Equilibria without the survival assumption*

Alexander Konovalov [†] Valeri Marakulin[‡]

Abstract

It is well known that an equilibrium in the Arrow-Debreu model may fail to exist if a very restrictive condition called the survival assumption is not satisfied. We study two approaches that allow for the relaxation of this condition. Danilov and Sotskov (1990), and Florig (2001) developed a concept of a generalized equilibrium based on a notion of hierarchic prices. Marakulin (1988), (1990) proposed a concept of an equilibrium with non-standard prices. In this paper, we establish the equivalence between non-standard and hierarchic equilibria. Furthermore, we show that for any specified system of dividends the set of such equilibria is generically finite. As a consequence, we have generic finiteness of Mas-Colell's equilibria with slack, uniform dividend equilibria, and other special cases of our concept.

Key words: competitive equilibrium, hierarchic price, non-standard analysis, survival assumption, dividend equilibrium

JEL Classification: D50

1 Introduction

It is well known that an equilibrium in the Arrow—Debreu model may fail to exist if a very restrictive condition called the survival assumption is not satisfied. Its most widely used and widely criticized version requires every consumer to have a positive initial endowment of every good existing in the economy.

To illustrate the problem consider an example (cf. Gale (1976)) of a market with two traders and two commodities: apples and oranges. The first trader owns apples and oranges, but has a positive utility only for apples, the second trader cares for both, but owns only oranges. If the price of oranges is positive then the first agent sells his oranges in order to buy more apples, but he already has all the apples. If prices of oranges is zero then the second agent demands an

*The research was supported by the Council for Grants (under RF President) and State Aid of Fundamental Science Schools (grant No. SS-80.2003.6) and by the Russian Foundation for Humanities grant No. 05-02-02005a. We would like also to thank an anonymous referee for helpful comments and suggestions.

†Corresponding author: Econometric Institute, Faculty of Economics, Erasmus University Rotterdam, P.O. Box 1738, 3000 DR Rotterdam, The Netherlands. E-mail: konovalov@few.eur.nl, phone: 31-10-4081253

‡Institute of Mathematics, Russian Academy of Science, ac. Koptyug pr.4, Novosibirsk, 630090, Russia. E-mail: marakul@math.nsc.ru

infinite amount of oranges. Thus, no equilibrium results. The reason for this is that the second trader's budget correspondence is not lower hemicontinuous. As the price for oranges falls to zero, the budget set and the demand "explode". An idea that suggests itself is to redefine a budget correspondence by appropriate refining the notion of prices in order to get an equilibrium that always exists. In particular, an equilibrium in Gale's example would be restored if one manages to define prices for oranges so small that no apples can be bought for any amount of oranges, but still non-zero.

Two realizations of this idea were proposed so far. Gay (1978), Danilov and Sotskov (1990), Mertens (1996), and Florig (1998, 2001) developed an approach based on a notion of a hierarchic price. At equilibrium, all commodities (or commodity bundles treated as separate goods) are divided into several disjoint classes and traded against commodities of the same class according to prices which are an element of some set called a hierarchic price. Moreover, the set of such classes is ordered, superior class commodities cost infinitely much compared to the inferior class ones. Marakulin (1990) uses non-standard prices in the sense of Robinson's infinitesimal analysis (Robinson (1966)). A similar hierarchic structure of submarkets arises.

This paper is a continuation of the study of generalized equilibria in a model without the survival assumption. Its first contribution is the reconciliation of standard and non-standard approaches. Starting with non-standard equilibrium, we derive a characterization of non-standard budget sets in pure standard terms, which brings us the equivalence between non-standard equilibria and Florig's hierarchic equilibria. The second contribution of the paper is the generic finiteness result for the set of hierarchic equilibria for any specified system of dividends. This result also applies to the special cases of this concept such as equilibrium with slack (Mas-Colell (1992)), equilibrium with individual slack (Kajii (1996)), uniform dividend equilibrium (Aumann and Drèze (1986)), etc.

Section 2 introduces the model and the basic definitions and contains the equivalence result. Section 3 is devoted to the generic finiteness problem. The proofs of all results can be found in the appendix.

2 The Model and the Equivalence Result

We work with an exchange economy $\mathcal{E}$ defined by $L = \{1, \ldots, l\}$ — the set of commodities; $Q \subseteq \mathbb{R}^l$ — the set of admissible prices; $N = \{1, \ldots, n\}$ — the set of agents, where each agent $i \in N$ is characterized by his net trade set $X_i \subset \mathbb{R}^l$, $0 \in X_i$, and preferences given by a correspondence $\mathcal{P}_i : X_i \to 2^{X_i}$, where $\mathcal{P}_i(x^i)$ denotes the set of net trades *strictly* preferred to x^i. Denote the Cartesian product of individual net trade sets $\prod_{i \in N} X_i$ by X. An *allocation* is a vector $(x^i)_{i \in N} \in X$ such that $\sum_{i \in N} x^i = 0$.

A notion of dividend equilibrium was proposed by several authors (Makarov (1981), Aumann and Drèze (1986), Mas-Colell (1992)) in order to analyze economies with possibly satiated preferences of consumers. In a dividend equilibrium, each agent i's budget constraint is relaxed by some slack variable d_i in order to allow for redistribution of a budget excess created by satiated agents among non-satiated ones. Such a slack variable can be interpreted as an agent's endowment of coupons (as in Drèze and Müller (1980)) or paper money (as in Kajii (1996)).

Consider the $*$-image ${}^*Q$ of the set Q as the set of all admissible non-standard prices and a non-standard system of dividends $d \in {}^*\mathbb{R}^n_+$. Define by analogy with the standard case non-standard dividend budget sets of consumers:

$$^*B_i(p, d_i) = \{x \in {}^*X_i \mid px \leq d_i\}, \quad p \in {}^*Q, \quad i \in N.$$

By definition, these sets consist of non-standard net trades. Consider their standard parts $\overline{B}_i(p, d_i)$:

$$\overline{B}_i(p, d_i) = \mathrm{st}\, {}^*B_i(p, d_i) = \{x \in X_i \mid \exists \tilde{x} \in {}^*B_i(p, d_i) : \tilde{x} \approx x\},$$ [1]

where $\tilde{x} \approx x$ denotes infinitesimality of the difference $\tilde{x} - x$: $\|\tilde{x} - x\| \approx 0$.

Definition 2.1 An allocation $\bar{x} \in X$ is a *non-standard dividend equilibrium*, if there exist vectors $d = (d_1, \ldots, d_n) \in {}^*\mathbb{R}^n_+$ and $p \in {}^*Q$, such $\bar{x}^i \in \overline{B}_i(p, d_i)$, and $\mathcal{P}_i(\bar{x}^i) \cap \overline{B}_i(p, d_i) = \emptyset$, $i \in N$.

Next we introduce a notion of hierarchic equilibrium due to Florig (2001). An ordered orthonormal set $\{q_1, \ldots, q_k\}$ of non-zero vectors in $\mathbb{R}^l$ is called a *hierarchic price*. Consider a hierarchic price $q = (q_1, \ldots, q_k)$ and define the *q-value* of a net trade x as a vector $qx = (q_1 x, \ldots, q_k x, +\infty, \ldots, +\infty) \in (\mathbb{R} \cup \{+\infty\})^l$. A *hierarchic revenue* r is an element of $(\mathbb{R} \cup \{+\infty\})^l$. For a hierarchic price q, revenue r and $i \in N$ consider

$$s_i(q, r) = \min \{s \in \{1, \ldots, l\} \mid \exists x \in X_i : (q_1 x, \ldots, q_s x) \prec (r_1, \ldots, r_s)\},$$

where $\prec$ denotes the lexicographic ordering. In principle, s_i is the first level at which a consumer i is not at minimum wealth. Given $s_i(q, r)$, consider an augmented revenue vector

$$\rho_i(q, r) = (r_1, \ldots, r^{s_i(q,r)}, +\infty, \ldots, +\infty) \in (\mathbb{R} \cup \{+\infty\})^l$$

and the budget set of i's consumer $B_i(q, r) = \{x \in X_i \mid qx \preceq \rho_i(q, r)\}$. This construction guarantees the completeness of the budget set $B_i(q, r)$.

Definition 2.2 An allocation x is a *hierarchic equilibrium* of the economy $\mathcal{E}$ if there exist a hierarchic price q and positive hierarchic revenues $r^i \in \mathbb{R}^l_+$, $i \in N$ such that $x^i \in B_i(q, r^i)$ and $\mathcal{P}_i(x) \cap B_i(q, r^i) = \emptyset, \quad i \in N$.

The first main result of the paper is the following theorem:

Theorem 2.3 *Suppose that all net trade sets X_i are polyhedral. Then the set of hierarchic equilibria of an economy $\mathcal{E}$ coincides with the set of non-standard dividend equilibria.*

The proof is based on the representation of non-standard prices by an orthonormal set of standard vectors and is given in the appendix.

[1] The subscript i is often dropped if a single arbitrary agent is considered.

3 Generic finiteness of non-standard equilibria

Gerard Debreu (1970) was the first to establish the finiteness of equilibria for "almost all" exchange economies. His approach was based on a variation of initial endowments while all other parameters of the model were fixed. Using Sard's theorem applied to the aggregate excess demand function, Debreu obtained finiteness of equilibria for an open set containing almost all – in the sense of Lebesgue measure – allocations of initial resources. In subsequent contributions to the issue, Smale (1974), Dubey (1980) and others used also variations of utility functions, which required the use of Thom's theorems of openness and density of transversal intersections. In the present paper we follow the latter approach, i.e., only utility functions vary, while initial endowments are kept fixed. This is done because of the fact that for almost all initial allocations of resources, the survival condition is satisfied, in which case non-standard equilibria coincide with usual Walrasian equilibria. Their local uniqueness follow then by the Debreu's (1970) theorem. Another relative advantage of this approach is that no convexity condition on agents' preferences are needed. Finally, for the considered problem, generic arguments on the interior of the consumption set are not very relevant economically since in a "generic" real life situation, nobody has a mathematically generic initial endowments, i. e. in the interior of the consumption set.

In this section, we assume that preferences of agents are given by utility functions defined over the set $X = \prod_{i\in N} X_i$, $u_i : \prod_{i\in N} X_i \to \mathbb{R}, \quad i \in N.$

We introduce now a new concept of non-standard dividend equilibrium where a system of dividends is specified. If the survival assumption is satisfied in the model, then this concept boils down to a notion of equilibrium with individual slacks proposed by Kajii (1996). Fix a *standard* strictly positive vector $\delta \in \mathbb{R}^n_{++}$. For a given non-standard number $\varepsilon \in {}^*\mathbb{R}_+$ consider the dividend budget sets

$$\overline{B}_i(p, \varepsilon\delta_i) = \mathrm{st}\,\{x \in X_i \ \mid \ px \le \varepsilon\delta_i\}, \quad i \in N.$$

Definition 3.1 An allocation $\bar{x}$ is a *δ-equilibrium* of an economy $\mathcal{E}$, if there exist $\varepsilon \in {}^*\mathbb{R}_+$ and $p \in {}^*Q$ such that $\bar{x}^i \in \overline{B}_i(p, \varepsilon\delta_i)$, and

$$u_i(\bar{x}) = \max_{x^i \in \overline{B}_i(p,\varepsilon\delta_i)} u_i(\bar{x}|x^i), \quad i \in N.$$

The specifics of δ-equilibria are twofold. First, the ratio of individual dividends is assumed to be given a priori and fixed, as in the case of Kajii's equilibria with individual slacks, or, for instance, Mas-Colell's (1992) equilibria with slack, where the uniform dividend scheme was applied. Second, dividends of all consumers have the same "order of smallness" ε. Therefore, income is redistributed at most at one infinitesimality level. Interpretation of the components δ_i depends on the further specification of the model. For instance, they may represent initial stocks of coupons or paper money or express market shares of individuals.

We continue with an example which illustrates that the system of dividends has to be specified for the number of non-standard equilibria to be finite. Let

$$X_1 = \{(x_1, x_2) \mid x_1 \ge -2, x_2 \ge -1\},$$

$$X_2 = X_3 = \{(x_1, x_2) \mid x_1 \geq -2, x_2 \geq 0\},$$

$$Q = \{p \in \mathbb{R}^l \mid \ \|p\| \leq 2\},\ u_1(x) = -(x_1+1)^2 - (x_2-1)^2,\ u_2(x) = u_3(x) = x_1.$$

There are neither Walrasian nor dividend equilibria in this example. Allo-

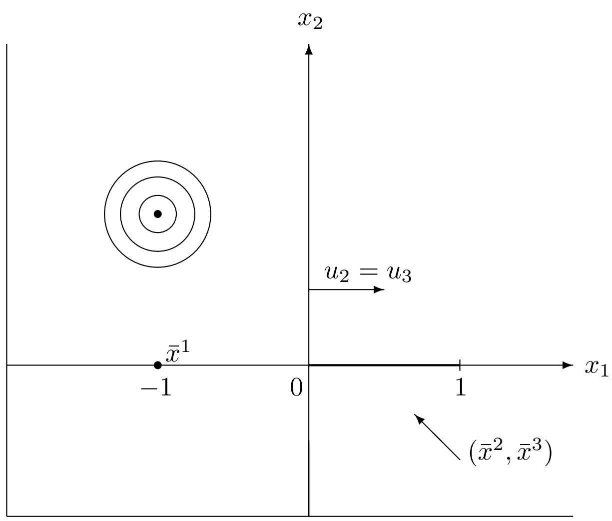

Figure 1: Continuum of non-standard dividend equilibria.

cations $\bar{x}^1 = (-1, 0)$, $\bar{x}^2 = (\lambda, 0)$, $\bar{x}^3 = (1-\lambda, 0)$, $0 \leq \lambda \leq 1$ constitute a continuum of non-standard dividend equilibria for $p = (\varepsilon, 1)$, $\varepsilon \approx 0$, $\varepsilon > 0$, and dividends $d = (0, \lambda\varepsilon, (1-\lambda)\varepsilon)$. This example is robust against sufficiently small perturbations of utility functions.

In what follows, we will focus our attention on a class of δ-equilibria, for which there exists at least one agent who consumes an element of the interior of his net trade set.

Definition 3.2 A δ-equilibrium $\bar{x}$ is *proper* if there exists $i_0 \in N$ such that $\bar{x}^{i_0} \in \operatorname{int} X_{i_0}$.

Properness is a technical assumption which makes sure that the tangent correspondence of the mapping which characterizes δ-equilibria is surjective. Let e_j be the jth unit vector and consider $w_j = -\sum_{i=1}^n \inf e_j X_i$, an upper bound to the amount of good j an agent can obtain. Let $W = \prod_{j=1}^l (-\infty, w_j]$. The following assumption (not needed in the proof of the finiteness result) guarantees the existence of a proper δ-equilibrium.

A1. There exists $i_0 \in N$ such that $0 \in \operatorname{int} X_{i_0}$ and $\mathcal{P}_{i_0}(0) \cap W \cap \operatorname{bd} X_{i_0} = \emptyset$.

A proper δ-equilibrium exists if A1 together with a standard set of conditions excluding the survival assumption are satisfied. The proof is similar to that of

Theorem 3.6.2 in Konovalov (2001) and hence omitted. Condition A1 implies that at equilibrium agent i_0 chooses an element in the interior of the set X_{i_0}.

The finiteness result is established under the following assumptions:

A2. For all $i \in N$ the set X_i is a bounded from below polyhedron with a non-empty interior.

A3. Utility functions u^i are defined and twice differentiable on an open neighborhood $\widetilde{X}$ of the set X.

Denote by U the linear space $C^2(\widetilde{X}, \mathbb{R}^n)$ and endow it with the standard topology of C^2 uniform convergence on compacts: if $\{f_t\}_{t=1}^{\infty} \subset C^2(\widetilde{X}, \mathbb{R}^n)$, then $f_t \to f_0 \in C^2(\widetilde{X}, \mathbb{R}^n)$ if and only if $f_t \mid_K \to f_0$ if $t \to \infty$ in the norm $\| \cdot \|_{C^2}$ of the vector space $C^2(K, \mathbb{R}^n)$ for every compact set $K \subset \widetilde{X}$. The norm $\| \cdot \|_{C^2}$ is defined by $\|g\|_{C^2(K,\mathbb{R}^n)} = \max\{\|g_i\|_{C(K,\mathbb{R}^n)}, \|\frac{\partial g_i}{\partial x_j}\|_{C(K,\mathbb{R}^n)}, \|\frac{\partial^2 g_i}{\partial x_j \partial x_s}\|_{C(K,\mathbb{R}^n)}, i \in N, j, s \in N \times L\}$, where $\|g\|_{C(K,\mathbb{R}^n)} = \max\{|g(x)| : x \in K\}$.[2] Thus we can think of an economy as of given by an element u of the set U.

Theorem 3.3 *For any strictly positive vector δ, there exists a residual (of the second category and hence dense) set $G \subseteq U$ such that for each $u \in G$ the set of proper δ-equilibria is finite.*

The proof of this Theorem is organized into a number of steps. We give now a brief guide how the finiteness of δ-equilibria will be obtained. First we consider the set of equilibrium hierarchic prices Θ_F that correspond to δ-equilibria from the relative interior of an arbitrary face F of the polyhedron X. After that, we construct a mapping Ψ_u, $u \in U$, which is defined on $\operatorname{ri} F \times \Theta_F$ and takes its values in some finite-dimensional space. This mapping characterizes δ-equilibria from $\operatorname{ri} F$. Then, in the range of Ψ_u, we find a manifold Δ_F such that $\Psi_u^{-1}(\Delta_F)$ contains all δ-equilibria from $\operatorname{ri} F$. Thus, to establish finiteness of δ-equilibria, it is sufficient to show that the manifold $\Psi_u^{-1}(\Delta_F)$ has dimension zero (is discrete). We show that it is indeed so if Ψ_u is transversal to Δ_F. Finally, it follows by Thom's theorems of density and openness of transversal sections that the mapping Ψ_u is transversal to Δ_F for a residual set of economies $u \in U$.

We conclude this section with an example illustrating generic non-existence of a standard dividend equilibrium and the need to use non-standard prices and dividends. Consider a market with two goods and three agents, where net trade sets of agents are given as follows: $X_i = \{x \in \mathbb{R}^2_+ : x_1 + x_2 \geq 4\} - \{w^i\}$, where $w^1 = w^3 = (3, 2)$, $w^2 = (3, 1)$. Agents' preferences are represented by utility functions $u_1(x) = x_1$, $u_2(x) = x_2$, $u_3(x) = -\|x\|_2$.

An allocation $\bar{x}^1 = (2, -2)$, $\bar{x}^2 = (-2, 2)$, $\bar{x}^3 = (0, 0)$ is the only Pareto optimum of this economy. Nevertheless, it can not be obtained through the market mechanism unless a refined price system and dividends are employed. First, show that for all utility functions sufficiently close (in the C^2 topology of uniform convergence) to the original ones neither Walrasian nor dividend equilibrium exists in this economy. Let $d = (d_1, d_2, d_3) \in \mathbb{R}^3_+$ be an arbitrary system of

[2] $| \cdot |$ stands for the absolute value.

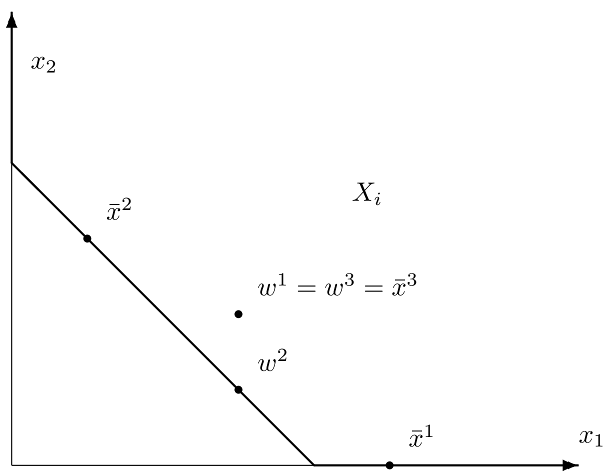

Figure 2: Non-existence of equilibrium without non-standard prices and dividends.

dividends. Whatever the price system is, consumer 3's demand is near zero. Negative prices can be excluded, so we can normalize p_1 to 1. If $p_2 \leq 1$, then consumer 2 will demand a point close to $(-3, \frac{3+d_2}{p_2})$, which is not feasible given consumer 3's demand. If $p_2 > 1$, the first consumer will demand a point close to $(2p_2+d_1, -2)$, while the second consumer will select a trade close to $(\frac{-d_2}{p_2-1}, \frac{d_2}{p_2-1})$ if $d_2 \leq 3(p_2-1)$, and a trade which is not feasible, otherwise. For the allocation to be feasible d_2 needs to be close to $2(p_2-1)$. Then, the excess demand for the first good is close to $2p_2 - 2 + d_1$ and is strictly positive for all utilities sufficiently close to the original ones. Thus, a dividend equilibrium does not generically exist in this example. For non-standard prices $p = (1+\varepsilon, 1-\varepsilon)$, where $\varepsilon > 0$, $\varepsilon \approx 0$, there is excessive demand for good 2. If $p = (1-\varepsilon, 1+\varepsilon)$, then the excess demand for good 1 is strictly greater than zero. It is easy to prove that ${}^\circ p = (1,1)$ is the only case where non-standard prices make difference. Therefore, a non-standard equilibrium without dividends also does not exist in this economy. This is also true for any sufficiently small perturbations of the original utility functions. However, $\bar{x}$ is a non-standard dividend equilibrium if prices are $p = (1-\varepsilon, 1+\varepsilon)$ and dividends are $d = (0, 4\varepsilon, 0)$. (This equilibrium is also proper, since condition A1 is trivially satisfied for the third consumer). Essentially, in addition to the standard price system $(1,1)$, a commodity bundle $(-1,1)$ is priced by an infinitesimal value. One could say that infinitesimal prices and dividends determine an additional constraint, which plays a role of a rationing scheme.

4 Appendix

4.1 Proof of Theorem 2.3

The first proposition projects the set of non-zero non-standard prices ${}^*\mathbb{R}^l \backslash \{0\}$ onto the set of all possible hierarchic prices.

Proposition 4.1 *For each $p \in {}^*\mathbb{R}^l \backslash \{0\}$, there exists an orthonormal set of standard vectors $\{q_1, q_2, \dots, q_k\} \in \mathbb{R}^{lk}$ such that p has a unique representation*

$$p = \lambda_1 q_1 + \cdots + \lambda_k q_k, \tag{1}$$

with positive coefficients $\lambda_j \in {}^*\mathbb{R}_{++}$ satisfying $\lambda_{j+1}/\lambda_j \approx 0$, $j \in \{1, \dots, k-1\}$.

Proof. The proof of the proposition is by induction in the dimension l of the space containing vector $p \in {}^*\mathbb{R}^l$. The proposition is trivially verified for $l = 1$. Suppose that it is true for $l \leq m$ for some natural m and show that it holds for $l = m+1$. Assuming $p \neq 0$, let $p' = p/{}^*\|p\|$. Since p' is near-standard (due to compactness of the unit ball in a finite dimensional space and the non-standard criterion of compactness), one may find $q_1 = {}^\circ(p')$ and $\lambda_1 = pq_1$. Let $p'' = p - \lambda_1 q_1$ and consider a subspace $\mathcal{L} = \{x \in \mathbb{R}^l \mid xq_1 = 0\}$. Note that by the Transfer Principle (see Anderson (1991), p. 2158), ${}^*\mathcal{L}$ has a similar structure as $\mathcal{L}$: ${}^*\mathcal{L} = \{x \in {}^*\mathbb{R}^l \mid xq_1 = 0\}$. By the definition of p'', $p''q_1 = (p, q_1) - (p, q_1)(q_1, q_1) = 0$, (since $(q_1, q_1) = 1$), which implies that $p'' \in {}^*\mathcal{L}$. However, $\dim \mathcal{L} \leq l$, hence, by the induction agreement, there exists an orthonormal system of standard vectors $\{q_2, \dots, q_k\}$, such that $p'' = \lambda_2 q_2 + \cdots + \lambda_k q_k$, where positive coefficients λ satisfy $\lambda_{j+1}/\lambda_j \approx 0$, $j = 2, \dots, k-1$. Consequently,

$$p = \lambda_1 q_1 + p'' = \lambda_1 q_1 + \lambda_2 q_2 + \cdots + \lambda_k q_k.$$

It is clear that the system $\{q_1, \dots, q_k\}$ is orthonormal. Let us show that $\lambda_2/\lambda_1 \approx 0$. To this end, put $\delta = \|p' - q_1\|^2 \approx 0$. Then

$$\begin{aligned} {}^*\|p' - (p', q_1)q_1\|^2 &= (p', p') - 2(p', q_1)^2 + (p', q_1)^2(q_1, q_1) = \\ &= 1 - (p', q_1)^2 = 1 - (1 - \frac{\delta}{2})^2 = \delta(1 - \frac{\delta}{4}), \end{aligned}$$

which implies ${}^*\|p''\| = {}^*\|p - \lambda_1 q_1\| = {}^*\|p\|({}^*\|p' - (p', q_1)q_1\|) = \varepsilon\, {}^*\|p\|$, where $\varepsilon = \sqrt{\delta(1 - \frac{\delta}{4})} \approx 0$. On the other hand,

$${}^*\|p''\| = {}^*\|p - \lambda_1 q_1\| = {}^*\|\lambda_2 q_2 + \cdots + \lambda_k q_k\| = \sqrt{\lambda_2^2 + \cdots + \lambda_k^2}.$$

Taking into account $(p', q_1) \approx 1$, one obtains

$$0 \leq \frac{\lambda_2}{\lambda_1} \leq \frac{\sqrt{\lambda_2^2 + \cdots + \lambda_k^2}}{\lambda_1} = \frac{{}^*\|p''\|}{\lambda_1} = \frac{\varepsilon\, {}^*\|p\|}{(p, q_1)} = \frac{\varepsilon}{(p', q_1)} \approx 0.$$

The uniqueness of the representation follows by construction. □

The next proposition gives a characterization of the set $\overline{B}(p, 0)$ and is crucial for our analysis. It asserts that if X_i is a polyhedral set then there is a number $m \in \{1,\dots,k\}$ such that the set $\overline{B}(p,0)$ consists of elements x such that the vector $(q_j x)_{j=1,\dots,m}$ is lexicographically less than zero. Denote the set $\{x \in X_i \mid q_1 x = 0,\dots,q_m x = 0\}$ by $X_i(q_1,\dots,q_m)$ and put $X_i(\emptyset) = X_i$. Consider the sets

$$B_m(p) = \{x \in X_i(q_1,\dots,q_{m-1}) \mid q_m x \le 0\}, \quad m \le k, \tag{2}$$

and $B_{k+1}(p) = X_i(q_1,\dots,q_k)$.

Proposition 4.2 *Suppose that $p \in {}^*\mathbb{R}^l$, and the set X_i is a polyhedron, then there exists a natural number $m \in \{1,\dots,k+1\}$ such that $\overline{B}(p) = B_m(p)$. Moreover, for $m \le k$ there exists $y \in \overline{B}(p)$ such that $q_m y < 0$.*

The proof is based on the following lemma.

Lemma 4.3 *Suppose that $X_i \in \mathbb{R}^l$ is a polyhedral set and $p \in {}^*\mathbb{R}^l$. Then $pX_i \ge 0$ implies $p({}^*X_i) \ge 0$.*

Note first that the conclusion of the lemma does not hold if X is not polyhedral. Take for instance $X_i = \{x \in \mathbb{R}^2_+ \mid x_2 \ge (x_1)^2\}$ and $p = (-\varepsilon, 1)$, $\varepsilon \approx 0$, $\varepsilon > 0$. Then for each $x \in X_i$, $px = -\varepsilon x_1 + x_2 \ge 0$. But once an element $\tilde{x} = (\varepsilon/2, \varepsilon^2/4) \in {}^*X_i$ is taken, $p\tilde{x} = -\varepsilon^2/2 + \varepsilon^2/4 < 0$.

Proof. The set X_i consists of all vectors in $\mathbb{R}^l$ that satisfy some system of linear inequalities: $X = \{x \in \mathbb{R}^l \mid d_\alpha x \le g_\alpha,\ \alpha \in A\}$, where $d_\alpha \in \mathbb{R}^l \backslash \{0\}$, $g_\alpha \in \mathbb{R}$, and A is finite. X_i can be represented as a sum of a compact polyhedron Y and a convex cone Z with a finite number of generators (see Schrijver (1986), Theorem 7.16). In other words, $X_i = Y + Z$, and for some finite sets $B \subset \mathbb{R}^m$, $C \subset \mathbb{R}^m$

$$Y = \ \mathrm{conv} B = \{\sum_{b\in B} \beta_b b \mid \beta_b \in \mathbb{R}_+ \ \&\ \sum_{b\in B} \beta_b = 1\ \},$$

and $Z = \ \mathrm{con}\, C = \{\sum_{c\in C} \gamma_c c \mid \gamma_c \in \mathbb{R}_+\}$. Then by definition

$$pX_i \ge 0 \quad \Longleftrightarrow \quad pb \ge 0 \quad \& \quad pc \ge 0 \quad \forall b \in B, \quad \forall c \in C.$$

By the Transfer Principle, ${}^*X_i = {}^*Y + {}^*Z$, where ${}^*Y$ and ${}^*Z$ are defined by substitution $\mathbb{R}_+$ for ${}^*\mathbb{R}_+$ in the definitions of Y and Z, respectively. Therefore,

$$pb \ge 0 \quad \& \quad pc \ge 0 \quad \forall b \in B, \quad \forall c \in C \quad \Longleftrightarrow \quad p\,({}^*X_i) \ge 0.$$

□

Proof of Proposition 4.2. Let $q^p = (q_1,\dots,q_k)$ for some $k \le l$. If $p = 0$ then $q_1 = 0$, $k = 1$ and $\overline{B}(p,0) = X_i = B_1(p)$. Suppose that there exist a number $m \in \{1,\dots,k\}$ and an element $y \in X_i(q_1,\dots,q_{m-1})$ such that $q_m y < 0$. Moreover, assume that m is the smallest such a number, which guarantees that

$$q_j X_i(q_1,\dots,q_{j-1}) \ge 0, \quad j \in \{1,\dots,m-1\}. \tag{3}$$

Note that by construction $y \in B_m(p)$. If $m = 1$ then the survival assumption is satisfied for prices $q_1 = {}^\circ(p/\|p\|)$ and it is easy to show that the budget set is Walrasian: $\overline{B}(p,0) = \{x \in X_i \mid q_1 x \le 0\} = B_1(p)$. Assume $m \in \{2,\dots,k\}$

and show that $\overline{B}(p,0) = B_m(p)$. First, we shall prove that $\overline{B}(p,0) \subseteq B_m(p)$. Take some arbitrary $x \in \overline{B}(p,0)$ and suppose that $x \notin B_m(p)$. If so, then by the choice of m and the system of inequalities (3) there exists a number $j \in \{1,\ldots,m\}$ such that $q_j x > 0$, $q_t x = 0$, $t = 1,\ldots,j-1$. Consider some arbitrary non-standard element $\tilde{x} \in {}^*X_i$ such that $\tilde{x} \approx x$. Then $q_j x > 0$ implies that $q_j \tilde{x}$ is greater than some strictly positive real number. Take now any standard $x' \in X_i$ and consider the first non-zero element in the ordered set $\{q_1 x', \ldots, q_{j-1}x'\}$. Since $j \leq m$, it follows again from (3) and the choice of m that such an element, if it exists, is strictly positive. Therefore, a non-standard linear functional $\lambda_1 q_1 + \cdots + \lambda_{j-1} q_{j-1}$ takes only positive values on X_i : $\Big(\lambda_1 q_1 + \cdots + \lambda_{j-1} q_{j-1}\Big) X_i \geq 0$. Then by Lemma 4.3

$$\Big(\lambda_1 q_1 + \cdots + \lambda_{j-1} q_{j-1}\Big)\, {}^*X_i \geq 0. \tag{4}$$

Consider

$$\frac{1}{\lambda_j} p\tilde{x} = \frac{1}{\lambda_j}[\lambda_1 q_1 + \cdots + \lambda_{j-1} q_{j-1}]\tilde{x} + q_j \tilde{x} + \frac{1}{\lambda_j}[\lambda_{j+1} q_{j+1} + \cdots + \lambda_k q_k]\tilde{x}.$$

The first component of the sum in the right-hand side is positive (it vanishes if $j = 1$), the second component exceeds 0 by a non-infinitesimal amount and the third component is infinitesimal. Therefore, $(1/\lambda_j)p\tilde{x}$ is strictly positive, so that $p\tilde{x} > 0$ for all $\tilde{x} \in {}^*X_i$ such that $\tilde{x} \approx x$. This contradicts $x \in \overline{B}(p,0)$. We have shown that $\overline{B}(p,0) \subseteq B_m(p)$. Let $x \in B_m(p,0)$, $y \in X_i(q_1,\ldots,q_{m-1})$, $q_m y < 0$. Consider a sequence $x_n = (1/n)y + (1-1/n)x$. By convexity, $x_n \in {}^*X_i$ for any $n \in {}^*\mathbb{N}$. Moreover, $px_n < 0$ for all $n \in \mathbb{N}$. Show that $px_{\tilde{n}} < 0$ for some hyperfinite $\tilde{n}$. Suppose that the set $A = \{n \in {}^*\mathbb{N} \mid px_n \geq 0\}$ is non-empty. This set is internal as a definable subset of an internal set ${}^*\mathbb{N}$ (cf. Davis (1977), Theorem 1-8.1). Therefore it has a least element $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$. Take $\tilde{n} = \nu - 1$, then $x_{\tilde{n}} \approx x$ and $px_{\tilde{n}} < 0$, which proves that $x \in \overline{B}(p,0)$. To complete the proof it suffices to show that

$$\forall\, m \in \{1,\ldots,k\} \quad q_m X_i(q_1,\ldots,q_{m-1}) \geq 0$$

implies $\overline{B}(p,0) = B_{k+1}(p)$. If $x \in B_{k+1}(p)$ then $px = 0$, so $x \in \overline{B}(p,0)$. The proof of the inclusion $\overline{B}(p,0) \subseteq B_{k+1}(p)$ goes along the same lines as in the case $m \leq k$. □

Next, we turn our attention to the characterization of non-standard dividend budget sets. For $\gamma \in {}^*\mathbb{R}_{++}$ and $p \in {}^*\mathbb{R}^l$ consider the set $\overline{B}(p,\gamma) = \mathrm{st}\{x \in {}^*X_i \mid px \leq \gamma\}$. The following auxiliary lemma is useful. It says that small changes in prices and dividends do not alter a standardized dividend budget set.

Lemma 4.4 *Let X_i be a closed convex set, $0 \in X_i$. Suppose that $p, p' \in {}^*\mathbb{R}^l$ and that the non-standard numbers $\gamma > 0$, $\gamma' > 0$ satisfy $|p - p'| \,/\, \gamma \approx 0$, $\gamma/\gamma' \approx 1$, then $\overline{B}(p,\gamma) = \overline{B}(p',\gamma')$.*

Proof. Show first that $\overline{B}(p,\gamma) = \overline{B}(p,\gamma')$. To this end assume $\gamma' > \gamma$ and show that the inclusion

$$\overline{B}(p,\gamma') \subseteq \overline{B}(p,\gamma) \tag{5}$$

holds. Let $x \in \overline{B}(p, \gamma')$. Then one can find $\tilde{x} \approx x, \tilde{x} \in {}^*X_i$ such that $p\tilde{x} \le \gamma'$. Suppose that $p\tilde{x} > \gamma$ (otherwise there is nothing to prove), and consider $y = (1-\varepsilon)\tilde{x}$ where $\varepsilon \approx 0$ satisfies $\gamma' = (1+\varepsilon)\gamma$. It is clear that $y \approx \tilde{x} \approx x$, $y \in {}^*X_i$ by convexity, and $py = p\tilde{x} - \varepsilon p\tilde{x} \le \gamma' - \varepsilon\gamma = \gamma$. Thus inclusion (5) follows. Next we shall establish $\overline{B}(p, \gamma') = \overline{B}(p', \gamma')$. Let $p'' = p - p'$. Since $|p''| \,\big/\, \gamma' \approx 0$, one can find $\varepsilon \approx 0$, $\varepsilon > 0$ such that $|p''| \,\big/\, \varepsilon\gamma' \approx 0$. Then for every near-standard $y \in {}^*X_i$, $p'y - \varepsilon\gamma' \le py \le p'y + \varepsilon\gamma'$. Therefore $py \le \gamma'$ implies $p'y \le \gamma' + \varepsilon\gamma'$ and $\overline{B}(p, \gamma') \subseteq \overline{B}(p', \gamma' + \varepsilon\gamma') = \overline{B}(p', \gamma')$. Similarly, $\overline{B}(p', \gamma') \subseteq \overline{B}(p, \gamma')$. □

Using representation (1) for $p \in {}^*\mathbb{R}^l$, assign to each non-standard $\gamma > 0$ its *infinitesimality level*, that is a number $j = j(p, \gamma) \in \{1, \dots, k+1\}$ such that

$$j(p,\gamma) = \begin{cases} \min\{m \mid \gamma/\lambda_m \not\approx 0\}, & \text{if } \gamma/\lambda_k \not\approx 0, \\ k+1, & \text{otherwise.} \end{cases}$$

For $j \le k$, denote by $\mu = \mu(p, \gamma)$ a standard part of the ratio γ/λ_j: $\mu(p, \gamma) = {}^\circ(\gamma/\lambda_j)$. Thus μ is an element of $\mathbb{R}_{++} \cup \{+\infty\}$. Put $\mu = +\infty$ if $j = k+1$. The next statement gives a complete characterization of a non-standard dividend budget set for a polyhedral set X_i.

Proposition 4.5 *Let X_i be polyhedral, $p \in {}^*\mathbb{R}^l$, $\gamma \in {}^*\mathbb{R}_{++}$. Assume that $0 \in X_i$. Then one of the following alternatives is true:*

(i) $\overline{B}(p, \gamma) = X_i(q_1, \dots, q_{j-1})$ *and* $\mu = +\infty$*;*

(ii) $\overline{B}(p, \gamma) = \{x \in X_i(q_1, \dots, q_{j-1}) \mid q_j x \le \mu\}$ *and* $\mu < +\infty$;

(iii) $\overline{B}(p, \gamma) = B_m(p)$ *for some* $m < j$ *and there exists* $y \in \overline{B}(p, \gamma)$ *such that* $q_m y < 0$.

Here $q = (q_1, \dots, q_k)$ is a hierarchic price representing p, $j = j(p, \gamma)$ is the infinitesimality level of γ.

Proof. Consider an $(l+1)$-dimensional set $X_i^* = X_i \times \{1\}$, and a "budget" set

$$\{x \in {}^*X_i^* \mid p^* x \le 0\}, \tag{6}$$

where the price vector $p^* \in \mathbb{R}^{l+1}$ is given by

$$p^* = \begin{cases} (p', -\gamma), & \text{if } \gamma/\lambda_j \approx +\infty, \quad p' = p - \sum\limits_{t \ge j} \lambda_t q_t, \\ (p', -\lambda_j \mu), & \text{if } \gamma/\lambda_j < +\infty, \quad p' = \sum\limits_{t \le j} \lambda_t q_t. \end{cases}$$

By construction $|p - p'|/\gamma \approx 0$ and $\lambda_j \mu/\gamma \approx 1$. Since $0 \in X_i$, Lemma 4.4 implies that the projection of the standardization of the set defined in (6) onto the first l components of the set X_i^* coincides with $\overline{B}(p, \gamma)$. At the same time, $p^* = \sum_{t<j} \lambda_t q_t^* + \lambda_j^* q_j^*$, where $q_t^* = (q_t, 0)$ if $t < j$,

$$q_j^* = \begin{cases} (0, -1), & \text{if } \gamma/\lambda_j \approx +\infty, \\ (q_j, -\mu), & \text{if } \gamma/\lambda_j < +\infty, \end{cases}$$

and

$$\lambda_j^* = \begin{cases} \gamma, & \text{if } \gamma/\lambda_j \approx +\infty, \\ \lambda_j, & \text{if } \gamma/\lambda_j < +\infty. \end{cases}$$

Since a hierarchic price $\{q_t^*\}_{t\le j}$ represents p^* (strictly speaking, the vector q_j^* should be normalized), Proposition 4.2 is applicable. The alternatives $(i)-(iii)$ follow immediately. The case $m < j$ corresponds to alternative (iii). If $m = j$ and $\gamma/\lambda_j < +\infty$, then $\mu < +\infty$, and the budget restriction in the definition of $B_m(p^*)$ has the form $\Big((x,1),(q_j,-\mu)\Big) \le 0 \Rightarrow q_j x \le \mu$, so alternative (ii) follows. If $m = j$ and $\gamma/\lambda_j \approx +\infty$, then alternative (i) is true. The case $m = j+1$ will not occur because the assumption $0 \in X_i$ guarantees that $0 \in \overline{B}(p,\gamma)$. □

To show that non-standard dividend equilibria coincide with hierarchic equilibria one only needs to observe that non-standard prices and dividends on the one hand, and hierarchic prices and revenues on the other, provide an individual with the same budget opportunities. Suppose that x is a hierarchic equilibrium, and that q and r are the corresponding hierarchic price and revenue, respectively. Then each consumer maximizes his preferences on the set

$$B_i(q, r^i) = \{x \in X_i(q_1, \ldots, q_{s_i-1}) \mid q_{s_i} x \le r_{s_i}\},$$

which immediately implies that x is a non-standard dividend equilibrium at prices $p = q_1 + \varepsilon q_2 + \cdots + \varepsilon^{k-1} q_k$ for some $\varepsilon > 0$, $\varepsilon \approx 0$, and dividends $d_i = \varepsilon^{s_i-1} r_{s_i}, \quad i \in N$. Conversely, if x is a non-standard dividend equilibrium at $p \in {}^*\mathbb{R}^l$ and $d \in {}^*\mathbb{R}^n_+$, then the representation q^p will be a hierarchic equilibrium price, and for each $i \in N$ the components of a hierarchic equilibrium revenue r^i will be

$$r_t^i = \begin{cases} 0, & \text{if } t < j(p, d_i), \\ {}^\circ(d_i/\lambda_{j(p,d_i)}), & \text{if } t = j(p, d_i), \\ +\infty, & \text{if } t > j(p, d_i). \end{cases}$$

□

4.2 Proof of Theorem 3.3

In our argument we will rely on Proposition 4.1 and Proposition 4.5. When using Proposition 4.5 to characterize δ-equilibria, one should remember that all dividend terms $\varepsilon\delta_i$ have the same infinitesimality level $j(p, \varepsilon\delta_i) = j(p, \varepsilon)$. Moreover, it follows from Lemma 4.4 that all components q_s of an equilibrium hierarchic price with s higher than j do not matter. Hence, we can assume without loss of generality that for any hierarchic representation $q = \{q_1, \ldots, q_k\}$ of non-standard equilibrium prices p, $k = j(p, \varepsilon)$ and $\varepsilon/\lambda_k < +\infty$. The case $\varepsilon/\lambda_k \approx +\infty$ is not interesting, since it can be reduced to the prices $p' = \sum_{j<k} \lambda_t q_t$ for which $\varepsilon/\lambda_{j-1} \approx 0$. Proposition 4.5 implies that each agent i faces $m-1$ budget constraints in the form of equalities:

$$q_t x = 0, \quad t < m, \quad x \in X_i, \tag{7}$$

and one restriction in the form of inequality $q_m x \le 0$, $x \in X_i$, for some natural number $m < k$. Moreover, there always exists $y \in X_i$ for which the last inequality is strict. For $m = k$ the last restriction transforms to $q_k x \le \delta_i \mu$, $\mu = {}^\circ(\varepsilon/\lambda_k)$

and can be realized as an equality if $\mu = 0$. Equations (7) determine a face $F_i^m(p)$ of the polyhedron X_i such that $0 \in F_i^m(p)$. In other words,

$$\overline{B}_i(p, \varepsilon\delta_i) = \{x \in F_i^m(p) \mid q_m x \leq 0\}$$

for $m < k$, and $\overline{B}_i(p, \varepsilon\delta_i) = \{x \in F_i^k(p) \mid q_k x \leq \delta_i \mu\}$ otherwise. Consider a partition $\mathcal{N}(p) = \{N_1, \ldots, N_k\}$ of the set N such that each set N_m contains the agents, whose last budget constraint involves q_m. For all such agents we have $\overline{B}_i(p - \sum_{t>m} \lambda_t q_t, d_i) = \overline{B}_i(p, d_i)$. Fix some face $F \subseteq X$ and consider a δ-equilibrium x that belongs to $\operatorname{ri} F$ — the relative interior of F. Let p be the corresponding non-standard equilibrium price vector. It is easy to see that $\operatorname{ri} F_i \subseteq F_i^m(p)$, $i \in N$, where F_i are faces of X_i that compose the face F : $F = \prod_{i \in N} F_i$. Faces $F_i^m(p) \subseteq X^i$ are determined by prices p and equalities (7) for the appropriate $m \leq k$. The definition of a δ-equilibrium implies that each x^i maximizes utility $u_i(.)$ on the set $\{x \in \operatorname{ri} F_i \mid q_m x \leq 0\}$ if $m < k$, $i \in N_m$ and on the set $\{x \in \operatorname{ri} F_i \mid q_k x \leq \delta_i \mu\}$ if $i \in N_k$. In the first auxiliary result we prove that if all net trade sets in an economy are polyhedral, then the set of all relevant hierarchic equilibrium prices $\bar{\Theta}$ can be described as a finite union of manifolds of dimension l. "Relevant" here means that each non-standard dividend equilibrium is supported by a hierarchic equilibrium price from $\bar{\Theta}$. Thus, the situation is similar to what we have in the purely standard case, where a price vector is an element of $\mathbb{R}^l$. This is somewhat surprising, since it means that introducing non-standard or hierarchic prices does not change the dimension of the set of all possible equilibrium prices.

Proposition 4.6 *Suppose that X_i is a polyhedral set for every $i \in N$. Then there exists a set $\bar{\Theta}$ such that (i) for each non-standard dividend equilibrium $\bar{x}$ there exists $q \in \bar{\Theta}$ such that q is an equilibrium hierarchic price for $\bar{x}$; and (ii) $\bar{\Theta}$ is a union of manifolds of dimension l.*

Proof. Suppose that $\bar{x}$ is a non-standard dividend equilibrium and $p \in {}^*\mathbb{R}^l$ and $d \in {}^*\mathbb{R}^n_+$ are corresponding prices and dividends. Consider an arbitrary agent i, $i \in N_m$ for some m. His/her budget set is a subset of the set $X_i(q_1, \ldots, q_{m-1})$. Each vector q_t, $t = 1, \ldots, m-1$, supports $X_i(q_1, \ldots, q_{t-1})$, (though it does not have to support X_i), which implies that the sets

$$X_i \supset X_i(q_1) \supset \ldots \supset X_i(q_1, \ldots, q_{m-1})$$

form a finite sequence of faces of X_i contained in each other. Denote the face $X_i(q_1, \ldots, q_{t-1})$ by $F_i^t(p)$ (note that the superscript t specifies that there are $t-1$ equalities). Let us construct a set Θ_p which contains a hierarchic price $q^p = (q_1, \ldots, q_k)$ representing p. Consider the set N_k. Without loss of generality it is not empty (otherwise one can throw away the last component of q^p and consider a new equilibrium price $p' = p - \lambda_k q_k$). Let L_k be the linear hull of faces F_i^k, $i \in N_k$,

$$L_k = \operatorname{span}\left(\bigcup_{i \in N_k} F_i^k(p)\right).$$

It is clear that the vector q_k must belong to this subspace (if necessary, q_k can be replaced by its projection on L_k). Take $L_k \setminus \{0\}$ as the last component of Θ_p. Secondly, consider $F_i^{k-1}(p)$ — superfaces of $F_i^k(p)$ for $i \in N_k$ — that is the sets $\{x \in X_i \mid q_t x = 0, t \leq k-2\}$, and faces $F_i^{k-1}(p)$ for $i \in N_{k-1}$. Taking a linear hull of the union of the sets $F_i^{k-1}(p)$ for all i from N_k and N_{k-1}, we obtain a linear space M_{k-1} which contains L_k. Denote by L_{k-1} the orthogonal complement to L_k in the space M_{k-1}, $L_{k-1} = M_{k-1} \cap (L_k)^{\perp}$, and take $L_{k-1} \setminus \{0\}$ as the next component of Θ_p. (Note that since the vectors q_t are mutually orthonormal, the vector q_{k-1} always has a non-zero projection on L_{k-1}). This procedure is reiterated $k-1$ times. As a result, a system of mutually orthogonal subspaces $L_1, \ldots, L_k$ is constructed, where

$$L_m = M_m \cap (L_{m+1})^{\perp}, \quad M_m = \operatorname{span}\left(\bigcup_{i \in N_t, t \geq m} F_i^m(p) \right).$$

By construction, $M_1 = \mathbb{R}^l$, $M_1 = L_1 \oplus \cdots \oplus L_k$. The set Θ_p is defined as the product of $L_m \setminus \{0\}$, $m = 1, \ldots, k$, so its dimension is equal to l. Look now at the set

$$\Upsilon = \{\Theta_p \mid p \in {}^*\mathbb{R}^l \text{ is an equilibrium price of an economy } \mathcal{E}\}.$$

This set is finite because each X_i has only a finite number of faces. Finally, let $\bar{\Theta} = \bigcup_{\Theta_p \in \Upsilon} \Theta_p$. By construction, for every non-standard dividend equilibrium $\bar{x}$ there exists a corresponding non-standard equilibrium price vector p, such that its hierarchic representation q^p belongs to $\bar{\Theta}$. □

Let Θ_F be the set that contains all hierarchic equilibrium prices that correspond to δ-equilibria from F. This set can be represented as a finite union of manifolds Θ_F^{ξ} such that for each Θ_F^{ξ} there exists $k \in \{1, \ldots, l\}$ such that $\Theta_F^{\xi} = \prod_{m=1}^{k} (L_m^{\xi} \setminus \{0\})$, where $L_1^{\xi}, \ldots, L_k^{\xi}$ are mutually orthogonal subspaces of $\mathbb{R}^l$. From now on, we fix an arbitrary element Θ of this finite union and denote by Θ^m its components $L_m \setminus \{0\}$, $m = 1, \ldots, k$, $\Theta = \Theta^1 \times \cdots \times \Theta^k$. We proceed with introducing a mapping Ψ_u that characterizes δ-equilibria from the face F. This mapping will be constructed as a product of the following correspondences. Mappings $\Psi_i^u : \widetilde{X} \to \mathbb{R}^{L \times \{i\}}$, $i \in N$ are defined by fragments of the gradient vectors of agents' utility functions related to their own consumption:

$$(\Psi_i^u)_j(x) = \frac{\partial u^i}{\partial x_j^i}(x), \quad j \in L.$$

Mappings Ψ_i^F, $i \in N$ reflect a condition that a δ-equilibrium allocation belongs to the face $F = \prod_{i \in N} F_i$. Recall that we consider only $x^i \in \operatorname{ri} F_i$. We replace here this condition with the milder requirement $x^i \in \operatorname{span} F_i$. Choose vectors $c_t^i \in \mathbb{R}^l$ such that $x^i \in \operatorname{span} F_i$ iff x^i is a solution of a system of $t(i)$ linear equations $c_t^i x^i = 0$, $t \in T_i$, $T_i = \{1, \ldots, t(i)\}$, and suppose that all rows of a matrix

$$C_i = \begin{pmatrix} c_1^i \\ \vdots \\ c_{t(i)}^i \end{pmatrix}$$

are linearly independent for each $i \in N$. A mapping $\Psi_i^F : \widetilde{X} \to \mathbb{R}^{T_i}$ is defined by $\Psi_i^F(x) = C_i x^i$. Thus, the condition $\Psi_i^F(x) = 0$ is necessary for x^i to belong to ri F_i. Let $N_1, \dots, N_k$ be an arbitrary partition of the set N. A mapping $\Psi^m : \widetilde{X} \times \Theta^m \to \mathbb{R}^{N_m \setminus \{i_0\}}$ responds to the budget restriction $q_m x^i = 0$ for $i \in N_m$:

$$(\Psi^m(x, q_m))_i = q_m x^i, \quad i \in N_m \setminus \{i_0\}, \quad m = 1, \dots, k.$$

The budget constraint for agent i_0 is removed since it follows from those of other individuals and feasibility of an equilibrium allocation. A mapping $\Psi^{mc} : \widetilde{X} \to \mathbb{R}^{L \times \{n+1\}}$ represents the equilibrium market clearing condition: $(\Psi^{mc}(x))_j = \sum_{i \in N} x^i_j, \quad j \in L$. Finally, we need the identity mapping $\Psi^q : \Theta \to \Theta$, $\Psi^q(q_1, \dots, q_k) = (q_1, \dots, q_k)$. A mapping Ψ_u is defined as a product of the correspondences described above:

$$\Psi_u = \prod_{i \in N} \Psi^u_i \times \Psi^{mc} \times \prod_{i \in N} \Psi^F_i \times \prod_{m=1}^{k} \Psi^m \times \Psi^q.$$

This mapping has domain $Z = \tilde{X} \times \Theta$ and takes its values in a finite-dimensional space $\mathbb{R}^T \times \Theta$, where

$$T = \left(\bigcup_{i=1}^{n+1} L \times \{i\} \right) \cup (N \setminus \{i_0\}) \cup \left(\bigcup_{i \in N} T_i \right).$$

A mapping $\Psi : U \times Z \to \mathbb{R}^T \times \Theta$ is defined by $\Psi(u, z) = \Psi_u(z)$.

At the next step, we describe a submanifold Δ_F such that for each proper δ-equilibrium $(x, q_1, \dots, q_k)$ its value $\Psi_u(x, q_1, \dots, q_k)$ belongs to Δ_F for some choice of parameters $i_0, \Theta, N_1, \dots, N_k$. Partition the set N_m into two subsets N'_m and N''_m, where N'_m consists of those agents for whom the last budget restriction is binding and N''_m contains those agents who are locally satiated on the face F_i. The necessary first-order conditions of a local extremum for the agents of the first type can be formulated as follows: there exist $y \in \mathbb{R}^{T_i}$ and $\lambda^m \in \mathbb{R}_{++}$ such that

$$\Psi^u_i(x) = \partial u_i / \partial x^i(x) = \lambda^m q_m + y C_i.$$

For agents of the second type those conditions have a simpler form: there exists $y \in \mathbb{R}^{T_i}$ such that $\Psi^u_i(x) = \partial u_i / \partial x^i(x) = y C_i$. When defining a manifold Δ_F, we take into consideration that $\Psi^m_i(x, q)$ is equal to zero at equilibrium for agents from N'_m whenever $m < k$, or to $\mu \delta_i$ if $m = k$; and corresponds to a free variable for $i \in N''_m$, $m = 1, \dots, k$. Define a submanifold $\Delta_F \subset \mathbb{R}^T \times \Theta$ as follows:

$$\Delta_F \stackrel{def}{=} \Big\{ (\nu_1, \dots, \nu_n, \beta_1, \dots, \beta_k, \sigma, q, \varphi_1, \dots, \varphi_n) \in \mathbb{R}^T \times \Theta \mid$$

$$\nu_i = \begin{cases} \lambda^m_i q_m + y C_i, \quad y \in \mathbb{R}^{T_i}, & \text{if } i \in N'_m, \quad m = 1, \dots, k, \\ \quad y C_i, \quad y \in \mathbb{R}^{T_i}, & \text{if } i \in N''_m, \quad m = 1, \dots, k, \end{cases} \tag{8}$$

$$(\beta_m)_i = \begin{cases} 0, & \text{if } i \in N'_m \setminus \{i_0\}, \quad m = 1, \dots, k-1, \\ \delta_i \mu, & \text{if } i \in N'_k \setminus \{i_0\}, \quad m = k, \end{cases} \tag{9}$$

$$\varphi_i = 0, \quad i \in N, \quad \sigma = 0, \quad \|q_m\| = 1, \quad m = 1, \dots, k \Big\}.$$

Here μ, λ^m_i, and y are free variables; ν_i is an l-dimensional vector that corresponds to a fragment of agent i's gradient of utility function related to his own consumption, $i \in N$; $\beta_m \in \mathbb{R}^{N_m \setminus \{i_0\}}$, $m = 1, \dots, k$ correspond to the budget constraints; $\sigma \in \mathbb{R}^l$ reflects the market clearing condition; $(q_1, \dots, q_k) \in \Theta$ corresponds to a hierarchic price representing non-standard equilibrium prices p; $\varphi_i \in \mathbb{R}^{T_i}$ relates to the condition $x^i \in F^i$. Equations (8) are necessary conditions for the utility maximization problem under the restrictions imposed by the face F and the budget. Denote by $\mathcal{L}_i$ the linear hull of vectors $\{c_t^i\}$, $t \in T_i$ that constitute the rows of the matrix C_i.

Lemma 4.7 *Consider an element $(u_0, z_0) = (u_0, x_0, q_1^0, \dots, q_k^0) \in U \times Z$ such that*

$$q_m^0 \notin \mathcal{L}_i, \quad i \in N_m, \quad m = 1, \dots, k. \tag{10}$$

Then $\Psi : U \times Z \to \mathbb{R}^T \times \Theta$ is transversal at (u_0, z_0) to any submanifold of $\mathbb{R}^T \times \Theta$.

Proof. It is sufficient to show that the tangent correspondence $T_{(u_0,z_0)}\Psi$ is surjective. Let $\upsilon = (\upsilon_1, .., \upsilon_n, \upsilon'_1, .., \upsilon'_k, \upsilon'', \upsilon''', \upsilon^1, .., \upsilon^n) \in \mathbb{R}^T \times \Theta$. We need to find a differentiable path $(u(\tau), z(\tau))_{\tau \in [0,1]}$ such that

$$u(0) = u_0, \quad z(0) = z_0 = (x_0, q_1^0, .., q_k^0), \quad \frac{\partial}{\partial \tau} \Psi(u(\tau), z(\tau))|_{\tau=0} = \upsilon.$$

Components $x(\tau)$, $q_m(\tau)$, $m = 1, \dots, k$ of a path $z(\tau)$ can be found in the following form:

$$x(\tau) = x_0 + \bar{x}\tau, \quad q_m(\tau) = q_m^0 + \bar{q}_m \tau, \quad m = 1, \dots, k, \tag{11}$$

where $(\bar{x}, \bar{q}_1, \dots, \bar{q}_k)$ are determined from the system of linear equations

$$\begin{cases} (\sum_{i \in N} x^i(\tau))'|_{\tau=0} = \upsilon'', \\ (x^i(\tau), q_m(\tau))'|_{\tau=0} = \upsilon'_{mi}, \quad i \in N_m \setminus \{i_0\}, \quad m = 1, \dots, k, \\ (C_i x^i(\tau))'|_{\tau=0} = \upsilon^i, \quad i \in N, \\ (q_m(\tau))'|_{\tau=0} = \upsilon'''_m, \quad m = 1, \dots, k. \end{cases}$$

After substituting representation (11) into this system and taking the first derivatives at $\tau = 0$, we can find elements $\bar{q}_1, \dots, \bar{q}_k$ from the last kl equations of this system. Take as $\bar{x}$ any solution of a system

$$\begin{cases} \sum_{i \in N} \bar{x}^i = \upsilon'', \\ (\bar{x}^i, q_m^0) = \tilde{\upsilon}_{mi}, \quad i \in N_m \setminus \{i_0\}, \quad m = 1, \dots, k, \\ C_i \bar{x}^i = \upsilon^i, \quad i \in N, \end{cases} \tag{12}$$

where $\tilde{\upsilon}_{mi} = \upsilon'_{mi} - (x_0^i, \bar{q}_m)$. By the conditions of the lemma, all rows of the matrix of this system are linearly independent, hence a solution exists. Let us determine a path $u(\tau)$ assuming $u_i(\tau, x) = u_0(x) + b_i x^i \tau$, where vectors $b_i \in \mathbb{R}^l$ are chosen to satisfy

$$\frac{\partial}{\partial \tau} \Big(\frac{\partial u_i}{\partial x_j^i}(x(\tau)) \Big) \Big|_{\tau=0} = \upsilon_{ij}. \tag{13}$$

Since $x(\tau) = x_0 + \bar{x}\tau$, the equation (13) is transformed to

$$b_{ij} = \upsilon_{ij} - \sum_{s \in N \times L} \frac{\partial^2 (u_0)_i}{\partial x_j^i \partial x_s}(x_0) \bar{x}_s.$$

Since the vector v has been chosen arbitrarily, we have shown that the derivative mapping $T_{(u_0,z_0)}\Psi$ is surjective. To complete the proof, we need to establish that $(T_{(u_0,z_0)}\Psi)^{-1}(T_yW)$ splits. This follows from the surjectivity of $T_{(u_0,z_0)}\Psi$ and from the finite dimensionality of its range. □

Linear dependence of the rows of a matrix $\begin{pmatrix} q_m \\ C_i \end{pmatrix}$ or, which is the same, the condition $q_m \in \mathcal{L}_i$, $i \in N_m$ implies that the budget constraint $q_m x^i = 0$ follows from the condition that the trade x^i belongs to the face F_i. We are going now to remove the budget constraints for agents from some set $H \subseteq N$, delete corresponding components $(\Psi^m)_i$ of the mapping Ψ and accordingly restrict the domain of Ψ in the part concerning prices. The modified mapping Ψ^H will be transversal to any submanifold $W \subseteq \mathbb{R}^l \times \Theta$ at any point $(u, x, q_1, \dots, q_k) \in U \times \tilde{X} \times \Theta^H$ — the domain of Ψ^H. Namely, let Θ^H be a set of all elements $(q_1, \dots, q_k)$ in Θ such that $q_m \in \mathcal{L}_i$ *only* for $i \in H \cap N_m$. Denote $H \cap N_m$ by H_m and consider sets

$$W_m^H = \begin{cases} (\bigcap_{i \in H_m} \mathcal{L}_i) \backslash (\bigcup_{i \in N_m \backslash H_m} \mathcal{L}_i), & \text{if } H_m \neq \emptyset \ , \\ \mathbb{R}^l \backslash (\bigcup_{i \in N_m} \mathcal{L}_i), & \text{otherwise.} \end{cases} \tag{14}$$

Take sets $\Theta_m^H = \Theta^m \cap W_m^H$, $m = 1, \dots, k$ as components of the manifold Θ^H : $\Theta^H = \Theta_1^H \times \cdots \times \Theta_k^H$. Notice that sets Θ^H, $H \subseteq N$ are relatively open and form a partition of the manifold Θ. Let $\overline{T} = T \backslash H$ and consider the mapping

$$\Psi_u^H = \prod_{i \in N} \Psi_i^u \times \Psi^{mc} \times \prod_{i \in N} \Psi_i^F \times \prod_{m=1}^{k} \Psi_H^m \times \Psi_H^q : \widetilde{X} \times \Theta^H \to \mathbb{R}^{\overline{T}} \times \operatorname{cl} \Theta^H,$$

where the mappings $\Psi_H^m : \widetilde{X} \times \Theta_m^H \to \mathbb{R}^{N_m \backslash (H_m \cup \{i_0\})}$ are defined by

$$(\Psi_H^m(x, q_m))_i = q_m x^i, \quad i \in N_m \backslash (H_m \cup \{i_0\}), \quad m = 1, \dots, k,$$

and $\Psi_H^q(q_1, \dots, q_k) = (q_1, \dots, q_k)$ is an identity embedding from Θ^H to $\operatorname{cl}\Theta^H$. All other mappings are defined as before. Put $\Psi_H(u, z) = \Psi_u^H(z)$. The manifold Δ_{HF} is the analogue of the manifold Δ_F :

$$\Delta_{HF} \stackrel{def}{=} \Big\{ (\nu_1, \dots, \nu_n, \beta_1, \dots, \beta_k, \sigma, q, \varphi_1, \dots, \varphi_n) \in \mathbb{R}^{\overline{T}} \times \operatorname{cl} \Theta^H |$$

$$\nu_i = \begin{cases} \lambda_i^m q_m + y C_i, \quad y \in \mathbb{R}^{T_i}, & \text{if } i \in N'_m \backslash H_m, \quad m = 1, \dots, k, \\ y C_i, \quad y \in \mathbb{R}^{T_i}, & \text{otherwise,} \end{cases} \tag{15}$$

$$(\beta_m)_i = \begin{cases} 0, & \text{if } i \in N'_m \backslash (H_m \cup \{i_0\}), \quad m = 1, \dots, k-1, \\ \delta_i \mu, & \text{if } i \in N'_k \backslash (H_k \cup \{i_0\}), \quad m = k, \end{cases} \tag{16}$$

$$\varphi_i = 0, \quad i \in N, \quad \sigma = 0, \quad \|q_m\| = 1, \quad m = 1, \dots, k \Big\}.$$

Lemma 4.8 *Suppose that $u \in U$ is such that for every $H \subseteq N$, Θ, every partition $(N_1, \dots, N_k)$ of the set N, and for all possible choices of subsets of satiated agents $N_1'', \dots, N_k''$, the mapping Ψ_u^H is transversal to the manifold Δ_{HF}. Then the number of proper δ-equilibria that belong to the face F is finite.*

Proof. The finiteness of proper δ-equilibria follows from the finiteness (with respect to a choice of parameters Θ, $N_1, \ldots, N_k$, $N_1'', \ldots, N_k''$, and $H \subseteq N$) of the possibly different arrangements of the correspondence Ψ_u^H and the manifold Δ_{HF}, as well as from the discreteness of the inverse images $(\Psi_u^H)^{-1}(\Delta_{HF})$ that cover all proper δ-equilibria from F. Therefore, it will suffice to establish finiteness of the sets $(\Psi_u^H)^{-1}(\Delta_{HF})$ to prove the lemma. By a well-known property of transversal correspondences, $\operatorname{codim} \Delta_{HF} = \operatorname{codim} (\Psi_u^H)^{-1}(\Delta_{HF})$. We show that

$$\operatorname{codim} \Delta_{HF} \geq \dim \widetilde{X} \times \Theta^H = nl + \dim \Theta^H. \tag{17}$$

Consider first only such equilibria that $N_1 = N_1'$ (i. e., there is no satiation in the usual sense). By the construction and the assumptions made, $\operatorname{codim} \Delta_{HF}$ is equal to the difference between a number of restrictions and a number of free variables. Each restriction of type (16) or $\varphi_i = 0$, $i \in N$ corresponds to the free variable λ or y, respectively. The budget restriction $q_1 x^{i_0} = 0$ is omitted (i_0 necessarily belongs to the set N_1). Taking into account relations (15), $\sigma = 0$, $\|q_m\| = 1$, $m = 1, \ldots, k$, and a free variable μ, one gets $\operatorname{codim} \Delta_{HF} = nl + l + k - 2$. Since $\dim \Theta^H \leq l$

$$\operatorname{codim} \Delta_{HF} \geq nl + \dim \Theta^H \tag{18}$$

if $k > 1$. Note, that $\operatorname{codim} \Delta_{HF}$ does not depend on the choice of H. If $k = 1$ then $N = N_1$, so we can put the free variable μ equal to zero (there are no satiated agents). This increases the codimension of the manifold by 1, and (18) is established again. In the case $N_1'' \neq \emptyset$ we proceed in the same way with the only difference that μ is not equated to zero but expressed through the values of the budget correspondences of the satiated agents. By transversality of Ψ_u^H and relative openness of Θ_m^H in $\cap_{i \in H_m} \mathcal{L}_i$, we conclude that

$$\dim (\Psi_u^H)^{-1}(\Delta_{HF}) < 0, \quad \text{if } \dim \Theta^H \neq l;$$

$$\dim (\Psi_u^H)^{-1}(\Delta_{HF}) = 0, \quad \text{if } \dim \Theta^H = l.$$

Thus, $(\Psi_u^H)^{-1}(\Delta_{HF})$ is discrete whenever $\dim \Theta^H = l$, $k \leq 2$ or is empty otherwise. One can easily see that there is only one subset $\overline{H} \subseteq N$ such that $\dim \Theta^{\overline{H}} = l$. Since Δ_F is closed, and X is, without the loss of generality, compact in $\mathbb{R}^{ln}$, the intersection $(\Psi_u)^{-1}(\Delta_F) \cap (X \times \Theta)$ is compact in $\mathbb{R}^{ln} \times \Theta$. We have the following relation

$$(\Psi_u)^{-1}(\Delta_F) \subseteq \bigcup_H (\Psi_u^H)^{-1}(\Delta_{HF}) = (\Psi_u^{\overline{H}})^{-1}(\Delta_{\overline{H}F}).$$

But $(\Psi_u^{\overline{H}})^{-1}(\Delta_{\overline{H}F})$ is discrete in $\widetilde{X} \times \Theta^H$. Therefore, $(\Psi_u)^{-1}(\Delta_F) \cap (X \times S)$ is a discrete compact, which implies the finiteness of proper δ-equilibria from F. $\square$

For each $i \in N$ choose a compact $K_i \in \mathbb{R}^{L \times \{i\}}$ such that $\operatorname{int} K_i \supset X_i$, $K = \prod_{i \in N} K_i \subset \widetilde{X}$, and let $\{S_t^H\}_{t=1}^{\infty}$ be a sequence of compact sets approximating Θ^H from within:

1. $S_t^H \subset \Theta^H, \quad t = 1, 2 \ldots,$

2. $\bigcup_{t=1}^{\infty} S_t^H = \Theta^H$.

Consider a sequence of compact sets $K_H^t = K \times S_t^H \subset \widetilde{X} \times \Theta^H$, $t = 1, 2, \ldots$, and apply the theorems of density and openness of transversal sections (see Abraham and Robbin (1967)) to the case $\mathcal{V} = U$, $X = \widetilde{X} \times \Theta^H$, $Y = \mathbb{R}^{\overline{T}} \times \mathrm{cl}\,\Theta^H$, $\rho_v = \Psi_u^H$, $\omega_\rho = \Psi_H$, $W = \Delta_{HF}$. By construction, Ψ_H is transversal to Δ_{HF}, and all other conditions of Thom's theorems are satisfied as well. Therefore, the set $\mathcal{V}_{KW}^t = \{u \in U \mid \Psi_u^H \pitchfork_z \Delta_{HF},\ z \in K_H^t\}$ is open and dense in U. Let $G = \bigcap_{t=1}^{\infty} \mathcal{V}_{KW}^t$, where the intersection is taken over $t = 1, 2, \ldots$ and over all admissible F, Θ^H, H, $N_1', \ldots, N_k'$, i_0. Since G is a countable intersection of open dense sets, it is residual. Direct application of Lemma 4.8 completes the proof. □

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