On the Edgeworth conjecture for production economies with public goods: A contract-based approach*

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Abstract

The paper applies and elaborates a contractual approach to study economies with a production of public goods. The barter contractual approach was developed in Marakulin (2003, 2011) for exchange economies; it is now modified and extended to the production economy. This includes hereby the introduction of a production contract and the adoption of known earlier notions: a web of contracts, coalitional domination for webs, a partial breaking of contracts and so on. Thus specific notions of properly contractual and fuzzy contractual allocations for an economy with public goods are introduced and their equivalence with Lindahl equilibria is stated. These theorems can be interpreted as a new way of perfect competition presentation.

Keywords and Phrases: public goods, contract, Walrasian and Lindahl equilibria, Foley core.
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Introduction

Modern views on the theory of financing of public goods stem from Samuelson's papers and the results of some other authors; see the survey Milleron, (1972) and Ruys, (1974). A public good is a product of joint consumption of all economic agents. The Pareto efficient mechanism of value regulation of public goods is based on the individual valuations calculated as a product of the individual price and total consumption.[1] Clearly, there can be ordinary commodities in the economy: their exchange and production are governed by usual market rules. An appropriate theoretical concept of Pareto efficient equilibrium defined in the literature is known as the Lindahl equilibrium. However practical implementation of the individual price apparatus being applied to public goods raises a problem of calculation of these prices (and taxes). This is in fact a difficult theoretical question that still does not have a clear answer in classical theory. One way to solve this problem could be based on the cooperative description of equilibrium, in such a way as is done in models with purely private goods, via a theorem on the coincidence of the core and the equilibrium under perfect competition conditions. However, examples show that under ordinary replication[2] of an economy the Foley core does not shrinks to equilibrium (see Muench, (1972), Milleron, (1972), and Buchholz and Peters, (2007)). If one does not take into account such extremal results as Conley, (1994), the problem finds its resolution in the modern literature only through the transformation of the concept of coalitional domination. Instead of the familiar public goods so-called semi-public goods are introduced, as was done in Vasil'ev et al. (1995) (see also Weber and Wiesmeth, (1991), Vasil'ev, (1996), and Florenzano and Mercato (2006)). The difference is that now the utility of consumed goods depends of the total number of customers.[3] One can say that consumers are interested in the average level of consumption of public goods. In theory there appeared such concepts as "returns to group size", see Roberts, (1974), a congestion of goods etc. Vasil'ev et al. (1995) proved subtle theorems on the coincidence of the core and equilibria under the assumption of constant returns to the size of the coalition. The condition proves to be necessary for the coincidence between the core and equilibria, which is a remarkable result. In this paper, the theorems on the coincidence of the core and the Lindahl equilibrium are stated within the contractual approach and under assumptions that are similar those made in the papers mentioned above.

[1] There are also other, non-Pareto efficient, equilibrium notions, such as those based on private provision of public goods (e.g. Florenzano, (2009)).

[2] This is one of the most popular ways to model perfect competition conditions, which goes back to Debreu and Scarf. There are other methods, e.g. Aumann's approach, applying a non-atomic measure space of economic agents.

[3] Perception of the good is inversely proportional to the number of users of the same type; not only the total amount, but mainly the density of vehicles and traffic congestion affect the satisfaction of such a good as "transport infrastructure".

However, here we are not talking about coalition sizes or a measure of congestion in public goods: in the model individuals are engaged in ordinary economic activity; they sign and break (partially and asymmetrically) production and barter contracts, thereby producing a stable regime of functioning, which corresponds to a Lindahl equilibrium allocation.

The analysis is based on the author's approach developed in the series of papers of the recent decade in the context of exchange economies of various kinds and generality (e.g. Marakulin (2003, 2011) etc.). The idea of a barter exchange (contract) is by no means new in theoretical economics and evidently goes back to classical Edgeworth results (1881), but it usually appeared as an interpretation, in the form of net trade in a formal model. A Contract as a barter exchange of commodities appears in the works of other authors (though the theory of barter contracts was not elaborated in a proper way), including Russian ones: Polterovich, (1970) and Makarov, (1982). Then Kozyrev, (1981) suggested partial breaking of contracts and obtained some preliminary positive results. Incorporation of partially breaking contracts into the notion of stable webs provides an alternative description of the Walrasian equilibrium for the case of complete markets. The author's results (Marakulin, 2003, 2011) provide a basis of barter contract theory that can be considered as a cooperative replenishment of the classical views on the condition of perfect competition in a market economy.

In the consumption sector every contract is an elementary *permissible* exchange of commodities among consumers (barter): the members of a coalition implement the exchange of commodities. Contracts may be summed up and an allocation of resources may be put into correspondence to every (finite) set of contracts—as a result of summation of contracts and the initial endowment allocation. The presence of production affects the contract definition in an essential way. Now, a contract is not only commodity barter: it describes an allocation of individual inputs to a production plan. It is presumed that every feasible set of (permissible) contracts—let us call it a '*web of contracts*'—may be changed during periods of economic activity. Each consumer or a coalition of consumers can *break contracts* in which he/she participates, and a coalition can also *sign a new one*. Sometimes the partial breaking of contracts is also permissible. Our theory examines a stable web of contracts where the stability can take various forms depending on admissible ways of contract breaking: *total, partial (symmetric breaking), fuzzy (partial asymmetric breaking)*, etc. The different formal rules of handling with the sets of contracts correspond to different forms of web stability and therefore to different forms of stability of allocation implemented by a web. The kinds of these 'stabilities', together with the property of contracts to be permissible, reflect different behavioral, physical and institutional principles, formally given in a game-theoretical form. A peculiar property of our contractual approach is that all production and exchange processes operate without any kind of value parameters. So, in applying a contractual approach, we are interested in a stable web of contracts and this stability can vary. The paper presents a contractual analysis of equilibria for production economies with public goods: Lindahl equilibrium is described in purely contractual terms if aggregated production is determined by a convex cone (constant returns to scale); in general stable webs describe a wider set of Pareto optimal allocations.

The paper is organized as follows. A model of production economy with public goods and the notion of Lindahl equilibrium are presented in the Section 1. Section 2 is devoted to the introduction of the notions of Foley and fuzzy cores; the relationships among these cores and the equilibrium are revealed here. Section 3 presents the elements of a contractual approach for economies with public goods: a new specific notion of production contract is introduced here. The theorem on equivalence of properly contractual allocation (partial breaking of contracts is possible) and the Lindahl equilibrium is presented here. This is one of the major results of the paper. The specific notion of fuzzy contractual allocation is introduced and studied in the fourth section. Another theorem stating the equivalence of Lindahl equilibrium and fuzzy contractual allocations presents the second significant result of the paper. Conclusion finishes the main part of the paper. All long proofs are collected in the Appendix.

1 Public goods economy and the Lindahl equilibrium

An economy with public goods is specified by the presence of special commodities, which by their physical characteristics are the goods of public consumption. Examples of public goods are public television and radio, street lighting, roads, production of "security" (police, national defense, etc.). The list of examples can be continued, but clearly a public good is a commodity that is simultaneously consumed by many agents and there is a need to reproduce it (one needs to repair roads, produce and broadcast television programs) that has to be financed somehow. It is clear that the funding of production of a collective consumption commodity should be carried out by all its consumers. In the neoclassical theory of a decentralized economy the concept of individual valuations is considered as a basis for public goods financing; these valuations are calculated as the product of individual prices and a (total) consumption bundle. Of course, ordinary products may exist in the economy and processes of their allocation and production are carried out under usual market rules. In theory, an appropriate concept of equilibrium (by Lindahl) is defined and studied in such a way that the related allocation is a Pareto optimal one. The correct determination of individual prices is a difficult theoretical problem in practice. In the case of private commodities, this issue is resolved via the market mechanism, based on a large number of exchange transactions, by the method of price "tatônnement."

This method does not work for public goods, because individuals are incapable to exchange the parts of public goods.[4] From a theoretical point of view individual prices should be proportional to the marginal rates of substitution (exchange), but in terms of utility functions they should be proportional to a fragment of the gradient, corresponding to public goods. Thus, in order to "evaluate" the individual prices, one must have purely private information about the preferences of individuals, which is not practicable in real life. Below I suggest a theoretical way to resolve this problem.

The structure of an economy with public goods is similar to the Arrow–Debreu model: the main difference is in the commodity space, which also represents public goods. The mechanism of public goods decentralization is also different and specific. The model has n consumers forming a set $\mathcal{I} = \{1, \ldots, n\}$ and m producers (firms) $\mathcal{J} = \{1, \ldots, m\}$. There are l types of private good, their nomenclature being $\{1, \ldots, l\}$, and s kinds of public good, numbered by the indices $\{l+1, \ldots,\ l+s\}$. Thus, the total number of products is $l + s$. Consumers are equipped with individualized *private goods consumption sets* $X_i^p \subset \mathbb{R}^l$ and a *common* for all consumers set of *permissible* for consumption *public goods* $X^c \subset \mathbb{R}^s$. So, here $\mathbb{R}^{l+s} = E$ is a commodity space. In addition, consumers have initial endowments of private goods $\omega_i \in X_i^p$, $i \in \mathcal{I}$, and the economy as a whole has endowments of public goods $\omega^c \in X^c$. In general, firms can produce and spend private as well as public goods; their production capacities are presented by technological sets $Y_j \subset \mathbb{R}^{l+s}$, $0 \in Y_j$, $j \in \mathcal{J}$. Production plans $y_j \in Y_j$ are written in the form $y_j = (y_j^p, y_j^c)$, where $y_j^p \in \mathbb{R}^l$ is associated with the private goods and $y_j^c \in \mathbb{R}^s$ with the public ones. The set

$$\mathbb{Z} = \prod_{\mathcal{I}} X_i^p \times X^c \times \prod_{\mathcal{J}} Y_j$$

is identified with the set of all *admissible states* and the space $L = \mathbb{R}^{ln+s+m(l+s)} \supset \mathbb{Z}$ is a *space of allocations.* Consumers' preferences are defined and take values in $X_i^p \times X^c$, i.e. $\mathcal{P}_i : X_i^p \times X^c \rightrightarrows X_i^p \times X^c$. The sets $X_i^p \times X^c$ are associated with consumption sets of individuals; they are assumed to be convex. One can see that this model has externalities, concentrated in the area of public goods. In addition, as in the Arrow–Debreu model, $\theta_i^j \geq 0$ (a component of vector $\theta_i = (\theta_i^1, \ldots, \theta_i^m)$) is the *share* of consumer i in the profit of producer j. These quantities satisfy $\sum_{i\in\mathcal{I}} \theta_i^j = 1$ for all $j \in \mathcal{J}$ (i.e. profit is completely distributed among all shareholders).

Assume that the processes of exchange and production of goods are regulated by the individual prices for public goods $q_i \in \mathbb{R}^s$ and the market prices $p \in \mathbb{R}^l$ for the private commodities. In this case, the budget constraint for consumption plans

[4] Therefore, it is recommended in practice to transform the "public goods" into private ones whenever possible, through various specific techniques in order to enable the market mechanism. An example of this is the transition to the counters in water provision.

$(x_i, x^c) \in X_i^p \times X^c$ of individual $i \in \mathcal{I}$ are specified by

$$\langle x_i, p\rangle + \langle x^c, q_i\rangle \leq \langle \omega_i, p\rangle + \langle \omega^c, q_i\rangle + \sum_{j\in\mathcal{J}} \theta_i^j (py_j^p + \bar{q}y_j^c),$$

where $y \in \prod_{\mathcal{J}} Y_j$, $q = (q_1, \dots, q_n) \in [\mathbb{R}^s]^{\mathcal{I}}$, $p \in \mathbb{R}^l$ and $\bar{q} = \sum_{\mathcal{I}} q_i$. So, one can see that agents' incomes are formed from three sources: the sale of private endowments ω_i for market prices p, the individualized value of public goods $\langle q_i, \omega^c\rangle$, and the "sum of dividends" from the profits of producers. It is also worth repeating that the producers' profits are determined by "production" prices $(p, \bar{q})$. Now in its shortest form the model under study can be written as

$$\mathcal{E}^{pg} = \langle \mathcal{I}, \mathcal{J}, \mathbb{R}^l, \mathbb{R}^s, \{X_i^p, \mathcal{P}_i(\cdot), \theta_i, \omega_i\}_{i\in\mathcal{I}}, \{Y_j\}_{j\in\mathcal{J}}, X^c, \omega^c \rangle.$$

In the neoclassical setting, the Lindahl equilibrium is considered as the main solution concept.

Definition 1.1 *An allocation $z = (x, x^c, y) \in \mathbb{Z}$ is said to be a **Lindahl equilibrium** with a price bundle $(p, q_1, \dots, q_n) \in \mathbb{R}^{l+ns}$ if for $\bar{q} = \sum_{\mathcal{I}} q_i$ it satisfies*

$$py_j^p + \bar{q}y_j^c \geq \langle (p, \bar{q}), Y_j\rangle, \quad j \in \mathcal{J},^5 \tag{1.1}$$

$$\langle (p, q_i), \mathcal{P}_i(x_i, x^c)\rangle > \omega_i p + \omega^c q_i + \sum_{j\in\mathcal{J}} \theta_i^j (py_j^p + \bar{q}y_j^c) = x_i p + x^c q_i, \quad i \in \mathcal{I}, \tag{1.2}$$

$$\sum_{\mathcal{I}} x_i = \sum_{\mathcal{I}} \omega_i + \sum_{\mathcal{J}} y_j^p, \tag{1.3}$$

$$x^c = \omega^c + \sum_{\mathcal{J}} y_j^c. \tag{1.4}$$

*In the case of non-strict inequalities in (1.2) it is said to be a **quasi-equilibrium**.*

Requirements (1.1)–(1.4) have their usual substantial sense. Condition (1.1) implements the principle of producers' profit maximization, (1.2) states that (x_i, x^c) is an optimal budget acceptable plan, condition (1.3) presents the balance for private consumption commodities, and (1.4) is the balance for public goods.

Commenting on the equilibrium definition, one notes that individuals evaluate the consumption of public goods according to individual prices q_i, while in production specific industrial prices are applied: they are equal to the sum of individual prices, $\bar{q} = \sum_{\mathcal{I}} q_i$. Also note the specific requirement of the public goods balance:

[5]Here $\langle p, x\rangle = p(x) = px$ denotes the inner product of vectors $p \in E^*$, $x \in E$ and $\langle p, A\rangle = \{p(x) \mid x \in A\}$ & $\langle A, B\rangle = \{\langle x, y\rangle \mid x \in A,\ y \in B\}$, for $A \subset E$, $B \subset E^*$; $A > B \iff a > b$, $\forall a \in A,\ \forall b \in B$ for $A, B \subset \mathbb{R}$ and similarly for nonstrict inequality.

this is so because all individuals consume the same quantity of good, if it is a public one. Further, any allocation from $\mathbb{Z}$ satisfying (1.3) and (1.4) is called a *feasible* (valid) one, and the set of all such allocations is denoted $\mathcal{A}(\mathcal{E}^{pg})$.

Without going into detail, we note that for a convex economy the Lindahl equilibrium is always Pareto optimal and does exist under almost the same assumptions as the Walrasian equilibrium, see, for example, Ruys, (1974), Florenzano and Mercato (2006), Florenzano, (2009), and Marakulin, (2012) §1.2.3. Similarly to the usual Arrow–Debreu model in an economy with public goods there is an analog of the second Welfare theorem: every Pareto optimal allocation can be presented in the form of a Lindahl equilibrium for a specific redistribution of initial endowments. In other words, for a Pareto optimal allocation one can provide a dual characterization in value categories (a similar result can be found e.g. in Florenzano, (2009)). Insofar as this is involved in the subsequent analysis, I present a formalization of an appropriate mathematical result and give its proof. Recall that:

• *A feasible allocation* $(\bar{x}, \bar{x}^c, \bar{y}) = ((\bar{x}_i)_{i\in\mathcal{I}}, \bar{x}^c, (\bar{y}_j)_{j\in\mathcal{J}}) \in \mathbb{Z}$ *is said to be (weak)* ***Pareto optimal*** *if there is no allocation* $((x_i)_{i\in\mathcal{I}}, x^c, (y_j)_{j\in\mathcal{J}}) \in \mathcal{A}(\mathcal{E}^{pg})$ *such that* $(x_i, x^c) \succ_i (\bar{x}_i, \bar{x}^c)$ *for each* $i \in \mathcal{I}$.

Everywhere below we assume that the model under study satisfies the following assumption.

(**A**) *For each* $i \in \mathcal{I}$, $X_i \times X^c \subset E$ *is a convex solid closed set (non-empty interior),* $(\omega_i, \omega^c) \in X_i \times X^c$ *and for every* $(x_i, x^c) \in X_i \times X^c$ *there exists an open convex* $G_i \subset E$ *such that*

$$\mathcal{P}_i(x_i, x^c) = G_i \cap (X_i \times X^c) \quad \& \quad (x_i, x^c) \in \overline{\mathcal{P}_i(x_i, x^c)} \setminus \mathcal{P}_i(x_i, x^c).^6$$

For the convenience of further exposition, we also introduce a specific notion of a smooth economy.

• *An economy* $\mathcal{E}$ *has* ***smooth consumption sector*** *if for each* $i \in \mathcal{I}$

$$\mathcal{P}_i(x_i, x^c) = \{(x'_i, x^{c\prime}) \in X_i \times X^c \mid u_i(x'_i, x^{c\prime}) > u_i(x_i, x^c)\}, \ \forall (x_i, x^c) \in X_i \times X^c$$

for a differentiable concave function $u_i(\cdot)$ *defined on an open neighborhood of* $X_i \times X^c$.

In the following lemma and the subsequent analysis we shall also apply a specific notion of locally *non-satiated* preferences of each individual in the groups of *private* and (separately) *public* goods. The latter means, that changing the consumption bundle (x_i^p, x_i^c) only in part of the private or (separately) public goods while the

[6] Here $\overline{A}$ denotes the closure of A and $\setminus$ is set for the set-theoretical difference. This presents local non-satiation of agents' preferences.

consumption from another group of commodities is the same, it is possible to obtain a strictly preferred consumption bundle:

$$\mathcal{P}_i(x_i^p, x_i^c) \cap \mathbb{R}^l \times \{x_i^c\} \neq \emptyset \quad \& \quad \mathcal{P}_i(x_i^p, x_i^c) \cap \{x_i^p\} \times \mathbb{R}^s \neq \emptyset, \quad \forall i \in \mathcal{I}. \tag{1.5}$$

Lemma 1.1 *Let the allocation $\bar{z} = ((\bar{x}_i)_{i\in\mathcal{I}}, \bar{x}^c, (\bar{y}_j)_{j\in\mathcal{J}}) \in \mathcal{A}(\mathcal{E}^{pg})$ be Pareto optimal. Then there is a vector of prices for private goods $p \in \mathbb{R}^l$ and individualized price vectors for public goods $q_i \in \mathbb{R}^s$, $i \in \mathcal{I}$, which are* ***not all*** *equal to* ***zero*** *and are such that*

$$\langle (p, q_i), \mathcal{P}_i(\bar{x}_i, \bar{x}^c)\rangle \ \geq \langle (p, q_i), (\bar{x}_i, \bar{x}^c)\rangle, \ i \in \mathcal{I}, \tag{1.6}$$

$$\langle (p, \sum_{i\in\mathcal{I}} q_i), \bar{y}_j\rangle \geq \langle p, Y_j\rangle, \ j \in \mathcal{J}. \tag{1.7}$$

If for $\bar{z}$ in addition the consumption of each individual is ***non-satiated*** *by private and (separately) public goods, and if $(\bar{x}_i, \bar{x}^c) \in \mathrm{int}(X_i^p \times X^c)$ $\forall i \in \mathcal{I}$, then all these price vectors* ***are non-zero****, i.e. $p \neq 0$, $q_i \neq 0$ $\forall i \in \mathcal{I}$.*

Remark 1.1 If $\mathcal{E}^{pg}$ is an economy with a smooth consumption sector and all the lemma's assumptions are satisfied then for interior allocation and for every i there is a real $\lambda_i > 0$: $(p, q_i) = \lambda_i \cdot \nabla u_i(\bar{x}_i, \bar{x}^c)$. ■

2 Fuzzy core versus Foley core—what is the best?

In the model $\mathcal{E}^{pg}$ with public goods the concept of Foley core is usually considered in the literature and it has a familiar substantial sense: the set of all production allocations, which can be dominated by no coalition, i.e., no group of individuals would benefit to live as a separate economy.

• *An allocation $z = (x, x^c, y) \in \mathcal{A}(\mathcal{E}^{pg})$ is said to be dominated (blocked) by coalition $\emptyset \neq S \subseteq \mathcal{I}$, if there exist production $y^S = (y^{Sp}, y^{Sc}) \in Y_S = \sum_{i\in S}\sum_{j\in\mathcal{J}} \theta_i^j Y_j$ and consumption $((x_i^S)_S, x^{Sc}) \in \prod_S X_i \times X^c$ programs such that*

$$\sum_{i\in S} x_i^S = \sum_{i\in S} \omega_i + y^{Sp}, \quad x^{Sc} = \omega^c + y^{Sc} \quad \& \quad (x_i^S, x^{Sc}) \succ_i (x_i, x^c) \ \ \forall i \in S.$$

The set of all allocations that are ***dominated by no coalition*** *is denoted as $\mathcal{C}(\mathcal{E}^{pg})$ and is called* ***Foley core****.*

Foley introduced this concept (Foley,, 1970) and proved under certain assumptions that the Lindahl equilibrium belongs to the core: in our paper this result follows directly from the analogous fact for the fuzzy core; see Proposition 2.1. However, does the Foley core shrink to equilibria under infinity replication of the model? It is well known from the literature that an infinite replication of the model *does not* imply that the *Foley core shrinks* to Lindahl equilibria, unlike economies with only private goods. One can find appropriate examples e.g. in Muench, (1972), Buchholz and Peters, (2007). So, how is perfect competition to be presented in public goods economy? There are at least two approaches: the fuzzy core approach and the contractual approach.

Below, I introduce and study the concept of the fuzzy core for a model with public goods. In Section 4, fuzzy contractual allocations will be introduced and studied. As we shall see, these concepts are closely related to each other; they work fruitfully in the equilibrium theory and present an adequate solution of how the core can shrink to the Lindahl equilibria.

Let us start from the definition and analysis of the fuzzy core. Recall that a fuzzy coalition is identified with any vector

$$t = (t_1, t_2, \ldots, t_n) \neq 0, \quad 0 \leq t_i \leq 1 \ \ \forall i \in \mathcal{I},$$

where real t_i is interpreted as a measure of agent i's participation in the coalition activities. The key property that determines the efficiency of fuzzy coalitions is their ability to dominate a current allocation of the economy, and it defines the fuzzy core. In a model with public goods this is an especially peculiar thing.

• *Let $(x, x^c) = ((x_i)_{i\in\mathcal{I}}, x^c) \in \prod_{\mathcal{I}} X_i^p \times X^c$ be a family of private and public consumption plans, and $y = (y^p, y^c) \in \sum_{\mathcal{J}} Y_j = Y$ be an aggregated production program. Let triplet $(x, x^c, y) = z$ present a feasible allocation, i.e. $x^c = \omega^c + y^c$ and $\sum_{\mathcal{I}} x_i = \sum_{\mathcal{I}} \omega_i + y^p$ hold. A* ***fuzzy coalition*** *$t = (t_1, \ldots, t_n)$* ***blocks allocation*** *z if there is a triplet $((\xi_i)_{\mathcal{I}}, \xi^c, (\zeta^p, \zeta^c))$, such that*

$$\sum_{\mathcal{I}} t_i(\xi_i - \omega_i) = \zeta^p, \quad \xi^c = \zeta^c, \quad (\zeta^p, \zeta^c) \in Y \tag{2.1}$$

and

$$(\xi_i, \frac{\xi^c}{t_i} + \omega^c) \succ_i (x_i, x^c) \ \ \forall i \in \mathcal{I} : \ t_i \neq 0. \tag{2.2}$$

The set of all allocations that are dominated by no fuzzy coalition is denoted $\mathcal{C}^f(\mathcal{E}^{pg})$ and is called the ***fuzzy core****.*

Despite the fact that similar constructions have already appeared in the literature (e.g. see Vasil'ev, (1996)), further analysis is quite original.

The meaning of the fuzzy core and the blocking is as follows. Imagine that an index $i \in \mathcal{I}$ specifies only the type of economic agent which is represented by many identical copies (the same number for different types). Then $t_i \in (0, 1]$ is a share of type i individuals, entered in a blocking coalition. Being separated, the coalition passes to self-sufficiency of all its needs, which is expressed by relation (2.1). In this case, however, agents have to improve their situation and reach a more preferred consumption, which is expressed by (2.2), and this is the key peculiarity of (semi)public goods. Indeed, the agents estimate public goods produced within the coalition as the relative proportion of individuals of presented type. Florenzano and Mercato (2006) do not introduce a fuzzy core concept; however, they expressly postulate the appropriate type of domination in a replica of the original model (see Definitions 3.4, 3.5). One can say that an *average* level (for a type) of public goods consumption is crucial for agents instead of a common level of consumption. Thus, one can speak about semi-public goods, just as it is done in Vasil'ev et al. (1995). Moreover, in Vasil'ev et al. (1995), domination similar to that described above is interpreted in the terms of congestion or crowding in its provision (in these terms the results on the equivalence of the core and the equilibria are formulated). For example, (dis)pleasure from the consumption of such a good as the opportunity to attend a public skating-rink (park, road infrastructure, etc.) essentially depends on the number of visitors. Below, we will see that the domination and the fuzzy core elements can be interpreted in contractual categories, where the coalition has the opportunity to enter into contracts for the production of public goods for inter-coalitional consumption. Then the elements of the fuzzy core correspond to stable sets of contracts (webs), subject to approval of the possibility of an asymmetric partial breaking.

Below, the first important result on the fuzzy core is presented; it is proven under the following additional assumptions.

(**P**) *For each $j \in \mathcal{J}$ the set Y_j is a convex closed cone with a vertex at zero and public goods can only be produced, their amount cannot be reduced; i.e. $Y_j \subset \mathbb{R}^l \times \mathbb{R}^s_+$.*

(**M**) *The set of feasible public goods consumption programs obeys $X^c + \mathbb{R}^s_+ \subseteq X^c$ and all public goods are desirable*[7] *for each individual; i.e.*

$$(x_i, x^c + z) \succ_i (x_i, x^c) \quad \forall (x_i, x^c) \in X_i \times X^c, \ \forall z \in \mathrm{int}\mathbb{R}^s_+ \quad \forall i \in \mathcal{I}.$$

Proposition 2.1 *If $\mathcal{E}^{pg}$ satisfies* (**P**), (**M**), *then the Lindahl equilibrium belongs to the fuzzy core.*

In order to better understand the properties of the fuzzy core and to reveal the exact relationship between the core and equilibrium (equivalence?), I establish the following fuzzy core characterization. Define

$$\Omega_i(x_i, x^c) = \mathrm{co}(\mathcal{P}_i(x_i, x^c) \cup \{(\omega_i, \omega^c)\}), \quad i \in \mathcal{I} \qquad (2.3)$$

[7] This is monotonicity of preferences with respect to public goods.

Due to the convexity of $\mathcal{P}_i(x_i, x^c)$, for $\mathcal{P}_i(x_i, x^c) \neq \emptyset$ (we have it by (**A**)) we conclude that

$$\mathrm{co}(\mathcal{P}_i(x_i, x^c) \cup \{(\omega_i, \omega^c)\}) = \bigcup_{0 \leq \lambda \leq 1} [\lambda \mathcal{P}_i(x_i, x^c) + (1 - \lambda)(\omega_i, \omega^c)] =$$

$$= \Omega_i(x_i, x^c) = \bigcup_{0 \leq \lambda \leq 1} \lambda(\mathcal{P}_i(x_i, x^c) - (\omega_i, \omega^c)) + (\omega_i, \omega^c), \quad i \in \mathcal{I}.$$

Next, consider a set $\prod_{\mathcal{I}} \Omega_i(x_i, x^c) \times Y$, a vector

$$\tilde{\omega} = ((\omega_1, \omega^c), (\omega_2, \omega^c), \ldots, (\omega_n, \omega^c), 0) \in \mathbb{R}^{(n+1)(l+s)}$$

and a condition $z + \tilde{\omega} \in \prod_{\mathcal{I}} \Omega_i(x_i, x^c) \times Y$. By construction and analysis, one has a representation

$$z = (\lambda_1(\xi_1 - \omega_1, \xi_1^c - \omega^c), \lambda_2(\xi_2 - \omega_2, \xi_2^c - \omega^c), \ldots, \lambda_n(\xi_n - \omega_n, \xi_n^c - \omega^c), \zeta^p, \zeta^c) \quad (2.4)$$

considered relative to some $(\xi_i, \xi_i^c) \in \mathcal{P}_i(x_i, x^c)$, $i \in \mathcal{I}$, $(\zeta^p, \zeta^c) \in Y$. Now, let us consider a subspace which corresponds to the material balance conditions for the model with public goods:

$$\mathcal{L}^{pg} = \{((z_1^p, z_1^c), (z_2^p, z_2^c), \ldots, (z_n^p, z_n^c), (y^p, y^c)) \mid$$
$$z_1^p + z_2^p + \cdots + z_n^p = y^p + \sum_{\mathcal{I}} \omega_i, \quad z_1^c = z_2^c = \cdots = z_n^c = y^c + \omega^c\}. \quad (2.5)$$

Finally, if $z + \tilde{\omega} \in \prod_{\mathcal{I}} \Omega_i(x_i, x^c) \times Y \cap \mathcal{L}^{pg}$, then these balance conditions are added to constraint (2.4) for z:

$$\lambda_1(\xi_1 - \omega_1) + \cdots + \lambda_n(\xi_n - \omega_n) = \zeta^p \quad \& \quad \lambda_1(\xi_1^c - \omega^c) = \cdots = \lambda_n(\xi_n^c - \omega^c) = \zeta^c,$$

as in (2.1). For $\xi_i^c = \frac{\zeta^c}{\lambda_i} + \omega^c$, by construction, one has $(\xi_i, \frac{\zeta^c}{\lambda_i} + \omega^c) \in \mathcal{P}_i(x_i, x^c)$, which is equivalent to (2.2), $i \in \mathcal{I}$. Hence, fuzzy blocking occurs if and only if there exists *non-zero* z, satisfying all requirements (i.e. $z + \tilde{\omega}$ belongs to the intersection). The presented reasonings prove the following characteristic

Lemma 2.1 *A feasible allocation* $(x, x^c, y^p, y^c) \in \mathcal{C}^f(\mathcal{E}^{pg})$ *if and only if*

$$\prod_{\mathcal{I}} \Omega_i(x_i, x^c) \times Y \bigcap \mathcal{L}^{pg} = \{\tilde{\omega}\}. \quad (2.6)$$

Provided that in (2.6) all intersected sets are convex, it allows us to apply the separation theorem for the characterization of the elements of fuzzy core in value categories. One can do this applying the result and the mathematical technique used in Lemma 1.1 to intersection (2.6), and the following

Proposition 2.2 *Let the allocation* $\bar{z} = (\bar{x}, \bar{x}^c, \bar{y}^p, \bar{y}^c) \in \mathcal{C}^f(\mathcal{E}^{pg})$ *and preferences be non-satiated (via* (**A**)*). Then* $(\bar{x}, \bar{x}^c, \bar{y}^p, \bar{y}^c)$ *is Pareto optimal and therefore there exist prices* $p \in \mathbb{R}^l$, $q_i \in \mathbb{R}^s$, $i \in \mathcal{I}$, *not all equal to zero, such that relations (1.6) and (1.7) hold. Moreover if* (**P**) *holds (convex conic production) then in addition*

$$\langle (p, q_i), (\bar{x}_i, \bar{x}^c) \rangle = \langle p, \omega_i \rangle + \langle q_i, \omega^c \rangle, \; i \in \mathcal{I}, \tag{2.7}$$

i.e. all the budgets are balanced and therefore the allocation is a ***quasiequilibrium****. If the conditions of the second part of Lemma 1.1 are true, then it is a genuine Lindahl equilibrium.*

Now a theorem on the equivalence of the core and equilibria can be obtained as a consequence of the unconditional approval of the latter in conjunction with the Proposition 2.1 proved above.

Theorem 2.1 *Let* $\mathcal{E}^{pg}$ *satisfy* (**P**), (**M**), $(x_i, x^c) \in \mathrm{int}(X_i \times X^c)$, $i \in \mathcal{I}$ *and (1.5) be true (separated non-satiation). Then allocation* $z = ((x_i)_{\mathcal{I}}, x^c, y^p, y^c) \in \mathcal{C}^f(\mathcal{E}^{pg})$ *if and only if it is a* ***Lindahl equilibrium*** *allocation.*

Remark 2.1 In order to avoid such restrictive requirement as an interior point in the consumption of each individual, one can use the *irreducibility* assumption (specified for the public goods) and *non-trivial quasi-equilibrium* property, as in Florenzano and Mercato (2006) and Florenzano, (2009); see also Section 1.2.4 in Marakulin, (2012). Here and below, I restrict myself to the case of an interior point, in order to make the presentation not too overburdened. ■

3 Contractual approach: partial breaking and proper allocations

Now, I recall briefly the basic ideas and conceptual apparatus of the theory of barter contracts (see Marakulin (2003, 2011)) while adapting it to the model with the production sector.

If E denotes a private commodity space, then for a pure exchange economy any vector $v = (v_i)_{i \in \mathcal{I}} \in E^{\mathcal{I}}$ satisfying $\sum_{i \in \mathcal{I}} v_i = 0$ is called a barter *contract*. Clearly such barter (exchange) contracts can be also applied for the analysis of an economy with production. In what follows, we assume that any barter agreement is valid. With every finite collection V of (permissible) contracts one can associate allocation $x(V) = \omega + \sum_{v \in U} v$, where $\omega = (\omega_1, \ldots, \omega_n) \in X = \prod_{\mathcal{I}} X_i$ is an initial endowments allocation. If $\omega + \sum_{v \in U} v \in X \; \forall U \subseteq V$, i.e., if for any part of broken contracts one can get anyway a feasible allocation, then we call V a *web* of contracts.

Consider further the operations of breaking of existing contracts and signing of new contracts. It is assumed that any contract $v \in V$ may be *broken* by any trader in $S(v) = \mathrm{supp}(v) = \{i \in \mathcal{I} \mid v_i \neq 0\}$, since he/she can do not adhere his/her contractual obligations. Also, a non-empty group (coalition) of consumers can *sign* any number of new contracts. Being applied jointly, i.e., as a simultaneous procedure, these operations allow coalition $T \subseteq \mathcal{I}$ to yield new webs of contracts. The set of all such webs is denoted by $F(V, T)$. Formally, we require that each element $U \in F(V, T)$ has to satisfy the following properties.

(i) $v \in V \setminus U \Rightarrow S(v) \bigcap T \neq \emptyset$.

(ii) $v \in U \setminus V \Rightarrow S(v) \subset T$.

A web of contracts U dominates V via coalition T (written as $U \underset{T}{\succ} V$) if the following hold.

(i) $U \in F(V, T)$.

(ii) $x_i(U) \underset{i}{\succ} x_i(V) \quad \text{for all} \quad i \in T$.

Thereby the domination via coalitions is transmitted from the allocations to the webs of contracts whose stability is analyzed.

• *A web of contracts V is called stable if there is no web U and no coalition $T \subseteq \mathcal{I}$, $T \neq \emptyset$ such that $U \underset{T}{\succ} V$. An allocation x is called contractual if $x = x(V)$ for a stable web V.*

Stability with respect to only breaking of earlier concluded contracts is called lower stability; similarly upper stability corresponds only to the possibility to sign new contracts and so on.

More subtle stability concepts can be suggested when coalition possibilities to break contracts increase. In particular, a partial breaking of contracts can be admitted, which leads us to the notion of *properly* contractual allocation. The simplest way to introduce it is as follows. For a web V and some real α, define $\alpha V = \{\alpha \cdot v \mid v \in V\}$, i.e. αV is a web resulting from V by multiplying contracts by α. For $0 \leq \alpha \leq 1$, consider a web $U = \alpha V \cup (1 - \alpha)V$, which is called an α-partition of the web V and which obviously implements the same allocation, i.e. $x(U) = x(V)$. An allocation $x = x(V)$ is *properly contractual* if the α-partition of implementing web V is stable for every $\alpha \in [0, 1]$. Contracts can also be broken asymmetrically; this means the replacement of contract $v = (v_1, \dots, v_n)$ by a vector (it is not a contract at all) of the form $(\lambda_1 v_1, \dots, \lambda_1 v_n)$, for some real $0 \leq \lambda_i \leq 1$, $i \in \mathcal{I}$. This method leads us to the notion of *fuzzy contractual* allocation. In Marakulin (2003, 2011) it was proven (under some assumptions) that properly and fuzzy contractual allocations are exactly equilibrium ones. Further I will adapt these concepts and extend the results to an economy with public goods.

The main thing that is necessary to clarify is what and how a contract is concluded in the production sector and how is it broken. The essential feature of these contracts is that they are carrying out a joint production of collective consumption goods. Formally, the contract is

$$(r_1, \ldots, r_n, y^c) \in \mathbb{R}^{ln} \times \mathbb{R}^s : \quad (\sum_{\mathcal{I}} r_i, y^c) \in Y = \sum_{\mathcal{J}} Y_j.$$

With every production contract $w = (r, y^c)$, $r = (r_i)_{i \in \mathcal{I}}$ one can associate its support:

$$\operatorname{supp}(w) = \{i \in \mathcal{I} \mid r_i \neq 0\} = S(w).$$

It is formed by the agents which are involved to realize the production program $(\sum_{i \in \mathcal{I}} r_i, y^c)$. As follows from the definition, the main specific feature of production contract for the model with public goods is its cooperative nature. That is, unlike the classical Arrow–Debreu model where production can be individualized, it is a project consisting of some joint activities related to the production of goods $y^c = (y_1^c, y_2^c, \ldots, y_s^c)$ from resources $\bar{r} = \sum_{i \in S(w)} r_i$ obtained from the agents from $S(w)$. Notice also that its is not a necessity for private commodities to be consumed in production: private goods can be produced along with public goods, if the technology allows it. In other words, the vectors r_i and $\bar{r} = \sum_{i \in S(w)} r_i$ can also have positive components. The breaking of production contract w is possible by any of its members $i \in S(w)$ and this means that all mutual obligations among members of the coalition $S(w)$ are void.

Similarly to barter, production contracts may form a *web*, i.e. a finite set W of contracts, each subset $U \subseteq W$ of which forms a set of agreements, which correspond to a *feasible* production plan:

$$(\sum_{w \in U} \sum_{i \in \mathcal{I}} r_i^w, \sum_{w \in U} y_w^c) \in \sum_{j \in \mathcal{J}} Y_j.$$

Thus, the specific feature of production webs is that the breaking of a part of the contracts does not directly have any effect on the implementation of other contracts and the corresponding production programs. Notice also that for the *convex conic* total *technological set* every collection of feasible *contracts forms a feasible web* (and vice versa). Note that forming a joint web with a family V of barter contracts one needs also to require a feasibility in consumption:

$$\omega_i + \sum_{w \in U} r_i^w + \sum_{v \in U'} v_i \in X_i, \quad \omega^c + \sum_{w \in U} y_w^c \in X^c, \quad \forall U \subseteq W,\ U' \subseteq V,\ \forall i \in \mathcal{I}.$$

Now, let us assume that the production contracts of a web can be broken not only fully, but also partially—similarly to the case of pure barter contracts for private commodities. Moreover, this analogy is also extended to the specific concepts of web stability: to the concepts of lower stable (the breaking), upper (conclusion of a new contract) and just a stable web (simultaneous breaking and signing of new contracts); in addition, these concepts can be applied to the union of barter (exchange) and production webs. Hereby *admitting only the total breaking* of contracts we arrive at the concept of contractual allocation. A simple analysis of the definitions shows that the contractual allocations of this type in an economy with public goods are exactly the allocations of the Foley core; see the definition above. Indeed, the fact that a coalition $S \subseteq \mathcal{I}$ dominates (blocks) the current allocation can be expressed in contractual terms as follows: this coalition breaks *all* contracts and signs new inter-coalitional contracts (both barter and production ones), in which only members of the coalition and their technological set $Y_S = \sum_{i\in S}\sum_{j\in\mathcal{J}} \theta_i^j Y_j$ are involved. As a result of these activities, each member of the coalition should improve upon its position (get higher utility). Finally, the core is the set of feasible allocations implemented by a web of contracts that no coalition can improve upon. Allowing the possibility to break contracts partially leads to a more qualified type of stability and a number of new concepts, which is reflected in the subsequent definitions.

Now, let us consider a specific notion of properly contractual allocation. Similarly to a pure exchange economy, barter V and production W webs are replaced by their partitions $\alpha V \cup (1-\alpha)V$, $0 \le \alpha \le 1$ and $\beta W \cup (1-\beta)W$, $0 \le \beta \le 1$. For properly contractual allocation this operation does not imply that the implementing web $V \cup W$ loses stability. Below a narrative definition is presented.

Let V be a web of barter contracts and W be a web of production ones in the model $\mathcal{E}^{pg}$. Define $z(V,W) = (x, x^c, y) \in [\mathbb{R}^l]^{\mathcal{I}} \times \mathbb{R}^s \times \mathbb{R}^{(l+s)}$ as an allocation which these webs implement, i.e. for

$$z(V,W) = ((x_i(V,W))_{i\in\mathcal{I}}, x^c(W), y(V,W))$$

determine

$$x_i(V,W) = \omega_i + \sum_{v\in V} v_i + \sum_{w\in W} r_i^w, \quad i \in \mathcal{I}, \quad y(V,W) = (\sum_{\mathcal{I}}\sum_{w\in W} r_i^w, \sum_{w\in W} y_w^c) \tag{3.1}$$

$$x^c(W) = \omega^c + \sum_{w\in W} y_w^c. \tag{3.2}$$

Definition 3.1 *A triplet (x, x^c, y), where $((x_i)_{i\in\mathcal{I}}, x^c) \in \prod_{\mathcal{I}} X_i^p \times X^c$ is a family of private and public consumption plans, and $y \in \sum_{\mathcal{J}} Y_j$ is an aggregated production program, is called a* ***properly contractual allocation*** *if there exist a barter web V and a production W web such that the following hold.*

(i) $x_i = x_i(V,W) = \omega_i + \sum_{v\in V} v_i + \sum_{w\in W} r_i^w, \;\; i \in \mathcal{I}.$
$x^c = x^c(W) = \omega^c + \sum_{w\in W} y_w^c \;\;\&\;\; y = y(V,W) = (\sum_{\mathcal{I}}\sum_{w\in W} r_i^w, \sum_{w\in W} y_w^c).$

(ii) *There is no coalition* $S \subseteq \mathcal{I}$ *for which it is profitable (in a separate or simultaneous regime)*

  (α) *to partially break barter and production contracts;*

  (β) *to sign new barter and production contracts which together with preserved contracts form new feasible webs.*

In other words, the allocation is properly contractual if it is implemented by a pair of stable webs (barter and production) that do not lose stability with respect to any of their partial decompositions.

Remark 3.1 For some public goods, it is adequate to assume the possibility of partial breaking of production contracts, while for others it is not. For example, public skating rinks or picture galleries can be viewed as a good of the first type, because an individual contribution to the production contract can be understood as the money spent by the individual to visit a rink or a gallery (season tickets, etc.). Here, partial breaking of a contract means reducing the number of visits, which implies a decrease (in the same volume) of the consumption of the public good. A contract for goods such as street lighting or national security is fundamentally different: failure to pay taxes to finance these goods (assume such a form of financing) has almost no impact on their consumption. The described theory is related to the goods of the first kind. ■

The following results characterize properly contractual allocations in terms of values that allow establishing their relationship with the equilibria and stating the main result of this section: the theorem on coincidence (under appropriate assumptions) of Lindahl equilibria and properly contractual allocations.

Proposition 3.1 *Let* $\mathcal{E}^{pg}$ *obey* (**P**), (**M**). *Then the Lindahl equilibrium is a properly contractual allocation.*

Characteristic properties of properly contractual allocations are analyzed in the following propositions: they allow one to reverse the last statement.

Proposition 3.2 *Let* $z(V,W) = ((x_i(V,W))_{i\in\mathcal{I}}, x^c(W), y(W))$ *be an allocation implemented by a joint web of contracts: barter web* V *and production web* W*. Let* $z(V,W) \in \mathrm{int}(\prod_{\mathcal{I}} X_i^p \times X^c) \times \sum_J Y_j$ *and let* $\mathcal{E}^{pg}$ *be a convex model with a smooth consumption sector and each individual be* ***non-satiated*** *in private and public goods.*

Then, if $z(V,W)$ *is implemented as a properly contractual allocation, there exist* ***nonzero*** *vectors* $p \in \mathbb{R}^l$, $q_i \in \mathbb{R}^s$, $i \in \mathcal{I}$ *such that*

$$\langle (p,q_i), \mathcal{P}_i(x_i,x^c)\rangle \; > \langle (p,q_i),(x_i,x^c)\rangle, \;\forall i \in \mathcal{I}, \tag{3.3}$$

$$pv_i = 0, \ \forall v \in V, \quad \forall i \in \mathcal{I}, \tag{3.4}$$

$$pr_i^w + q_i y_w^c \geq 0, \quad \forall w \in W, \ \forall i \in \mathcal{I}, \tag{3.5}$$

$$\langle (p, \sum_{i\in\mathcal{I}} q_i), y(W)\rangle \geq \langle (p, \sum_{i\in\mathcal{I}} q_i), \sum_{j\in\mathcal{J}} Y_j\rangle. \tag{3.6}$$

The peculiar circulation of the results of Proposition 3.2 gives the following.

Proposition 3.3 *Let* (**P**), (**M**)[8] *and the conditions of Proposition 3.2 hold. Then every contractual allocation satisfying (3.3)–(3.6) is a properly contractual one.*

Equalities (3.4) mean that for each agent the cost of the consumption bundle of private goods is equal to the cost of initial endowments minus the cost of private goods transactions in the production sector, while (3.3) means that, if the consumption of public goods is valuated via individual prices, then there is no other strictly preferred bundle that is cheaper than this. Inequality (3.5) says that every production contract is individually profitable, and (3.6) shows that the production sector of the economy as a whole operates in a maximum profitable way.

The following theorem immediately follows from Propositions 3.2, 3.3 and actually presents an equivalent description of the Lindahl equilibrium in purely contractual terms. Note that the contractual allocations of any type, as well as properly contractual one, are not directly associated with value parameters: their stability has a cooperative nature, expressed in terms of product flows and the contractual obligations among agents. Thus our approach eliminates the main theoretical difficulty of the Pareto efficient equilibrium with public goods—the presence of individual prices in their implementing mechanism.

Theorem 3.1 *Let economy* $\mathcal{E}^{pg}$ *have a smooth consumption sector, obey* (**P**), (**M**) *and each individual be non-satiated in private and public goods. Then a feasible allocation* $z = (x, x^c, y) \in \mathbb{Z}$, $(x, x^c) \in \mathrm{int}(\prod_{\mathcal{I}} X_i^p \times X^c)$ *and a bundle of private and individualized prices* $(p, q_1, \ldots, q_n) \in \mathbb{R}^{l+ns}$ *present a* ***Lindahl equilibrium if and only if*** *there are the webs of barter* V *and production* W *contracts, implementing this allocation as a* ***properly contactual*** *one.*

Remark 3.2 Note that the only difference between the conditions of Theorem 3.1 and the assumptions of Theorem 2.1 consists in the fact that the consumption sector is smooth.

Proof of Theorem 3.1. Necessity follows from Proposition 3.1. Sufficiency: Under the theorem conditions Proposition 3.3 takes place, and hence (3.3)–(3.6) are

[8] Public goods are desirable, can be only produced but cannot be spent and technological set is a convex cone with vertex at zero.

fulfilled. Since the production sets are cones, the total profit is zero, and according to (3.6) the inequalities (3.5) can only be performed in the form of equalities. This provides budget equalities in the right side of (1.2). Other requirements of equilibrium definition obviously follow from (3.3)–(3.6). ■

4 Fuzzy contractual allocations

It was shown in previous studies (Marakulin, 2003, 2011) that the contractual approach and especially its methodology and concepts related to the partial breaking of contracts may be considered as a specific way to model perfect competition conditions. Being much simpler than the "classical" methods (non-atomic space of economic agents by Aumann, or replicas and Edgeworth equilibria by Debreu, Scarf and Aliprantis, etc.) well-known in the literature, the contractual approach demonstrates high efficiency and leads to the same conclusions as in the previously analyzed situations. However, it is also applicable to many other situations which have not yet been explored. Here I introduce a concept of fuzzy contractual allocation, which really can be considered as an alternative model of perfect competition. Consider first a meaningful scenario.

Imagine that at some intermediate moment of economic interaction, individuals intend to improve the structure of their contracts, partially breaking the old ones and entering into new contracts. Nobody controls their contractual activities and there is no coordinating body. Therefore, the breaking of contracts may take place asynchronously and secretly, with the result that in an intermediate planning stage individuals may operate with unrealistic asymmetrical agreements which are not contracts at all. However, during the search for a new contract agents can rely on the resources made through such bogus contractual options. This can motivate them to sign new contracts and really to break old ones, but now the contractual system as a whole breaks down, as the breaking always occurs at the highest possible option, because all contracts are concluded on a voluntary basis, and they are voluntarily prolonged. The above situation may occur when the current allocation is not fuzzy contractual in the sense described below. Fuzzy contractual allocation is resistant to such perturbations of the contractual agreements. A formalization of this scenario is now presented.

Suppose again that V is a web of barter contracts and W is a web of production contracts of the model $\mathcal{E}^{pg}$ and they implement allocation $z(V,W) = (x, x^c, y)$ according to (3.1) and (3.2). For simplicity and without loss of generality, one can assume that both webs are singletons, i.e. $V = \{v\}$ and $W = \{w\}$. Suppose that an individual $i \in \mathcal{I}$ intends to partially break the barter contract in the amount of $(1 - g_i^v)$, $0 \le g_i^v \le 1$ and production in the amount of $(1 - t_i^w)$, $0 \le t_i^w \le 1$. As a result, he/she will have the following bundle of private and public goods:

$$\tilde{x}_i(t_i^w, g_i^v) = \omega_i + g_i^v \cdot v_i + t_i^w \cdot r_i^w \quad \& \quad \tilde{x}^c(t_i^w) = \omega^c + t_i^w \cdot y_w^c.$$

Further the individual intends to sign new barter $\varsigma = (s_1, \dots, s_n)$ and production $\vartheta = (\vartheta_1, \dots, \vartheta_n, \eta^c)$ contracts, which together should lead to a preferred consumption: $(\xi_i, \xi^c) \succ_i (x_i, x^c)$, where

$$\xi_i = \xi_i(t_i^w, g_i^v) = \omega_i + g_i^v \cdot v_i + t_i^w \cdot r_i^w + s_i + \vartheta_i, \quad \xi^c = \xi^c(g_i^v) = \omega^c + t_i^w \cdot y_w^c + \eta^c.$$

The situation can be further simplified if one notes that due to the definition of barter contract $\sum s_k = 0$ and hence if ϑ was a *feasible* production contract, i.e. $(\sum \vartheta_k, \eta^c) \in Y$, then contract $(\vartheta_1 + s_1, \dots, \vartheta_n + s_n, \eta^c)$ is also feasible. Therefore, there is no need to use two new contracts *for domination*; it is sufficient to apply *only one production contract* ϑ. Moreover, it will also be valid for the original allocation: if instead of two contracts v and w one considers only a production contract $(r_1 + v_1, r_2 + v_2, \dots, r_n + v_n, y_w^c)$ then the stability of allocation and the web of contracts can only be strengthened due to the fact that now the partition of contracts is carried out only in *equal amounts*. In other words, without loss of generality, one may always assume that $v = 0$. Below a formal definition is presented.

Definition 4.1 *An allocation $(x, x^c, y^p, y^c) \in \mathcal{A}(\mathcal{E}^{pg})$ is called **fuzzy contractual** if a production contract $w = (r_1, r_2, \dots, r_n, y^c)$ implementing the allocation as*

$$x_i = \omega_i + r_i, \quad i \in \mathcal{I}, \quad x^c = \omega^c + y^c, \quad y^p = \sum_{\mathcal{I}} r_i, \quad (y^p, y^c) \in Y$$

is such that for every $t = (t_i)_{i \in \mathcal{I}}$, $0 \le t_i \le 1$, $\forall i \in \mathcal{I}$ there is no another feasible contract $\vartheta = (\vartheta_1, \dots, \vartheta_n, \eta^c)$ implementing a new allocation

$$\xi_i^p = \xi_i^p(t, w, \vartheta) = \omega_i + t_i r_i + \vartheta_i, \quad \xi_i^c = \xi_i^c(t, w, \vartheta) = \omega^c + t_i y^c + \eta^c$$

such that $\forall i : (\xi_i^p, \xi_i^c) = (x_i, x^c)$ implies $t_i = 1$ and

$$(\xi_i^p, \xi_i^c) \succ_i (x_i, x^c) \quad \forall i : \quad (\xi_i^p, \xi_i^c) \neq (x_i, x^c). \tag{4.1}$$

With respect to this definition, note that, by applying (4.1) with $\vartheta = 0$ it can be seen that no individual may be interested only in a partial breaking of contracts, i.e. in accordance with the terminology of Marakulin (2003, 2011) the production contract could be called proper.[9] In addition, the definition posits the absence of explicit contractual missense activity, leaving consumption bundle unchanged, but

[9] In Marakulin (2003, 2011), only pure exchange economies were studied.

the non-trivial current contract is broken, and the broken part is then returned to the individual through his/her participation in a new contract. Notice also that unlike the concept of proper contractual allocation here requirements regarding the joint admissibility of the old (partially asymmetrically broken) contract(s), and a new production contract are not imposed. However, notice again that for conic production sets this happens automatically: any set of admissible contracts constitutes a valid web.

Proposition 4.1 *Let $\mathcal{E}^{pg}$ obey* (**P**), (**M**). *Then the Lindahl equilibrium is a fuzzy contractual allocation.*

Proof of Proposition 4.1. The proof completely repeats the arguments of the proof Proposition 3.1; the fact that the production contract is broken asymmetrically does not matter. ■

The following lemma characterizes fuzzy contractual allocation in "geometrical" categories.

Lemma 4.1 *An allocation $\bar{z} = (\bar{x}, \bar{x}^c, \bar{y}^p, \bar{y}^c) \in \mathcal{A}(\mathcal{E}^{pg})$ is fuzzy contractual if and only if it is (lower) stable relative to the partial breaking of contracts and*

$$\mathcal{L}^{pg} \bigcap \Upsilon = \{\tilde{\omega}\} \tag{4.2}$$

for

$$\Upsilon = \prod_{i \in \mathcal{I}} [(\mathcal{P}_i(\bar{x}_i, \bar{x}^c) + \mathrm{co}\{0, (\omega_i - \bar{x}_i, \omega^c - \bar{x}^c)\}) \cup \{(\omega_i, \omega^c)\}] \times Y.$$

Here (as above),

$$\tilde{\omega} = ((\omega_1, \omega^c), (\omega_2, \omega^c), \ldots, (\omega_n, \omega^c), 0) \in \mathbb{R}^{(n+1)(l+s)},$$

and $\mathcal{L}^{pg}$ is the subspace corresponding to the balance constraints of economy with public goods; see (2.5):

$$\mathcal{L}^{pg} = \{((z_1^p, z_1^c), (z_2^p, z_2^c), \ldots, (z_n^p, z_n^c), (y^p, y^c)) \mid$$
$$z_1^p + z_2^p + \cdots + z_n^p = y^p + \sum_{\mathcal{I}} \omega_i, \quad z_1^c = z_2^c = \cdots = z_n^c = y^c + \omega^c\}.$$

By Definition (2.3), one has

$$\Omega_i(x_i, x^c) \subset (\mathcal{P}_i(x_i, x^c) + \mathrm{co}\{0, (\omega_i - x_i, \omega^c - x^c)\}) \cup \{(\omega_i, \omega^c)\}, \quad i \in \mathcal{I}.$$

Hence, applying Lemmas 2.1, 4.1 implies the following

Corollary 4.1 *Every fuzzy contractual allocation belongs to the fuzzy core.*

Remark 4.1 One can note that in general (4.2) itself implies that the allocation is (lower) stable relative to the partial breaking of contracts. This is so (by the above corollary) when all elements of the fuzzy core are equilibria. Thus in general (4.2) is solely sufficient for an allocation to be fuzzy contractual. ■

The central result of the paper is the following theorem on the equivalence of Lindahl equilibria and fuzzy contractual allocations.

Theorem 4.1 *Let $\mathcal{E}^{pg}$ satisfy* (**P**)*,* (**M**)*, $(x_i, x^c) \in \mathrm{int}(X_i \times X^c)$, $i \in \mathcal{I}$ and (1.5) be true (separate non-satiation). Then allocation $z = ((x_i)_{\mathcal{I}}, x^c, y^p, y^c) \in \mathcal{A}(\mathcal{E}^{pg})$ is fuzzy contractual if and only if it is a Lindahl equilibrium allocation.*

Notice that Remark 2.1 is valid in this theorem context, i.e. for irreducible economies this result is true without the interior point assumption.

Proof of Theorem 4.1. The proof of necessity follows immediately from the fact that the conditions of this theorem are identical to the conditions of Theorem 2.1 on the coincidence of equilibria with the elements of the fuzzy core: therefore, being an element of the fuzzy core, a fuzzy contractual allocation is an equilibrium one. The proof of sufficiency is due to Proposition 4.1 and repeats the arguments of Proposition 3.1. ■

The following final statement of the section fully reveals the relationship between the elements of the fuzzy core and fuzzy contractual allocations.

Lemma 4.2 *Let $z = (x, x^c, y) \in \mathcal{A}(\mathcal{E}^{pg})$ and $\mathcal{P}_i(x_i, x^c) \neq \emptyset$ for all $i \in \mathcal{I}$. Then $(x, x^c, y) \in \mathcal{C}^f(\mathcal{E}^{pg})$ implies that*

$$\mathcal{L}^{pg} \bigcap \prod_{i \in \mathcal{I}} (\mathcal{P}_i(x_i, x^c) + \mathrm{co}\{0, (\omega_i - x_i, \omega^c - x^c)\}) \times Y = \emptyset. \tag{4.3}$$

Here (as before) $\mathcal{L}^{pg}$ is a subspace corresponding to the balance constraints of the economy with public goods; see (2.5).

Comparison of formulas (4.2) and (4.3) clarifies the difference between the fuzzy core allocations and fuzzy contractual ones. It is evident that this difference is not too large, which allows us to interpret the allocations of the fuzzy core as fuzzy contractual. Moreover, now the fact that every element of the fuzzy core is a quasi-equilibrium (this is the main reason why the fuzzy core is so popular in existence theory) can be easily deduced from the formula (4.3).

5 Conclusion

A contractual approach for an economic model with public goods and convex production was proposed and analyzed in the paper. The study revealed the high potential of this approach, which presents a contractual description of important theoretical concept of the Lindahl equilibrium. Moreover this is realized in several forms and via a number of new contractual concepts. These concepts characterize equilibria in cooperative terms and without value categories, and this is the basic advantage of the contractual approach. The main results are the following.

(i) Theorem 2.1 on the equivalence of Lindahl equilibria and the fuzzy core.

(ii) Theorem 3.1 for smooth economics on the equivalence of Lindahl equilibria and specific properly contractual allocations.

(iii) Theorem 4.1 on the equivalence of Lindahl equilibria and fuzzy contractual allocations.

These theorems characterize Lindahl equilibria do not involving the individual prices apparatus, which is difficult to implement in practice; here, using an appropriate concept of contract, purely cooperative properties of the economic model are embodied into the results, and it does not address such notions as congested public goods and crowding in its provision. However, the results can be applied only for public goods that can be provided by production contracts admitting partial breaking.

All this contributes to the further development of the contractual approach as a universal tool for constructing and analyzing the models of economic processes.

6 Appendix: proofs

Proof of Lemma 1.1. To establish the lemma, one expresses the Pareto optimality of the allocation $\bar{z}$ in a form suitable for the application of the separation theorem. For this purpose, let us define the following affine space:

$$\mathcal{L} = \{(x_1^p, x_1^c, \dots, x_n^p, x_n^c, y_1^p, y_1^c, \dots, y_m^p, y_m^c) \in \mathbb{R}^{(l+s)(n+m)} :$$

$$x_1^p + \dots + x_n^p - (y_1^p + \dots + y_m^p) = \sum_{\mathcal{I}} \omega_i; \tag{6.4}$$

$$x_2^c - x_1^c = 0,\ x_3^c - x_1^c = 0, \dots,\ x_n^c - x_1^c = 0; \tag{6.5}$$

$$x_1^c - (y_1^c + \dots + y_m^c) = \omega^c\}. \tag{6.6}$$

Now, Pareto optimality can be written as

$$\prod_{\mathcal{I}} \mathcal{P}_i(\bar{x}_i, \bar{x}^c) \times \prod_{\mathcal{J}} Y_j \bigcap \mathcal{L} = \emptyset. \tag{6.7}$$

Since the intersecting sets are convex and non-empty (by (**A**)), then, by the separation theorem, there exists a linear functional $f \neq 0$, which separates these sets, i.e. if one presents this functional in the form of inner product, then there exists a vector $f = (f_1, \dots, f_n, g_1, \dots, g_m)$ such that

$$\langle f, \prod_{\mathcal{I}} \mathcal{P}_i(\bar{x}_i, \bar{x}^c) \times \prod_{\mathcal{J}} Y_j \rangle \geq \langle f, \mathcal{L} \rangle.$$

Further, one reveals the structure of the functional (vector) f. Since f is bounded from above on $\mathcal{L}$, then it must be constant on $\mathcal{L}$ and, hence, its representing vector must be located in the subspace orthogonal to $\mathcal{L}$. However, $\mathcal{L}^\perp$ is represented as a linear hull of the normal vectors to the hyperplanes defined by relations (6.4)–(6.6) (all equations in vector form). The analysis of these relations gives the following representation: $\exists\, p \in \mathbb{R}^l$, $\exists\, \bar{q}, q_2, \dots, q_n \in \mathbb{R}^s$ such that for $q_1 = \bar{q} - \sum_{i=2}^n q_i$ one has $f_i = (p, q_i)$, $g_j = (-p, -\bar{q})$ $\forall i, j$, i.e.

$$f = (p, q_1, p, q_2, \dots, p, q_n, (-p, -\bar{q}), \dots, (-p, -\bar{q})).$$

Further, to establish (1.6), consider the value of the functional f on the vector

$$(\bar{x}_1^p, \bar{x}^c, \dots, \bar{x}_n^p, \bar{x}^c, \bar{y}_1^p, \bar{y}_1^c, \dots, \bar{y}_m^p, \bar{y}_m^c) \in \mathcal{L} \tag{6.8}$$

and compare it with the value on a similar vector, where at the place of i's consumption a vector $(x_i^p, x_i^c) \in \mathcal{P}_i(\bar{x}_i^p, \bar{x}_i^c)$ is written. By (**A**) one has $(\bar{x}_k, \bar{x}^c) \in \overline{\mathcal{P}_k(\bar{x}_k, \bar{x}^c)}$, $\forall k \in \mathcal{I}$ and, therefore, constructed vector belongs to the closure of the set, recorded in the left side of the intersection (6.7). Consequently, the value of the functional on the constructed vector must be not less than its value on the vector (6.8), hence, eliminating the common terms in the right-hand and the left-hand sides, one arrives at (1.6).

Inequalities (1.7) are proved in a similar way: the value of the functional on the vector (6.8) should be compared with the value on a similar vector, where instead of production plan $(\bar{y}_j^p, \bar{y}_j^c)$ (arbitrary chosen) plan $(y_j^p, y_j^c) \in Y_j$ is written. In so doing, one arrives at

$$\langle (-p, -\bar{q}), (y_j^p, y_j^c) \rangle \geq \langle (-p, -\bar{q}), (\bar{y}_j^p, \bar{y}_j^c) \rangle, \ \forall (y_j^p, y_j^c) \in Y_j \iff$$

$$\langle (p, \bar{q}), \bar{y}_j \rangle \geq \langle (p, \bar{q}), Y_j \rangle,$$

which proves the first part of the lemma. We now prove the second part.

Consider an inequality of (1.6) such that $(p, q_i) \neq 0$. Suppose, for example, that $p = 0$. Now substitute consumption bundle $\bar{x}_i$ by x_i so that $(x_i, \bar{x}^c) \succ_i (\bar{x}_i, \bar{x}^c)$ is true. Next, find x^c such that $q_i x^c < q_i \bar{x}^c$ but still $(x_i, x^c) \succ_i (\bar{x}_i, \bar{x}^c)$. The assumptions of the lemma (interior point and non-satiation separately for private and public goods) allow us to do this. However, now one obtains $\langle (p, q_i), (x_i, x^c) \rangle < \langle (p, q_i), (\bar{x}_i, \bar{x}^c) \rangle$, which contradicts (1.6). Therefore $q_i \neq 0$ implies that $p \neq 0$. It can be proven similarly that $p \neq 0 \Rightarrow q_i \neq 0$ for each i. Lemma 1.1 is proved. ■

Proof of Proposition 2.1. Let the triplet (x, x^c, y) satisfy Definition 1.1 for $(x, x^c) \in \prod_{\mathcal{I}} X_i^p \times X^c$, $y \in Y = \sum_{\mathcal{J}} Y_j$. Assume that there is a dominating coalition $t = (t_i)_{\mathcal{I}} \neq 0$. Then estimating (2.2) by prices and applying (1.2), we find that, for $\forall i \in \operatorname{supp}(t)$

$$\langle p, \xi_i \rangle + \langle q_i, \frac{\xi^c}{t_i} + \omega^c \rangle > \langle p, x_i \rangle + \langle q_i, x^c \rangle = \langle p, \omega_i \rangle + \langle q_i, \omega^c \rangle + \sum_{j \in \mathcal{J}} \theta_i^j (p y_j^p + \bar{q} y_j^c).$$

Due to (1.1) and assumption (**P**) for technological sets (they are convex cones with the vertex at zero), we conclude that $p y_j^p + \bar{q} y_j^c \geq 0$, $\forall j \in \mathcal{J}$. Substituting this in the last formula and multiplying inequalities on t_i, one finds that

$$t_i \langle p, \xi_i \rangle + \langle q_i, \xi^c \rangle > t_i \langle p, \omega_i \rangle, \;\; i \in \operatorname{supp}(t) \;\; \Rightarrow \;\; \langle p, \sum_{\operatorname{supp}(t)} t_i(\xi_i - \omega_i) \rangle + \langle \sum_{\operatorname{supp}(t)} q_i, \xi^c \rangle > 0.$$

However, it follows now from (**P**), (**M**) that $\xi^c \geq 0$ and $q_i \geq 0$ for all i, which implies that $\langle \sum_{\mathcal{I}} q_i, \xi^c \rangle \geq \langle \sum_{\operatorname{supp}(t)} q_i, \xi^c \rangle$ and, therefore,

$$\langle p, \sum_{\operatorname{supp}(t)} t_i(\xi_i - \omega_i) \rangle + \langle \sum_{\mathcal{I}} q_i, \xi^c \rangle \geq \langle p, \sum_{\operatorname{supp}(t)} t_i(\xi_i - \omega_i) \rangle + \langle \sum_{\operatorname{supp}(t)} q_i, \xi^c \rangle > 0.$$

Due to (2.1), this means that by equilibrium prices $(p, \bar{q})$, $\bar{q} = \sum_{\mathcal{I}} q_i$ production plan $(\sum_{\operatorname{supp}(t)} t_i(\xi_i - \omega_i), \xi^c) = (\zeta^p, \zeta^c)$ yields a strictly positive profit, which is impossible in view of (1.1) and the right-hand side of (2.1) for conical technological sets. ■

Proof of Proposition 2.2. Let $\bar{z} = (\bar{x}, \bar{x}^c, \bar{y}^p, \bar{y}^c) \in \mathcal{C}^f(\mathcal{E}^{pg})$, which due to Lemma 2.1 is equivalent to (2.6). Now applying separation theorem,[10] one can find a non-zero linear functional non-strictly separating these sets. Since

$$\prod_{\mathcal{I}} \mathcal{P}_i(x_i, x^c) \times Y \subset \prod_{\mathcal{I}} \Omega_i(x_i, x^c) \times Y,$$

it is clear that this functional also separates the sets of (6.7); this relation is equivalent to Pareto optimality (a small difficulty with the fact that $Y = \sum_{\mathcal{J}} Y_j$ can be easily bypassed), and therefore all conclusions regarding to the separating functional from Lemma 1.1 can be applied to our functional. This proves the first part of the

[10] Strictly speaking, to apply the theorem, one has to extract point $\tilde{\omega}$ from the set, located in the left side and notice that it does not imply the loss of convexity. Instead of the set one can also take its interior, which is convex, and clearly has empty intersection with a subspace.

statement: there are prices $p \in \mathbb{R}^l$, $q_i \in \mathbb{R}^s$, $i \in \mathcal{I}$, not all equal to zero and such that the relations (1.6) and (1.7) are true. We now prove (2.7).

The vector representing the functional that separates the sets of (2.6) has the following structure:

$$f = (p, q_1, p, q_2, \ldots, p, q_n, (-p, -\bar{q})), \quad \bar{q} = \sum_{\mathcal{I}} q_i.$$

Now, from the local non-satiation, we conclude that $(\bar{x}_i, \bar{x}^c) \in \overline{\mathcal{P}_i(\bar{x}_i, \bar{x}^c)}$, $i \in \mathcal{I}$. Hence $\text{co}\{(\bar{x}_i, \bar{x}^c), (\omega_i, \omega^c)\} \subset \overline{\Omega_i(\bar{x}_i, \bar{x}^c)}$, $i \in \mathcal{I}$ and certainly $\text{co}\{(\bar{y}^p, \bar{y}^c), (0, 0)\} \subset Y$. Therefore functional f non-strictly separates the set

$$\text{co}\{(\bar{x}_1, \bar{x}^c), (\omega_1, \omega^c)\} \times \cdots \times \text{co}\{(\bar{x}_n, \bar{x}^c), (\omega_n, \omega^c)\} \times \text{co}\{(\bar{y}^p, \bar{y}^c), (0, 0)\}$$

and affine subspace $\mathcal{L}^{pg}$. Calculating further the value of the functional at the point $\tilde{\omega} = ((\omega_1, \omega^c), \ldots, (\omega_n, \omega^c), 0, 0) \in \mathcal{L}^{pg}$ and for an appropriate element of the product, one finds that

$$\sum_{j \neq i} (p\omega_j + q_j \omega^c) + p\bar{x}_i + q_i \bar{x}^c \geq \sum_{j \neq i} (p\omega_j + q_j \omega^c) + p\omega_i + q_i \omega^c \quad \Rightarrow \quad p\bar{x}_i + q_i \bar{x}^c \geq p\omega_i + q_i \omega^c.$$

Summing inequalities over $i \in \mathcal{I}$ and taking into account the balance relations $\sum_{\mathcal{I}} \bar{x}_i = \sum_{\mathcal{I}} \omega_i + \bar{y}^p$ and $\bar{x}^c = \omega^c + \bar{y}^c$, we find that $p\bar{y}^p + \bar{q}\bar{y}^c \geq 0$, which can only be executed in the form of equality (since Y is a convex cone with the vertex at zero). Consequently, each added inequality can be fulfilled only in the form of equality. ■

Proof of Proposition 3.1. Let the triplet (x, x^c, y) satisfy Definition 1.1 for $(x, x^c) \in \prod_{\mathcal{I}} X_i^p \times X^c$, $y \in Y = \sum_{\mathcal{J}} Y_j$. One needs to construct two webs of contracts: a barter web V and a production web W, which implement the equilibrium allocation and satisfy Definition 3.1. Having this in mind define $\bar{y} = \sum_{\mathcal{J}} y_j = (\bar{r}, \bar{y}^c)$, i.e. one specifies a vector $\bar{r}$ of total production inputs and outputs of private goods as $\bar{r} = \sum_{\mathcal{J}} y_j^p$. Consider a production web W, consisting of a single contract $w = (r_1, \ldots, r_n, \bar{y}^c)$, where by definition $r_i = x_i - \omega_i$, $i \in \mathcal{I}$. Then as soon as allocation is balanced one has $\sum_{\mathcal{I}} r_i = \sum_{\mathcal{J}} y_j^p = \bar{r}$ and therefore contract w implements production program $y = \sum_{\mathcal{J}} y_j$. Furthermore, by definition $p(x_i - \omega_i - r_i) = 0$, $\forall i \in \mathcal{I}$. Next define $v_i = x_i - \omega_i - r_i = 0$ and form a formal web $V = \{v\}$, consisting of a single contract $v = 0$. Let us show that the constructed web implements a properly contractual allocation.

First, notice that the bundle $(r_i, \bar{y}^c)$ has zero value by equilibrium prices, i.e. $pr_i + q_i \bar{y}^c = 0$. This is implied by budget balance (equality) in the equilibrium and zero profit in production: due to (**P**), the technological set is a cone. Further, one argues going to contradiction. Assume that there is $0 \leq t \leq 1$ and a production contract $\vartheta = (\vartheta_1, \ldots, \vartheta_n, \eta^c)$, $(\sum \vartheta_k, \eta^c) \in Y$, such that, with contract w being broken in an amount $(1-t)$, a new contract ϑ is concluded by coalition $S = \text{supp}\,(\vartheta)$ whose members are better off with respect to their consumption plans: $(\xi_i, \xi^c) = (\omega_i + tr_i + \vartheta_i, \omega^c + t\bar{y}^c + \eta^c) \succ_i (x_i, x^c)$, $i \in S$. Now, estimating these consumptions by equilibrium prices and via (1.2), we find that $\langle (p, q_i), (\xi_i, \xi^c)\rangle > \langle (p, q_i), (x_i, x^c)\rangle$, i.e. for all $i \in \text{supp}\,(\vartheta)$, we conclude that

$$p(\omega_i + tr_i + \vartheta_i) + q_i(\omega^c + t\bar{y}^c + \eta^c) > p(\omega_i + r_i) + q_i(\omega^c + \bar{y}^c) \quad \Rightarrow \quad p\vartheta_i + q_i\eta^c > 0.$$

Summing these inequalities, for $\vartheta \neq 0$ one obtains

$$p \sum_{\text{supp}\,(\vartheta)} \vartheta_k + (\sum_{\text{supp}\,(\vartheta)} q_i)\eta^c > 0 \quad \Rightarrow \quad p \sum_{\text{supp}\,(\vartheta)} \vartheta_k + (\sum_{\mathcal{I}} q_i)\eta^c > 0,$$

which due to (1.1) for a conic production is impossible. For $\vartheta = 0$ domination is possible only for a singleton coalition and only then current contract is (partially) broken, i.e, for $t < 1$. Here, similar reasonings yield $0 > 0$. Everything leads to a contradiction. Proof is completed. ■

Proof of Proposition 3.2. The fact that $z(V, W)$ is a properly contractual allocation implies that allocation $((x_i(V, W))_{i\in\mathcal{I}}, x^c(W), (y^p(W), y^c(W))) = z(V, W)$, where

$$x_i(V, W) = \omega_i + \sum_{v\in V} v_i + \sum_{w\in W} r_i^w, \; i \in \mathcal{I}, \quad x^c(W) = \omega^c + \sum_{w\in W} y_w^c$$

and

$$(y^p(W), y^c(W)) = (\sum_{w\in W}\sum_{i\in\mathcal{I}} r_i^w, \sum_{w\in W} y_w^c) \in \sum_{j\in\mathcal{J}} Y_j$$

is feasible and Pareto optimal. Hence, by Lemma 1.1, there exist non-zero vectors $p \in \mathbb{R}^l$, $q_i \in \mathbb{R}^s$, $i \in \mathcal{I}$, satisfying (1.6), where all inequalities are fulfilled in strict form and therefore (3.3) is true, and

$$\langle (p, \sum_{i\in\mathcal{I}} q_i), (y^p(W), y^c(W))\rangle \geq \langle (p, \sum_{i\in\mathcal{I}} q_i), \sum_{j\in\mathcal{J}} Y_j\rangle.$$

Next, one uses these price vectors and proves (3.4)–(3.6). The requirement (3.6) is true by construction. Let us prove (3.4). It is sufficient to show by definition, that for each individual and every contract $v \in V$ one has $pv_i \geq 0$, $i \in \mathcal{I}$. Assuming the contrary, one finds contract v and an individual i such that $pv_i < 0$. Due to Remark 1.1, it follows that $\langle \nabla u_i(x_i(V, W), x^c(W)), (-v_i, 0)\rangle > 0$, i.e. the derivative of the utility in the direction $-(v_i, 0)$ is *positive*. Hence, it would be advantageous to this individual to partially break contract v in a possibly small volume $\alpha > 0$, because in this case locally his/her change of utility is calculated as

$\alpha\langle\nabla u_i(x_i(V,W),x^c(W)),(-v_i,0)\rangle > 0$. This contradicts to the definition of properly contractual allocation.

To prove (3.5), let us consider the value of the consumption bundle of the individual i at prices (p,q_i) after a partial breaking of the production contract $w\in W$, $i\in\operatorname{supp}(w)$ in a volume $\alpha>0$. It is easy to see that after the break one has

$$\langle(p,q_i),(x_i(V,W),x^c(W))\rangle - \alpha\langle(p,q_i),(r_i^w,y_w^c)\rangle.$$

In other words, the value is changed by $-\alpha\langle(p,q_i),(r_i^w,y_w^c)\rangle$, which for $pr_i^w+q_iy_w^c<0$ is positive. Applying Remark 1.1 to this case, one obtains $\langle\nabla u_i(x_i(V,W),x^c(W)),(-r_i^w,-y_w^c)\rangle>0$, i.e., the derivative of i's utility along the direction corresponding to the breaking of the contract w is strictly greater than zero. Therefore it is beneficial for the individual to break this contract at least in a small volume. Hence, the assumption $pr_i^w+q_iy_w^c<0$ leads us to a contradiction with the definition of a properly contractual allocation. ∎

Proof of Proposition 3.3. Let conditions (3.3)–(3.6) be fulfilled. Arguing by contradiction, it is easy to see that (3.3) and (3.6) together imply Pareto optimality of the allocation implemented by a web of contracts (one comes to a contradiction by comparing the aggregate cost balance of the current and dominating allocations).

Without loss of generality, assume that there is only one barter contract v and one production contract w. Suppose there exists a coalition $S\subseteq\mathcal{I}$, interested in a partial breaking of existing contracts in amounts $1\geq 1-\alpha\geq 0$, $1\geq 1-\beta\geq 0$ and in the conclusion of new contracts $\tilde{v}$ and $\tilde{w}$. Consumption bundles obtained in this way should be preferred for the agents $i\in S$. These bundles are

$$\tilde{x}_i=\omega_i+\alpha v_i+\beta r_i+\tilde{v}_i+\tilde{r}_i,\ i\in S,\ \ \&\ \ \tilde{x}^c=\omega^c+\beta y^c+\tilde{y}^c,$$

$$x_i=\omega_i+v_i+r_i,\ i\in S,\ \ \&\ \ x^c=\omega^c+y^c.$$

Estimating the bundles by prices (p,q_i), and applying (3.3) and (3.4), one finds that

$$p(\beta r_i+\tilde{v}_i+\tilde{r}_i)+q_i(\beta y^c+\tilde{y}^c) > pr_i+q_iy^c.$$

Summing up these inequalities over $i\in S$, via $\sum_S\tilde{v}_i=0$, one concludes that

$$\beta p\sum_S r_i+p\sum_S\tilde{r}_i+(\sum_S q_i)(\beta y^c+\tilde{y}^c) > p\sum_S r_i+(\sum_S q_i)y^c\ \Rightarrow$$

$$p\sum_S\tilde{r}_i+(\sum_S q_i)\tilde{y}^c > (1-\beta)[p\sum_S r_i+(\sum_S q_i)y^c]. \qquad (6.9)$$

On the other hand, the updated production program must be technologically acceptable, i.e., it has to be

$$(\sum_S\tilde{r}_i+\beta\sum_{\mathcal{I}}r_i,\tilde{y}^c+\beta y^c)\in\sum_{\mathcal{J}}Y_j,$$

which in view of (3.6) yields

$$p(\sum_S \tilde{r}_i + \beta \sum_{\mathcal{I}} r_i) + (\sum_{\mathcal{I}} q_i)(\tilde{y}^c + \beta y^c) \leq p \sum_{\mathcal{I}} r_i + (\sum_{\mathcal{I}} q_i) y^c \;\Rightarrow$$

$$p \sum_S \tilde{r}_i + (\sum_{\mathcal{I}} q_i)\tilde{y}^c \leq (1-\beta)[p \sum_{\mathcal{I}} r_i + (\sum_{\mathcal{I}} q_i) y^c].$$

Further, remember that all public goods are desirable for all agents (an assumption of the theorem), i.e., preferences are monotone in this commodity group. Hence, by (3.3) it is easy to conclude that $q_i \geq 0$, $\forall i \in \mathcal{I}$. In addition, it was assumed in the theorem that public goods can be produced but not expended, i.e., $\tilde{y}^c \geq 0$, which together with the previous gives $\sum_{\mathcal{I} \setminus S} q_i \tilde{y}^c \geq 0$. Now applying (3.5) (summing the inequalities over $i \in S$) one concludes that the right-hand side of (6.9) is non-negative, and finally one concludes that

$$0 \leq (1-\beta)[p \sum_S r_i + (\sum_S q_i) y^c] < \sum_{\mathcal{I} \setminus S} q_i \tilde{y}^c + p \sum_S \tilde{r}_i + (\sum_S q_i)\tilde{y}^c \leq$$

$$\leq (1-\beta)[p \sum_{\mathcal{I}} r_i + (\sum_{\mathcal{I}} q_i) y^c].$$

Now $\beta = 1$ implies $0 < 0$, which is impossible. For $\beta < 1$, one concludes that

$$p \sum_{\mathcal{I}} r_i + (\sum_{\mathcal{I}} q_i) y^c > 0,$$

which contradicts (3.6) and the theorem assumption on production sets which are cones with the vertex at zero: in this case, firms' profits (the value in the left-hand side of inequality (3.6)) have to be zero. ■

Proof of Lemma 4.1. Let $\bar{z}$ be a fuzzy contractual allocation according to Definition 4.1. Assume that (4.2) is false, and therefore that there is $z = ((z_i^p, z_i^c)_{\mathcal{I}}, y^p, y^c) \neq \tilde{\omega}$, belonging to the left part of (4.2). Consider a coalition $S = \{i \in \mathcal{I} \mid z_i \neq (\omega_i, \omega^c)\}$. Notice that $\mathcal{P}_i(\bar{x}_i, \bar{x}^c) \neq \emptyset$, $i \in S$ and find $(\xi_i^p, \xi_i^c) \in \mathcal{P}_i(\bar{x}_i, \bar{x}^c)$, $i \in S$ such that $z_i = \xi_i + t_i[(\omega_i, \omega^c) - (\bar{x}_i, \bar{x}^c)]$ for some $0 \leq t_i \leq 1$, $i \in S$ and $z_i = (\omega_i, \omega^c)$, $i \notin S$ (because $z \in \mathcal{L}^{pg}$, the possibility $S \neq \mathcal{I}$ occurs only at zero production of public goods, which is one of the options). Define $\vartheta_i = z_i^p - \omega_i$, $i \in \mathcal{I}$. Now $\sum_{i \in \mathcal{I}} z_i^p = \sum_{i \in \mathcal{I}} \omega_i + y^p$ implies $\sum_{\mathcal{I}} \vartheta_i = y^p$ and, therefore, for $\eta^c = y^c$, the vector $\vartheta = (\vartheta_1, \ldots, \vartheta_n, \eta^c)$ presents a feasible production contract. Now for $i \in S \neq \emptyset$ one can write

$$\xi_i^p = z_i^p - \omega_i + t_i(\bar{x}_i - \omega_i) + \omega_i = \vartheta_i + t_i r_i + \omega_i$$

and, for the public sector,

$$z_i^c = \xi_i^c + t_i(\omega^c - \bar{x}^c) = y^c + \omega^c \quad \Rightarrow \quad \xi_i^c = y^c + \omega^c + t_i\bar{y}^c = \eta^c + t_i\bar{y}^c + \omega^c.$$

We have $(\xi_i^p, \xi_i^c) \succ_i (\bar{x}_i, \bar{x}^c)$, $i \in S$. Thus, in accordance with Definition 4.1, one finds a vector $t = (t_1, \ldots, t_n)$ and contract $\vartheta = (\vartheta_1, \ldots, \vartheta_n, \eta^c)$ satisfying (4.1) (here $t_i = 1$ and $\vartheta_i = 0$ for $i \notin S$). One comes to a contradiction.

We show that if a stable allocation $\bar{z} = (\bar{x}, \bar{x}^c, \bar{y}^p, \bar{y}^c) \in \mathcal{A}(\mathcal{E}^{pg})$ relative to the partial breaking obeys (4.2), then it is fuzzy contractual with respect to the production web $V = \{(r_1, r_2, \ldots, r_n, \bar{y}^c)\}$, where $r_i = \bar{x}_i - \omega_i$, $i \in \mathcal{I}$. Assume to the contrary, and find $t = (t_1, \ldots, t_n)$ and contract $\vartheta = (\vartheta_1, \ldots, \vartheta_n, \eta^c)$, $(y^p, y^c) = (\sum_{\mathcal{I}} \vartheta_i, \eta^c) \in Y$, $\vartheta \neq 0$ such that, $\forall i$: $(\xi_i^p, \xi_i^c) \neq (\bar{x}_i, \bar{x}^c)$, $\xi_i^p = \omega_i + t_i r_i + \vartheta_i$, $\xi_i^c = \omega^c + t_i\bar{y}^c + \eta^c$,

$$(\xi_i^p, \xi_i^c) \succ_i (\bar{x}_i, \bar{x}^c) \iff$$

$$z_i = (\omega_i, \omega^c) + (\vartheta_i, \eta^c) \in \mathcal{P}_i(\bar{x}_i, \bar{x}^c) + t_i((\omega_i, \omega^c) - (\bar{x}_i, \bar{x}^c)). \tag{6.10}$$

Here, either $(\xi_i^p, \xi_i^c) \neq (\bar{x}_i, \bar{x}^c)$ $\forall i \in \mathcal{I}$ or $\exists i \in \mathcal{I}$: $(\xi_i^p, \xi_i^c) = (\bar{x}_i, \bar{x}^c)$.

In the first case, (6.10) is realized for all $i \in \mathcal{I}$, and $z = ((z_i)_{\mathcal{I}}, y^p, y^c) \neq \tilde{\omega}$ belongs to the intersection on the left side of (4.2); this is a contradiction.

In the second case, by Definition 4.1 for some i one has $t_i = 1$ and $\omega^c + t_i\bar{y}^c + \eta^c = \xi_i^c = \bar{x}^c = \omega^c + \bar{y}^c$, which implies that $\eta^c = 0 = y^c$. Take $z_i = (\omega_i, \omega^c)$ for $i \notin \operatorname{supp}(\vartheta)$. Now via the contract's definition one can conclude that $\sum_{i\in\mathcal{I}} z_i^p = \sum_{i\in\mathcal{I}} \omega_i + y^p$, $z_i^c = \omega^c$ $\forall i \in \mathcal{I}$. Thus, there is found an allocation $z = ((z_i)_{\mathcal{I}}, (y^p, y^c)) \neq \tilde{\omega}$, belonging to the left side of (4.2), which is a contradiction. ■

Proof of Lemma 4.2. The proof is based on the Lemma 2.1 and relation (2.6). One needs to show that (2.6) implies (4.3). Assume that $z = (\bar{x}, \bar{x}^c, \bar{y}^p, \bar{y}^c)$ satisfies (2.6), but (4.3) is false. Then there is a vector $t = (t_1, \ldots, t_n)$, $0 \leq t_i \leq 1$, production plan $(y^p, y^c) \in Y$ and bundles $\zeta_i = (\zeta_i^p, \zeta_i^c)$, $\zeta_i^p = \xi_i^p + t_i(\omega_i - \bar{x}_i)$, $\zeta_i^c = \xi_i^c + t_i(\omega^c - \bar{x}^c)$ satisfying $\xi_i = (\xi_i^p, \xi_i^c) \succ_i (\bar{x}_i, \bar{x}^c)$, $i \in \mathcal{I}$ and such that

$$\sum_{\mathcal{I}} \xi_i^p + \sum_{\mathcal{I}} t_i(\omega_i - \bar{x}_i) = \sum_{\mathcal{I}} \omega_i + y^p \quad \& \quad \xi_i^c + t_i(\omega^c - \bar{x}^c) = \omega^c + y^c. \tag{6.11}$$

Define $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n, y^p, y^c)$. By construction, $\zeta \in \mathcal{L}^{pg}$. Further, for a real $0 < \beta \leq \frac{1}{2}$, consider a vector $\beta\zeta + (1-\beta)z = \rho(\beta) = \rho$, where for $i \in \mathcal{I}$ one has by construction

$$\rho_i^p(\beta) = \beta[\xi_i^p + t_i(\omega_i - \bar{x}_i)] + (1-\beta)\bar{x}_i, \quad \& \quad \rho_i^c(\beta) = \beta[\xi_i^c + t_i(\omega^c - \bar{x}^c)] + (1-\beta)\bar{x}^c.$$

In view of $z \in \mathcal{A}(\mathcal{E}^{pg}) \subset \mathcal{L}^{pg}$, one has $\rho(\beta) \in \mathcal{L}^{pg}$ for every β. Moreover,

$$(\beta y^p + (1-\beta)\bar{y}^p, \beta y^c + (1-\beta)\bar{y}^c) \in Y,$$

i.e. a feasible production program corresponds to the vector $\rho(\beta)$. Further, let us present vectors $\rho_i(\beta)$ in the form

$$\rho_i(\beta) = (1-\beta t_i)(\bar{x}_i, \bar{x}^c) + \beta t_i(\omega_i, \omega^c) + (1-\beta t_i)\frac{\beta}{1-\beta t_i}[(\xi_i^p, \xi_i^c) - (\bar{x}_i, \bar{x}^c)], \quad i \in \mathcal{I},$$

where by choice of β one has $\mu_i = \frac{\beta}{1-\beta t_i} \leq 1$. For $i \in \mathcal{I}$ the last expression due to (**A**) entails

$$\mu_i(\xi_i - (\bar{x}_i, \bar{x}^c)) \in \mathcal{P}_i(\bar{x}_i, \bar{x}^c) - (\bar{x}_i, \bar{x}^c) \ \Rightarrow$$
$$\exists\, \eta_i \in \mathcal{P}_i(\bar{x}_i, \bar{x}^c) : \ \mu_i(\xi_i - (\bar{x}_i, \bar{x}^c)) = \eta_i - (\bar{x}_i, \bar{x}^c).$$

Hence, from the previous formula, one concludes that

$$\rho_i = (1-\beta t_i)\eta_i + \beta t_i(\omega_i, \omega^c),$$

which implies that $\rho_i \in \Omega_i(\bar{x}_i, \bar{x}^c)$, $i \in \mathcal{I}$. Now one can apply (2.6) and conclude that $\rho = \rho(\beta) = \tilde{\omega}$ for *all* real $0 < \beta \leq \frac{1}{2}$. We write this equation componentwise, and, by definition of $\rho_i(\beta)$, we find that

$$\beta[\xi_i + t_i((\omega_i, \omega^c) - (\bar{x}_i, \bar{x}^c))] + (1-\beta)(\bar{x}_i, \bar{x}^c) = (\omega_i, \omega^c) \quad \Rightarrow$$
$$\xi_i + t_i[(\omega_i, \omega^c) - (\bar{x}_i, \bar{x}^c)] = (\bar{x}_i, \bar{x}^c) + \frac{(\omega_i, \omega^c) - (\bar{x}_i, \bar{x}^c)}{\beta},$$

which has to be true for all $i \in \mathcal{I}$ and *all* $0 < \beta \leq \frac{1}{2}$. However, these equalities hold for *different* β, which is possible only if $(\bar{x}_i, \bar{x}^c) = (\omega_i, \omega^c) = \xi_i$, $i \in \mathcal{I}$, which by the choice of ξ_i implies that $(\bar{x}_i, \bar{x}^c) \succ_i (\bar{x}_i, \bar{x}^c)$. This contradicts (**A**) (preferences are irreflexive ones). Lemma 4.2 is proved. ■

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