cdot (z - y) \quad for\ each \quad y \leq z, \quad y \in Z. \tag{2.6}$$

Replacing the condition $Z + K_+ \subset Z$ by $Z - K_+ \subset Z$ and applying the previous lemma to the sets $-V$, $-Z$ and to the point $-z$, we get immediately the following corollary:

Corollary 2.1 *Let (L, τ) be a Hausdorff locally convex topological vector space and let K be a vector subspace of L. Let Z be a convex subset K and V be a convex τ-open subset of L such that $V \cap Z \neq \emptyset$. If p is a linear functional on K satisfying for some $z \in \overline{V} \cap Z$*

$$p \cdot z \geq p \cdot y \quad \forall y \in V \cap Z$$

then there exists a τ-continuous linear functional $\pi \in (L, \tau)'$ and a linear functional h on L, such that $p = \pi_{|K} + h_{|K}$ and

$$\pi \cdot z \geq \pi \cdot y \quad \forall y \in V, \quad h \cdot z \geq h \cdot y \quad \forall y \in Z. \tag{2.7}$$

Let us assume in addition that (L, τ) is an ordered vector space and set $K_+ = L_+ \cap K$. If $Z - K_+ \subset Z$, then $h_{|K} \geq 0$, $\pi_{|K} \leq p$ and

$$p \cdot (z - y) = \pi \cdot (z - y) \quad for\ each \quad y \geq z, \quad y \in Z. \tag{2.8}$$

Exploiting the previous lemma and its corollary, the following proposition describes the important properties of F-proper economies. Its proof makes extensive use of the Riesz decomposition property of L and the Riesz-Kantorovich formula applied to continuous linear functionals on L, both guaranteed by the structural assumptions *(SA)* on the commodity-price duality [5]. In what follows, $\mathcal{E}_{|K}$ denotes the economy truncated to K as defined above.

[5] For more on these two properties, see [7]

Proposition 2.1 *Let K be some order ideal of L containing all ω_i and let (x, y, p) be a quasiequilibrium of $\mathcal{E}_{|_K}$ with a price $p \in K^*$, the algebraic dual of K. Under Assumption (C(iii)), if $\mathcal{E}$ is F-proper relative to K at (x, y) and if the sets $V^i_{x_i}$, $Z^i_{x_i}$, $V^j_{y_j}$, $Z^j_{y_j}$ are taken from the definition of properness, then there exist, for every $t \in I \cup J$, τ-continuous functionals $\pi_t \in (L, \tau)'$ such that $\pi_{t|_K} \leq p$, and*

$$\pi_i \cdot \overline{V^i_{x_i}} \geq \pi_i \cdot x_i, \quad \pi_j \cdot \overline{V^j_{y_j}} \leq \pi_j \cdot y_j, \quad i \in I, \quad j \in J. \tag{2.9}$$

Moreover, if $\pi = (\vee_{i \in I} \pi_i) \vee (\vee_{j \in J} \pi_j)$ then $\pi \in (L, \tau)'$ and

$$\forall i \in I, \ \forall z_i \leq x_i, \quad z_i \in Z^i_{x_i}, \quad \pi_i \cdot (x_i - z_i) = \pi \cdot (x_i - z_i) = p \cdot (x_i - z_i) \tag{2.10}$$

$$\forall j \in J, \ \forall z_j \geq y_j, \quad z_j \in Z^j_{y_j}, \quad \pi_j \cdot (z_j - y_j) = \pi \cdot (z_j - y_j) = p \cdot (z_j - y_j) \tag{2.11}$$

$$\pi \cdot (\omega - z) = p \cdot (\omega - z) \quad \textit{for each} \quad z \leq \omega, \quad z \in \sum_{i \in I} Z^i_{x_i} - \sum_{j \in J} Z^j_{y_j} \tag{2.12}$$

and, finally, for every $i \in I$

$$\pi \cdot x_i = \pi \cdot \omega_i + \sum_{j \in J} \theta_{ij} \pi \cdot y_j. \tag{2.13}$$

Proposition 2.2 *Assume, in the conditions of Proposition 2.1, that for some order ideal K' of K, $\sum_{i \in I}(Z^i_{x_i} \cap K') - \sum_{j \in J}(Z^j_{y_j} \cap K')$ is a radial at ω subset of K'. Then $\pi_{|_{K'}} = p_{|_{K'}}$. Consequently, if in the conditions of Proposition 2.1, $\mathcal{E}$ is nontrivially F-proper relative to K at (x, y) with v_i, $i \in I$, v_j, $j \in J$ as nontrivial properness vectors, and if $p_{|_{L(u)}} \neq 0$, then $\pi \cdot v_i = p \cdot v_i > 0$ or $\pi \cdot v_j = p \cdot v_j > 0$, for some $i \in I$, $j \in J$.*

The following theorem is the first main result of this paper. It states that a feasible allocation being price supported in K as a nontrivial quasiequilibrium can, because of E-properness relative to K, also be price supported in L as a nontrivial quasiequilibrium. More precisely,

Theorem 2.1 *If, in the conditions of Proposition 2.1, $\mathcal{E}$ is E-proper relative to K at (x, y) and if there is $i_0 \in I$ such that $p \cdot x'_{i_0} < p \cdot x_{i_0}$ for some $x'_{i_0} \in X_{i_0} \cap Z^{i_0}_{x_{i_0}}$, then (x, y, π) is a nontrivial quasiequilibrium of $\mathcal{E}$. If $\mathcal{E}$ is nontrivially E-proper relative to K at (x, y) and if $p_{|_{L(u)}} \neq 0$, then (x, y, π) is a quasiequilibrium of $\mathcal{E}$. This quasiequilibrium is nontrivial if, in addition,*

$$v_i \in \{\omega\} + \sum_{j \in J} Y_j - \sum_{i \in I} X_i, \ v_j \in \{\omega\} + \sum_{j \in J} Y_j - \sum_{i \in I} X_i, \ i \in I, \ j \in J. \tag{2.14}$$

for some collection of nontrivial properness vectors at (x, y).

As a by-product, replacing K by L in the previous result, it should be noticed that any feasible allocation sustainable as a nontrivial quasiequilibrium with a price vector $p \notin (L, \tau)'$ such that $p \cdot x'_{i_0} < p \cdot x_{i_0}$ for some $x'_{i_0} \in X_{i_0} \cap Z^{i_0}_{x_{i_0}}$ is also sustainable as a nontrivial quasiequilibrium with continuous price if $\mathcal{E}$ is E-proper at (x, y) relative to L. We extend here a result of Podczeck [24] (Theorem 2 with Assumption A.7′) to the case of a production economy with general consumption sets. The second part of the theorem shows, without assuming any monotonicity of preferences or free-disposal in production, that any feasible allocation (x, y), sustainable as a quasiequilibrium with a price vector $p \notin (L, \tau)'$ but such that $p_{|_{L(u)}} \neq 0$ can also be sustained as a quasiequilibrium with continuous price in case of nontrivial E-properness of $\mathcal{E}$ relative to L. As we will see in the last section, this assertion extends similar results of Tourky [27], [28].

Let us now come back to the case of F-proper economies. The next proposition gives sufficient conditions to get an analogue result under F-properness. In view of Proposition 2.2 and of the proof of Theorem 2.1, it may be stated without proof.

Proposition 2.3 *Assume, in the conditions of Proposition 2.1, that $\sum_{i \in I} Z^i_{x_i} - \sum_{j \in J} Z^j_{y_j}$ is a radial at ω subset of K and that each $P_i(x_i) \cap K$ and each $Y_j \cap K$ are τ-dense in (respectively) $P_i(x_i)$ and Y_j. Then $\pi_{|_K} = p$ and (x, y, π) is a quasiequilibrium of $\mathcal{E}$. Obviously, if (x, y, p) is a nontrivial quasiequilibrium, then (x, y, π) is also nontrivial.*

In view of applying Proposition 2.2, let us assume that the order ideal K is τ-dense in L. One can wonder how to guarantee that each $P_i(x_i) \cap K$ and each $Y_j \cap K$ are τ-dense in (respectively) $P_i(x_i)$ and Y_j. Let us first remark that under *(C (iv))*, for the τ-density of preferred sets it is enough to require the τ-density of $X_i \cap K$ in X_i. One important example of set having this property is the positive cone L_+ of L, as proved by Podczeck [24] in his Lemma 3, using the same structural assumptions as our *(SA (i) – (iii))*[6]. It should be clear from this that any set of the form $\{k\} + L_+$ or $\{k\} - L_+$ for some $k \in K$ has the same property. Since K is assumed to be an order ideal of L, the same is true for any order interval with lower and upper bounds in K (note that, in view of *(SA)(ii)*, each such interval is also τ-closed). The following lemma and its obvious corollary give sufficient conditions for density which may have an economic interpretation.

Lemma 2.2 *Let some $Z \subset L$ be τ-closed, K be τ-dense in L and every $z \in Z$ satisfy*
– either there exists $a_z \in K$, $a_z \geq z$ and some τ-open, convex $U_z \subset L$ such that

$$\emptyset \neq U_z \cap (\{a_z\} - L_+) \subset Z, \quad \& \quad z \in \overline{U}_z \cap (\{a_z\} - L_+), \tag{2.15}$$

– or there exists $a_z \in K$, $a_z \leq z$ and some τ-open, convex $U_z \subset L$ such that

$$\emptyset \neq U_z \cap (\{a_z\} + L_+) \subset Z, \quad \& \quad z \in \overline{U}_z \cap (\{a_z\} + L_+). \tag{2.16}$$

Then $\overline{K \cap Z} = Z$.

[6] The proof is given in Podczeck [24] in the case where K is the order ideal generated by a positive element of L, but it is easily checked that the same proof can be given for any other order ideal τ-dense in L.

Corollary 2.2 *Let some closed $Z \subset L$ admit a representation of the form*

$$Z = \overline{U} \cap (B - L_+)$$

or of the form

$$Z = \overline{U} \cap (B + L_+),$$

for some τ-open convex set U and $B \subset K$ satisfying for every $b \in B$, $(\{b\} - L_+) \cap U \neq \emptyset$ in the first case and $(\{b\} + L_+) \cap U \neq \emptyset$ in the second case. Then $\overline{K} = L$ implies $\overline{K \cap Z} = Z$.

The conditions (2.15), (2.16) imposed in the lemma can be interpreted as a kind of "upper-properness" at the point z relative to K ("lower-properness" respectively), required for consumption sets and production sets in addition to the properness of preferences and production. In the properties postulated in Lemma 2.2, one restrictive point is the fact that every $z \in Z$ is supposed to have an upper (lower) bound in K. But being applied to consumption or production sets, this requirement may be economically interpreted. In fact, taking into account that $\omega_i \in K$ for each i, one may postulate that firms' inputs (outputs) have to be chosen from $L(\overline{\omega}) \subset K$, the order ideal generated in K by $\overline{\omega} = \sum_i |\omega_i|$. Analogous hypothesis may be taken for the admissible consumption plans of consumers[7].

As shown in Corollary 2.2, consumption sets of the form $X_i = Z_i \cap (\{a_i\} + L_+)$, with $a_i \in K$, Z_i convex, τ-closed, with an interior point in $(\{a_i\} + L(\overline{\omega})_+)$, satisfy (2.16). This kind of assumption is evoked by Podczeckin [24] as allowing, for an exchange economy, an extension of his equilibrium existence result under F-properness. A similar hypothesis for production sets seems more questionable.

3 Application to the quasiequilibrium existence problem

As already noticed, under Assumptions *(C(i)–(iv)), (P), (B)*, we have $\mathcal{C}^f(\mathcal{E}) \neq \emptyset$. We now start from an element $(x, y) \in \mathcal{C}^f(\mathcal{E})$ and assume *(SA)* and that K is a principal order ideal in L[8]. The next proposition proves the existence of $p \in (K, \leq)^\sim$ such that (x, y, p) is a quasiequilibrium of $\mathcal{E}_{|K}$.

Proposition 3.1 *Let K be a principal ideal of L and let $(x, y) \in \mathcal{C}^f(\mathcal{E})$. Under Assumptions (SA), (C (i), (iii), (iv)) and (P), if $\mathcal{E}$ is F-proper relative to K at (x, y), then there exists $p \in (K, \leq)^\sim$ such that (x, y, p) is a quasiequilibrium of $\mathcal{E}_{|K}$.*

If we restrict ourself to the case $K = L(u)$, combining Proposition 3.1 with Theorem 2.1, we get immediately the main result of this section.

[7] Note that such a kind of property automatically holds true if $X_i \subset L_+$ (since $0 \in K$).

[8] In other words, K is a solid Riesz subspace of L with an order unit e, equivalently $K = L(e)$, the principal order ideal generated in L by e.

Theorem 3.1 *Let $(x, y) \in \mathcal{C}^f(\mathcal{E})$. Under Assumptions (SA), (C (i), (iii), (iv)) and (P), if $\mathcal{E}$ is nontrivially E-proper relative to $L(u)$ at (x, y), then there exists $\pi \in (L, \tau)'$ such that (x, y, π) is a nontrivial quasiequilibrium of $\mathcal{E}$. This quasiequilibrium is nontrivial if nontrivial properness vectors can be chosen so as to verify (2.14). Consequently, under all assumptions made in Section 2.1 on $\mathcal{E}$, and if $\mathcal{E}$ is nontrivially E-proper, relative to $L(u)$, at every $(x, y) \in \mathcal{C}^f(\mathcal{E})$, then there exists a nontrivial quasiequilibrium $(\bar{x}, \bar{y}, \bar{\pi})$ with $\bar{\pi} \in (L, \tau)'$ and this quasiequilibrium is nontrivial if the nontrivial properness vectors at each $(x, y) \in \mathcal{C}^f(\mathcal{E})$ can be chosen so as to verify (2.14).*

Theorem 3.1 can be considered as extending an existence result (Theorem 1, with Assumption A.7) in Podczeck [24] to a production economy with general consumption sets. Indeed, for an exchange economy with consumption sets equal to the positive orthant, $L(u) = L(\omega)$ and we have already noticed that E-properness relative to $L(\omega)$, as defined in [24], implies E-properness relative to $L(\omega)$, as defined in this paper, with comprehensive lattices $Z^i_{x_i}$ all equal to $L(\omega)_+$. If $\omega > 0$, it is obvious that $L(\omega)_+$ is radial at ω in $L(\omega)$. The properness vectors, belonging to $L(\omega)$, are nontrivial and the reader will easily verify that $\mathcal{E}$ has also nontrivial properness vectors satisfying (2.14).

With production and general consumption sets, contrary to the previous case, the ideal $L(u)$ depends on the particular allocation at which are stated the properness assumptions. We now give conditions which imply that an economy F(resp. E)-proper at a feasible allocation (x, y) with properness vectors in $L(u)$ is also nontrivially F(resp. E)-proper at (x, y).

Lemma 3.1 *Assume that $\omega > 0$ and that $0 \in Z^i_{x_i}$, for every $i \in I$. Then*

$$[\omega - \frac{1}{4}u, \omega + \frac{1}{4}u] \subset \sum_{i \in I}(Z^i_{x_i} \cap L(u)) - \sum_{j \in J}(Z^j_{y_j} \cap L(u))$$

and the set $\sum_{i \in I}(Z^i_{x_i} \cap L(u)) - \sum_{j \in J}(Z^j_{y_j} \cap L(u))$ is a radial at ω subset of $L(u)$.

Finally, let us let us assume, as suggested above, that firms' inputs (outputs) have to be chosen from $L(\overline{\omega})$, the principal order ideal generated by $\overline{\omega} = \sum_{i \in I} |\omega_i|$, while admissible consumption plans of consumers are to have their positive (negative) part in $L(\overline{\omega})$. As already noticed, such a condition is automatically satisfied in an exchange economy with consumption sets included in L_+ or having an inferior bound in $L(\overline{\omega})$. Then $\mathcal{A}(\mathcal{E}) = \mathcal{A}(\mathcal{E}_{|_{L(\overline{\omega})}})$ and $L(u) = L(\overline{\omega})$ does not depend on the particular attainable allocation (x, y). The following proposition is a corollary of Theorem 3.1.

Proposition 3.2 *Let us assume (SA), (C), (P) (B) and that $\mathcal{A}(\mathcal{E}) = \mathcal{A}(\mathcal{E}_{|_{L(\overline{\omega})}})$. If $\mathcal{E}$ is nontrivially E-proper, relative to $L(\overline{\omega})$, at every $(x, y) \in \mathcal{C}^f(\mathcal{E})$, then there exists a nontrivial quasiequilibrium $(\bar{x}, \bar{y}, \bar{\pi})$ with $\bar{\pi} \in (L, \tau)'$.*

The same conclusion holds true if the order ideal $L(\overline{\omega})$ is τ-closed[9], since then $\mathcal{C}^f(\mathcal{E}_{|_{\overline{L(\overline{\omega})}}})$ is nonempty.

Proposition 3.3 *Let us assume (SA), (C), (P) (B) and that the order ideal $L(\overline{\omega})$ is τ-closed. If $\mathcal{E}$ is nontrivially E-proper, relative to $L(\overline{\omega})$, at every $(x, y) \in \mathcal{C}^f(\mathcal{E}_{|_{L(\overline{\omega}}})$, then there exists a nontrivial quasiequilibrium $(\bar{x}, \bar{y}, \bar{\pi})$ with $\bar{\pi} \in (L, \tau)'$.*

More important is the following result where we assume that $L(\overline{\omega})$ is τ-dense in L. We omit its immediate proof which combines Proposition 3.1 with Proposition 2.3.

Theorem 3.2 *Let us assume (SA), (C), (P) (B) and that $L(\overline{\omega})$ is τ-dense in L. Let us assume also either that*

(i) $\forall i$, $X_i = (\{a_i\} + L_+) \cap Z_i$, with $a_i \in L(\overline{\omega})$, Z_i convex, τ-closed, with an interior point in $(\{a_i\} + L_+)$

(ii) and, in case of production, $\forall j$, $Y_j = (\{b_j\} - L_+) \cap Z_j$, with $b_j \in L(\overline{\omega})$, Z_j convex, τ-closed, with an interior point in $(\{b_j\} - L_+)$

or that

(iii) $\forall i$, $X_i = (\{a_i\} - L_+) \cap Z_i$, with $a_i \in L(\overline{\omega})$, Z_i convex, τ-closed, with an interior point in $(\{a_i\} - L_+)$

(iv) and, in case of production, $\forall j$, $Y_j = (\{b_j\} + L_+) \cap Z_j$, with $b_j \in L(\overline{\omega})$, Z_j convex, τ-closed, with an interior point in $(\{b_j\} + L_+)$.

Then $\mathcal{A}(\mathcal{E}) = \mathcal{A}(\mathcal{E}_{|_{L(\overline{\omega})}})$ and if $\mathcal{E}$ is nontrivially F-proper relative to $L(\overline{\omega})$ at every $(x, y) \in \mathcal{C}^f(\mathcal{E})$, then $\mathcal{E}$ has a nontrivial quasiequilibrium $(\bar{x}, \bar{y}, \bar{\pi})$ with $\bar{\pi} \in (L, \tau)'$.

It is worth noticing that Theorem 3.2 extends Theorem 1 with Assumption A.6 in Podczeck [24], thus Araujo and Montero [9], to a production economy with general consumption sets. Obviously, the quasiequilibrium obtained in Propositions 3.2, 3.3 and Theorem 3.2 is nontrivial if at each $(x, y) \in \mathcal{C}^f(\mathcal{E})$ ($\mathcal{C}^f(\mathcal{E}_{|_{L(\overline{\omega})}})$ for Proposition 3.3), some collection of nontrivial properness vectors satisfies (2.14).

4 Concluding remarks

Our main concern in this paper was to obtain extension and quasiequilibrium existence results without assuming monotonicity of preferences or free-disposal in production. As already noticed, our results extend Podczeck's results to a production economy with general consumption sets. These results are now to be compared with those obtained by Tourky [27], [28] in the same framework

[9]Note that depending on the properties of L and of $\overline{\omega}$ relative to L, $L(\overline{\omega})$ may be τ-closed, τ-dense in L (obviously, it may have both properties). In most of cases, it is neither closed nor τ-dense.

but with a technique of proof which heavy relies on both assumptions of strict monotonicity of preferences and free disposal in production.

We have already observed that the E-properness used in this paper is similar to Tourky's M-properness, as far as E-properness is relative to L, the whole commodity space. More precisely, at $(x, y) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$, Tourky's M-properness implies E-properness relative to L with ω as a common properness vector and it follows form Lemma 3.1 that if $\omega > 0$ and if $0 \in Z^i_{x_i}, \forall i \in I$, then ω is a common nontrivial properness vector. Let then (x, y, p) be an equilibrium of $\mathcal{E}$. As, in view of F-properness at (x, y), $p \cdot \omega > 0$, it follows from Theorem 2.1 that there exists $\pi \in (L, \tau)'$ such that (x, y, π) is a quasiequilibrium of $\mathcal{E}$, as in Corollary 2.2 of [27] and Theorem 2.2 of [28]. Contrary to [27], the conclusion of Tourky's Corollary 2.2 is obtained here without assuming that preferences are monotone at each component of x; contrary to [27], the conclusion of Tourky's Theorem 2.2 is obtained here without assuming free-disposal in production.

On the other hand, in Theorem 3.1 our main existence result, we required at $(x, y) \in \mathcal{C}^f(\mathcal{E})$, nontrivial E-properness relative to $L(u)$, an assumption which is, in general, neither stronger nor weaker than M-properness with nontrivial properness vectors in $L(u)$. Defining *M-properness at (x, y) relative to some order ideal K* by the conditions:

- $\forall i \in I,\ P_i(x) = \widehat{P}_i(x_i) \cap (Z^i_{x_i} + L_+)$ where
  $\widehat{P}_i(x_i)$ is a convex set such that $x_i \in \overline{\widehat{P}_i(x_i)}$ and $x_i + \omega \in \widehat{P}_i(x_i)$;
  $Z^i_{x_i}$ is a sublattice of K containing 0, x_i, ω_i, such that $Z^i_{x_i} + K_+ \subset Z^i_{x_i}$
- $\forall j \in J,\ Y_j = \widehat{Y}_j(y_j) \cap (Z^j_{y_j} - L_+)$ where
  $\widehat{Y}_j(y_j)$ is a convex set such that $y_j - \omega \in \text{int}\widehat{Y}_j(y_j)$;
  $Z^j_{y_j}$ is a sublattice of K containing y_j, 0, such that $Z^j_{y_j} - K_+ \subset Z^j_{y_j}$

Theorem 3.1 has, as corollary, the following Tourky-like theorem:

Corollary 4.1 (to Theorem 3.1) *Let $(x, y) \in \mathcal{C}^f(\mathcal{E})$. Under Assumptions (SA), (C (i), (iii), (iv)) and (P), if $\mathcal{E}$ is M-proper at (x, y) relative to $L(u)$ and if $\omega > 0$, then there exists $\pi \in (L, \tau)'$ such that $\pi \cdot \omega > 0$ and (x, y, π) is a quasiequilibrium of $\mathcal{E}$.*

Here M-properness relative to $L(u)$ is a stronger assumption than M-properness relative to L, but the existence of quasiequilibrium is obtained without assuming monotonicity or free-disposal. For this reason, our main existence result, which allows to get the existence of quasiequilibrium in a case not covered by Tourky, is a complement rather than a substitute to Tourky's existence results. It is still for us an open question to know if Tourky's equilibrium existence results could be deducted from the ours.

5 Proofs

Proof of Lemma 2.1. As already said, the first part of this lemma follows from Lemma 2 in Podczeck [24][10]. Let us prove the second part.

Since $Z + K_+ \subset Z$, from (2.5) we conclude $h \cdot y \geq 0$ $\forall y \in K_+$, hence $h_{|_K} \geq 0$ and $\pi_{|_K} \leq p$. To prove (2.6), take any $y \in Z$, $y \leq z$. Now (2.5) gives $h \cdot y \geq h \cdot z$, but since $z - y \geq 0$, $z - y \in K$ and $h_{|_K} \geq 0$, we have $h \cdot z \geq h \cdot y$ and conclude $h \cdot (z - y) = 0$. From $p = \pi_{|_K} + h_{|_K}$, we deduce

$$p \cdot (z - y) = \pi \cdot (z - y) + h \cdot (z - y) = \pi \cdot (z - y).$$

□

Proof of Proposition 2.1. Since (x, y, p) is a quasi-equilibrium of $\mathcal{E}_{|_K}$, formulas (i)-(ii) from the definition of quasiequilibrium are fulfilled for $P_i(x) \cap K$ and $Y_j \cap K$ respectively. Now let us specify the cones

$$\Gamma_i = \{\lambda(v - x_i) \mid v \in V^i_{x_i}, \lambda > 0\}, \quad i \in I,$$

$$\Gamma_j = \{\lambda(v - y_j) \mid v \in V^j_{y_j}, \lambda > 0\}, \quad j \in J.$$

These cones are obviously convex and open. In view of (2.1), (2.3), since A_{x_i} and A_{y_j} are assumed to be radial in L at the points x_i, y_j respectively, we see that

$$\emptyset \neq (\{x_i\} + \Gamma_i) \cap Z^i_{x_i} \subset \{x_i\} + \{\lambda(v - x_i) \mid v \in \operatorname{co} P_i(x_i) \cap K, \lambda > 0\}$$

and

$$\emptyset \neq (\{y_j\} + \Gamma_j) \cap Z^j_{y_j} \subset \{y_j\} + \{\lambda(v - y_j) \mid v \in Y_j \cap K, \lambda > 0\}.$$

Therefore, using (i), (ii) from the definition of quasiequilibrium, we can conclude that

$$p \cdot ((\{x_i\} + \Gamma_i) \cap Z^i_{x_i}) \geq p \cdot x_i, \quad i \in I,$$
$$p \cdot ((\{y_j\} + \Gamma_j) \cap Z^j_{y_j}) \leq p \cdot y_j, \quad j \in J.$$

From $x_i \in \overline{V^i_{x_i}}$ and $y_j \in \overline{V^j_{y_j}}$, we deduce $x_i \in \overline{(\{x_i\} + \Gamma_i) \cap Z^i_{x_i}}$ and $y_j \in \overline{(\{y_j\} + \Gamma_j) \cap Z^j_{y_j}}$. Applying our *Main auxiliary lemma* (Lemma 2.1) and its corollary, we get the existence of τ-continuous functionals $\pi_i, \pi_j \in (L, \tau)'$ such that

$$\pi_i \cdot (\{x_i\} + \Gamma_i) \geq \pi_i \cdot x_i, \;\; \pi_i \cdot (x_i - z_i) = p \cdot (x_i - z_i), \; \forall z_i \leq x_i, \; z_i \in Z^i_{x_i}, \;\; i \in I \quad (5.1)$$

and

$$\pi_j \cdot (\{y_j\} + \Gamma_j) \leq \pi_j \cdot y_j, \;\; \pi_j \cdot (z_j - y_j) = p \cdot (z_j - y_j), \; \forall z_j \geq y_j, \; z_j \in Z^j_{y_j}, \;\; j \in J. \quad (5.2)$$

[10] Podczeck assumes in this lemma that (L, τ) is locally convex but does not use this extra assumption in his proof.

Clearly by the Γ_i and Γ_j specification and due to the continuity of π_i, π_j, the latter ones implies (2.9). Also, due to assumptions $Z^i_{x_i}+K_+\subset Z^i_{x_i}$ and $Z^j_{y_j}-K_+\subset Z^j_{y_j}$, these functionals satisfy

$$\pi_{t_{|K}}\leq p,\ \forall t\in I\cup J.$$

Now we set $\pi=(\vee_{i\in I}\pi_i)\vee(\vee_{j\in J}\pi_j)$ and note that, in view of *(SA)(iii)*, π is a τ-continuous linear functional. Also, since K was assumed to be an order ideal of L, we have

$$p\geq\pi_{|K},\quad \pi\geq\pi_t,\quad t\in I\cup J. \tag{5.3}$$

This, in view of (5.1), gives for $z_i\leq x_i$, $z_i\in Z^i_{x_i}$

$$\pi\cdot(x_i-z_i)\geq\pi_i\cdot(x_i-z_i)=p\cdot(x_i-z_i)$$

and

$$p\cdot(x_i-z_i)\geq\pi\cdot(x_i-z_i),$$

that yields (2.10). Analogously (5.2) and (5.3) yields (2.11).

To prove (2.12), let us recall that (x,y) is feasible, i.e.

$$\sum_{i\in I}x_i=\sum_{i\in I}\omega_i+\sum_{j\in J}y_j.$$

Taking any $z_i\leq x_i$, $z_i\in Z^i_{x_i}$ and any $z_j\geq y_j$, $z_j\in Z^j_{y_j}$, we can write

$$\sum_{i\in I}(x_i-z_i)+\sum_{j\in J}(z_j-y_j)=\sum_{i\in I}\omega_i+\sum_{j\in J}z_j-\sum_{i\in I}z_i.$$

Applying p to the left and to the right sides of this equality and using (2.10), (2.11), we conclude

$$\begin{aligned}\pi\cdot(\omega+\sum_{j\in J}z_j-\sum_{i\in I}z_i)=p\cdot(\omega+\sum_{j\in J}z_j-\sum_{i\in I}z_i),\\ \forall z_i\leq x_i, z_i\in Z^i_{x_i}\text{ and }\forall z_j\geq y_j, z_j\in Z^j_{y_j}.\end{aligned} \tag{5.4}$$

Now let $u=\sum_{i\in I}v_i-\sum_{j\in J}u_j$, $v_i\in Z^i_{x_i}$, $u_j\in Z^j_{y_j}$ and $u\leq\omega$. Let $z_i=x_i\wedge v_i$, $z_j=y_j\vee u_j$ for all i,j. Since Z^i_x and Z^j_y were assumed to be lattices, we have $z_i\in Z^i_{x_i}$ and $z_j\in Z^j_{y_j}$ for all i,j and also

$$u'\leq u\leq\omega,\ u'=\sum_i z_i-\sum_j z_j,$$

and

$$\sum_i(x_i-z_i)+\sum_j(z_j-y_j)=\omega-u'\geq u-u'\geq 0.$$

Using Riesz's decomposition property (see for example th.1.2, p.3 in Aliprantis and Burkinshaw [6]), we may find $v'_i,\ i\in I$ and $u'_j,\ j\in J$ such that

$$0\leq v'_i\leq x_i-z_i,\ 0\leq u'_j\leq z_j-y_j,$$

and $\sum v_i' + \sum u_j' = u - u'$. Remembering that K is an ideal and that $Z^i_{x_i} + K_+ \subset Z^i_{x_i}$, $Z^j_{y_j} - K_+ \subset Z^j_{y_j}$, one gets $v_i', u_j' \in K$ and

$$x_i \geq v_i' + z_i \in Z^i_{x_i},\ y_j \leq z_j - u_j' \in Z^j_{y_j},\ \forall i \in I,\ \forall j \in J. \tag{5.5}$$

Now

$$u = u' + \sum_{i \in I} v_i' + \sum_{j \in J} u_j' = \sum_{i \in I}(z_i + v_i') - \sum_{j \in J}(z_j - u_j') \Rightarrow$$

$$\omega - u = \sum_{i \in I}(x_i - (z_i + v_i')) - \sum_{j \in J}(y_j - (z_j - u_j')),$$

that, in view of (5.4), (5.5), gives us the result.

To end the proof, we need to verify (2.13). Since we assumed $\omega_i \in Z^i_{x_i}$ for each $i \in I$, we can find $z_i \in Z^i_{x_i}$ such that $z_i \leq x_i \wedge \omega_i$. Now subtracting $p \cdot z_i$ from the left and the right sides of equalities $p \cdot x_i = p \cdot \omega_i + \sum_j \theta_i^j p \cdot y_j$, and using (2.10), we obtain for every $i \in I$

$$\pi \cdot (x_i - z_i) = p \cdot (\omega_i - z_i) + \sum_{j \in J} \theta_i^j p \cdot y_j.$$

Analogously, since $0 \in Z^y_{y_j}$, , we have $y_j^+ \in Z^y_{y_j}$ for each $j \in J$. Now adding $\sum_j \theta_i^j p \cdot y_j^+$ to the right and the left sides of latter equalities and using (2.11), we get

$$\pi \cdot (x_i - z_i) + \sum_{j \in J} \theta_i^j \pi \cdot (y_j^+ - y_j) = p \cdot (\omega_i - z_i) + \sum_{j \in J} \theta_i^j p \cdot y_j^+,\ \forall i \in I.$$

But $\omega_i \geq z_i$, $y_j^+ \geq 0$ and $p \geq \pi_{|K}$ imply

$$p \cdot (\omega_i - z_i) + \sum_{j \in J} \theta_i^j p \cdot y_j^+ \geq \pi \cdot (\omega_i - z_i) + \sum_{j \in J} \theta_i^j \pi \cdot y_j^+,\ \forall i \in I$$

hence

$$\pi \cdot x_i \geq \pi \cdot \omega_i + \sum_{j \in J} \theta_i^j \pi \cdot y_j,\ \forall i \in I.$$

That all these inequalities are actually equalities comes from the fact that (x, y) is feasible. □

Proof of Proposition 2.2. Let us apply Proposition 2.1 and consider $\pi = (\vee_{i \in I} \pi_i) \vee (\vee_{j \in J} \pi_j)$, such that π, π_i and π_j satisfy the relations (2.9)–(2.13). By *(SA)(iii)*, we have $\pi \in (L, \tau)'$. We first show that $\pi_{|_{K'}} = p$.

Take any $y \in K'$. Since K' is an ideal of L we have $y^+, y^- \in K'$. Now since $Z = \sum_{i \in I}(Z^i_{x_i} \cap K') - \sum_{j \in J}(Z^j_{y_j} \cap K')$ is radial in K' at the point ω, we may find real $\lambda > 0$ such that $\omega - \lambda y^+, \omega - \lambda y^- \in Z$, that due to (2.12) gives

$$\pi \cdot (\omega - (\omega - \lambda y^+)) = p \cdot (\omega - (\omega - \lambda y^+)) \Rightarrow \lambda \pi \cdot y^+ = \lambda p \cdot y^+ \Rightarrow \pi \cdot y^+ = p \cdot y^+$$

and analogously $\pi \cdot y^- = p \cdot y^-$. Since $y = y^+ - y^-$, we conclude $\pi_{|_{K'}} = p_{|_{K'}}$.

To prove the second part of the proposition, recall that for every $i \in I$, $\pi_i \cdot \overline{V^i_{x_i}} \geq \pi_i \cdot x_i$, $x_i + v_i \in V^i_{x_i} \cap Z^i_{x_i}$ and for every $j \in J$, $\pi_j \cdot \overline{V^j_{y_j}} \leq \pi_j \cdot y_j$, $y_j - v_j \in V^j_{y_j} \cap Z^j_{y_j}$. In view of the openness of $V^i_{x_i}$, the relation $\pi_i \cdot (x_i + v_i) = \pi_i \cdot x_i$ implies $\pi_i = 0$. Analogously, the relation $\pi_j \cdot (y_j - v_j) = \pi_i \cdot y_j$ implies $\pi_j = 0$. As $\pi_i = 0$, $\forall i \in I$ and $\pi_j = 0$, $\forall j \in J$ imply, by the definition of π, $\pi = 0$ which contradicts $\pi_{|_{L(u)}} = p_{|_{L(u)}} \neq 0$, it follows that there exists some π_i or π_j such that $\pi \cdot v_i > 0$ or $\pi_j \cdot v_j > 0$.

Let, without loss of generality, $i_0 \in I$ be such that $\pi_{i_0} \cdot v_{i_0} > 0$. Define $z_{i_0} = (x_{i_0} + v_{i_0}) \wedge x_{i_0}$. We note that $z_{i_0} \in Z^{i_0}_{x_{i_0}}$. Applying Proposition 2.1, we get: $\pi \cdot v_{i_0} = p \cdot v_{i_0} = p \cdot (x_{i_0} + v_{i_0} - z_{i_0}) + p \cdot (z_{i_0} - x_{i_0}) \geq \pi_{i_0} \cdot (x_{i_0} + v_{i_0} - z_{i_0}) + \pi_{i_0} \cdot (z_{i_0} - x_{i_0}) = \pi_{i_0} \cdot v_{i_0} > 0$.

Proof of Theorem 2.1. For each $i \in I$, take and fix any $x'_i \in P_i(x_i) \cap A_{x_i}$, where the set A_{x_i}, radial at x_i in L, is chosen from (2.2). Due to (2.2) and to the convexity of $P_i(x_i)$, one can find $u_i \in Z^i_{x_i}$ such that $u_i \leq x'_i$ and define $v_i = u_i \wedge x_i$. In view of the assumptions, we conclude $v_i \in Z^i_{x_i}$. Now applying (2.9), we obtain $\pi_i \cdot (x'_i - v_i) \geq \pi_i \cdot (x_i - v_i)$. This, (2.10) and $\pi \geq \pi_i$ allows us to write

$$\pi \cdot (x'_i - x_i) + p \cdot (x_i - v_i) = \pi \cdot (x'_i - x_i) + \pi \cdot (x_i - v_i) = \pi \cdot (x'_i - v_i) \geq \\ \pi_i \cdot (x'_i - v_i) \geq \pi_i \cdot (x_i - v_i) = p \cdot (x_i - v_i)$$

which implies $\pi \cdot (x'_i - x_i) \geq 0$. We have thus proved for each $i \in I$

$$\pi \cdot (\operatorname{co} P_i(x_i) \cap A_{x_i}) \geq \pi \cdot x_i.$$

Since A_{x_i} is radial at x_i in L, we conclude

$$\pi \cdot P_i(x_i) \geq \pi \cdot x_i, \ \forall i \in I,$$

that proves condition *(i)* of the definition of quasiequilibrium.

The proof of *(ii)* can be given symmetrically. Let j and $y'_j \in Y_j \cap A_{y_j}$ be fixed. Due to (2.4), one can find $u_j \in Z^j_{y_j}$ such that $u_j \geq y'_j$. Define $v_j = u_j \vee y_j$. Now (2.9) gives $\pi_j \cdot (y'_j - v_j) \leq \pi_j \cdot (y_j - v_j)$. This, (2.11) and $\pi \geq \pi_j$ allow us to write

$$\pi \cdot (y'_j - y_j) + p \cdot (y_j - v_j) = \pi \cdot (y'_j - y_j) + \pi \cdot (y_j - v_j) = \pi \cdot (y'_j - v_j) \leq \\ \pi_j \cdot (y'_j - v_j) \leq \pi_j \cdot (y_j - v_j) = p \cdot (y_j - v_j)$$

which implies $\pi \cdot (y'_j - y_j) \leq 0$. We have thus proved for every $j \in J$,

$$\pi \cdot (Y_j \cap A_{y_j}) \leq \pi \cdot y_j.$$

Since each A_{y_j} is radial at y_j in L, we conclude

$$\pi \cdot Y_j \leq \pi \cdot y_j, \ \forall j \in J.$$

At this point, let us remark that we did not prove that $\pi \neq 0$. It will follow from the nontriviality of (x, y, π). To finish the proof, we first show the nontriviality of (x, y, π) under the additional assumption that $p \cdot x'_{i_0} < p \cdot x_{i_0}$ for some $i_0 \in I$ and some $x'_{i_0} \in X_{i_0} \cap Z^{i_0}_{x_{i_0}}$. Indeed, let us set $z = x_{i_0} \wedge x'_{i_0}$, getting $z \in Z^{i_0}_{x_{i_0}}$. Using $x'_{i_0} - z \geq 0$, that, by $p \geq \pi_{|K}$, implies $p \cdot (x'_{i_0} - z) \geq \pi \cdot (x'_{i_0} - z)$ and remembering (2.10), one can write

$$\pi \cdot (x'_{i_0} - x_{i_0}) = \pi \cdot (x'_{i_0} - z) + \pi \cdot (z - x_{i_0}) \\ \leq p \cdot (x'_{i_0} - z) + p \cdot (z - x_{i_0}) = p \cdot (x'_{i_0} - x_{i_0}) < 0,$$

which proves in this case that (x, y, π) is nontrivial.

In the other case, it suffices to apply Proposition 2.2 and to show that (x, y, π) is a quasiequilibrium of $\mathcal{E}$ and to use (2.14) to see that (x, y, π) is nontrivial if nontrivial properness vectors at (x, y) satisfy (2.14). □

Proof of Lemma 2.2. Let us suppose (2.15) for some $z \in Z$, that is,

$$\emptyset \neq U_z \cap (\{a_z\} - L_+) \subset Z \text{ and } z \in \overline{U_z} \cap (\{a_z\} - L_+)$$

and prove that

$$z \in \overline{U_z \cap (\{a_z\} - L_+) \cap K} \subset \overline{Z \cap K}.$$

For all $z' \in U_z \cap (\{a_z\} - L_+)$, for all $\lambda : 0 < \lambda \leq 1$, using the fact that U_z is convex and τ-open, we get $z + \lambda(z' - z) \in U_z \cap (\{a_z\} - L_+)$. Thus $z \in \overline{U_z \cap (\{a_z\} - L_+)} \subset Z$. Now, as proved in Podczeck, $\overline{(\{a_z\} - L_+) \cap K} = \{a_z\} - L_+$. Since U_z is τ-open, one has also

$$\overline{U_z \cap \overline{(\{a_z\} - L_+) \cap K}} = \overline{U_z \cap (\{a_z\} - L_+) \cap K},$$

hence $z \in \overline{U_z \cap (\{a_z\} - L_+) \cap K} \subset \overline{Z \cap K}$.
The case of (2.16) can be considered symmetrically.

Now since z is an arbitrary point of Z, we have $Z \subset \overline{Z \cap K}$. The reverse inclusion $\overline{Z \cap K} \subset Z$ follows from the assumption that Z is τ-closed. □

Proof of Proposition 3.1. By assumption, K is a principal ideal of L generated by (for example) $e \in L$, $e > 0$, which, in view of the F-properness of $\mathcal{E}$ relative to K at (x, y), contains each x_i, ω_i, $i \in I$, y_j, $j \in J$. Let us recall that $K = L(e)$ can be endowed with the Riesz norm $\|x\|_e = \inf\{\lambda > 0 \mid |x| \leq \lambda e\}$, so that the unit-ball is the order interval $[-e, +e]$. Moreover, as remarked by Podczeck [24], the fact that on $L(e)$ the norm topology is finer than the topology induced by τ still holds true when the assumptions *(SA)* replace the classical assumption that L is locally convex-solid. In fact, due to $SA(iii)$, the unit-ball is $\sigma(L', L)$-bounded, thus τ-bounded. In what follows, we will write $L(e)'$ for $(L(e), \|\cdot\|_e)'$.

Let

$$G \stackrel{\text{def}}{=} \text{co}\Big(\bigcup_{i \in I} (P_i(x_i) \cap L(e) - \sum_{j \in J} \theta_i^j (Y_j \cap L(e)) - \{\omega_i\})\Big).$$

From F-properness at x_i of preference P_i together with the assumptions on $\mathcal{E}$, we deduce that $G \neq \emptyset$ and that $x_i \in \overline{\text{co}\, P_i(x_i) \cap L(e)}^{\|\cdot\|_e}$. It is easily verified that $(x, y) \in \mathcal{C}^f(\mathcal{E})$ implies $(x, y) \in \mathcal{C}^f(\mathcal{E}_{|_{L(e)}})$, hence $0 \notin G$. On the other hand, it

follows also from the F-properness at x_i of P_i that G has a nonempty $\|\cdot\|_e$-interior. The rest of the proof is routine. From the classical separation theorem, there exists $p \in L(e)'$ such that $x'_i \in P_i(x_i) \cap L(e)$ implies $p\cdot x'_i \geq p\cdot x_i = p\cdot\omega_i + \sum_j \theta_i^j p\cdot y_j$ and $y'_j \in Y_j \cap L(e)$ implies $p\cdot y'_j \leq p\cdot y_j$. Then (x, y, p) is a quasiequilibrium of $\mathcal{E}_{|K}$. Since $L(e)$, endowed with the Riesz norm, is a Banach lattice, $p \in (L(e), \leq)^{\sim}$.
□

Proof of Lemma 3.1. Recall that $u = \sum_{i\in I} |\omega_i| + \sum_{i\in I} |x_i| + \sum_{j\in J} |y_j|$. Let us define

$$v = \frac{1}{2}(\sum_{i\in I} \omega_i^+ + \sum_{i\in I} x_i^+ + \sum_{j\in J} y_j^-).$$

Easy calculations show that $\omega - v = -\frac{1}{2}(\sum_{i\in I} \omega_i^- + \sum_{i\in I} x_i^- + \sum_{j\in J} y_j^+) = \frac{1}{2}\Big((-\sum_{i\in I} x_i^- - \sum_{j\in J} y_j^+) + (-\sum_{i\in I} \omega_i^-)\Big)$. Now, observe that, in view of the lattice properties of $Z^i_{x_i}$ and $Z^j_{y_j}$ and the fact that $L(u)$ is an ideal, the points $-\sum_{i\in I} x_i^- - \sum_{j\in J} y_j^+$ and $-\sum_{i\in I} \omega_i^-$ both belong to $\sum_{i\in I}(Z^i_{x_i} \cap L(u)) - \sum_{j\in J}(Z^j_{y_j} \cap L(u))$. All lattices, being comprehensive, are convex, so that $\omega - v \in \sum_{i\in I}(Z^i_{x_i} \cap L(u)) - \sum_{j\in J}(Z^j_{y_j} \cap L(u))$. By comprehensivity of this set we have also

$$[\omega - v, \omega + v] \subset \sum_{i\in I}(Z^i_{x_i} \cap L(u)) - \sum_{j\in J}(Z^j_{y_j} \cap L(u)).$$

Furthermore, from $\omega > 0$, it follows that $\sum_{i\in I} \omega_i^+ > \sum_{i\in I} \omega_i^-$ and $\sum_{i\in I} x_i^+ + \sum_{j\in J} y_j^- > \sum_{i\in I} x_i^- + \sum_{j\in J} y_j^+$, that implies

$$2\sum_{i\in I} \omega_i^+ > \sum_{i\in I} |\omega_i|$$

and

$$2(\sum_{i\in I} x_i^+ + \sum_{j\in J} y_j^-) > \sum_{i\in I} |x_i| + \sum_{j\in J} |y_j|.$$

Summing the last relations, we obtain $4v > u$ and can conclude that $[\omega - \frac{1}{4}u, \omega + \frac{1}{4}u] \subset \sum_{i\in I}(Z^i_{x_i} \cap L(u)) - \sum_{j\in J}(Z^j_{y_j} \cap L(u))$. As $[\omega - \frac{1}{4}u, \omega + \frac{1}{4}u]$ is a ball centered at ω in $L(u)$ endowed with the Riesz norm $||\cdot||_\omega$, the set $\sum_{i\in I}(Z^i_{x_i} \cap L(u)) - \sum_{j\in J}(Z^j_{y_j} \cap L(u))$ is radial at ω in $L(u)$. □

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