Contracts and domination in incomplete markets

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A domination concept, based on the notion of an exchange contract, is proposed and studied in this paper. Doing so, the classical notion of domination via coalitions is transmitted onto systems (webs) of contracts and onto allocations, whose stability is investigated. This way, the proposed concept of a core for incomplete markets is described as a set of allocations realized by the webs of contracts that have a special kind of stability relative to the breaking of existing contracts and relative to the ability to sign new contracts. This concept converts into classical core when the market turns complete. Under perfect competition conditions, core allocations are equilibria. These properties prove that the studied core concept is valid.

Keywords and Phrases: incomplete markets, core, contract, contractual allocation, competitive equilibrium.

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Introduction

The principal objective of economic theory and general equilibrium theory is to study the allocation of resources achievable via a system of markets. In the classical version it is (indirectly) presumed that the whole activity of an economy could be viewed as taking place in a single time period, in which the physical parameters are (more or less) stable, agents have full enough information about the economic variables to make their own rational decisions, bargains are realized for infinitely short periods and so on. Such types of settings are said to be *complete markets*, described by the classical theory of resource allocation, which finds its most completed form in the Arrow–Debreu theory (model). However in the world of real economy, individuals are forced to make decisions under uncertainty conditions arising from incomplete information and the objective uncertainty of future events. As a result, in a modern economy one can observe not only ordinary commodity markets but also a rich array of markets of specific financial tools, so-called assets. The functioning of these markets is directly aimed at solving problems of this kind, problems deeply related to the uncertainty of the future. Examples of these markets are the insurance business, the markets of futures contracts, trade with options[1] of different kinds, etc. This problem, related with the uncertainty of the future, was well understood by the classical economic theorists (see survey by Radner (1981)), but this subject has received new attention in the literature of the early 80's, when opportunities to develop the classical Arrow–Debreu theory ended. The result was the development, in an extended Arrow–Debreu model framework, of the *incomplete market theory* (e.g., see Geanakoplos (1990), Magill and Shafer (1991) for a general overview). The term *incomplete* appeals to the fact that the potentially infinite set of possible realizations of the future is surely wider than those created by people, 'insurance variants,' expressed in the form of financial assets. Thus the incomplete market theory models an economic environment in which economic agents live and function under the constraints related to the possible differences in times when goods appear on markets and the objectively defined uncertainty of the future with respect to the present. Moreover, from the big array of ways to model uncertainty and time, this theory chooses those that reflect the specific financial features of real market economies, covering simultaneously the classical theory of resource allocation. However, the modern version of incomplete market theory has one essential gap — there is no satisfactory concept of domination by coalitions (of allocations) and consequently an appropriate core notion is lacking. In fact, in a classical setting, the competitive equilibrium concept, having descriptive power, is also supported by the fact that

[1] This is the trade of property rights for future optional contracts, where 'call' is to buy and 'put' is to sell some commodity.

there is no group of agents (coalition) that has incentives to form an autonomous subeconomy (it is said that the equilibrium belongs to the core — i.e., this is not dominated by any coalition). Moreover in the conditions of perfect competition, *every allocation* from the core allows price decentralization, i.e., it is an equilibrium relative to some prices — this is Edgeworth's well known conjecture. So in the ideal world of an Arrow–Debreu economy, competitive equilibrium, primary defined in a purely descriptive way, obtains the normative foundation as an ideally stable (in a given sense) allocation. This is why it seems quite natural to rise the question about the core definition in an incomplete market environment and to clarify its relations with the financial equilibrium. Moreover, the answer to this question will allow us to better understand what kind of (coalition) stability real observed financial markets have and what are the obstacles to stabilizing them. In further perspective, the obtained answer may allow us to clarify the role of state (regulating body) in financial market stability, which seems to be important in practice for undeveloped markets and states with its economy in transition, in which the economic situation is far from equilibrium.

The main theoretical goal of this paper is to suggest and to investigate a *core concept in incomplete (financial) markets* — a difficult quest for economic theory, which still has not fond a satisfactory solution. In the author's strong opinion, a 'correct' core concept has to inherit the main properties of classical markets and has to satisfy the following two requirements:

• Let the economy be described as an incomplete market but in fact be complete, i.e., it is mathematically equivalent to a standard pure exchange model, in which equilibria correspond to financial equilibria. Formally, this means that the rank of the matrix of value returns from assets is equal to the number of future events. Then the classical concept of the core and a new concept, introduced for incomplete markets, can be applied simultaneously. In such a case, the set of allocations for the core of an incomplete market should coincide with the set of standard core allocations.

• Under perfect competition conditions, the core and equilibria have to coincide — for a standard exchange economy this is the coincidence of Edgeworth's equilibria (the allocations that belong to the core of each replicated economy) with competitive equilibria.

So I take these properties as the main criterion for a correct definition for a core and coalition domination in financial markets.

I suggest that the notion of contract be a cornerstone of the definition of domination and a core for an economic model and, of course, for incomplete markets. The original idea of a contract is attributed to Makarov (1980), (1982). In the framework of an ordinary pure exchange model, every contract is simply an elementary, possible and permissible exchange of commodities among consumers. Contracts may be added to one another and with every (finite) set of contracts, an allocation of resources can be associated — as a result of the summation of contracts and the initial endowments allocation. It is presumed that every feasible set of (permissible) contracts — let us call it *'a web of contracts'* — may be changed during the economic life. Each consumer or their coalition can *break contracts* in which it participates, and each coalition of consumers can also *sign a new contract(s)*. In the framework of a standard pure exchange economy this approach was developed by Kozyrev (1981), (1982), where the author suggested to partially break contracts, and by Vasil'ev (1984). In these papers the first positive results were obtained regarding the coincidence (under some technical assumptions) of equilibrium and the so-called (in our terminology) *'proper contractual allocations.'*[2] These are the allocations that can be realized by a web of contracts and which are stable relative to the procedure of both parties *partially breaking* existing contracts and the signing of new contracts. At the same time, core allocations were described in the terms of *'contractual allocations.'* These are the allocations that can be realized by a web of contracts and which are stable relative to the procedure of both parties (fully) breaking contracts and the signing new contracts. Thus the only difference between these two notions of contractual allocation is that in the first case the partial breaking of contracts is allowed, while in the second case, only complete breaking is possible. This way, an equilibrium is described in pure game-theoretical terms and does not address any kind of value parameters. The mathematical nature of this phenomena is the same as in the case of the coincidence of equilibrium allocations with fuzzy core elements (or Edgeworth's equilibria), which is one possible way to model the conditions of perfect competition.

In the analysis of modern economic models, the contract-based approach has, in the author's strong opinion, serious advantages in comparison with the classical approach. First, this approach provides more precise and clear language, avoiding the particular specifications of the studied model. This language allows easier to express ideas, in economic as well as in mathematical terms, to be applied in cumbersome constructions of modern economic models. In fact, the diversity of models[3] and the difficulties arising from their analysis are caused, on one hand, by the complexity of the object (economy) and, on the other hand, by the absence of sufficiently universal tools for model investigation. The latter resulted in a variety of solution concepts primarily related to the notion of domination (via coalition) and therefore to the concept of a core. The reason for this is that following the classical tradition, primary attention is paid to the analysis of the final resource allocation. The commonly missed fact is that in a real economy this allocation is a result of many exchange dealings among economic agents (coalitions). It is important that not every exchange is permissible in a real economy. There are many reasons for this: institutional, physical, informational, ethical, behavioral, etc. I believe that the focus of the theory should be shifted to concentrate directly on the exchange bargains of commodities (contracts), which should be included in the model as primitives and form (together with the other model elements) the basis for theoretical constructions instead of allocations. Applying the contract-based approach, one can more clearly describe the transition process to a stable (non-dominated) resource allocation (from a core). This approach is nearer to the intuitive imagination of

[2] I changed notations and terminology. My approach to the contract based economic model as a whole was essentially revised and generalized.

[3] In modern economic theory, one can see a number of *non-perfect* market models, not only incomplete markets; they are the markets with informational asymmetry (about future events, etc.), sequential markets (time factor and trust), and so on; see my list of references.

the real processes of price and consumption formations; in particular, it provides additional links between cooperative and individualistic views on agents' behavior. From the mathematical point of view the contract-based approach adds the analysis of direct variables, the exchange bargains of commodities, to the analysis of dual variables (prices). This, being weak investigated, allows us to produce new interesting results. I hope that this approach, as a supplement to the classical one, can help to clarify and solve many difficult problems of economic theory arising in the analysis of non-perfect markets.

This paper *develops the theory of contracts*, the elements of which were founded by Makarov (1980), (1982) and Kozyrev (1981), (1982), first in the framework of an abstract economy[4] and then applies the theory to incomplete markets. I consider and study the formal rules of operating with the sets of contracts. The difference in these rules corresponds to the difference in the types of a web's stability and therefore in the stability of allocations realized by given webs. The types of these 'stabilities,' together with the property of allowable contracts, reflect the different behavioral, physical and institutional principles formally given in game-theoretical form, which one can find in real life and in neoclassical economic theory. So the different types of web stabilities correspond to the different types of contractual allocations, as well as their modifications, which can relax or strengthen the property for an allocation to be stable. This way, depending on the structure of permissible contracts, one can describe notions well known in economic theory such as the core, competitive equilibria, Pareto boundary and so on in the terms of a stable web of contracts.

Applying my contractual approach in the incomplete market framework, I give the *description of financial equilibria* (see my definition in the third section). The fact that not every exchange of commodities can be realized in an incomplete market, i.e., not every contract is permissible, is very important in this context. For example, the direct exchange of commodities between the different states of the world is impossible (how one can imagine the exchange contract between the different events of the future so that only one or none of these events can be true in reality?). However some of these exchanges can be realized via assets, which together with the multiplicity of events characterizes in a most specific form the incomplete market since trade or exchanges by these 'standard contracts' is gone in the only common state of the world, that is in the present. Thus for incomplete markets, it is reasonable to presume that only contracts signed relative to some given (fixed) state of the world are permissible. Therefore, a new contract which some coalition is going to sign has to correspond to only one state of the world — to the present or future state. The agents can also partially or fully break contracts in the present and can break every contract (perhaps even equivalent ones in some sense) in each of the future states of the world. Suppose there is no such coalition, that the members of coalition can increase utility when these procedures are applied simultaneously. Then, using different abilities, one can yield contractual allocations of different types, which one can conventionally call *complex contractual*. Using this approach, I not only describe financial equilibria but also identify the kind of complex contractual allocations, which one can take as core allocations of an incomplete market. So one can see a similarity in the relations between the core and equilibria,

[4]This part of the paper was previously published by Marakulin (2002).

being defined in terms of contracts, a similarity between standard exchange economy and incomplete market. Moreover, it is not simply similarity, but it means that in perfect competition conditions, the core and equilibria of an incomplete market coincides.

This paper contains two parts and is organized as follows. The first part is an essay on the theory of exchange contracts and contains three sections. The model and assumptions are described in the first section; the main definitions of the contract-based approach in a general framework of *contractual* exchange economy is presented in the second section; the third section is devoted to the study of a standard pure exchange economy as one possible application of the contractual one. The second part is devoted to the study of an incomplete market economy in the context of a contractual economy. The formal definition of the core, its main properties and relations with equilibrium allocations are given here. The main and most difficult, long proofs of this part are given in a special section.The conclusion to the main investigation and its possible policy applications forms the last section.

Chapter 1

The model of a contractual exchange economy

1.1 The model

I consider a typical exchange economy in which E denotes the (finite dimensional) *space of commodities.* Let $\mathcal{I} = \{1, \ldots, n\}$ be a set of agents (traders or consumers). A consumer $i \in \mathcal{I}$ is characterized by a consumption set $X_i \subset E$, an initial endowment $\omega_i \in E$, and a preference relation described by a point-to-set mapping $\mathcal{P}_i : X_i \rightrightarrows X_i$, where $\mathcal{P}_i(x_i)$ denotes the set of all consumption bundles strictly preferred by the i-th agent to the bundle x_i. I also use the notation $y_i \succ_i x_i$, which is equivalent to $y_i \in \mathcal{P}_i(x_i)$. So, the pure exchange model may be represented as a triplet:

$$\mathcal{E} = \langle \mathcal{I}, E, (X_i, \mathcal{P}_i, \omega_i)_{i \in \mathcal{I}} \rangle.$$

Let us denote by $\omega = (\omega_i)_{i \in \mathcal{I}}$ the vector of initial endowments of all traders of the economy. Denote $X = \prod_{i \in \mathcal{I}} X_i$ and let

$$\mathcal{A}(\mathcal{X}) = \{\, x \in X \mid \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \omega_i \,\}$$

be the set of all *feasible allocations* and $\mathcal{A}_{X_i}(\mathcal{E})$ be its projection on X_i, $i \in \mathcal{I}$.

Everywhere below I assume that the model $\mathcal{E}$ satisfies the following assumption.

(A) *For each $i \in \mathcal{I}$, X_i is a convex closed set, $\omega_i \in X_i$, and for every $x_i \in X_i$ there exists an open convex $G_i \subset E$ such that $\mathcal{P}_i(x_i) = G_i \cap X_i$ and $x_i \in \overline{\mathcal{P}}_i(x_i) \setminus \mathcal{P}_i(x_i)$*[1] *for every $x_i \in \mathcal{A}_{X_i}(\mathcal{E})$.*

Let $L = E^{\mathcal{I}}$ denote the space of states of the economy $\mathcal{E}$. In the framework of model $\mathcal{E}$, I am going to introduce and study a formal mechanism of contracting and recontracting. This mechanism reflects the idea that any group of agents can find and realize some (permissible) within-the-group exchanges of commodities, referred to as contracts.

By the formal definition, any reallocation of commodities $v = (v_i)_{i \in \mathcal{I}} \in L$, i.e., any vector $v \in L$ satisfying $\sum v_i = 0$, is called a *contract.*

[1] The symbol $\overline{A}$ denotes the closure of A and $\setminus$ is set for the set-theoretical difference.

Not every kind of possible reallocation may be realized in the economy; there are some institutional, physical, and behavioral restrictions in the economic models of different types. This is why I equip the abstract contractual economy model with a new element, the set of *permissible* contracts $\mathcal{W} \subset L$. Thus, the contractual (exchange) economy under study may be concisely represented by the quadruplet

$$\mathcal{E}^c = \langle \mathcal{I}, E, \mathcal{W}, (X_i, \mathcal{P}_i, \omega_i)_{i \in \mathcal{I}} \rangle.$$

In addition to **(A)**, I assume everywhere below that for a contractual economy

(C) *The set $\mathcal{W}$ is starshaped at zero in L, i.e,*

$$v \in \mathcal{W} \implies \lambda v \in \mathcal{W} \quad \forall\, 0 \le \lambda \le 1.$$

The economy $\mathcal{E}^c$ as well as the economy $\mathcal{E}$ is said to be *smooth* if for every $i \in \mathcal{I}$,

$$\mathcal{P}_i(x_i) = \{y \in X_i \mid u_i(y) > u_i(x_i)\} \quad \forall\, x_i \in X_i$$

for some *differentiable* function u_i defined on an open neighborhood of X_i.

1.2 Main contract-based concepts

In the framework of a contractual economy, I study the sets of contracts which represent *feasible* allocations and introduce the operation of breaking a part of a given set of contracts. This motivates the next important definition.

A finite collection V of permissible contracts is called *a web of contracts relative to $y \in \mathcal{A}(\mathcal{X})$* if

$$y + \sum_{v \in U} v \in X \quad \forall\, U \subseteq V.$$

I denote by $x_y(U)$ the feasible allocation sustained by U relative to y, i.e., we put

$$x_y(U) := y + \sum_{v \in U} v.$$

Similarly, $V_y(x)$ denotes the web which realizes x relative to y.

A web of contracts V relative to ω is called *a web of contracts* or simply *a web*. Note that $V = \emptyset$ is a web relative to every $y \in \mathcal{A}(\mathcal{X})$. Denoting

$$\Delta(V) = \sum_{v \in V} v,$$

where V is an arbitrary collection of contracts (by convention, we write $\Delta(\emptyset) = 0$), we can write

$$x_y(V) = y + \Delta(V), \quad x(V) = x_\omega(V) = \omega + \Delta(V)$$

so that V being a web simply means that

$$x_\omega(U) \in X \quad \forall\, U \subseteq V.$$

Now we are going to introduce the operations of breaking existing contracts and signing new ones. For any contract $v \in V$, let us set

$$S(v) = \mathrm{supp}(v) = \{i \in \mathcal{I} \mid v_i \neq 0\},$$

the support of the contract v. It is assumed that contract $v \in V$ may be *broken* by any trader in $S(v)$, since he/she simply may not keep his/her contractual obligations. Also a non-empty group (coalition) of consumers can *sign* any number of new contracts. Being applied jointly, i.e., as a simultaneous procedure, these operations allow coalition $T \subseteq \mathcal{I}$ to yield new webs of contracts. The set of all such webs is denoted by $F(V,T)$. Formally, I require that each element $U \in F(V,T)$ has to satisfy the following properties:

(i) $v \in V \setminus U \;\Rightarrow\; S(v) \cap T \neq \emptyset$,

(ii) $v \in U \setminus V \;\Rightarrow\; S(v) \subset T$,

(iii) $\sum_{v \in U \setminus V} \lambda_v v \in \mathcal{W}$ for all $0 \le \lambda_v \le 1,\ v \in U \setminus V$.

Condition (i) means that only members of T can break contracts in V, condition (ii) means that only members of T may sign new contracts and (iii) is a kind of joint permissibility of new contracts, which is useful in applications of contractual economy and whose role will be clear later. Notice also that due to the definition of a web of contracts, a coalition can break any subset of contracts of a given web that satisfies (i).[2]

Next, for the webs of contracts, I introduce the notion of domination via a coalition. This property, being written as $U \underset{T}{\succ} V$ (read: U dominates V via coalition T), means that

(i) $U \in F(V,T)$,

(ii) $x_i(U) \underset{i}{\succ} x_i(V)$ for all $i \in T$.

Notice that one can strengthen the property of domination via a coalition if in (i) one additionally requires $S(u) \subseteq T$ or $S(u) \subseteq \mathcal{I} \setminus T$. This fits well with the idea that the members of T are going to be separated from the non-members of T and therefore they have to break all non-zero contracts with the members of $\mathcal{I} \setminus T$. Clearly, this modification extends the set of non-dominated allocations and relaxes their stability property.

Definition 1.1 *A web of contracts V is called stable if there is no web U and no coalition $T \subseteq \mathcal{I}$, $T \neq \emptyset$ such that $U \underset{T}{\succ} V$.*

An allocation x is called contractual if $x = x(V)$ for a stable web V.

The requirement that a web of contracts be stable may be relaxed as well as strengthened. The most important possibilities are described below.

Definition 1.2 *A web of contracts V is called lower stable if there is no web U and no coalition $T \subseteq \mathcal{I}$, $T \neq \emptyset$ such that $U \underset{T}{\succ} V$ and $U \subseteq V$.*

A web of contracts V is called upper stable if there is no web U and no coalition $T \subseteq \mathcal{I}$, $T \neq \emptyset$ such that $U \underset{T}{\succ} V$ and $V \subseteq U$.

An upper and lower stable web of contracts V is called weakly stable.

[2] Otherwise, it would occur that an allocation realized via breaking contracts is not feasible.

An allocation x is called lower, upper, or weakly contractual if $x = x(V)$ for some lower, upper, or weakly stable web V, respectively.

It has to be clear that all the above notions of stability and domination may be considered as "relative to some given feasible allocation", simply use this allocation instead of the initial ω. Also it has to be clear that the notion of a weakly stable web (weakly contractual allocation) is really weaker than the corresponding notion of a stable web (contractual allocation). The difference is that in the first case the operations of breaking existing contracts and signing new contracts are applied separately, whereas in the second case they are applied simultaneously. In the framework of a market economy, I consider below the relations among the sets of contractual, lower, upper, and weakly contractual allocations. They correspond to notions well known in economic theory.

How can the process of recontracting (breaking existing contracts and signing new ones) be expressed in economic terms? We can assume that this is something like a tâtonnement process (cooperative tâtonnement), which, for example, may be as follows. To simplify the argument let us imagine that there is an ordered list of all coalitions. At the first stage (iteration), the coalitions, in the given order of appearance, start to sign and/or break contacts (transitioning to webs in $F(V_\xi, T_\xi)$, where ξ is the order number of coalition T_ξ). Here the first coalition "starts" from the given initial endowment allocation ω and since there were no contracts signed before from the web $V_1 = \emptyset$. The stage, iterative loop, is finished when the last coalition has made its choice. Next, the second stage starts where the same process is going on assuming that the first coalition in the list deals with the web of contracts realized at the end of the first stage. The fixed points of this iterative process correspond to the contractual allocations and to the stable webs of contracts. Clearly, the order of coalitions' "appearance" during a stage is not essential. Moreover, a coalition can appear several times during one stage and the order of coalitions' appearance can vary from stage to stage. What is really important is that each coalition has a chance to appear in infinitely many iterations. In general, this scenario does not impose any time restrictions on the duration of an iteration and the number of iterations is potentially unlimited (therefore the duration of a stage is infinitely small). Informally, it is just presumed that the process finishes in "a reasonable time" and the economy transits to a stationary state. It is these potentially possible stationary states that are the subject of my analysis. Notice also that if in the iterative process, starting from some stage, one forbids the signing of new contracts for coalitions, one can realize stationary states and webs of contracts which are lower stable (in one of the sense described above or below: it depends on which kind of contract breaking is permissible, i.e., whether one can break a contract only in its entirety, partially, or even with a transition to equivalent contracts). Similarly, by forbidding breaking contracts or by breaking contracts in a mixed regime (at one stage forbidding signing new contracts, at another stage forbidding breaking them, and so on), one can realize upper and weakly stable webs and contractual allocations, respectively. A conventional presentation of the contracting and recontracting process, a kind of timing, is presented in Figure 1.1. Here coalition $T_\xi = \{1, 4\}$ breaks a part of contracts $W \subseteq V_{\xi-1}$ from the web $V_{\xi-1}$ and signs a new contract $w = (w_1, 0, 0, w_4, 0, \ldots, 0) \in \mathbb{R}^{ln}$ (l is a number of commodities), forming a

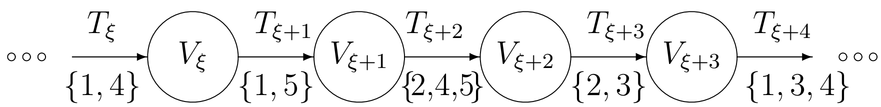

Figure 1.1: Contracting and recontracting process

new web V_ξ such that

$$\omega_1 + \sum_{v^{\xi-1} \in V_{\xi-1} \setminus W} v_1^{\xi-1} + w_1 \succ_1 \omega_1 + \sum_{v^{\xi-1} \in V_{\xi-1}} v_1^{\xi-1},$$

$$\omega_4 + \sum_{v^{\xi-1} \in V_{\xi-1} \setminus W} v_4^{\xi-1} + w_4 \succ_4 \omega_4 + \sum_{v^{\xi-1} \in V_{\xi-1}} v_4^{\xi-1}.$$

On the left hand side of these relations the summation is taken over contracts from $V_{\xi-1}$, which are not broken by coalition $\{1, 4\}$. Conversely, on the right hand side of these relations, the summation is taken over all contracts from the web $V_{\xi-1}$. Further the web V_ξ is transformed by coalition $T_{\xi+1} = \{1, 5\}$ in a similar way and so on.

Now we continue the list of stability property modifications, strengthening the stability property relative to the procedure of breaking contracts. It is clear that a web which is not lower stable cannot be long-living in a market. This is why we restrict our attention below only to the lower stable webs. First let us introduce an equivalence relation on the set of all such webs. This equivalence relation will allow us to partially divide some contracts. To this end, we can define a partial ordering on the set of all webs as follows:

$$U \geq V \iff \exists \text{ a map } \textbf{onto} \;\; f : U \to V, \text{ such that}$$

(i) $\lambda f(u) = u$ for some $0 \leq \lambda \leq 1$ and for every $u \in U$,
(ii) $\sum_{u \in f^{-1}(v)} u = v$ for every $v \in V$.

One can easily see that the set of contracts $f^{-1}(v)$ is a *partition* of contract v and so the web U consists of (finite) *partitions* of contracts in V. The minimal elements of the set of all webs may be called *root webs*. Note that it follows from this definition that $\Delta(U) = \Delta(V)$. Now the equivalence relation may be defined as follows:

$$U \simeq V \iff \exists \text{ a web } W \text{ such that } V \geq W \;\&\; U \geq W.$$

Clearly, $U \simeq V$ simply means that these webs have a common root web.

Definition 1.3 *An allocation x is called proper contractual (resp. lower proper contractual, weakly proper contractual) if there exits a web V such that $x = x(V)$ and for every $U \simeq V$ the allocation $x = x(U)$ is contractual (resp. lower contractual, weakly contractual).*

The economic meaning of proper contractuality of an allocation is, as it was already noted above, that we allow the agents to partially break contracts as well as to sign new contracts (simultaneously or separately). This may be interpreted in two ways.

First, speaking *in behavioral terms*, being experienced enough the agents intend to sign many small volume contracts instead of signing one contract of large volume. This way they gain more economic freedom through the ability to break some of the small contracts if a necessity arises. The second way is to treat a proper contract as a kind of a *preliminary agreement.* In this agreement only the rates of exchange are rigidly defined in contrast to the volume of the contract which is flexible and will be defined rigidly at the end of the contracting procedure. It should be clear that due to the last definition, the property of an allocation to be stable (in any of the senses) is essentially strengthened when we add the word "proper" to the term "contractual" allocation. Below I refine and define the term "proper" for a single contract and for a web.

Definition 1.4 *Let V be a web. A contract $v \in V$ is coherent if every web U such that $U \simeq \{v\}$ is lower stable relative to $(x(V)-v)$ is taken as the initial endowments or, equivalently for ordered preferences, if*

$$x_i(V) \succeq_i x_i(V) - \lambda v_i \quad \forall\, 0 \le \lambda \le 1, \quad \forall\, i \in \mathcal{I}.$$

A subweb $U \subseteq V$ consisting of coherent contracts is called coherent.

A subweb $U \subseteq V$ is called proper if for every web $W \simeq U$ the web $(V \setminus U) \cup W$ is lower stable.

An allocation $x = x(V)$ realized by a coherent web V is called (lower) coherent.

Notice that the only difference between coherent and proper webs is that in the first case the web is stable relative to the partial breaking of any *single* contract, whereas in the second case the agents may partially break *any number* of contracts in the web. In general these notions are not equivalent (see Example 1.2). Moreover, even the notions of coherent and proper contracts are not equivalent; in the latter case (for proper contracts) breaking more than one contract in the web is also allowed. The case of a proper web of contracts is geometrically presented in Figure 1.2 in the coordinate system of consumer $i \in \mathcal{I}$. Figure 1.3 gives the geometry of stable but non-coherent and non-proper webs. In fact, by partially breaking contracts u and

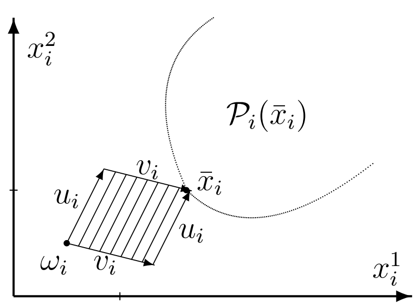

Figure 1.2: The web $\{u, v\}$ *is proper*

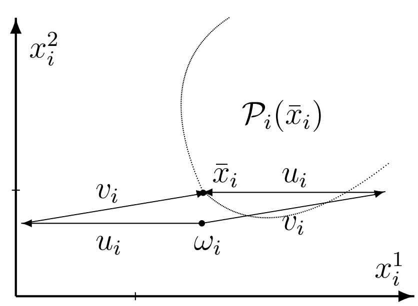

Figure 1.3: The web $\{u, v\}$ *is not proper*

v, agent i can realize any consumption bundle of the form

$$\omega_i + \lambda v_i + \mu u_i = \bar{x}_i - (1-\lambda)v_i - (1-\mu)u_i, \quad \lambda, \mu \in [0,1],$$

i.e., he/she can transit to any consumption plan from the *convex hull* of four points:

$$\omega_i, \quad \omega_i + v_i, \quad \omega_i + u_i, \quad \bar{x}_i = \omega_i + v_i + u_i.$$

However in the case when only complete breaking of contracts is permissible, just these four points can be realized after the breaking of contracts. Thus in geometrical terms the difference is that while in the first case *the whole "parallelogram of contracts"* does not *intersect* $\mathcal{P}_i(\bar{x}_i)$, in the second case it *does intersect*, but *no vertex* belongs to $\mathcal{P}_i(\bar{x}_i)$.

The next proposition fully characterizes coherent contracts for convex contractual economies. For a particular smooth case, this proposition establishes that contract $v \in V$ is coherent only if the derivative of utility by direction v_i is non-negative for each $i \in \mathrm{supp}(v)$.

Proposition 1.1 *Let V be a web. A contract $v \in V$ is coherent iff there exist linear functionals $p_i^v \neq 0$ such that*

$$\langle p_i^v, \mathcal{P}_i(x_i(V)) \rangle > \langle p_i^v, x_i(V) \rangle^3 \quad \& \quad \langle p_i^v, v_i \rangle \geq 0 \tag{1.1}$$

for every $i \in \mathcal{I}$. Moreover, if the utility functions are smooth, (1.1) is fulfilled for $p_i^v = \mathrm{grad}\, u_i(x_i(V))$, $i \in \mathcal{I}$.

Proof of Proposition 1.1. Let us show that (1.1) is sufficient. Assume that for every $i \in \mathcal{I}$, inequalities (1.1) are true, but the contract v is not coherent. Then after partially breaking v, the broken part being $0 \leq \alpha_v \leq 1$, the agents realize the new allocation

$$x_i^\alpha = x_i - \alpha_v v_i, \quad i \in \mathcal{I}$$

such that $x_i^\alpha \succ_i x_i$ for some $i \in S(v)$ so that the first part of (1.1) gives $p_i^v x_i^\alpha > p_i^v x_i$. But then using the second part of (1.1) for this i, we obtain $p_i^v x_i^\alpha \leq p_i^v x_i$, which contradicts the previous inequality.

To establish that (1.1) is necessary, assume that the contract $v \in V$ is coherent, i.e., the web V is stable relative to the partial breaking of contract v. For each consumer $i \in \mathcal{I}$, let us consider the set

$$\mathcal{U}_i(x) = \{\, x_i - \alpha_v v_i \mid 0 \leq \alpha_v \leq 1 \,\} \subset X_i.$$

By definition, the sets $\mathcal{U}_i$ are nonempty, convex, and closed. Now, since the contract $v \in V$ is coherent, it follows that

$$\mathcal{P}_i(x_i) \cap \mathcal{U}_i(x) = \emptyset \quad \forall\, i \in \mathcal{I}.$$

By assumption, there exists an open convex set $G_i \subset E$, such that $G_i \cap X_i = \mathcal{P}_i(x_i) \neq \emptyset$, $x_i \in \overline{G}_i$ for the given $x \in \mathcal{A}(\mathcal{X})$ and i. Now the last relations imply

$$G_i \cap \mathcal{U}_i(x) = \emptyset \;\; \& \; x_i \in \overline{G}_i \quad \forall\, i \in \mathcal{I}.$$

[3] $\langle A, p \rangle$ denotes the set $\{\langle a, p \rangle \mid a \in A\}$ and $A > b$ ($A \geq b$) means $a > b$ ($a \geq b$) for all $a \in A$.

Therefore, by the separation theorem, for each $i \in \mathcal{I}$ there exists $p_i^v \in E'$ such that $p_i^v \neq 0$ and

$$\langle p_i^v, G_i \rangle > \langle p_i^v, x_i \rangle \geq \langle p_i^v, \mathcal{U}_i(x) \rangle.$$

Clearly, one can put $p_i^v = \operatorname{grad} u_i(x_i(V))$, $i \in \mathcal{I}$ for differentiable utility functions. Now the first inequality implies the first part of (1.1), and the second one gives

$$p_i^v x_i \geq p_i^v x_i - \alpha_v \langle v_i, p_i^v \rangle, \quad 0 \leq \alpha_v \leq 1.$$

Therefore, $\langle v_i, p_i^v \rangle \geq 0$. □

The following corollaries characterize proper contractual allocations in terms of coherent webs.

Corollary 1.1 *Let $\mathcal{E}^c$ be a smooth contractual economy. Then x is lower proper contractual iff there exists a coherent web V such that $x = x(V)$. Moreover, relation (1.1) is fulfilled for $p_i = p_i^v = \operatorname{grad} u_i(x_i(V))$ and for every $v \in V$ and $i \in \mathcal{I}$.*

Proof of Corollary 1.1. It follows from the definition of lower proper contractual allocation that x can be realized by a web $V = V_\omega(x)$ which is stable relative to the procedure of partially breaking (any number of) contracts. This implies that every contract in V is coherent and, therefore, V is a coherent web. Thus it is enough to check that every coherent web U, such that $x = x(U)$, is also stable relative to the procedure of partially breaking *any number* of contracts, i.e., U is a proper web.

In fact, due to Proposition 1.1 for differentiable utilities, for *every* $v \in V_\omega(x)$ the functional (vector) $p_i^v = \operatorname{grad} u_i(x_i) = p_i$ satisfies condition (1.1) for all $i \in \mathcal{I}$. This implies

$$G_i \cap \mathcal{M}_i(x) = \emptyset \quad \forall\, i \in \mathcal{I},$$

where

$$\mathcal{M}_i(x) = \{\, x_i - \sum_{v \in V} \alpha_v v_i \ \mid\ 0 \leq \alpha_v \leq 1,\ v \in V \,\} \subset X_i,$$

which completes the proof. □

In particular, the last corollary states that for smooth economies, every coherent web is proper, i.e., stable relative to the procedure of partially breaking any number of contracts. Note that the assumption of differentiability of utilities cannot be dropped here; appropriate examples can be easily constructed (see Example 1.2, second part).

The previous corollary directly implies

Corollary 1.2 *Let $\mathcal{E}^c$ be a smooth contractual economy. Then x is weakly proper contractual iff there exists an upper stable coherent web V such that $x = x(V)$. Moreover, relation (1.1) is fulfilled for $p_i = p_i^v = \operatorname{grad} u_i(x_i(V))$ and for every $v \in V$ and $i \in \mathcal{I}$.*

The next important property of proper and coherent contracts is that they can be replaced by another proper web, keeping the lower stable property for the new web. This fact also follows from Proposition 1.1. Recall that $V_y(x) = V$ denotes a web realizing the allocation x relative to the initial endowments y, i.e., $x = y + \Delta(V)$.

Corollary 1.3 *Let $\mathcal{E}^c$ be a smooth economy and $x \in \mathcal{A}(\mathcal{E})$. Then any coherent web $V_\omega(x)$ has the following inheritance property: for every (coherent) contract $v \in V_\omega(x)$ and every coherent web $W_{x-v}(x)$, the new web $U = (V_\omega(x) \setminus \{v\}) \cup W_{x-v}(x)$ is lower stable relative to partially breaking (any number of) contracts and therefore is proper.*

Proof of Corollary 1.3. Due to Proposition 1.1 (necessity), the functionals $p_i = \operatorname{grad} u_i(x_i)$ satisfy (1.1) for any (coherent) contract $v \in V_\omega(x)$, as well as for contracts in the coherent web $W_{x-v}(x)$, since this web is stable relative to the partial breaking of contracts and $\sum_{w \in W_{x-v}(x)} w + x - v = x$. In other words, for every i there is a $p_i \neq 0$ such that

$$\langle p_i, \mathcal{P}_i(x_i)\rangle > \langle p_i, x_i\rangle, \quad \langle p_i, v'_i\rangle \geq 0 \quad \forall\, v' \in V_\omega(x)$$

and

$$\langle p_i, w_i\rangle \geq 0 \quad \forall\, w \in W_{x-v}(x).$$

Hence, joining the second relation with the first one for contracts in U and applying Proposition 1.1 in the part of sufficiency yields the result via Corollary 1.1. □

The following proposition gives the characterization of proper webs in dual terms for the non-smooth case. The proof of this proposition is similar to the proof of Corollary 1.1.

Proposition 1.2 *Let V be a web of contracts and $x = x(V)$. Then web V is proper, i.e., x is a lower proper contractual allocation, if and only if, there exist linear functionals $p_i \neq 0$, such that*

$$\langle p_i, \mathcal{P}_i(x_i(V))\rangle > \langle p_i, x_i(V)\rangle \quad \& \quad \langle p_i, v_i\rangle \geq 0 \;\; \forall\, v \in V \tag{1.2}$$

for each $i \in \mathcal{I}$.

Proof of Proposition 1.2. Under assumption (**A**) the fact that V is a proper web is equivalent to the condition

$$G_i \cap \mathcal{M}_i(x) = \emptyset \quad \forall\, i \in \mathcal{I},$$

where

$$\mathcal{M}_i(x) = \{x_i - \sum_{v \in V} \alpha_v v_i \;\mid\; 0 \leq \alpha_v \leq 1,\; v \in U\} \subset X_i.$$

Now applying the separation theorem to the sets G_i and $\mathcal{M}_i(x)$, we establish the necessity of (1.2). Its sufficiency can be checked directly. □

In applications of contractual economies, we will also use contracts with a stronger stability property, so-called *perfect* contracts. To introduce this notion, let us first consider another kind of equivalence relation defined on the set of all proper webs. This (weak) equivalence relation may be define as follows: Let U and V be proper webs, then

$$U \sim V \iff \sum_{u \in U} u = \sum_{v \in V} v.$$

Clearly, $U \sim V$ simply means that these webs are proper and realize the same allocation. It also has to be clear that $U \simeq V$ implies $U \sim V$ for all proper webs U and V. Given a proper web V, a proper web U such that $U \sim V$ may be referred to as a *virtual* web (relative to V).

Definition 1.5 *An allocation x is called perfectly contractual if there exits a proper web V such that $x = x(V)$, and for every proper web U such that $U \sim V$ the allocation $x = x(U)$ is contractual.*

The economic meaning of perfect contractuality of an allocation reflects the idea that when signing contracts the agents should take care that not only these contracts be small enough, as for the proper contractual behavior, but also be *differently directed* (i.e. have as many different exchange proportions as possible) to provide the opportunity to break the "unluckily directed" ones. So agents are allowed not only to partially break contracts but also to change (in a sense) the "directions" of contracts without loss of the low stable property. One can see an analogy with the hedge policy,[4] the main difference being that here we are speaking about the exchange-of-goods based contracts, by the signing of which and moving into agreements via "tacks or traverses", the agents reach a final resource reallocation.

This may be also treated in terms of an *optional agreement.* In fact, in a perfectly contractual allocation the society is protected from the possibility that some coalition initiates a new recontracting process. A coalition may hope that via recontracting, it will 'gather more profitable harvest,' i.e., find better consumption programs for its members. The following scenario may take place. A coalition, acting through its members may suggest to the non-members of the coalition, who are involved in the coalition contracting, that the contracts be rewritten so that

(i) the same allocation is realized,

(ii) nobody has incentives to *partially break* new contracts, i. e., the new web inherits the lower stable property of the initial web relative to the partial breaking of contracts.

In such a case, the non-members of the coalition may sign these new agreements as long as they have no revealed incentives to refuse from doing so (possibly the coalition members are simply good negotiators). But, once these new agreements are signed, the coalition breaks a part of the contracts and signs a new contract that as a whole provides the coalition members with better consumption bundles. However, for a perfectly contractual allocation, this hypothetical behavior of every coalition cannot be profitable.

The described scheme of agents' interaction during the contract process is illustrated in the following Example 1.1. Moreover this example presents a proper contractual allocation, which *is not perfect* contractual.

Example 1.1 Consider an exchange economy with *two* commodities and with *three* agents having the following characteristics. Let $X_i = \mathbb{R}^2_+$, $i = 1, 2, 3$ be the agents' consumption sets, and let preferences be defined via utility functions $u_i : X_i \to \mathbb{R}$. Let endowments $\omega_i \in X_i$ and utilities have the form

$$\begin{aligned} u_1(z) &= \min\{4z_1 + 4z_2,\ z_1 + 7z_2\}, & \omega_1 &= (2, \tfrac{1}{2}),\\ u_2(z) &= \min\{4z_1 + 6z_2, 3z_1 + 7z_2\}, & \omega_2 &= (\tfrac{5}{4}, \tfrac{9}{4}),\\ u_3(z) &= \min\{20z_1 + z_2, z_1 + 20z_2\}, & \omega_3 &= (\tfrac{3}{4}, \tfrac{5}{4}). \end{aligned}$$

[4] Hedging is a policy of investment risk neutralization conducted via using mutually exclusive contracts, which compensates potential benefits and costs.

Let us consider an allocation $x = (x_1, x_2, x_3) \in \mathbb{R}^6_+$, where

$$x_1 = (1, 1),\ \ x_2 = (2, 2),\ \ x_3 = (1, 1).$$

This allocation can be realized by a web consisting of the only contract $w = x - \omega$. Since $u_1(x_1) = 8 > 5\frac{1}{2} = u_1(\omega_1)$, $u_2(x_2) = 20 > 18\frac{1}{2} = u_2(\omega_2)$, $u_3(x_3) = 21 > 16\frac{1}{4} = u_3(\omega_3)$, then allocation x is individually rational. As in our case for each agent, as soon as the sets $\mathcal{P}_i(x_i)$ of strictly better consumption bundles is represented as the intersection of some (open) cone with the vertex at the point x_i and positive orthant, then the individual rationality of x implies that the web $W = \{w\}$ is *proper* relative to ω. Moreover, in the next section a stronger property will be established — this allocation $x(W)$ is proper contractual.

Further, coalition $\{1, 2\}$ can propose to agent 3 to rewrite contract $w = x - \omega$, dividing it due to an allocation $y \in X$ into two contracts: $u = x - y$ and $v = y - \omega$, forming a *virtual* proper web $\{u, v\} \sim \{w\}$. Agent 3 can accept this proposal, since he/she realizes the same consumption program, and moreover, via the partial breaking of new contracts the agent spreads his/her playing abilities to dominate current allocation. Let us consider an allocation $y = (y_1, y_2, y_3)$ satisfying

$$h = x_1 + x_2 - (y_1 + y_2) = (-3\varepsilon, \varepsilon), \quad \varepsilon > 0.$$

This condition compromises the requirement that the web $\{u, v\}$ be proper. For example one can take

$$y_1 = (\tfrac{7}{4}, \tfrac{2}{3}), \quad \varepsilon = \tfrac{1}{24} \implies y_2 = (\tfrac{11}{8}, \tfrac{55}{24}), \quad y_3 = (\tfrac{7}{8}, \tfrac{25}{24}).$$

To check that the web $\{u, v\}$ is proper, apply Proposition 1.2. Now for agent 1 consider functional $p_1 = (1, 7)$ supporting $\mathcal{P}_1(x_1)$ at point x_1, obtaining

$$p_1(x_1 - y_1) = 1, \quad p_1(y_1 - \omega_1) = \tfrac{11}{12}.$$

For agent 2 consider supporting functional $p_2 = (4, 6)$, having

$$p_2(x_2 - y_2) = p_2(y_2 - \omega_2) = \tfrac{3}{4}.$$

For agent 3 take supporting functional $p_3 = (20, 1)$, for which

$$p_3(x_3 - y_3) = 2\tfrac{11}{24}, \quad p_3(y_3 - \omega_3) = 2\tfrac{7}{24}.$$

All described functionals satisfy the sufficient condition (1.2) of Proposition 1.2, which proves the web $\{u, v\}$ to be proper. This analysis is illustrated by Figure 1.4, where the left figure presents Edgeworth' box,[5] constructed separately for coalition $\{1, 2\}$, the right one describes the case for agent 3 in his/her coordinate system. Here $\tilde{\omega}_2 = x_1 + x_2 - \omega_2$, $\widetilde{\mathcal{P}}_2(x_2) = x_1 + x_2 - \mathcal{P}_2(x_2)$, $\tilde{y}_2 = x_1 + x_2 - y_2$ are the presentations of initial endowments, preferences and consumption vector y_2 in 1st agent's coordinate system, correspondingly. Notice that $\tilde{\omega}_2 \neq \omega_1$ and $\tilde{y}_2 \neq y_1$, since $\omega_1 + \omega_2 \neq x_1 + x_2 \neq y_1 + y_2$.

[5] On Edgeworth's box, see also Example 1.2 from next section.

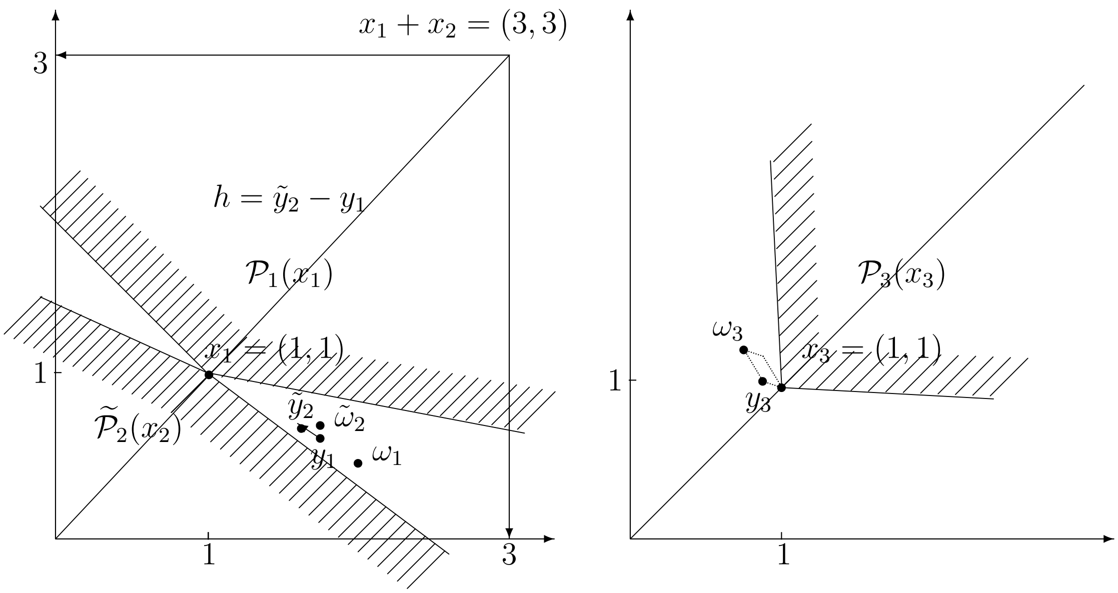

Figure 1.4: Edgewort's box for coalition $\{1, 2\}$ and agent 3's consumption separately

Finally, when agent 3 accepts the proposition of coalition $\{1, 2\}$, the members of this coalition can break contract $u = x - y$, realizing the allocation y as a kind of "new initial one," and can sign a new contract $g = (g_1, g_2)$, where $g_1 = x_1 - y_1 + \delta(1, 1)$, $g_2 = x_2 - y_2 - h - \delta(1, 1)$, $\delta > 0$. Due to the definition of h, this is a contract in fact such that new consumption bundles are $z_1 = x_1 + \delta(1, 1)$, $z_2 = x_2 + (3\varepsilon - \delta, -\varepsilon - \delta)$. The utility of the first agent for the new bundle increases due to preference monotonicity. The utility of the second agent also increases if δ is small enough:

$$u_2(z_2) = u_2(x_2) + \min\{6\varepsilon - 10\delta, 2\varepsilon - 10\delta\} = u_2(x_2) + 2\varepsilon - 10\delta > u_2(x_2) \quad \text{for} \quad \delta < \varepsilon/5.$$

So, acting in the described way, coalition $\{1, 2\}$ can reach a higher consumption level for its members (let all contracts be permissible), and studied allocation x *is not perfect* contractual. □

Certainly perfect contractuality is the strongest kind of allocation stability. The next definition extends the notion of being *perfect* to a single contract.

Definition 1.6 *Let V be a web. A coherent contract $v \in V$ is perfect if every web U such that $U \sim \{v\}$ is stable relative to $(x(V) - v)$, which is taken as the initial endowments.*

A web (subweb) which contains only perfect contracts is called perfect.

Remark 1.1 I would like also to mention an alternative definition of a web's perfect subweb of contracts.

Let V be a web. A subweb $U \subset V$ is called perfect if it is proper and, for every web W proper relative to $(x(V) - \sum_{u \in U} u)$ and such that $W \sim U$, the web $(V \setminus U) \cup W$ is stable. A contract $v \in V$ is called perfect if the subweb $\{v\}$ is perfect.

Note that due to this definition every web containing at least one perfect contract is stable. Notice also the difference between the two following statements for $U \subset V$

which arises in this case: "U is a perfect subweb" and "U is a perfect web relative to $(x(V) - \sum_{u \in U} u)$". The first one implies the second, but in general the reverse is not true (since in the first case it is allowed to break contracts in $V \setminus U$, but in the second one it is not).

If one assumes this definition, the contracts and their webs (subwebs), perfect in the sense of Definition 1.6, may be renamed as perfect coherent. Note also that due to Proposition 1.1 and its corollaries, for *smooth economies* both *variants of the definitions of a perfect contract and of a perfect web (subweb) are equivalent.* □

The graphical presentation of logical relationships among various stability concepts of the webs of contacts and related contractual allocations is given in Figure 1.5. In this figure the arrows show a stronger property from the nearest notions.

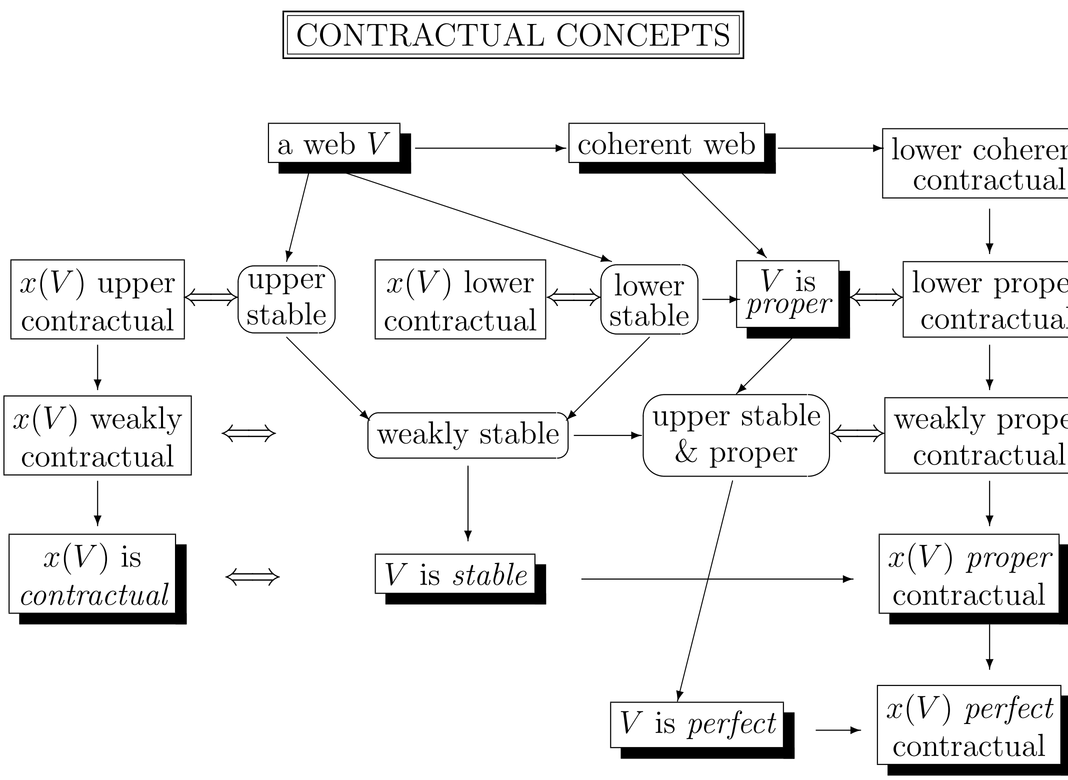

Figure 1.5: The relationships and and logical connections among various contract based concepts

Finishing the gallery of various kinds of allocation stability in a contractual economy, let us assume that the set of permissible contracts can be represented as a (finite) union of starshaped sets, i.e,

$$\mathcal{W} = \cup \mathcal{V}_\xi.$$

Note that $\mathcal{W}$ is then the starshaped set itself and $\mathcal{V}_\xi$ may, in particular, be convex sets or, moreover, subspaces of L (as it is the case for incomplete markets).

Now, for a given web V, we can associate with each $v \in V$ and $v \in \mathcal{W}$, certain sets in $\{\mathcal{V}_\xi\}$ and can require that if $v \in \mathcal{V}_\xi$ for a given ξ, the contract v has to be coherent and either proper or perfect or neither proper nor perfect. When such an association is established, the allocation $x(V)$ is called *complex* (composition) contractual. In other words, a complex contractual allocation is stable relative to both the procedure of appropriately breaking contracts (depending on the set to which the contract belongs) and the procedure of signing new (permissible) contracts. Moreover, some additional requirements may be imposed on a web realizing an allocation. These requirements always take the form of joint stability of contracts in the web. For example, in the case of on incomplete market economy we can identify the sets $\mathcal{V}_\xi$ with the subspaces of the commodity space which exactly correspond to the states of the world — trade exchanges (contracts) are allowed only in the present or at one and only one event in the future. For incomplete markets, one can establish that under some technical assumptions, the set of the GEI-equilibrium allocations coincide with the set of all complex contractual allocations such that their corresponding webs contain proper contracts in the present and perfect ones in future events. In these terms, a core allocation can be described as complex contractual for which perfect contracts are realized in future events and there are no restrictions in the present.

1.3 Contracts in a standard exchange economy

In the classical setting, it is indirectly assumed that for a pure exchange economy all kinds of commodity exchanges are allowed, the only restriction being that the realized consumption programs (bundles) have to belong to the agents' consumption sets, i.e., the allocations have to be feasible. This is why when one complements this model with a contract-based mechanism it is logical to think that all contracts are permissible, i.e., one may presume that $\mathcal{W} = L$, where L is the space of the economy's states. In fact it suffices to require a little bit less. So speaking of a *standard* exchange economy, we always assume that the corresponding contractual economy is such that the set $\mathcal{W}$ of all permissible contracts is *radial* (absorbing)[6] at zero in L. In all other aspects the standard model coincides with model $\mathcal{E}$. Next, let us recall some definitions.

A pair (x, p) is said to be a *quasi-equilibrium* of $\mathcal{E}$ if $x \in \mathcal{A}(\mathcal{X})$ and there exists a linear functional $p \neq 0$ onto E such that

$$\langle p, \mathcal{P}_i(x_i)\rangle \geq px_i = p\omega_i \quad \forall\, i \in \mathcal{I}.$$

A quasi-equilibrium such that $x'_i \in \mathcal{P}_i(x_i)$ actually implies $px'_i > px_i$ is a *Walrasian or competitive equilibrium.*

On the other hand, $x \in \mathcal{A}(\mathcal{X})$ is said to be dominated (blocked) by a nonempty coalition $S \subset \mathcal{I}$ if there exists $y^S \in \prod_{i\in S} X_i$ such that $\sum_{i\in S} y_i^S = \sum_{i\in S}\omega_i$ and $y_i^S \in \mathcal{P}_i(x_i)\ \forall\, i \in S$.

The *core* of $\mathcal{E}$, denoted by $\mathcal{C}(\mathcal{E})$, is the set of all $x \in \mathcal{A}(\mathcal{X})$ that are blocked by no (nonempty) coalition.

Weak Pareto boundary for $\mathcal{E}$, denoted by $\mathcal{PB}^w(\mathcal{E})$, is the set of all $x \in \mathcal{A}(\mathcal{X})$ that cannot be dominated by the coalition $\mathcal{I}$ of all agents.

[6] A set $A \subset L$ is radial at a point $a \in A$ if, for every $b \in L$, $\lambda b \in (A - a)$ for all real $0 \leq \lambda \leq \lambda_b$ and some $\lambda_b > 0$. Notice that a *convex* radial set is starshaped at its every point.

An allocation $x \in \mathcal{A}(\mathcal{X})$ is called *individual rational* if it cannot be dominated by singleton coalitions. $\mathcal{IR}(\mathcal{E})$ denotes the set of all these allocations.

The above definitions imply

$$\mathcal{C}(\mathcal{E}) \subset \mathcal{PB}(\mathcal{E}) \cap \mathcal{IR}(\mathcal{E}).$$

In general the reverse inclusion is true only for a two-consumer economy.

The next important notion, fruitfully working in the theory of economic equilibrium, is the concept of fuzzy core. Recall that any vector

$$t = (t_1, \ldots, t_n) \neq 0, \quad 0 \leq t_i \leq 1 \;\; \forall\, i \in \mathcal{I}$$

may be identified with a fuzzy coalition, the real number t_i being interpreted as the measure of agent i in the coalition. A coalition t is said to dominate (block) an allocation $x \in \mathcal{A}(\mathcal{X})$ if there exists $y^t \in \prod_{\mathcal{I}} X_i$ such that

$$\sum_{i \in \mathcal{I}} t_i y_i^t = \sum_{i \in \mathcal{I}} t_i \omega_i \iff \sum_{i \in \mathcal{I}} t_i (y_i^t - \omega_i) = 0 \tag{1.3}$$

and

$$y_i^t \succ_i x_i \quad \forall\, i \in \operatorname{supp}(t) = \{i \in \mathcal{I} \mid t_i > 0\}. \tag{1.4}$$

Conditions (1.3), (1.4) may be equivalently rewritten in the form[7]

$$0 \in \sum_{i \in \mathcal{I}} t_i (\mathcal{P}_i(x_i) - \omega_i).$$

The set of all feasible allocations which cannot be dominated by fuzzy coalitions is denoted by $\mathcal{C}^f(\mathcal{E})$ and is called the *fuzzy core* of the economy $\mathcal{E}$.

Next I would like to make several remarks on the concept of optimality by Pareto (Pareto boundary). First recall a stronger concept of optimality, sometimes called the *strong Pareto boundary*. Assume that every relation $\succ_i$ is *irreflexive and transitive*.[8] Then $x = (x_i)_{\mathcal{I}} \in \mathcal{A}(\mathcal{X})$ is a *strong Pareto optimum* if there is no $z = (z_i)_{\mathcal{I}} \in \mathcal{A}(\mathcal{X})$ such that

$$x_i \not\succ_i z_i \;\; \forall\, i \in \mathcal{I} \quad \& \quad \exists\, j \in \mathcal{I} : z_j \succ_j x_j.$$

For preordered preferences,[9] $\succeq_i$ this requirement is equivalent to

$$z_i \succeq_i x_i \;\; \forall\, i \in \mathcal{I} \quad \& \quad \exists\, j \in \mathcal{I} : z_j \succ_j x_j.$$

Let us denote by $\mathcal{PB}^s(\mathcal{E})$ the *strong Pareto boundary*, the set of all strong Pareto optimal allocations. The definitions imply $\mathcal{PB}^s(\mathcal{E}) \subset \mathcal{PB}^w(\mathcal{E})$.

There is one more possibility of defining the optimality concept in an economic model. It takes an intermediate position between the two notions considered above.

[7]Admitting some inaccuracy in the next and further relations, I will sometimes identify a singleton set with its element (a vector).

[8]It is known that every *strict* binary relation $\succ$ is irreflexive and transitive if it is defined as a *strict component* of a *non-strict relation* $\succeq$ that is *reflexive and transitive*.

[9]This is a reflexive, complete and transitive *non-strict* binary relation.

We will see below that exactly this kind of optimality is realized by upper contractual allocations.

Let us call an allocation $x = (x_i)_{\mathcal{I}} \in \mathcal{A}(\mathcal{X})$ *strictly Pareto optimal* if there is no coalition $S \subset \mathcal{I}$ for which there exists an $y^S \in \prod_{i \in S} X_i$ such that $\sum_{i \in S} y_i^S = \sum_{i \in S} x_i$ and $y_i^S \succ_i x_i$ for each $i \in S$. In other words, x is an allocation in the core of the other economy, which differs from the original one in only one aspect, namely, allocation x is taken as the initial endowments. In my opinion, the last concept of optimality presents the most precise form of optimality by Pareto.

Denote by $\mathcal{PB}(\mathcal{E})$ the *strict Pareto boundary*. It is easily seen from the definitions that

$$\mathcal{PB}^s(\mathcal{E}) \subset \mathcal{PB}(\mathcal{E}) \subset \mathcal{PB}^w(\mathcal{E}).$$

Therefore, if under some conditions one can show that an $x = (x_i)_{\mathcal{I}} \in \mathcal{PB}^w(\mathcal{E})$ is Pareto strongly optimal (this is the case if, for example, the preferences are locally non-satiated, which we have here due to (**A**), and $x \in \text{int}X$),[10] then the allocation x is strictly Pareto optimal as well.

The next theorem establishes relationships between the core and contractual allocations.

Theorem 1.1 *Let $\mathcal{E}^c$ be a contractual economy such that $\mathcal{W} = L$, and let x be an allocation. Then:*

(*i*) *x is contractual* $\iff$ $x \in \mathcal{C}(\mathcal{E}) \cap \mathcal{PB}(\mathcal{E})$,

(*ii*) *x is upper contractual* $\iff$ $x \in \mathcal{PB}(\mathcal{E})$,

(*iii*) *x is lower contractual* $\iff$ $x \in \mathcal{IR}(\mathcal{E})$,

(*iv*) *x is weakly contractual* $\iff$ $x \in \mathcal{IR}(\mathcal{E}) \cap \mathcal{PB}(\mathcal{E})$.

Proof of Theorem 1.1. The necessity of (*i*)–(*iv*) directly follows from the definitions. To check their sufficiency, for $x \in \mathcal{A}(\mathcal{X})$ let us consider the web $V_\omega(x)$ consisting of only one contract $v = x - \omega$. This is really a web since, by assumption, $(x - \omega) \in \mathcal{W}$. A routine checking of Definitions 1.1, 1.2 completes the proof. □

Remark 1.2 One might think that the assumption $\mathcal{W} = L$ made in the theorem is redundant and can be changed by a weaker requirement that the set $\mathcal{W}$ is radial at zero. In such a case one can realize the allocation x using the web consisting of contracts $(x - \omega)/r$ for a sufficiently large natural r. However, this is not enough; in order for $(x - \omega)/r$ to be in the core, we would have also to modify the rule of domination by coalition, to require that the dominating coalition "breaks all links" with the complementary coalition, i.e., the supports for all contracts in the dominating web have to be contained in this coalition or in its complement. Moreover, it is necessary to allow a coalition to sign new contracts and exclude requirement (iii) from the definition of sets $F(V, T)$. Also the problems with lower and weak contractuality can arise since by breaking some of the contracts, the coalition members can realize any allocation of the form $\omega + \frac{m}{r}(x - \omega)$, $m = 0, \ldots, r$. □

[10]Moreover, it is well known that if the preferences are strictly monotonic and $X_i = \mathbb{R}^l_+$ (l is a number of commodities) for all i, then the concepts of strong and weak Pareto optimality are equivalent.

The next theorem characterizes the equilibrium allocations in terms of the proper contractual ones.

Theorem 1.2 *Let $\mathcal{E}^c$ be a smooth contractual economy, the set $\mathcal{W}$ be radial at zero in L, and x be an allocation such that $x \in \mathrm{int}X$. Then the following statements are equivalent:*

(i) x is an equilibrium allocation.

(ii) x is optimal by Pareto and there exists a coherent web V realizing this allocation, i.e., $x = x(V)$ such that V is coherent and upper stable.

(iii) x is a proper contractual allocation.

(iv) x is a perfectly contractual allocation.

Moreover, if (x, p) is an equilibrium and V is a web realizing $x = x(V)$, then V is a coherent web if and only if $pv_i = 0\ \forall\, v \in V,\ \forall\, i \in \mathcal{I}$.

Remark 1.3 The analysis of the theorem's proof shows that the implication *(i)⇒(iii)* is true in the general case for the non-smooth preferences and without the requirement $x \in \mathrm{int}X$, i.e., in the standard exchange economy every equilibrium is a proper contractual allocation.

Notice also that the theorem's condition $x \in \mathrm{int}X$ is fulfilled automatically if $\overline{\mathcal{P}_i(\omega_i)} \subset \mathrm{int}X$ holds for all $i \in \mathcal{I}$.

Recall also that due to Theorem 1.1(ii), statement (ii) of Theorem 1.2, in which the optimality by Pareto of the allocation x is claimed, is equivalent to the existence of a coherent and upper stable web realizing this allocation. □

Proof of Theorem 1.2. To establish the equivalence of *(i)–(iv)* recall that under the theorem's conditions, $x \in \mathcal{A}(\mathcal{X})$ is optimal by Pareto iff there exists an $i \in \mathcal{I}$ such that for $p = \mathrm{grad}\, u_i(x_i)$

$$\langle \mathcal{P}_j(x_j), p\rangle > \langle p, x_j\rangle \quad \forall\, j \in \mathcal{I}.^{11} \tag{1.5}$$

Next let us notice that *(iv)⇒(iii)⇒(ii)*. To establish *(ii)⇒(i)*, apply Proposition 1.1 to obtain $pv_i \geq 0$ for all $v \in V$ and $i \in \mathcal{I}$. Summing up over $v \in V$ for every $i \in \mathcal{I}$, we arrive at

$$\langle p, \Delta_i(V)\rangle \geq 0 \quad \Longrightarrow \quad px_i \geq p\omega_i.$$

This, due to the feasibility of x, implies $px_i = p\omega_i$ for all i which by (1.5) yields the equilibrium properties of (x, p).

Now I prove *(i)⇒(iv)*. Let $v = v^r = (x - \omega)/r$, where the natural r is chosen based on the assumption that $\mathcal{W}$ is absorbing, and define the set V to consist of the r identical copies of the contract v. Clearly, V is a web. Now the equilibrium properties of the pair (x, p) imply $pv_i = 0\ \forall\, v \in V,\ \forall\, i \in \mathcal{I}$ and therefore (1.1) is true (one can take $p = \mathrm{grad}\, u_1(x_1)$ as the equilibrium price vector for x). Applying the

[11] In particular, it implies that $\mathrm{grad}\, u_i(x_i)$ and $\mathrm{grad}\, u_j(x_j)$ coincide up to a normalization for all $i \neq j$.

sufficiency part of Proposition 1.1, one concludes that V is a coherent and, moreover, proper web. Now let $U \sim V$ for a proper web U. Once again due to Proposition 1.1 (necessity), the properness of U implies $pu_i \geq 0 \; \forall\, u \in U, \forall\, i \in \mathcal{I}$. But then the contract specification ($\sum_{i \in \mathcal{I}} u_i = 0$) implies $pu_i = 0 \; \forall\, u \in U, \forall\, i \in \mathcal{I}$. Finally, if for a $T \subseteq \mathcal{I}$, $T \neq \emptyset$ and a web $W \in F(U,T)$ we have $W \succ_T U$, it follows from the equilibrium definition that

$$\langle p, y_i(W) \rangle > \langle p, x_i(U) \rangle \;\; \forall\, i \in T.$$

Consequently, summing up these inequalities over $i \in T$, we arrive at the contradiction with the contract specification of $w \in W \setminus U$ (since $S(w) \subset T$, because of $W \in F(U,T)$ and (ii)). The final part of theorem is also clear. □

Next let us consider several examples demonstrating the difference between the various notions of contractual allocation. Of course, for this difference to be realized when the partial breaking of contracts is allowed, the conditions of Theorem 1.2 have to be invalid, and either the utilities have to be non-smooth or the allocation has to belong to the boundary of X.

The following example, borrowed from Kozyrev (1982), shows that for non-differentiable utility functions a proper contractual allocation may not be an equilibrium.

Example 1.2 Consider a two-commodities exchange economy with two consumers in which $X_i = \mathbb{R}^2_+$ and the preferences are defined onto $\mathbb{R}^2_+$ by the strict monotonic utility functions

$$u_1(x^1, x^2) = 2\sqrt{x^1 x^2} + x^1 + x^2, \quad u_2(x^1, x^2) = 2\sqrt{x^1 x^2} + x^1 + x^2 + \min\{x^1, x^2\},$$

where the upper index shows the number of a commodity. For the initial endowments, take the vectors

$$\omega_1 = (1,0), \quad \omega_2 = (0,1), \quad \bar{\omega} = \omega_1 + \omega_2 = (1,1), \quad \omega = (\omega_1, \omega_2) = ((1,0),(0,1)).$$

In further considerations, we make use of the "Edgeworth's box," the well known subset of $\mathbb{R}^2$,

$$EB(\bar{\omega}) = \{x \in \mathbb{R}^2 \mid 0 \leq x \leq (1,1) = \bar{\omega}\}.$$

One can interpret $x \in EB(\bar{\omega})$ as the consumption of the first consumer and $(\bar{\omega} - x)$ as the consumption of the second one. This point may be also associated with the allocation $(x, \bar{\omega} - x)$.

A simple analysis shows that the Pareto boundary in this example is the set

$$\mathcal{PB} = \mathrm{co}\{\,(0,0),(1,1)\,\} = \{\, x \in EB(\bar{\omega}) \mid x^1 = x^2 = \alpha, \; 0 \leq \alpha \leq 1\,\},$$

i.e., the diagonal of $EB(\bar{\omega})$.

Since every equilibrium allocation is optimal by Pareto and, if it is an interior point of the box, the price vector has to coincide up to a normalization with $\operatorname{grad} u_1(x_1)$, the vector $(1,1)$ has to be an equilibrium price vector. Clearly, the points $(1,1)$ and $(0,0)$ are not equilibrium allocations. Consequently, $p = (1,1)$ is *the only* (up to a normalization) *equilibrium price*. Using the budget constraints $px_i = p\omega_i$, one can easily find the unique equilibrium allocation which corresponds to the first agent's consumption bundle $(\frac{1}{2}, \frac{1}{2})$ in the Edgeworth's box.

The core in this economy with two consumers coincides with the set $\mathcal{PB} \cap \mathcal{IR}$ which, in turn, is the set of all contractual and weakly contractual allocations and can be easily calculated to be

$$\mathcal{PB} \cap \mathcal{IR} = \{\, x \in \mathcal{PB} \mid u_1(x) \geq u_1(\omega_1),\ u_2(\bar{\omega} - x) \geq u_2(\omega_2) \,\} = \mathrm{co}\{\, (\tfrac{1}{4}, \tfrac{1}{4})\,, (\tfrac{4}{5}, \tfrac{4}{5}) \,\}.$$

Next let us find the set of all proper contractual allocations. Clearly, this is the

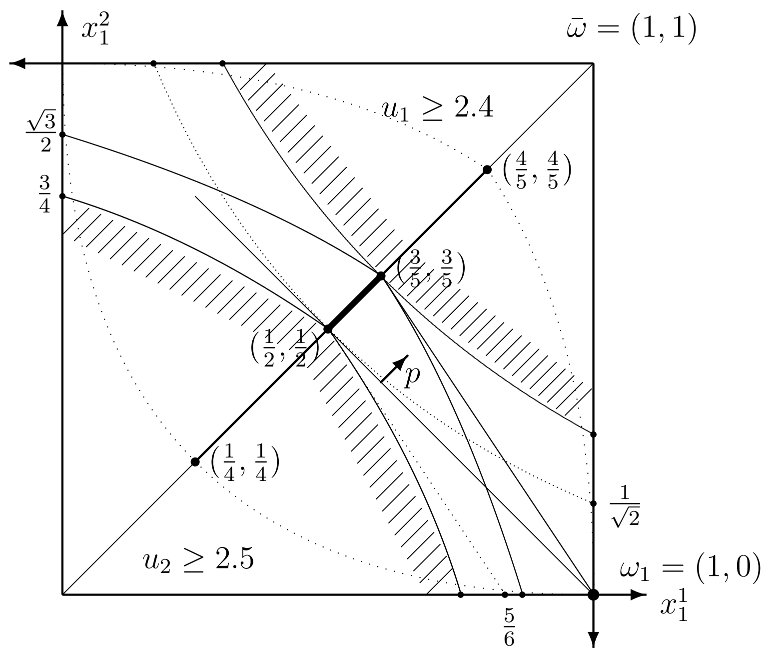

Figure 1.6: Non-smooth preferences

set of all points $x = (\alpha, \alpha)$ in $\mathcal{PB} \subset EB$ for which the derivatives of the utility functions u_1 and u_2 are not positive in the directions of $h_1 = \omega_1 - (\alpha, \alpha)$ and $h_2 = \omega_2 - (1-\alpha, 1-\alpha)$, i.e., we need to solve the system of equations

$$\partial_{h_1} u_1(\alpha, \alpha) \leq 0, \quad \partial_{h_2} u_2(1-\alpha, 1-\alpha) \leq 0.$$

A direct calculation gives

$$\mathrm{grad}\, u_1(\alpha, \beta) = (\sqrt{\beta/\alpha} + 1, \sqrt{\alpha/\beta} + 1)$$

for all $\alpha > 0$, $\beta > 0$ and

$$\mathrm{grad}\, u_2(1-\alpha, 1-\beta) = (\sqrt{(1-\beta)/(1-\alpha)} + 2\,,\ \sqrt{(1-\alpha)/(1-\beta)} + 1)$$

for $\alpha > \beta > 0$, $1 - \alpha > 0$. Calculating the inner products and substituting $\alpha = \beta$ (i.e., passing to the limit for $\beta \to \alpha$) yields

$$1 - 2\alpha \leq 0, \quad 5\alpha - 3 \leq 0.$$

As a result, the set of proper contractual allocations is described as $\mathrm{co}\{\,(\frac{1}{2},\frac{1}{2})\,,(\frac{3}{5},\frac{3}{5})\,\}$ and does *not coincide* with (but contains!) the set of *equilibrium* allocations. Figure 1.6 illustrates the analysis conducted.

In this example it seems to be interesting to clarify the structure of the set corresponding to another new theoretical concept, the set of all *lower* proper contractual allocations. With this in mind, let me describe allocations which cannot be dominated by the coalition $\{1\}$ relative to the partial breaking of the contract $x-\omega$ (i.e., the web consisting of only one contract). In the Edgeworth's box, they are the points (α,β) satisfying the condition

$$\partial_{h_1} u_1(\alpha,\beta)\le 0, \quad h_1=\omega_1-(\alpha,\beta).$$

Now calculating the directional derivative in the form of the inner product and substituting the value for ω_1, we obtain for $(\alpha,\beta)\gg 0$

$$(1-\alpha)(\sqrt{\beta/\alpha}+1)-\beta(\sqrt{\alpha/\beta}+1)\le 0,$$

which can be rewritten as

$$\sqrt{\beta/\alpha}\le(\sqrt{\alpha}+\sqrt{\beta})^2-1 \iff (\sqrt{\alpha}+\sqrt{\beta})\le(\sqrt{\alpha}+\sqrt{\beta})^2\sqrt{\alpha}.$$

Dividing both sides by $\sqrt{\alpha}+\sqrt{\beta}>0$ yields, after transformations,

$$\sqrt{\beta\alpha}\ge 1-\alpha \iff \beta+2\ge\frac{1}{\alpha}+\alpha.$$

Therefore, the set we are interested in is the part of the *epigraph* of the curve $x_1^2=\frac{1}{x_1^1}+x_1^1-2$ inside the rectangle $0\le(x_1^1,x_1^2)\le(1,1)$. This set is shown in Figure 1.7.

It is harder to carry out a similar analysis for agent 2 by purely analytical means. This is why we also turn to geometrical considerations. Let us choose an individual-rational point (α,β) for agent 2 in the coordinate system of agent 1, which lies in the interior of the box. If this point *belongs to the diagonal*, i.e., $\alpha=\beta$, then as we have seen above, it satisfies the required property only if $\alpha\le 3/5$. Now assume that the point (α,β) *does not belong* to the diagonal, i.e., $\alpha\ne\beta$ and consider the second agent's indifference curve (in the coordinate system of the 1st agent) passing through this point. This curve corresponds to the graph of a concave function; therefore, if our point is desired, i.e., $\partial_{h_2}u_2(1-\alpha,1-\beta)\le 0$ for $h_2=\omega_2-(1-\alpha,1-\beta)$, then *every* point of the curve to the *right* of (α,β) is also a desired one. Next, if $\alpha\le\beta$, the right hand side of the curve intersects the diagonal at a point (γ,γ) where, by the above argument, $\gamma\le 3/5$. Moreover, for $\alpha<\beta$, we have $u_2(1-\alpha,1-\beta)=u_1(1-\alpha,1-\beta)+1-\beta$. Now calculate the gradient and pass to the limit for $(\alpha,\beta)\to(\gamma,\gamma)$, $\gamma\ge 0$. The result is that (γ,γ) must satisfy the condition

$$\langle(2,3),(\gamma-1,\gamma)\rangle\le 0 \;\Rightarrow\; \gamma\le 2/5.$$

Consequently, in the coordinate system of agent 1 the desired set can be viewed as the *ordinate set* of the curve consisting of three parts: for the interval $[0,\frac{2}{5}]$, this is

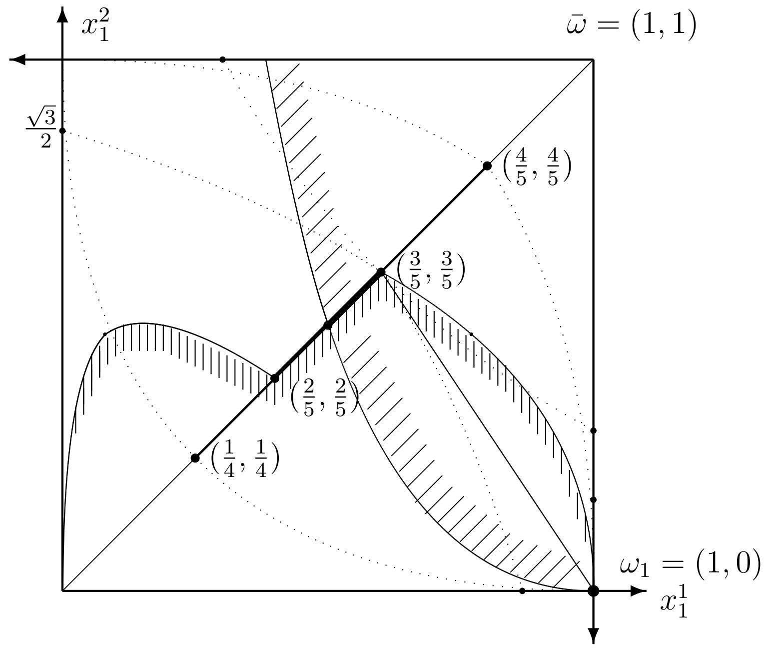

Figure 1.7: *Lower* proper contractual allocations

the graph of a concave function which is equal to 0 at 0 and $\frac{2}{5}$ at $\frac{2}{5}$; for the interval $[\frac{2}{5}, \frac{3}{5}]$ the curve coincides with the diagonal; for the interval $[\frac{3}{5}, 1]$ this is again the graph of a concave function which is equal to $\frac{3}{5}$ at $\frac{3}{5}$ and 0 at 1. Thus we arrive at a *nonconvex* set. The intersection of this second set with the first one represents the set of all *lower* proper contractual allocations in the Edgeworth's box. Being intersected with the Pareto boundary (the diagonal) in addition, this set represents the already known set of all proper contractual allocations; see Figure 1.7.

In the framework of this example, one can also demonstrate the difference between the notions of coherent and proper webs as well as between the corresponding stability properties. Let us consider an allocation, in which agent 1 consumes the bundle $(\frac{3}{5}, \frac{3}{5})$; see Figure 1.8. Set $\hat{h}_1 = (0,1) - (\frac{3}{5}, \frac{3}{5}) = (-\frac{3}{5}, \frac{2}{5})$ and calculate the derivative of the function u_1 at the point $(\frac{3}{5}, \frac{3}{5})$ in the direction of $\hat{h}_1$:

$$\partial_{\hat{h}_1} u_1(\tfrac{3}{5}, \tfrac{3}{5}) = \langle (2,2), \hat{h}_1 \rangle = -\frac{2}{5} < 0.$$

For the allocation considered, the second agent's consumption bundle is a point of non-differentiability of his/her utility function, but its derivative in the direction of $\hat{h}_2 = (1,0) - (\frac{2}{5}, \frac{2}{5}) = (\frac{3}{5}, -\frac{2}{5})$ can be easily calculated as the limit for $(\alpha, \beta) \to (\frac{3}{5}, \frac{3}{5})$ of the inner product of the vector-gradient of u_2 calculated at the point $(1-\alpha, 1-\beta)$, $\beta > \alpha > 0$, and the vector $\hat{h}_2$. Since for $\beta > \alpha > 0,\ 1 - \beta > 0$,

$$\operatorname{grad} u_2(1-\alpha, 1-\beta) = (\sqrt{(1-\beta)/(1-\alpha)} + 1, \sqrt{(1-\alpha)/(1-\beta)} + 2);$$

upon calculating the inner products and substituting $\alpha = \beta = \frac{3}{5}$, one obtains

$$\partial_{\hat{h}_2} u_2(\tfrac{2}{5}, \tfrac{2}{5}) = \langle (2,3), \hat{h}_2 \rangle = 0.$$

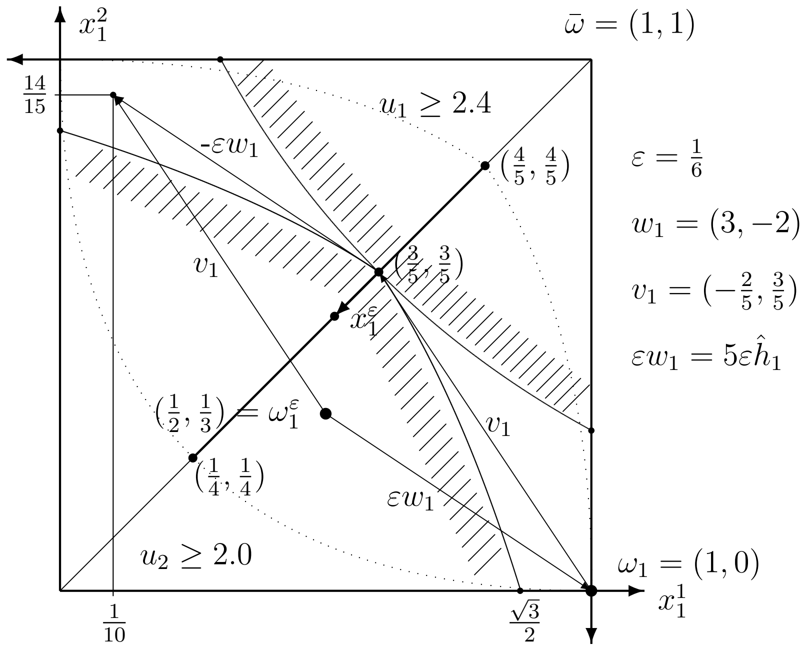

Figure 1.8: *Coherent non-proper* contractual allocations

Next, for a sufficiently small $\varepsilon > 0$, take allocation

$$\omega^\varepsilon = \omega - \varepsilon w$$

to be the initial endowments and consider web $V^\varepsilon = \{v, \varepsilon w\}$ where

$$v = (v_1, v_2), \quad v_1 = -v_2 = (\tfrac{3}{5}, \tfrac{3}{5}) - (1, 0) = (-\tfrac{2}{5}, \tfrac{3}{5})$$

and

$$w = (w_1, w_2), \quad w_1 = -w_2 = (3, -2)$$

is the contract in which the first consumer exchanges 2 units of commodity 2 for 3 units of commodity 1. For example, one can take $\varepsilon = \frac{1}{6}$. Obviously

$$\omega^\varepsilon + v + \varepsilon w = ((\tfrac{3}{5}, \tfrac{3}{5}), (\tfrac{2}{5}, \tfrac{2}{5})),$$

i.e., the allocation considered is realized by the web V^ε relative to the endowments ω^ε. The above calculations of the derivatives in the directions of $\hat{h}_1$, $\hat{h}_2$ together with the previous calculations of the derivatives in the directions of h_1 and h_2 (for $\alpha = \frac{3}{5}$), show that each contract in the web V^ε is *coherent* relative to ω^ε. However, for every $\varepsilon \in (0, \frac{1}{6}]$ the web V^ε is not proper since after the breaking of a half of εw and after the partial breaking of the contract v, the broken part being $\delta = \frac{5}{2}\varepsilon < 1$, it realizes the allocation for which the first agent's consumption program has the form

$$\tfrac{1}{2}\varepsilon(3, -2) + (1 - \tfrac{5}{2}\varepsilon)(-\tfrac{2}{5}, \tfrac{3}{5}) + (1, 0) - \varepsilon(3, -2) = (\tfrac{3}{5}, \tfrac{3}{5}) - \varepsilon(\tfrac{1}{2}, \tfrac{1}{2}) = x_1^\varepsilon.$$

Therefore, the proposed partial breaking of contracts is, in fact, profitable for the second agent since it results in an increase in her/his consumption by $\varepsilon(\frac{1}{2}, \frac{1}{2})$. This proves that the web V^ε *is not proper* relative to ω^ε. □

The above example stimulates a more careful study of the mathematical properties of proper (and perfectly) contractual allocations in situations where the conditions of Theorem 1.2 are invalid. To this end, let us proceed as follows.

Let us begin with the discussion of lower proper contractual allocations. Due to the definition, they are the allocations $x(V)$ which can be realized by a web of contracts V, which are stable relative to the procedure of partially breaking contracts. Since in the standard market model every contract is permissible, web V can be replaced by web $\Delta V = \{u\}$ consisting of only one contract $u = \sum_{v\in V} v = x - \omega$. It is easily seen that the lower stability of the original web implies the same type of stability for web ΔV. Therefore, one can restrict the analysis of proper (not only lower) contractual allocations to the webs having the form $\{x - \omega\}$. Further it will be clear that all the conclusions can be easily applied to the case of webs consisting of multiple contracts. However, in the case of webs consisting of a single contract, one can directly conclude from the definition that an allocation $x \in \mathcal{A}(\mathcal{X})$ is *lower proper contractual* if and only if

$$[x_i, \omega_i] \cap \mathcal{P}_i(x_i) = \emptyset \quad \forall\, i \in \mathcal{I}, \tag{1.6}$$

where

$$[x_i, \omega_i] = \{\lambda x_i + (1-\lambda)\omega_i \mid 0 \le \lambda \le 1\}.$$

Now the definitions of stability via Theorem 1.1 (*ii*) easily imply that *weakly proper contractual* allocations are exactly the allocations that are *optimal by Pareto and satisfy* (1.6).

The proper contractual allocations can be characterized in similar terms as follows. Breaking the $1-\lambda$ part of the contract $x-\omega$ and signing the new contract v, the members of $S \subset \mathcal{I}$ realize the collection of consumption bundles $y^S = (y_i^S)_S$ such that $\sum_S y_i^S = \sum_S(\lambda x_i + (1-\lambda)\omega_i))$. Since the definition of a proper contractual allocation forbids this type of domination, the allocation x must belong to the core of the economy with the initial endowments $\lambda x + (1-\lambda)\omega = \omega_x^\lambda = \omega_x \in [x, \omega]$, which I denote by $\mathcal{C}(\mathcal{E}_x^\lambda)$. Thus an allocation x is *proper contractual* if and only if it belongs to the core of each economy $\mathcal{C}(\mathcal{E}_x^\lambda)$, i.e.,

$$x \in \bigcap_{\omega_x \in [x,\omega]} \mathcal{C}(\mathcal{E}_x^\lambda).$$[12]

I continue by similarly describing the perfectly contractual allocations. In order to do this, we actually only need to understand to which kinds of the "initial endowment allocations" one can transit upon breaking a "virtual web" for $\{x-\omega\}$. First of all note that in this case we may restrict the analysis to the webs consisting of two contracts. To see this, let V be a proper virtual web realizing the allocation x, i.e., $V \sim \{x-\omega\}$, and let $U \subseteq V$ be the set of all contracts broken by some coalition. Form the web $W = \{\Delta(U), \Delta(V \setminus U)\}$ in which all broken contracts are aggregated into one contract and all preserved contracts into another one. Clearly,

[12]Notice that this formula also implies that in the Edgeworth's box the proper contractual allocations coincide with the weakly proper contractual ones. In fact, in an economy with just two agents the core is equal to $\mathcal{PB} \cap \mathcal{IR}$. However, lower stability implies $x_i \succeq (\lambda x_i + (1-\lambda)\omega_i)$, $i = 1, 2$, i.e., $x \in \mathcal{IR}(\mathcal{E}_x^\lambda)$, which together with the Pareto optimality gives the result.

web W is proper, $W \sim V$, and by breaking contract $\Delta(U)$ one realizes the allocation that coincides with the allocation obtained from V as a result of breaking contracts $U \subseteq V$, which completes the argument. So let $W = \{u, v\} \sim \{x - \omega\}$. Then by partially breaking contracts in this web, one can realize any point y in the convex hull of the set consisting of four points: x, ω, $\omega + v$, $\omega + u$. It is clear that the web $\{x - y, y - \omega\}$ constructed for this y is also proper. The opposite is also true: if this web is proper, $y \in \mathcal{A}(\mathcal{X})$ can be realized via breaking a part of the contracts in a virtual web realizing x. Thus if we define

$$PC = \{y \in \mathcal{A}(\mathcal{X}) \mid \text{web } \{x - y, y - \omega\} \text{ is proper}\},$$

the set of all perfectly contractual allocations can be described in the form

$$x \in \bigcap_{y \in PC} \mathcal{C}(\mathcal{E}_y) \iff x \text{ is perfectly contractual,} \tag{1.7}$$

where y is the vector of initial endowments in model $\mathcal{E}_y$. To better understand the meaning of this formula, the structure of the set PC needs to be clarified. Figure 1.11 below provides an illustration.

Next let us turn to the description of the objects under study in terms of dual cones. This description is of mathematical interest in its own right and, as we will see below, can considerably contribute to the understanding of various concepts of contractual allocation in the situations we are interested in.

The cone

$$K^* = \{p \in E' \mid \langle p, K \rangle \geq 0\}$$

is said to be the dual cone of set $K \subset E$. For every $i \in \mathcal{I}$, let us set

$$\Gamma(x_i) = \{p \in E' \mid \langle p, \mathcal{P}_i(x_i) - \{x_i\} \rangle \geq 0\}.$$

This is the dual cone of the $\mathcal{P}_i(x_i) - \{x_i\}$. It is well known (and easy to prove applying the separation theorem) that for every weakly optimal by Pareto allocation, there corresponds a (non-zero) linear price functional $p \in E'$ such that

$$\langle p, \mathcal{P}_i(x_i) \rangle \geq \langle p, x_i \rangle \quad \forall\, i \in \mathcal{I}.$$

The necessity part of this statement is always true (for convex, locally non-satiated preferences), whereas its sufficiency part is true for the interior points of the consumption sets (if in addition the sets $\mathcal{P}_i(x_i)$ are open in X_i). One can see that this description is very close to being a precise characterization of optimality by Pareto[13]. Accordingly, it is not inaccurate if we call the allocations which satisfy this property quasi-optimal by Pareto. In the above terms, they can be described as the allocations satisfying

$$\Gamma(x) = \bigcap_{\mathcal{I}} \Gamma(x_i) \neq \{0\}. \tag{1.8}$$

[13]In the framework of nonstandard analysis, i.e., making use of nonstandard prices, one can obtain a description of *strictly Pareto optimal* allocations in the form of strict inequalities and without the latter *interior point* assumption. However, I will not elaborate on this here and restrict myself to the standard analysis.

For every $i \in \mathcal{I}$, set

$$G(x_i - \omega_i) = \{p \in E' \mid \langle p, x_i - \omega_i \rangle \geq 0\}, \quad G(x) = \bigcap_{\mathcal{I}} G(x_i).$$

Now one can easily see that x is a *quasiequilibrium* $\iff$

$$\bigcap_{\mathcal{I}} [G(x_i - \omega_i) \cap \Gamma(x_i)] \neq \{0\} \iff G(x) \cap \Gamma(x) \neq \{0\}. \tag{1.9}$$

Assume that $x \in \operatorname{int} X$.[14] Then similarly applying the separation theorem and (1.6), we see that an allocation x is *lower proper contractual* $\iff$

$$G(x_i - \omega_i) \cap \Gamma(x_i) \neq \{0\} \;\; \forall\, i \in \mathcal{I} \iff$$

$$\forall\, i \in \mathcal{I} \;\; \exists\, p_i \in E', p_i \neq 0 : \;\; \langle p_i, \mathcal{P}_i(x_i) \rangle \geq \langle p_i, x_i \rangle \;\; \& \;\; p_i x_i \geq p_i \omega_i. \tag{1.10}$$

Thus, in this case an allocation is *weakly proper contractual* if (1.8) holds in addition. Next let us study the properties of the proper contractual allocations.

Lemma 1.1 *If an allocation x is proper contractual, then for each coalition $S \subseteq \mathcal{I}$ there exists $p_S \in E'$, $p_S \neq 0$ such that*

$$\langle p_S, \mathcal{P}_i(x_i) \rangle \geq \langle p_S, x_i \rangle \;\; \forall\, i \in S \;\; \& \;\; p_S \sum_S x_i \geq p_S \sum_S \omega_i. \tag{1.11}$$

If in addition $x \in \operatorname{int} X$, the opposite is true, i.e., if there exist linear $p_S \in E'$, $p_S \neq 0$ satisfying (1.11) $\forall\, S \subseteq \mathcal{I}$, then x is a proper contractual allocation.

In other words, the lemma states that for a proper contractual allocation, each coalition can find internal-coalition prices such that, first, they are "suitable" for every member of the coalition (the first inequality in (1.11) that may be treated as a form of the coalition efficiency) and, second, the contract $x - \omega$ is coalition-profitable relative to these prices (the second inequality in (1.11)). Thus, the proper contractual allocations are precisely the allocations which satisfy the condition of *coalition-profitability* (1.11). The statement of the lemma is similar to the description of the weakly proper contractual allocations given in (1.10), the only difference being that the lemma claims the existence of internal-coalition prices, satisfying (1.11) for *every coalition*, whereas in (1.10) just for *singleton coalitions*. This is why the weakly proper contractual allocations are just *individually profitable* and *Pareto optimal* (i.e., the coalition of all agents is profitable as well). Notice also that the statement of Lemma 1.1 may be rewritten in the equivalent form

$$\bigcap_S \Gamma(x_i) \bigcap G(\sum_S x_i - \sum_S \omega_i) \neq \{0\} \quad \forall\, S \subseteq \mathcal{I}.$$

Of course, in the general case, this requirement is weaker than (1.9). This is why in order to establish that a proper contractual allocation is a (quasi)equilibrium, we

[14] This condition together with (**A**) is essential for establishing sufficiency; as for necessity, it may be dropped (in view of (**A**)).

need to make additional assumptions to guarantee that (1.9) is equivalent to the last relation (for example, that the utilities are differentiable and $x \in \operatorname{int} X$), which is actually a considerably stronger assumption than we made in Theorem 1.2.

Proof of Lemma 1.1. It follows from the above analysis that an allocation x is proper contractual iff it cannot be improved upon by any coalition $S \subseteq \mathcal{I}$ relative to the endowments $x^\lambda = \lambda x + (1 - \lambda\omega)$ for all $\lambda \in [0,1]$. Let $\mathcal{P}_S(y^S) = \prod_S \mathcal{P}_i(y_i)$ for $y^S = (y_i)_{i \in S} \in \prod_S X_i = X^S$ and let $x_S^\lambda = (x_i^\lambda)_{i \in S}$. Then for a fixed λ, the last property can be written in the form

$$\mathcal{P}_S(x^S) \bigcap (E_S + x_S^\lambda) = \emptyset, \quad E_S = \{y^S \in E^S \mid \sum_S y_i = 0\}. \tag{1.12}$$

Therefore,

$$\mathcal{P}_S(x^S) \bigcap (E_S + [x^S, \omega^S]) = \emptyset, \quad x^S = (x_i)_{i \in S}, \quad \omega^S = (\omega_i)_{i \in S},$$

where $[x^S, \omega^S]$ is the segment in E^S connecting the points x^S and ω^S (the convex hull of two points). Due to assumption (**A**), since the set $E_S + [x^S, \omega^S]$ is convex one can apply the classical separation theorem, which gives the existence of a linear functional (vector) $p^S = (p_i)_{i \in S} \in (E')^S$, $p^S \neq 0$, such that

$$\langle p^S, \mathcal{P}_S(x^S) \rangle \geq \langle p^S, E_S + [x^S, \omega^S] \rangle.$$

The right-hand side of this inequality is a *bounded from above* subset of $\mathbb{R}$. Hence the inequality may be true only if the set $\langle p^S, E_S \rangle$ is bounded. Since E_S is a subspace of E^S, it follows that $\langle p^S, E_S \rangle = \{0\}$. A standard argument then implies that $p_i = p_j = p \; \forall\, i, j \in S$ (because $p^S z^S = 0$ for all $z^S \in E^S$ such that $z_i^S = -z_j^S \in E$ and $z_t^S = 0$ for $t \neq i, j,\ t \in S$). Moreover, $p \neq 0$ since $p^S = (p, \ldots, p) \neq 0$. Next, assumption (**A**) implies that $\mathcal{P}_S(x^S)$ is convex and $x^S \in \overline{\mathcal{P}_S(x^S)}$. Therefore, it follows from the last inequality that

$$\langle p^S, x^S \rangle \geq \langle p^S, [x^S, \omega^S] \rangle \iff p \sum_S x_i \geq p \sum_S \omega_i.$$

Moreover, arguing by contradiction, we obtain

$$\langle p^S, \mathcal{P}_S(x^S) \rangle \geq \langle p^S, x^S \rangle = \sup \langle p^S, [x^S, \omega^S] \rangle \iff \langle p, \mathcal{P}_i(y_i) \rangle \geq \langle p, x_i \rangle \quad \forall\, i \in S.$$

Now to complete the proof of the lemma's necessity, just substitute $p_S = p$.

The lemma's sufficiency follows from (**A**) and the condition $x \in \operatorname{int} X$, since in this case the inequalities in the first part of (1.11) are actually strict, which together with the second part of (1.11) implies (1.12) for all $\lambda \in [0, 1]$. □

Lemma 1.1 allows us to discover new interesting (and sometimes unexpected) properties specific to proper contractual allocations. For example, it follows from this lemma that every proper contractual allocation in a 2-replicated economy is an equilibrium if the economy contains just two agents or, alternatively, if there is one agent with differentiable preferences.[15]

First we would like to recall the concept of a replicated economy. Given a natural $r \in \mathbb{N}$, the r-fold replica of $\mathcal{E}$ is the model $\mathcal{E}^r$ in which every consumer of the original model defines a *type of economic agent* represented by her/his r precise copies. For convenience, the agents in $\mathcal{E}^r$ are numbered by double indexes (i, m), $i \in \mathcal{I}$, $m = 1, \ldots, r$. It is assumed that $X_{im} = X_i$, $\omega_{im} = \omega_i$, and the preferences, being defined on and taking values in X_{im}, are defined by $\mathcal{P}_{im} = \mathcal{P}_i$. Notice that for every allocation $x = (x_i)_{\mathcal{I}}$ in the initial model, there canonically corresponds an allocation in the replica according to the rule $x_{im} = x_i \ \forall\, i, m$. The opposite is also true if the allocation is *symmetric*, i.e., when identical agents consume equal bundles (this is so-called equal treatment).

Replicas play an important role in the analysis of perfect competition, especially for proving the well known Edgeworth's conjecture which states that under the conditions of perfect competition the core and equilibria coincide. It is the replica's symmetric allocations and the corresponding allocations in the original model that is the main subject of this analysis, with every coalition in the replica being allowed to dominate the original allocations by not necessarily symmetric inter-coalition allocations.

Theorem 1.3 (Kozyrev 1982) *Assume that an economy has two agents or, alternatively, there exists an agent with a smooth preference whose consumption choice is an interior point of his/her consumption set. Then every allocation, which is proper contractual in the 2-fold replica economy, is a quasiequilibrium.*

Proof of Theorem 1.3. Consider first the case of an economy with two agents. Let $\mathcal{I} = \{1, 2\}$ and let $x = (x_1, x_2)$ be a proper contractual allocation in the 2-fold replica economy. It suffices to establish (1.9) for x. To this end, apply Lemma 1.1 and relation (1.11) to the coalitions $S' = \{(1,1), (1,2), (2,1)\}$ and $S'' = \{(1,1), (2,1), (2,2)\}$. This results in the existence of *nonzero* vectors p' and $p'' \in E'$ such that

$$p', p'' \in \Gamma(x_1) \cap \Gamma(x_2)$$

and

$$p'(2x_1 + x_2) \geq p'(2\omega_1 + \omega_2), \quad p''(x_1 + 2x_2) \geq p''(\omega_1 + 2\omega_2).$$

Since $x_1 + x_2 = \omega_1 + \omega_2$, the last inequalities are equivalent to

$$p'x_1 \geq p'\omega_1 \ \ \& \ \ p''x_2 \geq p''\omega_2.$$

If one of these inequalities is actually an equality, due to the feasibility of x, either p' or p'' belongs to the intersection in (1.9). Suppose both inequalities are strict. Then, the first component of the *2-dimension* vector $(p'(x_1 - \omega_1), p'(x_2 - \omega_2))$ is strictly greatly than zero, the second one is strictly less than zero, and their summation is equal to zero. The same is true for the

[15] This result was first proved by Kozyrev (1982), who applied the technique of subdifferential calculus (to *concave* utility functions, etc.) which restricts the generality. Lemma 1.1 and the proof of the next theorem are new.

vector $(p''(x_1-\omega_1), p''(x_2-\omega_2))$, in which the first component is strictly less than zero. Next find a real $0<\alpha<1$ such that $\alpha p'(x_1-\omega_1) + (1-\alpha)p''(x_1-\omega_1) = 0$ (set $\alpha = -p''(x_1-\omega_1)/\ [p'(x_1-\omega_1) - p''(x_1-\omega_1)]$). Then it is clear that then $[\alpha p' + (1-\alpha)p''](x_2-\omega_2) = 0$. Now let us set $p = \alpha p' + (1-\alpha)p'' \neq 0$. Hence, by construction, we obtain $p(x_i-\omega_i) = 0 \Rightarrow p \in G(x_i-\omega_i)$, $i=1,2$, which due to the convexity of $\Gamma(x_i)$ implies $p \in \Gamma(x_i)$, $i=1,2$. This proves (1.9).

Now let us show (1.9), assuming that there exists an agent with a smooth preference in the economy. First of all note that due to this assumption, if there exists a nonzero vector $p \in \Gamma(x_i)\ \forall\ i \in \mathcal{I}$, it is *unique up to a normalization*. Next apply Lemma 1.1 and relation (1.11) to the coalitions $S^i = \{(i,2)\} \cup \mathcal{I} \times \{1\}$, $i \in \mathcal{I}$, all the coalitions in which the i-th type consumer is presented by two agents and all the other consumers just by one agent. It follows that there exists a nonzero vector $p \in \Gamma(x_i)\ \forall\ i \in \mathcal{I}$, which is *common* for all coalitions S^i, such that for every $i \in \mathcal{I}$

$$p[\sum_{j\in\mathcal{I}} x_j + x_i] \geq p[\sum_{j\in\mathcal{I}} \omega_j + \omega_i] \quad \Longrightarrow \quad px_i \geq p\omega_i.$$

Since x is feasible, $px_i = p\omega_i$ for all i and the proof is complete. □

Arguing along the lines of the proof of Lemma 1.1, one can obtain a dual description of perfectly contractual allocations. Having this in mind, let us make use of formula (1.7) to describe the proper web $\{x-y, y-\omega\}$ in dual terms. As in (1.6), we conclude that $\{x-y, y-\omega\}$ is proper $\iff$

$$\mathrm{co}\{x_i, y_i, \omega_i, x_i - y_i + \omega_i\} \cap \mathcal{P}_i(x_i) = \emptyset \quad \forall\, i \in \mathcal{I}.$$

Now applying the separation theorem, we see that this web is proper if and only if for each i there exists a nonzero $p_i \in E'$ such that

$$\langle p_i, \mathcal{P}_i(x_i)\rangle \geq p_i x_i \geq \langle p_i, \mathrm{co}\{x_i, y_i, \omega_i, x_i - y_i + \omega_i\}\rangle.$$

In this chain of inequalities, the second one is equivalent to $p_i x_i \geq p_i y_i \geq p_i \omega_i$. Figures 1.9, 1.10, 1.11 illustrate the above argument.

Thus we obtained a description of PC, the set of all allocations which can be realized via breaking a part of the contracts in a virtual web realizing x. We can apply this description to give a characterization of the perfectly contractual allocations.

In fact, being proper contractual, a perfectly contractual allocation has to satisfy (1.11) and, in addition, for every $y \in \mathcal{A}(\mathcal{X})$, the condition

$$\forall\ i \in \mathcal{I}\ \exists\, p_i \in E', p_i \neq 0\ :\ \langle p_i, \mathcal{P}_i(x_i) - \{x_i\}\rangle \geq 0\ \ \&\ \ p_i x_i \geq p_i y_i \geq p_i \omega_i \tag{1.13}$$

has to imply $x \in \mathcal{C}(\mathcal{E}_y)$. Moreover, since it has to be true for all $y' \in [x,y]$ (substitute y' for y in the last relations), i.e., since x is proper contractual relative to y, one can apply Lemma 1.1. Thus condition (1.13) has to imply that for each coalition $S \subseteq \mathcal{I}$, there exists a nonzero $p_S \in E'$ such that

$$\langle p_S, \mathcal{P}_i(x_i)\rangle \geq \langle p_S, x_i\rangle\ \forall\ i \in S\ \ \&\ \ p_S \sum_S x_i \geq p_S \sum_S y_i.$$

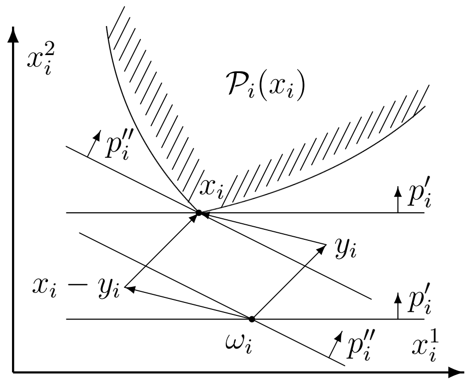

Figure 1.9: Domain of feasible variation of y_i for the web $\{x - y, y - \omega\}$ to be proper

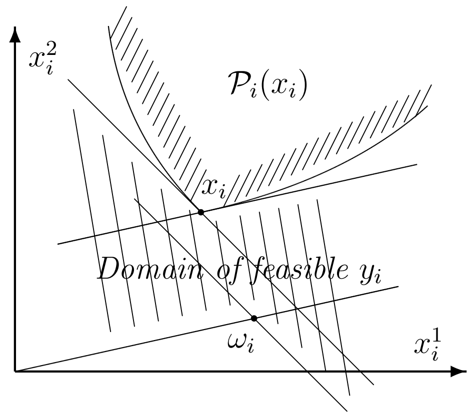

Figure 1.10: Domain of y_i for the web $\{x - y, y - \omega\}$ to be proper

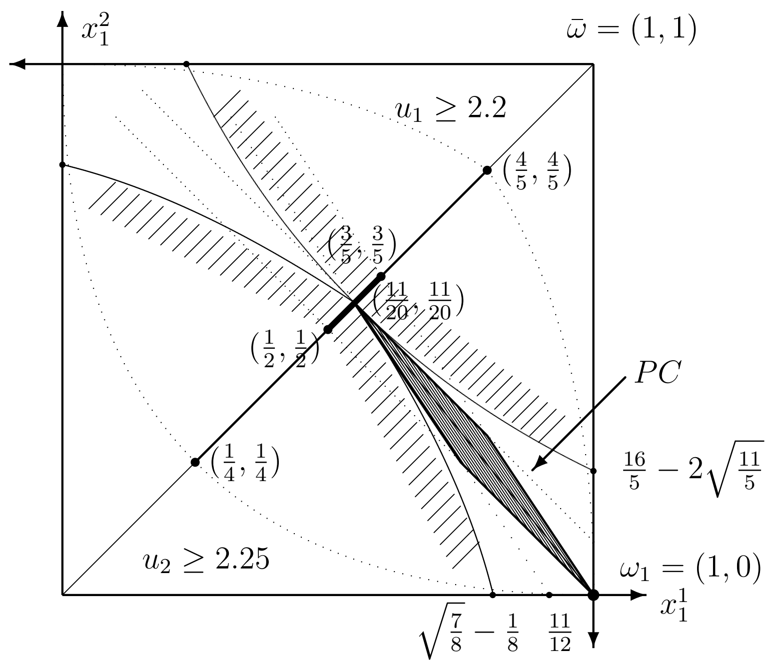

Figure 1.11: *Perfect contractual* allocations in the Edgeworth's box for non-smooth preferences

Next let us consider the elaborated characterization and Lemma 1.1 in the context of Example 1.1.

Example 1.1 (prolongation) Applying Lemma 1.1 let us show that the studied allocation $x = (x_1, x_2, x_3)$,

$$x_1 = (1, 1), \quad x_2 = (2, 2), \quad x_3 = (1, 1)$$

is, in fact, proper contractual. With this in mind, let us establish that relation (1.11) is true. For singleton coalitions this fact was proved above, and we need to check (1.11) for two-elements and for grand coalitions. To do this let us describe the dual cones corresponding to current allocation x:

$$\Gamma(x_1) = \text{cone}\{(4,4),(1,7)\},$$
$$\Gamma(x_2) = \text{cone}\{\,(4,6),\ (3,7)\,\},$$
$$\Gamma(x_3) = \text{cone}\{(20,1),(1,20)\},$$

where cone$A$ denotes the conic hull of set A. We obtain

$$\Gamma(x_1) \cap \Gamma(x_2) \cap \Gamma(x_3) = \Gamma(x_1) \cap \Gamma(x_2) = \text{cone}\{(2,3),(2,4\tfrac{2}{3})\}.$$

In particular we see that for a grand coalition of all agents, the vector $p_{\mathcal{I}} = (2,3)$ is suitable. For coalition $\{1,2\}$ this vector is also suitable, since then

$$\langle (2,3), x_1 + x_2 - \omega_1 - \omega_2 \rangle = \langle (2,3), (-\tfrac{1}{4}, \tfrac{1}{4}) \rangle > 0.$$

Since

$$(1,7) \in \Gamma(x_1) \cap \Gamma(x_3) \quad \& \quad \langle (1,7), x_1 + x_3 - \omega_1 - \omega_3 \rangle = \langle (1,7), (-\tfrac{3}{4}, \tfrac{1}{4}) \rangle > 0,$$

vector $(1,7)$ can be applied in (1.11) for coalition $\{1,3\}$. Finally, for coalition $\{2,3\}$, vector $(2,3)$ is suitable again, since then we obtain

$$\langle (2,3), x_2 + x_3 - \omega_2 - \omega_3 \rangle = \langle (2,3), (1, -\tfrac{1}{2}) \rangle > 0.$$

Thus, due to Lemma 1.1, the allocation x is proper contractual.

Ending, let us apply the characterization of perfect contractual allocation elaborated above and show once more that studied allocation x is not perfect contractual. We saw above that web $\{x - y, y - \omega\}$ is proper and, therefore, virtual for the web $\{x - \omega\}$ where $y = (y_1, y_2, y_3)$ is such that

$$y_1 = (\tfrac{7}{4}, \tfrac{2}{3}), \quad y_2 = (\tfrac{11}{8}, \tfrac{55}{24}), \quad y_3 = (\tfrac{7}{8}, \tfrac{25}{24}).$$

Consider coalition $S = \{1,2\}$ and check condition $p_S(x_1 + x_2) \geq p_S(y_1 + y_2)$ for $p_S \in \Gamma(x_1) \cap \Gamma(x_2)$. Normalizing p_S via condition $(p_S)_1 = 2$, find $3 \leq (p_S)_2 \leq 4\frac{2}{3}$. Now we can conclude

$$p_S(x_1 + x_2 - y_1 - y_2) = p_S(-3\varepsilon, \varepsilon) = -6\varepsilon + \varepsilon(p_S)_2 \leq -1\tfrac{1}{3}\varepsilon < 0, \quad \varepsilon = \tfrac{1}{24},$$

i.e., condition $p_S(x_1 + x_2) \geq p_S(y_1 + y_2)$ is *false* for every $0 \neq p_S \in \Gamma(x_1) \cap \Gamma(x_2)$, which is what we sought to prove. □

Next we turn to a comparative analysis of the proper contractual allocations and the fuzzy core allocations. Let us begin with a study of the specific properties of the fuzzy core allocations.

The definition of an allocation that cannot be dominated by the fuzzy coalitions (see relations (1.3) and (1.4)) implies that $x \in \mathcal{C}^f(\mathcal{E})$ is equivalent to[16]

$$0 \notin \text{co}[\bigcup_{\mathcal{I}} (\mathcal{P}_i(x_i) - \omega_i)],$$

[16] It is easy to see that domination by the fuzzy coalitions is equivalent to domination by the *normalized* coalitions corresponding to the weight coefficients of a convex combination, i.e., for a dominating coalition t one may always think that $\sum_{i \in \mathcal{I}} t_i = 1$.

which, by the separation theorem, implies that the elements of the fuzzy core are quasiequilibria. Below I propose a close but somewhat different description, given in "geometrical" terms. To this end, let us consider the sets

$$\Omega_i(x_i) = \mathrm{co}(\mathcal{P}_i(x_i) \cup \{\omega_i\}), \quad i \in \mathcal{I}.$$

Due to the convexity of $\mathcal{P}_i(x_i)$,

$$\mathrm{co}(\mathcal{P}_i(x_i) \cup \{\omega_i\}) = \bigcup_{0 \le \lambda \le 1} [\lambda \mathcal{P}_i(x_i) + (1-\lambda)\omega_i] = \bigcup_{0 \le \lambda \le 1} \lambda(\mathcal{P}_i(x_i) - \omega_i) + \omega_i, \quad i \in \mathcal{I}.$$

This implies that the condition $z + \omega \in \prod_{\mathcal{I}} \Omega_i(x_i)$, where $\omega = (\omega_1, \ldots, \omega_n)$, is equivalent to the existence of $0 \le \lambda_i \le 1$ and $y_i \in \mathcal{P}_i(x_i)$, $i \in \mathcal{I}$ such that

$$z = (\lambda_1(y_1 - \omega_1), \ldots, \lambda_n(y_n - \omega_n)).$$

Hence, due to (1.3), (1.4),

$$x \in \mathcal{C}^f(\mathcal{E}) \iff \nexists z \in E^{\mathcal{I}}, z \ne 0 : \ z + \omega \in \prod_{\mathcal{I}} \Omega_i(x_i) \ \ \& \ \sum_{i \in \mathcal{I}} z_i = 0$$

$$\iff \prod_{\mathcal{I}} \Omega_i(x_i) \bigcap \{(z_1, \ldots, z_n) \in E^{\mathcal{I}} \mid \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \omega_i\} = \{\omega\}.$$

In the case of a *2-agent economy*, this condition may be rewritten in the form

$$\Omega_1(x_1) \cap (\bar{\omega} - \Omega_2(\bar{\omega} - x_1)) = \{\omega_1\}, \quad \bar{\omega} = \omega_1 + \omega_2.$$

Moreover, in this case the fact that fuzzy coalition $(t_1, t_2) > 0$, $t_i \le 1$ dominates allocation (x_1, x_2) can be illustrated as follows. Consider the Edgeworth's box. Due to the definition of domination, in a nontrivial case it can occur only if $t_1 \ne 0 \,\&\, t_2 \ne 0$ and

$$\exists \ y_1, y_2 \in \mathbb{R}^2_+ \ : \ y_1 \succ_1 x_1, \ y_2 \succ_2 x_2 \ \ \& \ \ t_1(y_1 - \omega_1) = t_2(\omega_2 - y_2).$$

Let $z_2 = \bar{\omega} - y_2$ be the consumption bundle of the first agent when the second one consumes y_2. This vector represents y_2 in the natural coordinate system of the first agent's consumption bundle. Substituting z_2 in the right hand side of the last relation we obtain

$$t_1(y_1 - \omega_1) = t_2(\omega_2 - (\bar{\omega} - z_2)) = t_2(z_2 - \omega_1) \iff z_2 = \omega_1 + \frac{t_1}{t_2}(y_1 - \omega_1), \ t_2 \ne 0.$$

Geometrically, it means that the points y_1 and z_2 lie on one line with the point ω_1, and on one side of this line, with respect to ω_1 (i.e., they belong to a ray starting at ω_1). Moreover, due to the definition of domination, we have $y_1 \in \mathcal{P}_1(x_1)$ and $z_2 \in \bar{\omega} - \mathcal{P}_2(x_2)$. Hence,

$$(x_1, x_2) \notin \mathcal{C}^f(\mathcal{E}) \iff \exists \ \textit{ray} \text{ starting at the point } \omega_1, \text{ which intersects both sets, } \mathcal{P}_1(x_1) \text{ and } \bar{\omega} - \mathcal{P}_2(\bar{\omega} - x_1) = \widetilde{\mathcal{P}}_2(x_1).$$

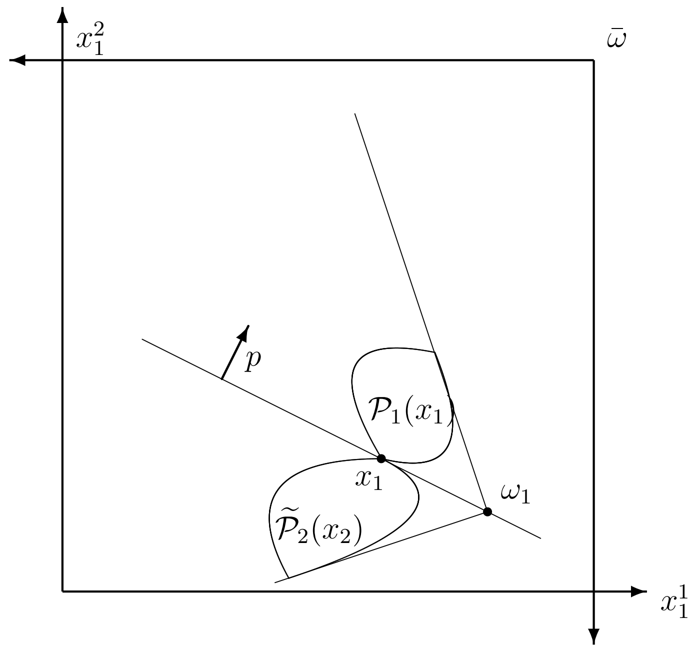

Figure 1.12: Fuzzy core

See Figure 1.12 for graphic illustration of the above analysis in the Edgeworth's box for a 2-goods economy. In this case, an allocation x lying in the fuzzy core is equivalent to the convex hulls of $\mathcal{P}_1(x_1) \cup \{\omega_1\}$ and of $[\bar{\omega} - \mathcal{P}_2(\bar{\omega} - x_1)] \cup \{\omega_1\}$ having only one point, ω_1, in common (alternatively, in the terms of cones with common vertex ω_1, which are going across the sets of all strictly preferred consumption bundles).

At this point, one may be curious to see in an Edgeworth's box illustration of Theorem 1.3 why the set of proper contractual allocations in the 2-fold replica economy $\mathcal{E}^2$ coincides with the set of (quasi)equilibrium allocations. To this end, let me show the abilities of domination by coalitions consisting of two agents of one type and of one agent of the other type. Let $v = x_1 - \omega_1 = \omega_2 - x_2$. By definition, the coalition $S' = \{(1,1),(2,1),(2,2)\}$ *dominates*[17] x if there exist $y_1 \in \mathcal{P}_1(x_1)$ and $y_2 \in \mathcal{P}_2(x_2)$ such that for some real $\lambda \in [0,1]$

$$y_1 - \omega_1 - \lambda v + 2(y_2 - \omega_2 + \lambda v) = 0 \implies y_1' + \frac{1}{2}(1+\lambda)v = \omega_2 - y_2, \quad y_1' = \frac{1}{2}(y_1 - x_1).$$

Let $\Upsilon = \frac{1}{2}(\mathcal{P}_1(x_1) - x_1)$; notice that it contains y_1'. Then the fact that the coalition S' *does not dominate* x can be written as

$$(\Upsilon + [\frac{1}{2}v, v]) \cap [\omega_2 - \mathcal{P}_2(\bar{\omega} - x_1)] = \emptyset.$$

Similarly, if coalition $S'' = \{(1,1),(1,2),(2,1)\}$ dominates x, one can find $y_1 \in \mathcal{P}_1(x_1)$ and $y_2 \in \mathcal{P}_2(x_2)$ such that for some real $\lambda \in [0,1]$

$$2(y_1 - \omega_1 - \lambda v) + y_2 - \omega_2 + \lambda v = 0 \implies 4y_1' + (2-\lambda)v = \omega_2 - y_2, \quad y_1' = \frac{1}{2}(y_1 - x_1).$$

Therefore, *the absence of domination* by S'' means

$$(4\Upsilon + [v, 2v]) \cap [\omega_2 - \mathcal{P}_2(\bar{\omega} - x_1)] = \emptyset.$$

[17]One may think without loss of generality that the second type agents consume the same bundle.

Notice that due to (**A**), we have $\Upsilon = \frac{1}{2}(\mathcal{P}_1(x_1) - x_1) \subset 2(\mathcal{P}_1(x_1) - x_1) = 4\Upsilon$. Hence, combining the above relation we arrive at the following condition, which is equivalent the fact that there is *no domination* by any of the two coalitions, S' and S'':

$$(\Upsilon + [\frac{1}{2}v, 2v]) \cap [\omega_2 - \mathcal{P}_2(\bar{\omega} - x_1)] = \emptyset. \tag{1.14}$$

Adding ω_1 to both sets being intersected, the latter can be rewritten as

$$(\Upsilon + [\omega_1 + \frac{1}{2}v, \omega_1 + 2v]) \cap [\bar{\omega} - \mathcal{P}_2(\bar{\omega} - x_1)] = \emptyset.$$

The second set in the intersection on the left-hand side is precisely the set of all second agent's strictly preferred consumption bundles in the coordinate system of the first agent. The last relation and the previous constructions can be illustrated in the Edgeworth's box, where $\widetilde{\mathcal{P}}_2(x_2) = \bar{\omega} - \mathcal{P}_2(\bar{\omega} - x_1)$ (see Figure 1.13). Moreover,

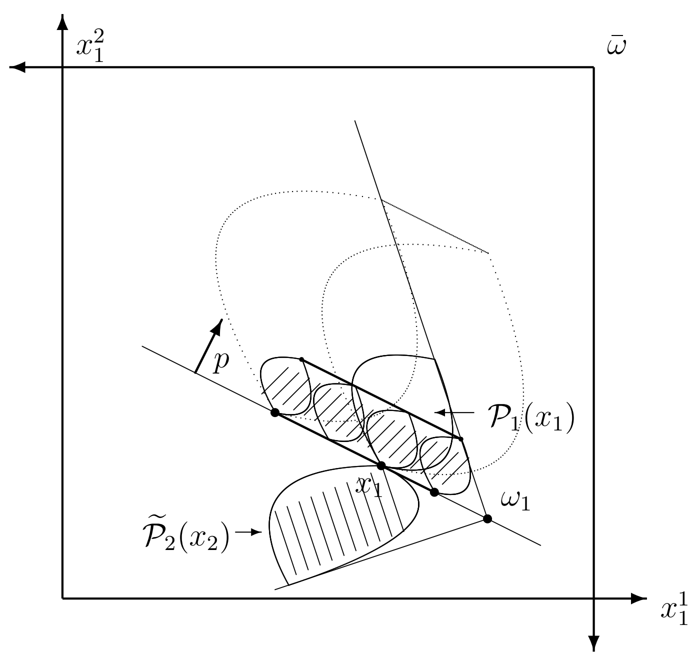

Figure 1.13: Proper-contractual in 2-fold replica

(1.14) provides an alternative way to see that under the conditions considered, a proper contractual allocation is a quasiequilibrium. Apply the separation theorem to the sets on the left-hand side of (1.14) to obtain a nonzero $p \in E'$ such that

$$\langle p, \Upsilon + [\frac{1}{2}v, 2v] \rangle \geq \langle p, [\omega_2 - \mathcal{P}_2(\bar{\omega} - x_1)] \rangle.$$

Next notice that $0 \in \operatorname{cl} \Upsilon$ and, in view of (**A**), $x_2 \in \operatorname{cl} \mathcal{P}_2(\bar{\omega} - x_1)$. Hence, due to the definition of $v = \omega_2 - x_2$, it follows that

$$\langle p, [\frac{1}{2}v, 2v] \rangle \geq pv \iff pv = 0 \iff px_i = p\omega_i, \ i = 1, 2.$$

The fact that p supports $\mathcal{P}_i(x_i)$ at the point x_i for $i = 1, 2$ can also be easily derived now.

Chapter 2

Contract-based incomplete markets

2.1 Incomplete market model

In the general framework of a pure exchange economy $\mathcal{E}$, let us consider a model with two periods $t = 0, 1$, in which there are l kinds of physically different (potentially) commodities available either today (with certainty) or tomorrow (contingent on each of a finite number s of possible future states of nature). So for this (market) economy, the *total* space of commodities E is associated with the space $\mathbb{R}^{l(s+1)}$. For convenience, we denote by $\sigma = 0$ the state of nature today. At each state $\sigma = 0, 1, \dots, s$, there is a spot market for each of the l commodities, whose price-vector is $p_\sigma \in \mathbb{R}^l$; at time 0, there exists also a financial market for k assets that deliver a random return across the states at $t = 1$. The price for j-s asset is represented by the value q_j and $q = (q_1, q_2, \dots, q_k)$ is the price-vector for assets. Let

$$\Pi = \{(p, q) \in \mathbb{R}^{l(s+1)} \times \mathbb{R}^k \mid \forall \sigma \ \ \|p_\sigma\| \leq 1, \ \|q\| \leq 1\}$$

denote the set of admissible prices for commodities and assets, the elements of which will be denoted by $\pi = (p, q)$. In a general setting, the asset structure is given by the map

$$\mathcal{A}(\cdot) = [a_j(\cdot)]_{j=1,\dots,k},$$

defined on $\mathbb{R}^{l(s+1)+k} \times X$; the image $\mathcal{A}(\pi, x)$ is a $(s \times k)$-matrix of which the j-th column vector $a_j(\pi, x)$ denotes, given p, q and x, the financial return of asset j across states of nature at period 1, denominated in units of account. In other words, the vector $a_j(\pi, x)$ is the promised monetary-valued payoff in all future states of nature associated with buying a unit of j-th asset. Now, if we denote

$$\lambda_j(x, \pi) = (-q_j, a_j(x, \pi)), \quad \Lambda = \begin{pmatrix} -q \\ \mathcal{A}(x, \pi) \end{pmatrix},$$

then the total transfer of wealth across different states of the world, which some agent can obtain from the market of assets with respect to his/her portfolio $z = (z^1, \dots, z^k)$ (trade program for assets), is described by the vector

$$\Lambda \cdot z = z^1 \begin{bmatrix} -q_1 \\ a_1(\pi, x) \end{bmatrix} + \dots + z^k \begin{bmatrix} -q_k \\ a_k(\pi, x) \end{bmatrix}.$$

In a general incomplete market setting, consumer i, being ordinary described by a consumption set $X_i \subset E$ and a preference correspondence $\mathcal{P}_i : X_i \rightrightarrows X_i$, is also in addition characterized by a portfolio set $Z_i \subset \mathbb{R}^k$ and vector-function

$$\alpha_i(\cdot) = (\alpha_i^\sigma(\cdot))_{\sigma=0}^{\sigma=s}, \quad \alpha_i^\sigma : \mathbb{R}^{l(s+1)+k} \times X \to \mathbb{R},$$

giving for each state the wealth of consumer i, given prices $p = (p_\sigma)_{\sigma=0}^{\sigma=s}$, q and actions of the other agents. Therefore each consumer i can choose the net of his/her consumption bundles under the following budget constraints, having the following vector-inequality form:

$$\mathrm{P}x_i \leq \alpha_i(\bar{x}, \pi) + \Lambda z_i, \quad x_i \in X_i, \quad z_i \in Z_i,$$

where the matrix

$$\mathrm{P} = \begin{bmatrix} p_0 & & 0 \\ & \ddots & \\ 0 & & p_s \end{bmatrix} = \begin{bmatrix} p_0 & \dots & 0 \\ \vdots & \mathrm{P}_1 & \\ 0 & & \end{bmatrix}$$

defines the consumption cost operator. Note that "squared product" (the standard notation $p \,\square\, x'$ denoting the vector $(p_\sigma \cdot x'_\sigma)_{\sigma=0}^{\sigma=s}$, $x' \in \mathbb{R}^{l(s+1)}$), commonly used in incomplete market theory, coincides with the ordinary matrix-vector product $\mathrm{P}x'$, $x' \in \mathbb{R}^{l(s+1)}$. The incomplete market model under study is also equipped with the vector of endowment $\bar{\omega} \in E$ of the whole economy or simply with the vectors of individualized consumers' initial endowments $\omega_i \in X_i$, $i \in \mathcal{I}$ and for this case I put $\bar{\omega} = \sum_{i\in\mathcal{I}} \omega_i$.

Now let us recall the equilibrium concepts applied in incomplete market theory.

Let us resume with $Z = \prod_{i\in\mathcal{I}} Z_i$, the list of data concerning the portfolio restrictions. Taking as given the actions of the other agents and a market system of commodity and asset prices, the budget set of i-th consumer is

$$B_i(p, q, x) = \{\ x'_i \in X_i \mid \exists z_i \in Z_i : \quad p \,\square\, x'_i \leq \alpha_i(p, x) + \Lambda(p, q, x) z_i\ \}.$$

Definition 2.1 *A financial Z-equilibrium is a pair of actions and admissible prices* $((\bar{x}_i, \bar{z}_i)_{i\in I}, (\bar{p}, \bar{q})) \in X \times Z \times \Pi$ *such that*

(i) for each $i \in \mathcal{I}$: $\bar{p} \,\square\, \bar{x}_i = \alpha_i(\bar{p}, \bar{x}) + \Lambda(\bar{p}, \bar{q}, \bar{x})\bar{z}_i$ *and*

$$\mathcal{P}_i(\bar{x}_i) \bigcap B_i(\bar{p}, \bar{q}, \bar{x}) = \emptyset,$$

(ii) $\sum_{i\in\mathcal{I}} \bar{x}_i = \bar{\omega}$ *and* $\sum_{i\in\mathcal{I}} \bar{z}_i = 0$.

Classically, (i) means that each $(\bar{x}_i, \bar{z}_i)$ is an optimal feasible budget plan for agent i, given $(\bar{p}, \bar{q}, \bar{x})$. Condition (ii) is a couple of market clearing conditions under the assumption that no production or intertemporal storage is possible[1] and assets are in zero net supply. In (ii), if $\sum_{i\in\mathcal{I}} \alpha_i(\bar{p}, \bar{x}) = \bar{p} \,\square\, \bar{\omega}$, the condition $\sum_{i\in\mathcal{I}} \bar{z}_i = 0$ is obviously redundant when the rank of $\Lambda(\bar{p}, \bar{q}, \bar{x})$ is equal to k.

[1] More exactly, informally intertemporal storage abilities are accumulated in agents' initial endowment vectors $\omega_i = (\omega_i^\sigma)$, and via it, in agents' profit functions.

The concept of consumption sets of the agents expresses the idea of sociological and physiological restrictions on consumption bundles, independently of any limitation of resources. A similar interpretation for portfolio sets seems more difficult and this is why economic theory is most interested in the particular case of *financial Z-equilibrium* in which there are *no restrictions on trade with assets.*

Definition 2.2 *A financial or GEI equilibrium is a pair of actions and admissible prices, representing financial Z-equilibrium for which* $Z_i = \mathbb{R}^k$ *for each* $i \in \mathcal{I}$.

Note that if $\sum_{i\in\mathcal{I}} \alpha_i(\bar{p},\bar{x}) = \bar{p}\,\Box\,\bar{\omega}$, even if the matrix $\Lambda(\bar{p},\bar{q},\bar{x})$ has a rank strictly less than k, the condition $\sum_{i\in\mathcal{I}} \bar{z}_i = 0$ is redundant in the following sense: by changing the portfolio of any one agent, it is easy to associate a financial equilibrium with any $((\bar{x}_i,\bar{z}_i)_{i\in\mathcal{I}},\bar{p},\bar{q})$ satisfying all the other conditions of Definition 2.2 but not necessarily $\sum_{i\in\mathcal{I}} \bar{z}_i = 0$.

There are three basic types of assets which are of practical significance and generally considered in the literature. The first one is described by *real* assets — the vectors: $a^1, a^2, \ldots, a^k \in \mathbb{R}^{ls}$, which as vector-columns form the matrix $\mathrm{A} = [a^j]_{j=1}^{j=k}$, i.e.,

$$\mathrm{A} = \begin{bmatrix} a_1^1 & a_1^2 & \ldots & a_1^k \\ \vdots & \vdots & \ddots & \vdots \\ a_s^1 & a_s^2 & \ldots & a_s^k \end{bmatrix}$$

— the $(sl \times k)$-matrix of commodity returns from assets which defines the matrix of financial returns across the future states of the world by formula

$$\mathcal{A}(x,p,q) = (p_\sigma \cdot a_\sigma^j)_{\substack{\sigma=1,\ldots,s \\ j=1,\ldots,k}}.$$

Note that the concept of financial equilibrium with real assets is inflation-proof, i.e., the changes of price levels (the type of its normalization) on future and present markets do not influence resource allocation (due to the homogeneity of budget constraints).

If a consumption bundle $e_\sigma \in \mathbb{R}^l$ is chosen as a unit of "numeraire" for $\sigma \geq 1$, *numeraire* assets are given by $a_\sigma^j = r_\sigma^j e_\sigma,\ r_\sigma^j \in \mathbb{R}$ and in this particular case of real assets, we have the second type of the matrix of returns from assets:

$$\mathcal{A}(x,p,q) = ((p_\sigma \cdot e_\sigma) r_\sigma^j)_{\substack{\sigma=1,\ldots,s \\ j=1,\ldots,k}}.$$

With purely financial securities, the matrix $\mathcal{A}(x,p,q)$ does not depend on $p,\ q$, and this is the third type of *nominal* assets.

It has to be clear that it makes sense to consider the notion of the core for an incomplete market economy with only real assets *and* so that the agents' profit functions are defined through the value of individualized vectors of initial endowments, i.e., for

$$\alpha_i^\sigma(p,q,x) = p_\sigma \omega_i^\sigma, \quad \omega_i = (\omega_i^0, \ldots, \omega_i^s)$$

on the domain of $\alpha_i^\sigma(\cdot)$ and for all $i,\ \sigma$. Note that in this case the i-th consumer budget constraints have the form

$$\mathrm{P}x_i \leq \mathrm{P}\omega_i + \begin{pmatrix} -\mathrm{q} \\ \mathrm{P}_1\mathrm{A} \end{pmatrix} z_i, \quad x_i \in X_i, \quad z_i \in \mathbb{R}^k, \tag{2.1}$$

where the matrix

$$\mathrm{P}_1 = \begin{bmatrix} p_1 & & 0 \\ & \ddots & \\ 0 & & p_s \end{bmatrix}$$

defines the consumption cost operator for future events, i.e., for $t = 1$. This matrix is the submatrix of P, which is formed by the rows $\sigma = 1$ to $\sigma = s$ and omitting the first l zero columns. Clearly in this considered case, we have $\mathcal{A}(x, p, q) = \mathrm{P}_1\mathrm{A}$.

As a result this model under study may be written in the following short form:

$$\mathcal{E}^{in} = \langle \mathcal{I}, E, (X_i, \mathcal{P}_i, \omega_i)_{i\in\mathcal{I}}, A\rangle.$$

The reader can find more on incomplete market theory in Geanakoplos (1990), Magill and Shafer (1991).

Being imposed for $\mathcal{E}^{in}$, given the above assumptions **(A)** and **(C)**, we can also assume that every X_i is "rectangular" over the states of the world, i.e., $X_i = \prod_{\sigma=0}^{s} X_i^\sigma$, $X_i^\sigma \subset E_\sigma$ and

(S) *Preferences are local nonsatiated in each spot market, i.e., for every σ and each $i \in \mathcal{I}$*

$$x_i^\sigma \in \overline{\mathcal{P}_i(x_i^\sigma, x_i^{-\sigma}) \cap E_\sigma}, \quad \forall\, x_i = (x_i^\sigma, x_i^{-\sigma}) \in \mathcal{A}_{X_i}(\mathcal{E}),$$

holds, where $x_i^{-\sigma} = (x_i^0, \ldots, x_i^{\sigma-1}, x_i^{\sigma+1}, \ldots, x_i^s)$ is a fragment-vector of x_i, complementing x_i^σ to x_i, and E_σ is a subspace of E, related with the event σ; $\mathcal{A}_{X_i}(\mathcal{E})$ denotes the projection of all feasible allocations on X_i.

2.2 Contractual approach in incomplete markets

In framework of this incomplete market model $\mathcal{E}^{in}$ next I consider the model of contractual economy, for which we define the set $\mathcal{W}$ of all permissible contracts as follows

$$\mathcal{W} = \mathcal{W}^{in} = \bigcup_{\sigma=0}^{\sigma=s} \mathcal{V}_\sigma,$$

where $\mathcal{V}_\sigma \subset E^{\mathcal{I}}$ are some subspaces of $E^{\mathcal{I}}$ corresponding to markets in all possible states of the world. More exactly these subspaces are defined by

$$\mathcal{V}_\sigma = \{v \in E^{\mathcal{I}} \mid v_i^m = 0,\ \forall\, m \neq \sigma,\ m = 0, \ldots, s,\ \forall\, i \in \mathcal{I}\}$$

for all $\sigma = 1, \ldots, s$, and for the present $(t = 0)$ I put

$$\mathcal{V}_0 = \{v \in E^{\mathcal{I}} \mid \exists\, z_i \in \mathbb{R}^k :\ v_i^\sigma = A_\sigma z_i,\ \forall\, i \in \mathcal{I},\ \forall\, \sigma = 1, \ldots, s\},$$

where A_σ are the submatrices of matrix A corresponding the future states of the world, i.e., $A_\sigma = (a_\sigma^j)_{j=1,\ldots,k}$. Note that if I were to apply incomplete market with portfolio constraints, then I would change the previous formula, and in addition require that $z_i \in Z_i$. Also from the last definition one can easily check that $v \in \mathcal{V}_0$ is a contract, i.e., $\sum_{i\in\mathcal{I}} v_i = 0$, if and only if there are exist such $z_i \in \mathbb{R}^k$, $i \in \mathcal{I}$, that $\sum_{i\in\mathcal{I}} z_i = 0$ and $v_i^\sigma = A_\sigma z_i$ holds for every $i \in \mathcal{I}$ and $\sigma = 1, \ldots, s$. This allows us, for the convenience of below considerations, to transit to the initial incomplete market terms and to work with portfolios for assets instead of with deviations of goods. Namely, to avoid misunderstanding, I shall apply the following specific notion of contract for the present, — this is the couple $w = (v, z)$, such that $v = (v_i)_{i\in\mathcal{I}} \in E^{\mathcal{I}}$, $z = (z_i)_{i\in\mathcal{I}} \in (\mathbb{R}^k)^{\mathcal{I}}$ and

$$\sum_{i\in\mathcal{I}} v_i = 0, \quad \sum_{i\in\mathcal{I}} z_i = 0 \ \ \& \ \ v_i^{\sigma} = 0, \quad \forall\, i \in \mathcal{I}, \quad \forall\, \sigma = 1, \dots, s$$

hold. Also, for convenience of notation, I will identify the contract $v \in \mathcal{V}_\sigma$ for $\sigma \geq 1$, which formally belongs to the space $E^{\mathcal{I}}$ with the vector v^σ from $(\mathbb{R}^l)^{\mathcal{I}}$.

It follows from above that for incomplete markets the contracts are classified according to the state of the world they belong and therefore each web V may be represented by the form

$$V = \bigcup_{\sigma=1}^{s} V^\sigma \bigcup W,$$

where V^σ is the set of all contracts relative to the state $\sigma \neq 0$, and W is the set of contracts in the present. The structure of permissible contracts in an incomplete market is presented in Figure 2.1.

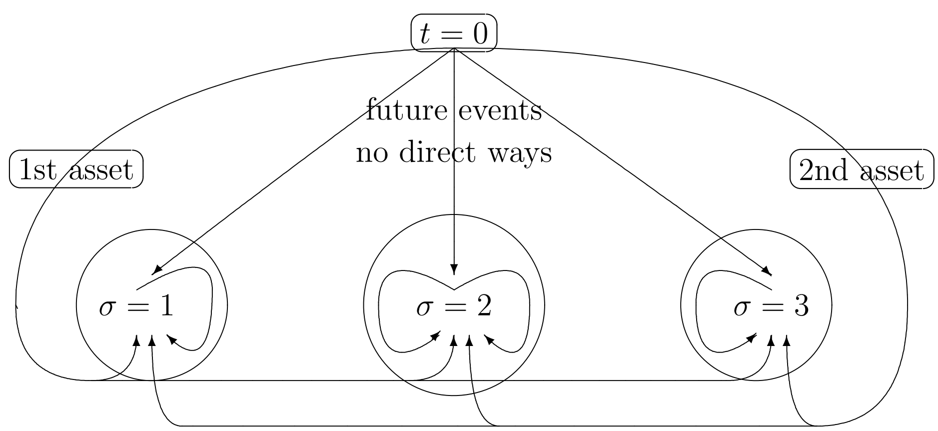

Figure 2.1: A contracts' structure for incomplete market

Let a web V be given. Then by definition consumer i's consumption bundle y_i corresponding to this web satisfies

$$\begin{aligned} y_i^0(V) &= \omega_i^0 + \sum_{(u,z)\in W} u_i^0, \\ y_i^\sigma(V) &= \omega_i^\sigma + \sum_{v\in V^\sigma} v_i^\sigma + \sum_{(u,z)\in W} A_\sigma z_i, \ \sigma = 1, \dots, s. \end{aligned} \tag{2.2}$$

Now, if we denote

$$\Delta_i^0(V) = \Delta_i^0(W) = \sum_{(u,z)\in W} u_i^0,$$

$$\Delta z_i = \Delta z_i(W) = \sum_{(u,z)\in W} z_i, \quad \Delta_i^\sigma(V^\sigma) = \sum_{v\in V^\sigma} v_i^\sigma, \quad \sigma = 1, \dots, s$$

and put $\Delta_i^\sigma(V) = \Delta_i^\sigma(V^\sigma) + A_\sigma \Delta z_i$ for $\sigma \geq 1$, then the total deviation of agent i's initial endowments may be represented by the vector

$$\Delta_i(V) = (\Delta_i^0(W), \Delta_i^1(V^1) + A_1\Delta z_i, \ldots, \Delta_i^s(V^s) + A_s\Delta z_i),$$

which by definition is the i's fragment-vector of the total deviation of initial allocation ω. Now relations (2.2) may be rewritten in the form

$$\begin{array}{l} y_i^0(V) = \omega_i^0 + \Delta_i^0(W), \\ y_i^\sigma(V) = \omega_i^\sigma + \Delta_i^\sigma(V^\sigma) + A_\sigma \Delta z_i, \ \sigma = 1, \ldots, s. \end{array} \tag{2.3}$$

The definition of a set of all permissible contracts $\mathcal{W}^{in}$ and also the rules of operating with webs, described in section 1, imply the following properties for breaking and signing contracts in an incomplete market:

- the agents can break any contracts;
- for given event $\sigma = 1, \ldots, s$ the agents can sign new contracts – commodity exchanges at this event;
- for event $\sigma = 0$ (i.e. in the present) they can sign new contracts by assets and by the commodity exchanges for date $t = 0$; the agents can do it in a common regime, as well as in a separate style.

Thus the situation with breaking contracts and with signing new ones is *non-symmetrical*, since consumers can break any kind of contract, but they can sign only the contracts relative to a *fixed* state of the world. This non-symmetry appears due to specific incomplete market properties and fits with item (iii) of the definition of $F(V, T)$: the set of possible webs, that may be realized by a coalition T after breaking some contracts and signing new ones.

Now in the context of incomplete market let us consider some notions of contractual allocations given in section 1. To characterize the core and equilibrium allocations, we shall use two kinds of complex contractual allocations, also called first and second contractual allocations.

Definition 2.3 *Let* $V = \bigcup_{\sigma=1}^{s} V^\sigma \bigcup W$ *be a weakly stable web such that all contracts from* V^σ *are perfect for all* $\sigma \geq 1$*. The allocation* $x = x(V)$ *is called semi-perfect contractual if for every virtual* $U^\sigma \sim V^\sigma$, $U^\sigma \subseteq \mathcal{V}_\sigma$, $\sigma \geq 1$, *there is no* $S \subseteq \mathcal{I}$ *and* $\widehat{V} \subseteq \bigcup_{\sigma=1}^{s} U^\sigma$, *satisfying*

$$\text{supp}(\hat{v}) \subseteq S, \quad \forall\, \hat{v} \in \widehat{V} \tag{2.4}$$

and such that for $t = \sigma = 0$, *there is a contract* $w' = (u', z')$, $\text{supp}(w') \subseteq S$ *such that*

$$y_i(V') \succ_i x_i(V), \quad \forall\, i \in S$$

is true for web $V' = \{w'\} \cup \widehat{V}$.

Let $\mathcal{D}^{sp}(\mathcal{E}^{in})$ denote the set of all *semi-perfect contractual* allocations in $\mathcal{E}^{in}$.

In reference to Definition 2.3, note that allocation x is semi-perfect contractual if there exists such a weakly stable web realizing this allocation, in which all contracts for the future states of the world are proper and, moreover, every one of these contracts can be changed by any weakly equivalent (virtual) web without the loss of stability in the following sense. In the present, a coalition considers the ability to create an autonomous subeconomy. To do this it has to break all contracts in the present and moreover, the coalition is forced *to break some contracts at every future event*, using a virtual equivalent web, such that *all contracts in which there is non-trivial exchange with some non-members of the coalition* are broken. The condition (2.4) realizes this requirement. When this deal is realized, the coalition can sign some new contract in the present. In so doing the breaking of given contracts and the signing of a new one is considered to be *simultaneous* procedure. Of course the fact that a coalition can use virtual contracts for future events is very important. Notice only that these contracts and the property of contracts from V^σ being perfect, one has to consider relative to contracts from $\mathcal{V}^\sigma$ — only in the limits of this set of permissible contracts one may pass to (weakly) equivalent contracts. Let us also be reminded that the fact that we apply perfect contracts for future events implies that there is no coalition which is able to increase its members' utility by breaking a part of equivalent contracts and the signing a new one relative to *any given future state of the world*. This is why this ability is not considered in the semi-perfect contractual allocation definition directly, but of course it is taken into account in further considerations.

It has to be clear that as in the general case, the stability of considered kind can increase if for a given fixed event, one changes the group of contracts in the web by their sum.[2] It follows that the subsystem of all contracts in the present, W, can be changed by $\sum_{w\in W} w$ and, therefore, condition (2.4) and the other requirements of Definition 2.3 have to be fulfilled, subject to breaking *the only contract* $w' = \sum_{w\in W} w$.

Definition 2.4 *Let* $V = \bigcup\limits_{\sigma=1}^{s} V^\sigma \bigcup W$ *be a weakly contractual web such that for* $\sigma \geq 1$ *all contracts from* V^σ *are perfect and (for* $\sigma = 0$*) all contracts from* W *are proper. Then the complex contractual allocation* $x = x(V)$ *is called proper-perfect contractual, if for every virtual* $U^\sigma \sim V^\sigma$, $\sigma \geq 1$, *and for every partition* $\widetilde{W} \simeq W$ *there is no* $S \subseteq \mathcal{I}$ *and* $\widehat{V} \subseteq \bigcup\limits_{\sigma=1}^{s} U^\sigma \bigcup \widetilde{W}$ *such that for* $t = \sigma = 0$ *there is a contract* $w = (u, z)$, $S(w) \subseteq S$, *such that*

$$y_i(V') \succ_i x_i(V), \quad \forall\, i \in S$$

is true for the web $V' = \{w\} \cup \widehat{V}$.

Denote $\mathcal{D}^{cp}(\mathcal{E}^{in})$ as the set of all *proper-perfect contractual* allocations from $\mathcal{E}^{in}$.

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[2] Note that it cannot be done for any subsystem of contracts, since one has to be sure the summed contract is permissible.

Notice that due to Definition 2.4, contracts in the present may be partially broken and, moreover, condition (2.4) is not imposed. Each of these differences increases the requirements for the stability of allocation. However remark (it will be clear later) that condition (2.4) does not play a special role in proper-perfect contractual allocations and may be added to the definition. The coalition of all agents, $\mathcal{I}$, plays the main role. This situation is similar to ordinary markets, where the most important thing is that the allocation be Pareto optimal and all contracts are proper. Thus a proper-perfect contractual allocation differs from a semi-perfect contractual allocation only in that we can partially break contracts in the present for the first kind of allocation, but for a semi-perfect contractual allocation non-proper contracts in the present are allowed and they may be broken only as a whole. In particular, note that $\mathcal{D}^{cp}(\mathcal{E}^{in})\subseteq\mathcal{D}^{sp}(\mathcal{E}^{in})$ is always true.

My analysis and the key properties of complex contractual allocations in incomplete markets is based on the following observation. Let some semi-perfect contractual allocation $\bar{x} \in \mathcal{D}^{sp}(\mathcal{E}^{in})$ be given. Consider and fix some event $\sigma \geq 1$ and fix consumption for other events. Further let us consider the reduced model $\mathcal{E}^{\sigma}$, the model in which only exchanges and the deviation of consumption bundles in state σ are allowed. If in this model one considers $\omega_i^\sigma + A_\sigma \Delta z_i(W)$ to be the vectors of agents' initial endowments, then one can transit to the standard exchange economy in which consumption sets are the appropriate sections of initial sets. Now if one presumes model $\mathcal{E}^{in}$ is smooth, then due to assumption $\bar{x}_i \in \operatorname{int}X_i$ and from the perfectness of contracts from V^σ and their upper stability in the reduced model, one can in a standard manner conclude that,[3] there is vector-price p_σ such that

$$p_\sigma = (p_\sigma^1, \dots, p_\sigma^l) = \lambda_i \mathrm{grad}_{|x_i^\sigma} u_i(\bar{x}_i), \quad p_\sigma \neq 0, \quad \lambda_i > 0, \quad i \in \mathcal{I}, \tag{2.5}$$

and $(\bar{x}_i^\sigma)_{i\in\mathcal{I}}$ is an equilibrium relative to x_i^σ and subject to fixed $\bar{x}_i^{-\sigma}$, where $\bar{x}_i^{-\sigma} = (\bar{x}_i^0, \dots, \bar{x}_i^{\sigma-1}, \bar{x}_i^{\sigma+1}, \dots, \bar{x}_i^s)$, and $\mathrm{grad}_{|x_i^\sigma} u_i(\bar{x}_i)$ denotes the subvector of the gradient of utility function, calculated at the point $\bar{x}_i$, and corresponding to the state $\sigma \geq 1$. Therefore, due to assumptions, the equalities in budget constraints are fulfilled for $\bar{x}_i^\sigma$, i.e.,

$$p_\sigma \bar{x}_i^\sigma = p_\sigma \omega_i^\sigma + p_\sigma A_\sigma z_i, \quad \sigma = 1, \dots, s,$$

for $z_i = \Delta z_i$ and each i. Now denote the total vector-price in future markets by

$$p^{\mathbf{1}} = (p_\sigma)_{\sigma=1}^{\sigma=s}, \quad p_\sigma \in \mathbb{R}^l, \quad \sigma \geq 1.$$

Define

$$H = H(p^{\mathbf{1}}) = \{x \in \mathbb{R}^{nl(s+1)} \mid \exists\, z \in \mathbb{R}^{nk} : \textstyle\sum_{i\in\mathcal{I}} z_i = 0 \quad \& \\ p_\sigma x_i^\sigma - p_\sigma \omega_i^\sigma = p_\sigma A_\sigma z_i, \ \forall\, \sigma = 1, \dots, s, \ \forall\, i \in \mathcal{I}\}.$$

By construction we have $\bar{x} \in H$. Now put

$$\mathcal{H}_i = \mathcal{H}_i(p^{\mathbf{1}}) = \{x_i \in \mathbb{R}^{l(s+1)} \mid \exists\, z_i \in \mathbb{R}^k : p_\sigma x_i^\sigma - p_\sigma \omega_i^\sigma = p_\sigma A_\sigma z_i, \ \sigma = 1, \dots, s\};$$

this is (in fact) the projection of subspace H onto a subspace corresponding to agent i's consumption bundles. Clearly,

$$\mathcal{H}_i = \mathcal{H} + \omega_i, \quad \mathcal{H} = \{y \in \mathbb{R}^{l(s+1)} \mid \exists\, z \in \mathbb{R}^k : p_\sigma y = p_\sigma A_\sigma z, \ \forall\, \sigma \geq 1\}. \tag{2.6}$$

[3]Note that for this assumption, **(S)** plays an important role.

takes place for all i.

The useful properties of incomplete market complex-contractual allocations (more exactly, for semi-perfect contractual and therefore for proper-perfect contractual) are stated in the following lemma, the proof of which is given in the next section.

Lemma 2.1 *Let $\mathcal{E}^{in}$ be a smooth incomplete market and $\bar{x} \in intX \cap \mathcal{D}^{sp}(\mathcal{E}^{in})$. Then*

(i) $\bar{x} \in H$ and after an appropriate breaking of contracts of goods and assets, the allocation does not leave space H,

(ii) $\bar{x}$ is not Pareto-dominated via an allocation from space H,

(iii) $\bar{x}$ is not Pareto-dominated via an allocation from space $E^{\sigma}_{\bar{x}} = \{y = (y_i)_{\mathcal{I}} \in E^{\mathcal{I}} \mid y_i^{-\sigma} = \bar{x}_i^{-\sigma},\ \forall\, i \in \mathcal{I}\},\ \ \forall\, \sigma \geq 0.$

The items (*ii*) and (*iii*) of this lemma and the above considerations induce the following terminology.

An allocation $x \in \mathcal{A}(\mathcal{X})$ is called *σ-Pareto optimal*, $\sigma = 0, \ldots, s$, if it is not Pareto dominated via an allocation $y \in \mathcal{A}(\mathcal{X})$ from the space

$$E^{\sigma}_{\bar{x}} = \{y = (y_i)_{\mathcal{I}} \in E^{\mathcal{I}} \mid y_i^{-\sigma} = \bar{x}_i^{-\sigma},\ \forall\, i \in \mathcal{I}\}.$$

An allocation, which is σ-Pareto optimal for every $\sigma \geq 0$, is called *partially Pareto optimal.*

Let $x = (x_\sigma)_{\sigma=0}^{\sigma=s} \in \mathcal{A}(\mathcal{X})$ be some σ-Pareto optimal allocation. The *nonzero* vector (functional) $p_\sigma \in \mathbb{R}^l$ is called *σ-Pareto prices* if

$$p_\sigma y_i^\sigma > p_\sigma \bar{x}_i^\sigma,\ \ \forall\, (y_i^\sigma, \bar{x}_i^{-\sigma}) \in \mathcal{P}_i(\bar{x}_i),\ \ \forall\, i \in \mathcal{I}. \tag{2.7}$$

Notice that for smooth preferences and if $x \in \operatorname{int} X$, relation (2.7) is equivalent to the existence of $\gamma_i^\sigma > 0$, satisfying

$$\operatorname{grad}_{|x_i^\sigma} u_i(x_i) = \gamma_i^\sigma p_\sigma,\ \ \forall\, i \in \mathcal{I}. \tag{2.8}$$

A collection of vectors $(p_\sigma)_{\sigma=0}^{\sigma=s}$, $p_\sigma \in \mathbb{R}^l$ is called (partial) *Pareto prices* if (2.7) is true for all $\sigma = 0, \ldots, s$.

An allocation from H is called *Pareto H-optimal* if it cannot be Pareto-dominated via another allocation from $H = H(p^{\mathbf{1}})$.[4] Using (2.6) in a standard manner, one can see that an allocation is Pareto H-optimal if and only if it cannot be Pareto-dominated via an allocation $y \in \mathcal{A}(\mathcal{X})$, for which $y - \omega \in \mathcal{H}^{\mathcal{I}}$, and this is the specific form of *constrained Pareto optimality.*

The next lemma, whose proof is given in the next section, states the key properties of H-optimal allocations.

[4]Notice that now prices $p^{\mathbf{1}}$ may not be partially Pareto optimal.

Lemma 2.2 *Let $\mathcal{E}^{in}$ be an incomplete market and $\bar{x} \in \operatorname{int} X$. Let $p^{\mathbf{1}} = (p_\sigma)_{\sigma=1}^{\sigma=s}$. Then $\bar{x} = (\bar{x}_i)_{\mathcal{I}} \in H(p^{\mathbf{1}})$ is Pareto $H(p^{\mathbf{1}})$-optimal if and only if the following property is true. Let $i_0 \in \mathcal{I}$ be an arbitrarily chosen and fixed agent. Then there exist such $\bar{p} = (\bar{p}_0, \bar{p}_1, \dots, \bar{p}_s)$ that $\bar{p}_\sigma \neq 0$ for all $\sigma = 0, \dots, s$ and*

$$\bar{p}y_i > \bar{p}\bar{x}_i \quad \forall\, y_i \in \mathcal{P}_i(\bar{x}_i) \;\mid\; \exists\, z_i \in \mathbb{R}^k \;:\; p_\sigma(y_i^\sigma - \omega_i^\sigma) = p_\sigma A_\sigma z_i,\ \forall\, \sigma \geq 1 \tag{2.9}$$

is true for all $i \in \mathcal{I}$, $i \neq i_0$. For i_0, a stronger property is true:

$$\bar{p}y_{i_0} > \bar{p}\bar{x}_{i_0}, \quad \forall\, y_{i_0} \in \mathcal{P}_{i_0}(\bar{x}_{i_0}). \tag{2.10}$$

Note that the analysis of the lemma proof shows that $\bar{x} \in \operatorname{int}X$ is essential just to obtain the strict inequality in relation (2.9). The next corollary gives us a convenient reformulation of Lemma 2.2 for a smooth case.

Corollary 2.1 *In Lemma 2.2 conditions, let us assume that $\mathcal{E}^{in}$ is smooth market and let $u_i(.)$ be a utility function for $i \in \mathcal{I}$. Then for allocation $\bar{x}$ to be Pareto $H(p^{\mathbf{1}})$-optimal, the following property is necessary and sufficient. Let*

$$\bar{p} = \operatorname{grad} u_{i_0}(\bar{x}_{i_0})$$

for some i_0. Then for all $i \neq i_0$ and each $\sigma \geq 1$, there exist real $\alpha_i > 0$ and some real λ_i^σ, $\sigma \geq 1$, such that

$$\operatorname{grad}_{|_{x_i^0}} u_i(\bar{x}_i) = \alpha_i \bar{p}_0,$$

$$\operatorname{grad}_{|_{x_i^\sigma}} u_i(\bar{x}_i) = \alpha_i \bar{p}_\sigma + \lambda_i^\sigma p_\sigma, \quad \forall\, \sigma \geq 1$$

hold and, moreover, $\sum_{\sigma=1}^{\sigma=s} \lambda_i^\sigma p_\sigma A_\sigma = 0$ is fulfilled.

Applying Lemma 2.2 and its corollary to a smooth economy, then $\bar{x} \in \operatorname{int}X$ is also Pareto optimal in each future market and (nonzero) spot prices satisfy $p_\sigma = \gamma_i^\sigma \operatorname{grad}_{|_{x_i^\sigma}} u_i(\bar{x}_i)$, $\gamma_i^\sigma > 0$, i.e., prices (uniquely) are derived from necessary optimal conditions (hence they are Pareto prices), one can immediately conclude

Corollary 2.2 *Let $\mathcal{E}^{in}$ be a smooth complete market and $\bar{x} \in \operatorname{int}X$. Suppose $\bar{x}$ is partially Pareto optimal and let $p^{\mathbf{1}} = (p_\sigma)_{\sigma=1}^{\sigma=s}$ be a bundle of σ-Pareto prices (i.e., (2.7) is true for $\sigma \geq 1$). Presume also that $\bar{x} = (\bar{x}_i)_{\mathcal{I}} \in H(p^{\mathbf{1}})$ and is Pareto $H(p^{\mathbf{1}})$-optimal. Then there exists a vector $\bar{p} = (\bar{p}_0, \bar{p}_1, \dots, \bar{p}_s)$ such that $\bar{p}_\sigma = \beta_\sigma p_\sigma$ for some $\beta_\sigma > 0$ and all $\sigma \geq 1$, so that for $\bar{q} = \sum_{\sigma=1}^{\sigma=s} \bar{p}_\sigma A_\sigma$ and every $i \in \mathcal{I}$*

$$\bar{p}_0 y_i^0 - \bar{p}_0 \omega_i^0 + \bar{q} z_i > \bar{p}_0 \bar{x}_i^0 - \bar{p}_0 \omega_i^0 + \bar{q}\bar{z}_i, \quad \forall\, y_i \in \mathcal{P}_i(\bar{x}_i) \tag{2.11}$$

is true for all $z_i, \bar{z}_i \in \mathbb{R}^k$, which satisfy

$$\bar{p}_\sigma(y_i^\sigma - \omega_i^\sigma) = \bar{p}_\sigma A_\sigma z_i \;\;\&\;\; \bar{p}_\sigma(\bar{x}_i^\sigma - \omega_i^\sigma) = \bar{p}_\sigma A_\sigma \bar{z}_i, \qquad \sigma = 1, \dots, s.$$

Proof of Corollary 2.2. To verify this corollary, first note that since $(p_\sigma)_{\sigma=1}^{\sigma=s}$ is a bundle of σ-Pareto prices, in corollary conditions (2.8) is true. Now let us take vector $\bar{p} = \operatorname{grad} u_{i_0}(x_{i_0})$ for $i_0 \in \mathcal{I}$ from the statement of Lemma 2.2 and via (2.8) put $\beta_\sigma = \gamma_{i_0}^\sigma > 0$. Now it is easy to see that in these lemma conditions we have $\mathcal{H}_i(p^{\mathbf{1}}) = \mathcal{H}_i(\bar{p}^{\mathbf{1}})$, i.e., in the right-hand side of (2.9) one can equivalently change vector p_σ by $\bar{p}_\sigma$ for all $\sigma \geq 1$. Now rewrite the inequality from the left hand side of (2.9) in the form $\bar{p}y_i - \bar{p}\omega_i > \bar{p}\bar{x}_i - \bar{p}\omega_i$ and substitute the following representations:

$$\sum_{\sigma=1}^{s}(\bar{p}y_i^\sigma - \bar{p}\omega_i^\sigma) = \sum_{\sigma=1}^{s}\bar{p}_\sigma A_\sigma z_i = \bar{q}z_i \ \ \& \ \ \sum_{\sigma=1}^{s}(\bar{p}\bar{x}_i^\sigma - \bar{p}\omega_i^\sigma) = \sum_{\sigma=1}^{s}\bar{p}_\sigma A_\sigma \bar{z}_i = \bar{q}\bar{z}_i.$$

This proves the result. □

The next theorem presents one of the most meaningful results of this paper. This theorem states the equivalence between proper-perfect contractual allocations of an incomplete market and GEI-equilibria.

Theorem 2.1 *Let $\mathcal{E}^{in}$ be a smooth incomplete market. Then*

$$\operatorname{int} X \cap \mathcal{D}^{cp}(\mathcal{E}^{in}) = W(\mathcal{E}^{in}) \cap \operatorname{int} X$$

holds, where $\mathcal{D}^{cp}(\mathcal{E}^{in})$ denotes the set of all proper-perfect contractual allocations and $W(\mathcal{E}^{in})$ is the set of GEI-equilibrium allocations.

Using the individual rationality property of equilibrium allocations one directly yields

Corollary 2.3 *If $\mathcal{E}^{in}$ is a smooth incomplete market, and $\overline{\mathcal{P}}_i(\omega_i) \subset \operatorname{int} X_i$ for all $i \in \mathcal{I}$, then*

$$\mathcal{D}^{cp}(\mathcal{E}^{in}) = W(\mathcal{E}^{in}),$$

i.e, an allocation is a GEI-equilibrium if and only if this allocation is proper-perfect contractual.

Further let us transit to the analysis of the core concept for incomplete markets. Let us identify, by definition, an allocation from the core of $\mathcal{E}^{in}$ with semi-perfect contractual allocation by Definition 2.3, i.e., put

$$\mathcal{C}(\mathcal{E}^{in}) = \mathcal{D}^{sp}(\mathcal{E}^{in}).$$

Below the main properties of $\mathcal{C}(\mathcal{E}^{in})$ are investigated and, in particular, it is shown that under some assumptions, which are not too strong in the context of incomplete market theory, the set $\mathcal{C}(\mathcal{E}^{in})$ fits with the ordinary notion of core as soon as the market becomes complete.

Consider $p^{\mathbf{1}} = (p_\sigma)_{\sigma=1}^{\sigma=s}$, $p_\sigma \in \mathbb{R}^l$, $\sigma = 1, \ldots, s$: some fixed vector of prices in the spot markets of the future states of the world.

Definition 2.5 *An allocation $x = (x_1, \ldots, x_n) \in X$ is called* $p^{\mathbf{1}}$-feasible *if there are portfolios $z = (z_1, \ldots, z_n)$, $z_i \in \mathbb{R}^k$, $\sum_{i\in\mathcal{I}} z_i = 0$ such that equalities*

$$p_\sigma x_i^\sigma = p_\sigma \omega_i^\sigma + p_\sigma A^\sigma z_i, \quad \forall\, i \in \mathcal{I},\ \forall\, \sigma = 1, \ldots, s$$

hold.

The definition of a $p^{\mathbf{1}}$-*feasible* allocation $x \in X$ can be written in an equivalent form:

$$\mathrm{P}_1(x_i^{\mathbf{1}} - \omega_i^{\mathbf{1}}) \in \mathcal{L}(\mathrm{P}_1\mathrm{A}), \quad i \in \mathcal{I}, \tag{2.12}$$

where $\mathcal{L}(\mathrm{P}_1\mathrm{A})$ is the linear hull of the vector-columns of the matrix of returns in future markets from assets $\mathrm{P}_1\mathrm{A}$ under prices $p^{\mathbf{1}}$.

Analogously one can define the notion of a $p^{\mathbf{1}}$-feasible allocation for an arbitrary (nonempty) coalition $S \subset \mathcal{I}$, substituting in Definition 2.5 the set $\mathcal{I}$ by means of S.

Let us denote by $\mathcal{A}_{p^{\mathbf{1}}}(S)$ (or by $\mathcal{A}_p(S)$) the set of all $p^{\mathbf{1}}$-feasible via coalition S allocations. Note that the set $\mathcal{A}_{p^{\mathbf{1}}}(S) \neq \emptyset$ for every $S \subset \mathcal{I}$ since the vector of initial endowments $\omega^S = (\omega_i)_{i \in S}$ always belongs to $\mathcal{A}_{p^{\mathbf{1}}}(S)$. Moreover, it has to be clear from the above definitions that $\mathcal{A}_{p^{\mathbf{1}}}(\mathcal{I}) = H(p^{\mathbf{1}}) \cap X$.

Definition 2.6 *p-core is the set* $C_{\mathbf{p}}(\mathcal{E}^{in})$ *of all* $p^{\mathbf{1}}$*-feasible allocations which cannot be dominated via coalitions, i.e.,*

$$x \in C_{\mathbf{p}}(\mathcal{E}^{in}) \iff x \in \mathcal{A}_p(\mathcal{I}) \ \& \ \nexists S \subset \mathcal{I} : \exists y \in \mathcal{A}_p(S) \ | \ y_i \succ_i x_i \ \forall i \in S.$$

Let agents' preferences be defined via utility functions, (which are presumed to be concave and continuous), and let price-vector $p^{\mathbf{1}}$ for future markets be fixed. Then for an incomplete market one can put into correspondence some cooperative game with non-transferable utility (to be short, a NTU-game). Recall that formally a NTU-game (for details, see Moulin (1988), for example) is a couple, $(\mathcal{I}, (V(S))_{S \subseteq \mathcal{I}})$, described by the set of players (agents) $\mathcal{I} = \{1, \ldots, n\}$, $(n \geq 2)$ and the sets of permissible vector-payoffs $V(S) \subseteq \mathbb{R}^S$ for every (nonempty) coalition $S \subset \mathcal{I}$, which have to satisfy the following properties:

- $V(S)$ is the nonempty closed subset in $\mathbb{R}^S$,
- $V(S)$ is comprehensive from below, i.e., $x \in V(S)$ and $y \leq x$ imply $y \in V(S)$,
- the set of all individual-rational vector-payoffs from $V(S)$, is by definition the set
$$Q(S) := \{v \in V(S) \ | \ v_i \geq V(\{i\}) \ \forall i \in S\},$$
which is nonempty and bounded from above in $\mathbb{R}^S$.

In our case the set of all permissible vector-payoffs for coalition S is determined by formula

$$V_p(S) = \bigcup_{x \in \mathcal{A}_p(S)} V_p^x(S),$$

where

$$V_p^x(S) = \{(v_i)_{i \in S} \leq (u_i(x_i))_{i \in S} \ | \ (x_i)_{i \in S} \in \mathcal{A}_p(S)\}.$$

Clearly that the sets $V_p(S)$ satisfy all the above described necessary conditions, it can be checked easily due to the compactness of $\mathcal{A}(\mathcal{X})$ and the continuity of utilities in the initial incomplete market model.

Recall that the family $\mathcal{B}$ of subsets in $\mathcal{I}$ is said to be *balanced*, if for every $S \in \mathcal{B}$ there is a real $\lambda_S \geq 0$, such that

$$\sum_{S \in \mathcal{B}: i \in S} \lambda_S = 1 \ \ \forall i \in \mathcal{I}$$

holds or, in an equivalent form,

$$\sum_{S\in\mathcal{B}} \lambda_S e_S = e_{\mathcal{I}}$$

takes place where, by definition, $e_S \in \mathbb{R}^{\mathcal{I}}$ is such a vector that $e_S^i = 1$ for $i \in S$ and $e_S^i = 0$ if $i \notin S$, i.e, this is the indicator-function of the set S.

A game $(\mathcal{I}, (V(S))_{S\subset\mathcal{I}})$ is said to be *balanced* if for every balanced family of coalitions $\mathcal{B}$

$$\bigcap_{S\in\mathcal{B}} \mathrm{pr}_{|_S}^{-1}(V(S)) \subseteq V(\mathcal{I}).$$

Here $\mathrm{pr}_{|_S}(.)$ is the projection map of space $\mathbb{R}^{\mathcal{I}}$ onto $\mathbb{R}^S$.

The famous Scarf's theorem states that the core of a balanced game $(\mathcal{I}, (V(S))_{S\subset\mathcal{I}})$ is nonempty. Applying this theorem and using standard arguments, one can prove the following

Proposition 2.1 *Let $\mathcal{A}(\mathcal{X})$ be a compact and agents' preferences be defined via concave continuous utility functions. Then $C_{\mathbf{p}}(\mathcal{E}^{in}) \neq \emptyset$.*

The next lemma presents a convenient tool for the study of an incomplete market core.

Lemma 2.3 *Let $\mathcal{E}^{in}$ be a smooth economy and $x \in \mathrm{int}X$. Then $x \in \mathcal{C}(\mathcal{E}^{in})$ if and only if*

(i) x is a partially Pareto optimal allocation, i.e., for every $\sigma \geq 0$ it cannot be dominated via allocations from $E_x^\sigma = \{y = (y_i)_{\mathcal{I}} \in E^{\mathcal{I}} \mid y_i^{-\sigma} = x_i^{-\sigma}\ \forall\, i \in \mathcal{I}\}$, and

(ii) $x \in C_{\mathbf{p}}(\mathcal{E}^{in})$, where $(p_\sigma)_{\sigma=1}^{\sigma=s}$ is a bundle of σ-Pareto prices that corresponds to item (i).

Note that if in addition $\overline{\mathcal{P}_i(\omega_i)} \subset \mathrm{int}X_i$ for all $i \in \mathcal{I}$ is true for the model, then in the previous lemma the assumption $x \in \mathrm{int}X$ can be omitted. So in this case this lemma gives the full description of set $\mathcal{C}(\mathcal{E}^{in})$.

In considerations and results immediately below, I always presume incomplete market $\mathcal{E}^{in}$ satisfies the strict monotonicity assumption for every spot market as follows:

(M) *For some $i \in \mathcal{I}$ and every $x \in \mathcal{A}(\mathcal{X})$*

$$(\{x_i\} + E^+) \setminus \{x_i\} \subset \mathcal{P}_i(x_i)^5$$

holds.

Let us call a market (i.e., model $\mathcal{E}^{in}$) *complete relative to prices* $p^{\mathbf{1}} = (p_\sigma)_{\sigma=1}^{\sigma=s}$ if the rank of matrix $\mathrm{P_1A}$ is equal to s, the total number of possible future states of the world.

[5] Remember symbol E denotes the *commodity space* of the economy, where $E = \mathbb{R}^{l(s+1)}$ and $E^+ = \mathbb{R}_+^{l(s+1)}$.

A market is *complete* if it is compete relative to every bundle of spot prices $p_1, \dots, p_s \in \mathbb{R}^l$ such that $p_\sigma \gg 0, \quad \sigma = 1, \dots, s$.

Clearly, this property of completeness (uniform relative to $p^{\mathbf{1}} \gg 0$) is a kind of restriction for matrix A, more exactly for financial markets of assets, the number of which under this hypothesis has to be not less then s.[6] An example of an incomplete market, satisfying this completeness assumption, is the above described market of *numeraire* assets for $e_\sigma > 0$,[7] $\sigma \geq 1$, in which the matrix $\mathrm{R} = (r_\sigma^j)_{\substack{\sigma=1,\dots,s \\ j=1,\dots,k}}$ has rank equal to s.

Slightly strengthening the assumptions of the model, we arrive at the description of the incomplete market core in familiar terms when the model is compete. This is stated the following important corollary of Lemma 2.3.

Corollary 2.4 *Let $\mathcal{E}^{in}$ be a* complete *smooth economy satisfying* (**M**). *Then*

$$\mathrm{int}X \cap \mathcal{C}(\mathcal{E}^{in}) = \bigcup_{p^{\mathbf{1}} \gg 0} (C_{\mathbf{p}}(\mathcal{E}^{in}) \cap \mathrm{int}X)$$

takes place. If, in addition, $\mathcal{E}^{in}$ is such that $\overline{\mathcal{P}_i(\omega_i)} \subset \mathrm{int}X_i$ for all $i \in \mathcal{I}$, then

$$\mathcal{C}(\mathcal{E}^{in}) = \bigcup_{p^{\mathbf{1}} \gg 0} C_{\mathbf{p}}(\mathcal{E}^{in}).$$

Proof of Corollary 2.4. Applying Lemma 2.3 on the side of necessity for $x \in \mathrm{int}X \cap \mathcal{C}(\mathcal{E}^{in})$, due to (i) one can conclude that the allocation x is partially Pareto optimal. Therefore via assumptions (**S**) and (**M**) there exists (and unique) σ-Pareto prices $\bar{p}^{\mathbf{1}} = (\bar{p}_\sigma)_{\sigma=1}^{\sigma=s}$, which satisfy $\bar{p}^{\mathbf{1}} \gg 0$. Now applying item (ii) of Lemma 2.3, we obtain

$$x \in C_{\bar{\mathbf{p}}}(\mathcal{E}^{in}) \subset \bigcup_{p^{\mathbf{1}} \gg 0} C_{\mathbf{p}}(\mathcal{E}^{in}).$$

To prove the inverse inclusion for *complete* model $\mathcal{E}^{in}$, let us chose any $x \in C_{\hat{\mathbf{p}}}(\mathcal{E}^{in}) \cap \mathrm{int}X$ for fixed $\hat{p}^{\mathbf{1}} \gg 0$. Next let us note that due to the completeness of $\mathcal{E}^{in}$ for every $p^{\mathbf{1}} \gg 0$ the next system of linear equations

$$\mathrm{P}_1 \hat{x}_i^{\mathbf{1}} = \mathrm{P}_1 \omega_i^{\mathbf{1}} + \mathrm{P}_1 \mathrm{A} \hat{z}_i \tag{2.13}$$

has a solution relative to $\hat{z}_i$ and other parameters of any kind. Note that these solutions satisfy $\sum_{i \in \mathcal{I}} \hat{z}_i = 0$ when $(\hat{x}_i)_{i \in \mathcal{I}}$ is feasible; in (2.13) instead of $\mathrm{P}_1\mathrm{A}$, one can take any square non-singular submatrix whose dimension is $s \times s$. Therefore *every* feasible $(\hat{x}_i)_{\mathcal{I}}$ is $p^{\mathbf{1}}$-feasible for *every* $p^{\mathbf{1}} \gg 0$. Therefore the condition $x \in C_{\hat{\mathbf{p}}}(\mathcal{E}^{in}) \cap \mathrm{int}X$ for $\hat{p}^{\mathbf{1}} \gg 0$ implies that x is Pareto optimal, which entails its partial Pareto optimality. From this, via (**S**) and (**M**), we can conclude the existence of (unique up to normalization) σ-Pareto prices $\bar{p}^{\mathbf{1}} = (\bar{p}_\sigma)_{\sigma=1}^{\sigma=s}$, which also satisfy $\bar{p}^{\mathbf{1}} \gg 0$. Now having in mind the application of Lemma 2.3 in the part of

[6] Of course this condition cannot be sufficient.

[7] This is a consumption bundle $e_\sigma \in \mathbb{R}^l$ chosen as a unit of “numeraire” for assets and for future spot market σ.

sufficiency, we have to show only that $x \in C_{\bar{\mathbf{p}}}(\mathcal{E}^{in})$. Let $y \in \mathcal{A}_{\bar{p}}(S)$ for $S \subset \mathcal{I}$. Once again, using the completeness of the market, we can conclude that system (2.13) may be solved with respect to $\hat{z}_i$ for all $i \in S$ when one substitutes y_i for $\hat{x}_i$ and after the substitution of $p^{\mathbf{1}}$ by $\hat{p}^{\mathbf{1}}$. Thus we obtain $\mathcal{A}_{\bar{p}}(S) \subset \mathcal{A}_{\hat{p}}(S)$ and, using $x \in C_{\hat{\mathbf{p}}}(\mathcal{E}^{in})$, may conclude that coalition S cannot dominate allocation x under prices $\bar{p}$. Now the application of Lemma 2.3 finishes the proof. □

The characterization of an incomplete market core for *complete* exchange economies gives the following

Theorem 2.2 *Let* $\mathcal{E}^{in}$ *be a smooth economy satisfying* (**M**) *such that* $\overline{\mathcal{P}_i(\omega_i)} \subset \operatorname{int} X_i$ *for each* $i \in \mathcal{I}$. *If* $\mathcal{E}^{in}$ *is* complete, *then* $\mathcal{C}(\mathcal{E}^{in}) = \mathcal{C}(\mathcal{E})$.

The asymptotic analog of Theorem 2.1 continues the analysis of the incomplete core concept. This result is expressed in the form of a replicated incomplete market that better fits with the classical representation of perfect competition conditions. The proof of this result is based on the reducing of the problem to the study of domination via fuzzy coalitions with the succeeding fuzzy core consideration. Of course the concept of fuzzy core has to be adopted into incomplete markets in a proper way. In what follows, the problem is reduced to the separation theorem being applied to separate some convex set from zero (zero cannot belong the set due to the fuzzy core property). In so doing my analysis is essentially based on the characteristic Lemma 2.3 and on the fact that rational numbers are dense in the set of all real ones.

An incomplete market replica of volume $r \in \mathbb{N}$ is called the economy $\mathcal{E}_r^{in}$, in which r exact copies of each consumer from initial model $\mathcal{E}^{in}$ is put into correspondence in $\mathcal{E}_r^{in}$. The agents from $\mathcal{E}_r^{in}$ are numbered by double index (i, m), $i \in \mathcal{I}$, $m = 1, \ldots, r$, and it is put $X_{im} = X_i$, $\omega_{im} = \omega_i$. Agents' preferences are defined and take values in X_{im} due to identification $\mathcal{P}_{im} = \mathcal{P}_i$. An assets structure for a replica exactly repeats the structure of the initial model. To an initial economy $\mathcal{E}^{in}$ allocation $x = (x_i)_{\mathcal{I}}$, we can put into correspondence the replicated economy allocation $x^r = (x_{im}^r)$ by the rule $x_{im} = x_i$, $\forall\, i, m$.

Definition 2.7 *An allocation* x *is called a GEI-Edgeworth equilibrium or incomplete market Edgeworth equilibrium if* $x^r \in \mathcal{C}(\mathcal{E}_r^{in})$ *for every natural* $r = 1, 2, \ldots$

$\mathcal{C}^e(\mathcal{E}^{in})$ *denotes the set of all Edgeworth equilibria for model* $\mathcal{E}^{in}$.

Next let us consider the most characteristic properties of the Edgeworth equilibria. This analysis is convenient to realize under assumptions for which Lemma 2.3 is true. So let $\mathcal{E}^{in}$ be a smooth economy and $x \in \operatorname{int} X$. Then due to Lemma 2.3, the property $x \in \mathcal{C}^e(\mathcal{E}^{in})$ is equivalent to the facts that allocation x is partially Pareto optimal and for partial Pareto prices $p^{\mathbf{1}} = (p_\sigma)_{\sigma \geq 1}$ the allocation belongs to the p-core of $\mathcal{E}_r^{in}$ for every natural r. Consider the last requirement in more detail. It is very important that a domination is admitted via *any coalitions and via any inter-coalition allocation*.

Presume that for some r a coalition $S \subseteq \mathcal{I} \times \{1, \ldots, r\}$ dominates the allocation x^r. Let $\mathcal{I}(S) \subseteq \mathcal{I}$ be the set of all agent types non-trivially presented in the coalition

S. Due to p-core specification, this domination means that for every $(i,m) \in S$ there is such $y_{im} \in \mathcal{P}_i(x_i)$ that for some $z_{im} \in \mathbb{R}^k$ and each $i \in \mathcal{I}(S)$

$$\mathrm{P}_1 y_{im} = \mathrm{P}_1 \omega_i + \mathrm{P}_1 \mathrm{A} z_{im}, \quad \forall\, m \,:\, (i,m) \in S$$

holds and in addition

$$\sum_{(i,m)\in S} y_{im} = \sum_{(i,m)\in S} \omega_{im}$$

takes place. Now if we "average out" the dominating consumption bundles and portfolios for each given type of agents, i.e., if we put

$$y_i = (\sum_{m|(i,m)\in S} y_{im})/s_i \quad \& \quad z_i = (\sum_{m|(i,m)\in S} z_{im})/s_i \quad \forall\, i \in \mathcal{I}(S),$$

where s_i is the number of elements (capacity) in the set $S^i = \{m \mid (i,m) \in S\}$ (we have $i \in \mathcal{I}(S) \iff S^i \neq \emptyset$), then former equalities yield

$$\mathrm{P}_1 y_i = \mathrm{P}_1 \omega_i + \mathrm{P}_1 \mathrm{A} z_i \quad \& \quad \sum_{\mathcal{I}(S)} s_i y_i = \sum_{\mathcal{I}(S)} s_i \omega_i.$$

Since $\mathcal{P}_i(x_i)$ is a convex set, we also obtain $y_i \in \mathcal{P}_i(x_i)$ for all $i \in \mathcal{I}(S)$. Next define a vector $t = (t_1, \ldots, t_n)$ by putting

$$t_i = s_i/r,\ i \in \mathcal{I}(S) \ \ \& \ \ t_i = 0,\ i \in \mathcal{I} \setminus \mathcal{I}(S).$$

Now it has to be clear that in the previous equality *natural* numbers s_i can be equivalently substituted by *rational* t_i. Moreover, under imposed assumptions the described logical chain can be inverted, i.e., one can show the sufficiency of described properties for some partially Pareto optimal allocation to be dominated via a coalition in some replica. Resuming the described arguments we are going to a fuzzy core concept for incomplete markets, which is described below.

Recall that any n-dimension vector $t = (t_1, \ldots, t_n) \neq 0, \ \ 0 \leq t_i \leq 1 \ \ \forall\, i \in \mathcal{I}$ is said to be a fuzzy coalition. Let $p^{\mathbf{1}} = (p_\sigma)_{\sigma \geq 1}$ be some fixed bundle of spot prices for future states of the world. Introduce now the notion of fuzzy p-domination.

A fuzzy coalition t is called p-dominating $p^{\mathbf{1}}$-feasible allocation $x \in \mathcal{A}_p(\mathcal{I})$, if there is $y^t \in \prod_{i\in\mathcal{I}} X_i$ such that

$$\sum_{i\in\mathcal{I}} t_i y_i^t = \sum_{i\in\mathcal{I}} t_i \omega_i \tag{2.14}$$

and

$$y_i^t \succ_i x_i \ \& \ \exists\, z_i \in \mathbb{R}^k \,:\, \mathrm{P}_1 y_i^{t\mathbf{1}} = \mathrm{P}_1 \omega_i^{\mathbf{1}} + \mathrm{P}_1 \mathrm{A} z_i \ \forall\, i \in \mathrm{supp}(t) = \{i \in \mathcal{I} \mid t_i > 0\} \tag{2.15}$$

holds.

Notice that if $\mathcal{E}^{in}$ is a smooth economy and allocation $x \in \mathrm{int}\, X$, then the fact $x \notin \mathcal{C}^e(\mathcal{E}^{in})$ is equivalent to the ability of its p-domination via some fuzzy coalition with *rational* components relative to a *partial Pareto prices*, corresponding to this allocation.

Definition 2.8 *The set $C^f_{\mathbf{p}}(\mathcal{E}^{in})$ of all $p^{\mathbf{1}}$-feasible allocations $x \in \mathcal{A}_p(\mathcal{I})$, for which there is no p-dominating fuzzy coalition, is called a* fuzzy p-core.

In accordance with this definition the concept of p-fuzzy core differs from ordinary requirements only in the right-hand side of (2.15), where the potential financial marketability of consumption bundles relative to the given prices is additionally required. If, moreover, one requires these prices to be partial Pareto, then one achieves the notion of incomplete market fuzzy core.

Definition 2.9 *The fuzzy core is the set $\mathcal{C}^f(\mathcal{E}^{in})$ of all feasible allocations satisfying the following properties:*

(i) x is partial optimal by Pareto, i.e., for every $\sigma \geq 0$ the allocation cannot be dominated by Pareto via an allocation from subspace $E^{\sigma}_x = \{y = (y_i)_{\mathcal{I}} \in E^{\mathcal{I}} \mid y_i^{-\sigma} = x_i^{-\sigma},\ \forall\, i \in \mathcal{I}\}$,

(ii) $x \in \mathcal{A}_p(\mathcal{I})$, i.e., it is $p^{\mathbf{1}}$-feasible, where $p^{\mathbf{1}} = (p_\sigma)_{\sigma=1}^{\sigma=s}$ is a bundle of σ-Pareto prices, existing due to item (i),

(iii) $x \in C^f_{\mathbf{p}}(\mathcal{E}^{in})$, i.e., it belongs to the fuzzy p-core of an incomplete market.

The next lemma states the key properties of a fuzzy p-core.

Lemma 2.4 *Let $p^{\mathbf{1}} = (p_\sigma)_{\sigma \geq 1}$ be a bundle of spot prices for future events and x is a $p^{\mathbf{1}}$-feasible allocation. Let $x \in C^f_{\mathbf{p}}(\mathcal{E}^{in})$ and $x_{i_0} \in \operatorname{int} X_{i_0}$ for some i_0. Then there is a vector $\bar{p} = (\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_s)$, such that $\bar{p}_\sigma \neq 0$ for all $\sigma \geq 0$ and*

$$\bar{p}y_i \geq \bar{p}\omega_i, \quad \forall\, y_i \in \mathcal{P}_i(\bar{x}_i) \mid \exists\, z_i \in \mathbb{R}^k \,:\, p_\sigma(y_i^\sigma - \omega_i^\sigma) = p_\sigma A_\sigma z_i,\ \forall\, \sigma \geq 1 \qquad (2.16)$$

is true for all $i \in \mathcal{I}$. Moreover, stronger property

$$\bar{p}y_{i_0} > \bar{p}\omega_{i_0}, \quad \forall\, y_{i_0} \in \mathcal{P}_{i_0}(\bar{x}_{i_0}) \qquad (2.17)$$

is true for agent i_0.

It is useful to compare the statement of this lemma with the statement of Lemma 2.2. The first difference is that the inequalities in Lemma 2.4 are non-strict. The second one is that in the right-hand side of the inequalities, the value of initial endowments is applied. One can find similarities between of these facts with classical market case, when Pareto optimality is compared with quasiequilibrium. Further notice that applying the local non-satiation assumption for agents' preferences in the present ($\sigma = 0$ and (**S**)) and passing to limits in the left-hand side of inequalities from (2.16), one can state $\bar{p}x_i \geq \bar{p}\omega_i \ \ \forall\, i \in \mathcal{I}$, that due to $\sum_{i\in\mathcal{I}} x_i = \sum_{i\in\mathcal{I}} \omega_i$ eventually yields

$$\bar{p}x_i = \bar{p}\omega_i \ \ \forall\, i \in \mathcal{I}.$$

Finally, let $x \in \operatorname{int} X$, the economy is smooth and prices $p^{\mathbf{1}} = (p_\sigma)_{\sigma \geq 1}$ are partially Pareto optimal. Then the first, non-strict inequalities from the left side of (2.16) are turned into strict ones. Therefore now the conditions of Lemma 2.2 and its Corollary 2.2 are true. This is why, due to similar arguments applied in Corollary 2.2 proof, one can state the following

Corollary 2.5 *Let $\mathcal{E}^{in}$ be a smooth incomplete market and $\bar{x} \in \mathrm{int}X \cap \mathcal{C}^f(\mathcal{E}^{in})$. Then there is a vector $\bar{p} = (\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_s)$ such that $\bar{p}_\sigma \neq 0$ for all $\sigma \geq 0$, and for $\bar{q} = \sum_{\sigma=1}^{\sigma=s} \bar{p}_\sigma A_\sigma$ and each $i \in \mathcal{I}$*

$$\bar{p}_0 y_i^0 > \bar{p}_0 \omega_i^0 + \bar{q} z_i \;\forall\, y_i \in \mathcal{P}_i(\bar{x}_i) \mid \;\exists\, z_i \in \mathbb{R}^k \;:\; \bar{p}_\sigma y_i^\sigma = \bar{p}_\sigma \omega_i^\sigma + \bar{p}_\sigma A_\sigma z_i \;\forall\, \sigma \geq 1 \quad (2.18)$$

is true.

Applying this corollary one can easy prove the asymptotic theorem, which states that every incomplete market Edgeworth equilibrium is a GEI-equilibrium.

Theorem 2.3 *Let $\mathcal{E}^{in}$ be a smooth incomplete market. Then*

$$\mathcal{C}^f(\mathcal{E}^{in}) = \mathcal{C}^e(\mathcal{E}^{in}) \quad \& \quad \mathrm{int}X \cap \mathcal{C}^e(\mathcal{E}^{in}) = W(\mathcal{E}^{in}) \cap \mathrm{int}X.$$

Proof of Theorem 2.3. First let us show $\mathcal{C}^e(\mathcal{E}^{in}) = \mathcal{C}^f(\mathcal{E}^{in})$. To do it let us state the inclusion $\mathcal{C}^e(\mathcal{E}^{in}) \subseteq \mathcal{C}^f(\mathcal{E}^{in})$ (the inverse inclusion is true due to definitions). Assuming contrary find an allocation $x \in \mathcal{C}^e(\mathcal{E}^{in})$, which is dominated via a fuzzy coalition $t \neq 0$. By definition this means the existence of $y^t \in \prod_{\mathcal{I}} X_i$, satisfying relations (2.14) and (2.15). We can show then that the allocation x is dominated via fuzzy coalition $q = (q_1, \ldots, q_n)$ with *rational* components q_i, $i \in \mathcal{I}$. With this in mind for $t_i > 0$, put

$$x_i' = (t_i/q_i) y_i + (1 - t_i/q_i)\omega_i \implies q_i(x_i' - \omega_i) = t_i(y_i - \omega_i),$$

where *rational* q_i satisfies the condition $t_i \leq q_i \leq 1$, and for $t_i = 0$ define $q_i = 0$ and $x_i' = y_i^t$. Since $\omega_i \in X_i$, then $x' = (x_i')_{\mathcal{I}} \in \prod_{\mathcal{I}} X_i$ and

$$\sum_{i \in \mathcal{I}} q_i(x_i' - \omega_i) = 0.$$

However due to (**A**), the scalars q_i can be chosen in such a way that $x_i' \in \mathcal{P}_i(x)$ is true for all i, satisfying $q_i > 0$. Moreover, for these i

$$\exists\, z_i' \in \mathbb{R}^k \;:\; \mathrm{P}_1 x_i'^{\mathbf{1}} = \mathrm{P}_1 \omega_i^{\mathbf{1}} + \mathrm{P}_1 \mathrm{A} z_i'$$

holds relative to σ-Pareto prices, corresponding to x, as soon as similar relations are true for y^t (put $z_i' = \frac{q_i}{t_i} z_i$). We obtain a contradiction with the choice of $x \in \mathcal{C}^e(\mathcal{E}^{in})$.

So, the coincidence of a fuzzy core with the set of Edgeworth equilibria has been proved for an incomplete market. Next let us apply Corollary 2.5 and using arguments fully equivalent to those described in the second part of Theorem 2.1, we can state $\mathrm{int}X \cap \mathcal{C}^f(\mathcal{E}^{in}) = W(\mathcal{E}^{in}) \cap \mathrm{int}X$. Theorem 2.3 is proved. □

In finishing this section, let us consider some incomplete market examples and describe our core concept in their context.

Example 2.1 (A MARKET WITH ONE ASSET) Let us consider an economic model with two consumers, two states of the world in the future and no present. Note, that the last feature (there is no present) is not an essential factor, since formally the present can always be added to the model description and moreover, to save the nonsatiation assumption, one can presume that agents are full antagonists in the present — let agents' preferences be separable and in the present let them be defined via linear monotonic and equal utility functions for $\sigma = 0$. Also in future events, $\sigma = 1, 2$, there are two commodities and let $x = (x^{\sigma=1}, x^{\sigma=2})$ correspond to the consumption of the 1st agent, but $y = (y^{\sigma=1}, y^{\sigma=2})$ be the consumption program for the 2nd one. Let $X_i = \mathbb{R}^4_+$, $i = 1, 2$, a total vector of initial endowments $\omega = (\omega_i)_{i=1,2} \in \mathbb{R}^8_+$ satisfy $\omega_i \gg 0$ $i = 1, 2$, and let utilities be described by functions

$$u_1(x) = \rho^1_{\sigma=1} U_1^{\sigma=1}(x^{\sigma=1}) + \rho^1_{\sigma=2} U_1^{\sigma=2}(x^{\sigma=2}),$$

$$u_2(y) = \rho^2_{\sigma=1} U_2^{\sigma=1}(y^{\sigma=1}) + \rho^2_{\sigma=2} U_2^{\sigma=2}(y^{\sigma=2}),$$

where for $i,\ \sigma = 1, 2$ real $\rho^i_\sigma > 0$, and U_i^σ are (logarithmed) Cobb–Douglas functions:

$$U_1^\sigma(z) = \alpha_\sigma \ln(z_1) + (1 - \alpha_\sigma) \ln(z_2), \quad 0 < \alpha_\sigma < 1,$$

$$U_2^\sigma(z) = \beta_\sigma \ln(z_1) + (1 - \beta_\sigma) \ln(z_2), \quad 0 < \beta_\sigma < 1.$$

The analysis of an incomplete market core will be based on key Lemma 2.3, which for Cobb–Douglas functions gives a complete description of core allocations. Now to apply item (i) of this lemma, we first need to give the constructive description of partially Pareto optimal allocations. As soon as utilities are separable relative to events, the σ-Pareto optimality of allocation (x, y) is completely determined by consumption bundles (x^σ, y^σ) for this event (in the general case it may depend on consumption at other events), i.e., by functions $U_1^\sigma(.)$, $U_2^\sigma(.)$ and via total initial endowments $\bar{\omega}^\sigma = \omega_1^\sigma + \omega_2^\sigma$. In other words, when utilities are separable, the set of partially Pareto optimal allocations may be represented as the Cart's product (by σ) of Pareto boundaries which correspond to spot markets. Therefore we first have to describe the Pareto boundary for a model reduced to σ.

Let us calculate in general form the Pareto boundary for a classical economy with Cobb–Douglas utility functions. Let there be two goods and two consumers, where as above x denotes the 1st agent consumption and y is the consumption of 2nd one. Due to individual rationality we are interested in allocations from the interior of consumption sets, i.e., $(x, y) \gg 0$. In such a case, for each Pareto optimal allocation one can put into correspondence (non-zero) price vector p, which has to be collinear to the gradients of the utility functions. This vector can be found unambiguously up to normalization; this is why the existence conditions of $p = (p_1, p_2) \gg 0$ and $\lambda > 0$ such that

$$p = \operatorname{grad} U_1 = (\tfrac{\alpha}{x_1}, \tfrac{1-\alpha}{x_2}) \iff x = (\tfrac{\alpha}{p_1}, \tfrac{1-\alpha}{p_2}) \ \&\ \langle p, x\rangle = 1,$$

$$p = \lambda \operatorname{grad} U_2 = \lambda(\tfrac{\beta}{y_1}, \tfrac{1-\beta}{y_2}) \iff y = \lambda(\tfrac{\beta}{p_1}, \tfrac{1-\beta}{p_2}) \ \&\ \langle p, y\rangle = \lambda,$$

are necessary and sufficient for allocation (x, y) to be Pareto optimal. Taking into account $x + y = \bar{\omega} = (\bar{\omega}^1, \bar{\omega}^2)$, from the right-hand side of last relations one can find

$$(\tfrac{\alpha}{p_1}, \tfrac{1-\alpha}{p_2}) + \lambda(\tfrac{\beta}{p_1}, \tfrac{1-\beta}{p_2}) = \bar{\omega} \implies p = (\tfrac{\alpha}{\bar{\omega}^1}, \tfrac{1-\alpha}{\bar{\omega}^2}) + \lambda(\tfrac{\beta}{\bar{\omega}^1}, \tfrac{1-\beta}{\bar{\omega}^2}).$$

Thus real $\lambda > 0$ parameterizes the Pareto boundary unambiguously (since x, y can be unambiguously found by p and λ). Moreover, it is clear that this analysis can be easily extended to a more general case, that is, for any (finite) number of goods and consumers. Notice only that then the number of (positive) parameters determining Pareto boundary is equal to the number of agents minus one.

Further let us turn to an incomplete economy and initially consider the case of a *unique* real asset. Let, for example, this asset a have the form

$$a = (a_{\sigma=1}, a_{\sigma=2}), \quad a_{\sigma=1} = -a_{\sigma=2} = (1, 0).$$

Then for given spot prices, financial returns matrix $\mathrm{P_1A}$ for trade portfolios has the form

$$\mathrm{P_1A} = \begin{pmatrix} \mathrm{p}^1_{\sigma=1} \\ \hline -\mathrm{p}^1_{\sigma=2} \end{pmatrix}.$$

Now let us turn to item (ii) from Lemma 2.3, which requires a current partial Pareto optimal allocation (x, y) to be $p^{\mathbf{1}}$-feasible and to belong to the p-core of the economy for partially Pareto prices p corresponding to this allocation.

To simplify further our analysis, let us assume without loss of generality that the total endowment of each commodity in every state of world is equal to 1, i.e., we put $\bar{\omega}_{\sigma=1} = \bar{\omega}_{\sigma=2} = (1, 1)$. Then partial Pareto optimality for prices means that for some $\lambda > 0$ and $\gamma > 0$ we have

$$p_{\sigma=1} = \left(\tfrac{\alpha_{\sigma=1}}{\bar{\omega}^1_{\sigma=1}}, \tfrac{1-\alpha_{\sigma=1}}{\bar{\omega}^2_{\sigma=1}}\right) + \lambda \left(\tfrac{\beta_{\sigma=1}}{\bar{\omega}^1_{\sigma=1}}, \tfrac{1-\beta_{\sigma=1}}{\bar{\omega}^2_{\sigma=1}}\right) \quad \overset{\bar{\omega}_\sigma=(1,1)}{\Longleftrightarrow}$$

$$p_{\sigma=1} = (\alpha_1 + \lambda\beta_1, 1 - \alpha_1 + \lambda(1 - \beta_1)), \tag{2.19}$$

$$p_{\sigma=2} = \left(\tfrac{\alpha_{\sigma=2}}{\bar{\omega}^1_{\sigma=2}}, \tfrac{1-\alpha_{\sigma=2}}{\bar{\omega}^2_{\sigma=2}}\right) + \gamma \left(\tfrac{\beta_{\sigma=2}}{\bar{\omega}^1_{\sigma=2}}, \tfrac{1-\beta_{\sigma=2}}{\bar{\omega}^2_{\sigma=2}}\right) \quad \overset{\bar{\omega}_\sigma=(1,1)}{\Longleftrightarrow}$$

$$p_{\sigma=2} = (\alpha_2 + \gamma\beta_2, 1 - \alpha_2 + \gamma(1 - \beta_2)). \tag{2.20}$$

The condition of $p^{\mathbf{1}}$-feasibility states that there is such real z that $\mathrm{P_1}x = \mathrm{P_1}\omega_1 + \mathrm{P_1A}z$ that via the structure of $\mathrm{P_1A}$ yields

$$p_{\sigma=1}x^{\sigma=1} = p_{\sigma=1}\omega_1^{\sigma=1} + p^1_{\sigma=1}z, \quad p_{\sigma=2}x^{\sigma=2} = p_{\sigma=2}\omega_1^{\sigma=2} - p^1_{\sigma=2}z.$$

This, due to $p_\sigma x^\sigma = 1$, is equivalent to

$$p^1_{\sigma=2} + p^1_{\sigma=1} = p^1_{\sigma=2}\langle p_{\sigma=1}, \omega_1^{\sigma=1}\rangle + p^1_{\sigma=1}\langle p_{\sigma=2}, \omega_1^{\sigma=2}\rangle.$$

Now applying (2.19), (2.20), we find

$$\begin{gathered} \alpha_2 + \gamma\beta_2 + \alpha_1 + \lambda\beta_1 = \\ = (\alpha_2 + \gamma\beta_2)\langle(\alpha_1 + \lambda\beta_1), 1 - \alpha_1 + \lambda(1 - \beta_1)), \omega_1^{\sigma=1}\rangle + \\ +(\alpha_1 + \lambda\beta_1)\langle(\alpha_2 + \gamma\beta_2, 1 - \alpha_2 + \gamma(1 - \beta_2)), \omega_1^{\sigma=2}\rangle. \end{gathered} \tag{2.21}$$

Thus an allocation (x, y) is $p^{\mathbf{1}}$-feasible if and only if, when determining parameters, $\lambda > 0$, $\gamma > 0$ satisfy equation (2.21).

Next let us study the property of an allocation (x, y) being dominated by no coalition. Since the list of coalitions contains only singleton and grand coalitions, then this property is equivalent to (*i*) $u_1(x) \geq u_1(\omega_1)$ & $u_2(y) \geq u_2(\omega_2)$,

(*ii*) the allocation (x, y) is Pareto $H(p^{\mathbf{1}})$-optimal relative to partial Pareto prices $p^{\mathbf{1}}$.

Condition (*ii*) requires subsequent analysis. To do this apply Corollary 2.1. In our context Corollary 2.1 states that an allocation is H-optimal iff there is such $\mu > 0$ that vector $\tilde{p} = \operatorname{grad} u_2(y) - \mu p$ satisfies

$$\tilde{p}_{\sigma=1} a_{\sigma=1} + \tilde{p}_{\sigma=2} a_{\sigma=2} = 0 \implies \tilde{p}^1_{\sigma=1} - \tilde{p}^1_{\sigma=2} = 0,$$

that due to the relationship between $\operatorname{grad} u_2(y)$ and p gives

$$(\frac{1}{\lambda} - \mu) p^1_{\sigma=1} - (\frac{1}{\gamma} - \mu) p^1_{\sigma=2} = 0 \iff \mu(p^1_{\sigma=2} - p^1_{\sigma=1}) = \frac{1}{\gamma} p^1_{\sigma=2} - \frac{1}{\lambda} p^1_{\sigma=1}.$$

The last relation is disintegrated into the following variants:

a) $p^1_{\sigma=2} = p^1_{\sigma=1} \implies \lambda = \gamma$,

b) $p^1_{\sigma=2} > p^1_{\sigma=1} \implies \lambda p^1_{\sigma=2} > \gamma p^1_{\sigma=1}$,

c) $p^1_{\sigma=2} < p^1_{\sigma=1} \implies \lambda p^1_{\sigma=2} < \gamma p^1_{\sigma=1}$.

So, taking into account relations (2.19) and (2.20), an allocation is H-optimal iff one of the below relations

$$\alpha_2 + \gamma\beta_2 = \alpha_1 + \lambda\beta_1 \;\;\&\;\; \lambda = \gamma, \tag{2.22}$$

$$\alpha_2 + \gamma\beta_2 > \alpha_1 + \lambda\beta_1 \;\;\&\;\; \lambda(\alpha_2 + \gamma\beta_2) > \gamma(\alpha_1 + \lambda\beta_1), \tag{2.23}$$

$$\alpha_2 + \gamma\beta_2 < \alpha_1 + \lambda\beta_1 \;\;\&\;\; \lambda(\alpha_2 + \gamma\beta_2) < \gamma(\alpha_1 + \lambda\beta_1) \tag{2.24}$$

is true.

Let us resume our given analysis. We supposed without loss of generality that $\bar{\omega}_{\sigma=1} = \bar{\omega}_{\sigma=2} = (1, 1)$. The allocations from the core are unambiguously determined via real parameters $\lambda > 0$, $\gamma > 0$, which have to satisfy (2.21) and one of the relations (2.22)–(2.24). Then the first agent's consumption is determined due to

$$x^{\sigma=1} = \left(\frac{\alpha_1}{p^1_{\sigma=1}}, \frac{1 - \alpha_1}{p^2_{\sigma=1}} \right), \quad x^{\sigma=2} = \left(\frac{\alpha_2}{p^1_{\sigma=2}}, \frac{1 - \alpha_2}{p^2_{\sigma=2}} \right),$$

where $p_{\sigma=1}$ and $p_{\sigma=2}$ are determined by λ, γ due to formulas (2.19), (2.20), and then the 2nd agent's consumption y is

$$y = (1, 1, 1, 1) - x.$$

Moreover, the following relations

$$u_1(x) \geq u_1(\omega_1) \;\;\&\;\; u_2(y) \geq u_2(\omega_2)$$

have to be true too. These requirements are not conflicting, since the equilibrium allocation, which does exist, satisfies all of them. In the general case, a core is represented as an image of all determining parameters, obtained as the intersection of some hyperbola defined by (2.21) and a set defined as the union of three sets, defined by (2.22)–(2.24). The properties of individual rationality of allocation have to be fulfilled in addition.

In conclusion let me say some words about the set that is determined via relations (2.22)–(2.24). It is clear that (2.22) can be true only for special parameters α_σ, β_σ, $\sigma = 1, 2$ (either both $\alpha_1 - \alpha_2$ and $\beta_1 - \beta_2$ are not zero simultaneously and have different signs, or $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ simultaneously), and in the general case, defines the empty or a singleton set. Constraints (2.23) and (2.24) are more involved. In fact, consider a straight line $\alpha_2 + \gamma\beta_2 = \alpha_1 + \lambda\beta_1$ and hyperbola $\lambda(\alpha_2 + \gamma\beta_2) = \gamma(\alpha_1 + \lambda\beta_1)$. The line intersects the positive orthant by some ray with a positive directing vector and separates the plane into two open half-planes, a left and right one. The hyperbola has asymptotes paralleled to coordinate aces which are intersected at point $(\lambda_0, \gamma_0) = (\frac{\alpha_1}{\beta_2 - \beta_1}, \frac{-\alpha_2}{\beta_2 - \beta_1})$. Since $(0, 0)$ satisfies the hyperbola equation, then a path going across this point, and only this path intersects the orthant. Moreover, note that the point of intersection of hyperbola and the line exactly corresponds to condition (2.22) (for $(\lambda, \gamma) \gg 0$). Next, let for example $\beta_2 - \beta_1 > 0$. Then the set determined by relation (2.23) can be described as the intersection of an open epigraph left hyperbola path with the left upper half-plane, defined by our line. Relation (2.24) is true at the points of the interior of a set, which supplements the epigraph of the left hyperbola path up to the positive orthant, being intersected with the right lower half-plane. The union of these two sets with the point of intersection of hyperbola and line completely describes the collection of all points $(\lambda, \gamma) \gg 0$, satisfying conditions (2.22)–(2.24). The case $\beta_2 - \beta_1 < 0$ is considered in a similar way. However now the point of hyperbola asymptotes intersection has a negative first component and only the right path of the hyperbola intersects the orthant's interior (it goes across the origin). This is why the set we are interested in is represented as a union of three sets. The fist one is the intersection of the right open half-plane, defined due to a line with a part of the orthant's interior restricted by the right hyperbola path (a part of epigraph). The second one is the intersection of the left open half-plane with a part of the orthant's interior, from which one has to remove the subgraph of the right hyperbola path. Finally, one needs to add the point of hyperbola intersection with the line if it does exist. □

Further let us consider a more complex incomplete market example, in which utilities are described in the same manner as in Example 2.1; however, there are two real assets. The particular case of this market is known in literature as Hart's example, in which *GEI*-equilibrium may not exist.

Example 2.2 (Hart's example) In the context of the economy described in Example 2.1, let us consider a financial market with *two* assets having the following structure. Let

$$a^1 = (a^1_{\sigma=1}, a^1_{\sigma=2}), \quad a^1_{\sigma=1} = a^1_{\sigma=2} = (1, 0)$$

be the first asset and let

$$a^2 = (a^2_{\sigma=1}, a^2_{\sigma=2}), \quad a^2_{\sigma=1} = a^2_{\sigma=2} = (0, 1)$$

be the second one. Thus the buying of the 1st asset unit promises the delivery of a unit of commodity 1 for the future (at every event). Analogous delivery of the second asset is a unit of the 2nd commodity for every future event. As a whole the matrix A of real returns has the form

$$\mathrm{A} = \left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] \begin{array}{l} = \mathrm{A}_{\sigma=1} \\ \\ = \mathrm{A}_{\sigma=2} \end{array}$$

From this one can find the matrix of financial returns $\mathrm{P}_1\mathrm{A}$ for the trade portfolios of the financial sector relative to given prices $p^{\mathbf{1}}$ for spot markets:

$$\mathrm{P}_1\mathrm{A} = \left[\begin{array}{c|c} p_{\sigma=1}\begin{pmatrix}1\\0\end{pmatrix} & p_{\sigma=1}\begin{pmatrix}0\\1\end{pmatrix} \\ \hline p_{\sigma=2}\begin{pmatrix}1\\0\end{pmatrix} & p_{\sigma=2}\begin{pmatrix}0\\1\end{pmatrix} \end{array}\right] = \left[\begin{array}{c|c} p^1_{\sigma=1} & p^2_{\sigma=1} \\ \hline p^1_{\sigma=2} & p^2_{\sigma=2} \end{array}\right] = \left[\begin{array}{c} p_{\sigma=1} \\ \hline p_{\sigma=2} \end{array}\right].$$

Similar to Example 2.1, the analysis of the core is based on Lemma 2.3. Due to item (i) of this lemma, a partial Pareto prices $p^{\mathbf{1}} = (p_\sigma)_{\sigma=1,2}$ may be put into correspondence to every core allocation (unambiguously). Now in view of Example 2.1 analysis for some partially Pareto prices, conveniently normed by (2.19) and (2.20) (here $\bar{\omega}_{\sigma=1} = \bar{\omega}_{\sigma=2} = (1,1)$ without loss of generality), we obtain

$$\mathrm{P}_1\mathrm{A} = \left[\begin{array}{c|c} \alpha_1 + \lambda\beta_1 & 1 - \alpha_1 + \lambda(1-\beta_1) \\ \hline \alpha_2 + \gamma\beta_2 & 1 - \alpha_2 + \gamma(1-\beta_2) \end{array}\right],$$

where $\lambda > 0$ and $\gamma > 0$ are some real parameters, which unambiguously determine the partial Pareto boundary. Due to item (ii) of Lemma 2.3 in order to current partially Pareto optimal allocation to be an element of the core it is also necessary (and sufficient) that the allocation be $p^{\mathbf{1}}$-feasible and be an element of p-core for its partial Pareto prices $p^{\mathbf{1}}$. The condition of $p^{\mathbf{1}}$-feasibility says that there is such *vector* $z = (z_1, z_2)$, that $\mathrm{P}_1 x = \mathrm{P}_1\omega_1 + \mathrm{P}_1\mathrm{A}z$. This, for the chosen normalization of prices (it implies $p_{\sigma=1}x_{\sigma=1} = p_{\sigma=2}x_{\sigma=2} = 1$), is equivalent to the system of linear equations

$$\mathrm{P}_1\mathrm{A}z = \begin{pmatrix}1\\1\end{pmatrix} - \mathrm{P}_1\omega_1, \tag{2.25}$$

which has a solution relative to z. First note that if square matrix $\mathrm{P}_1\mathrm{A}$ is non-degenerated, then a solution of system (2.25) does exist for every right-hand side, and therefore for a given one. On the contrary, if the matrix columns are linear dependent (degeneration), then a solution of the system may exist only if there is a solution for the system with only one unknown variable, where instead of matrix $\mathrm{P}_1\mathrm{A}$ one can take a matrix consisting of one (any) column of the initial matrix with the same right-hand side. In other words, if the matrix is degenerated, then we obtain a model with the only asset. Thus for non-degenerated matrix $\mathrm{P}_1\mathrm{A}$, the condition of $p^{\mathbf{1}}$-feasibility is true automatically, for the degenerated one it is not the case and it turns to be a non-trivial condition. A matrix $\mathrm{P}_1\mathrm{A}$ is degenerated if and only if its determinant is zero, i.e., $\det(\mathrm{P}_1\mathrm{A}) = 0 \iff$

$$(\alpha_1 + \lambda\beta_1)(1 - \alpha_2 + \gamma(1-\beta_2)) = (\alpha_2 + \gamma\beta_2)(1 - \alpha_1 + \lambda(1-\beta_1)). \tag{2.26}$$

In view of (2.25) the condition of $p^{\mathbf{1}}$-feasibility takes place only if for some real z

$$1 = p_{\sigma=1}\omega_1^{\sigma=1} + p^1_{\sigma=1}z, \quad 1 = p_{\sigma=2}\omega_1^{\sigma=2} + p^1_{\sigma=2}z$$

is true, which is equivalent to

$$p^1_{\sigma=2} - p^1_{\sigma=1} = p^1_{\sigma=2}\langle p_{\sigma=1}, \omega_1^{\sigma=1}\rangle - p^1_{\sigma=1}\langle p_{\sigma=2}, \omega_1^{\sigma=2}\rangle.$$

Now applying (2.19) and (2.20), this equation may be standardly rewritten as an equation relative to $\lambda > 0$ and $\gamma > 0$, similar to (2.21), which I omitted. It is important that this relation has to be fulfilled together with (2.26). Moreover, due to the fact that the allocation has to belong to the p-core, it is necessary to require the following relations to be true:

(i) $u_1(x) \geq u_1(\omega_1)$ & $u_2(y) \geq u_2(\omega_2)$,

(ii) allocation (x, y) is Pareto $H(p^{\mathbf{1}})$-optimal for partial Pareto prices $p^{\mathbf{1}}$.

The analysis of matrix $\mathrm{P_1A}$ shows that condition (ii) is always true since it is reduced to the existence of $\mu > 0$, which satisfies relation $\mu(p^1_{\sigma=2} + p^1_{\sigma=1}) = \frac{1}{\gamma}p^1_{\sigma=2} + \frac{1}{\lambda}p^1_{\sigma=1}$. Therefore only $p^{\mathbf{1}}$-feasibility and individual rationality (i) conditions are essential.

Now let us turn to the case when (2.26) is false, i.e., the *matrix of financial returns* for partial Pareto prices is *non-degenerated*. In this case an allocation belongs to the incomplete core only if it belongs to the classical core, i.e., requirements (i) and (ii) are true, where (ii) is transformed into *ordinary optimality by Pareto*. The last requirement can be expressed in a standard way as the requirement of utilities' gradients to be collinear: due to their relationship with the partial Pareto prices relative to chosen normalization,

$$\mathrm{grad}\, u_1(x) = (\rho^1_{\sigma=1}p_{\sigma=1}, \rho^1_{\sigma=2}p_{\sigma=2}) \ \ \& \ \ \mathrm{grad}\, u_2(y) = (\tfrac{\rho^2_{\sigma=1}}{\lambda}p_{\sigma=1}, \tfrac{\rho^2_{\sigma=2}}{\gamma}p_{\sigma=2})$$

yields

$$\lambda\rho^1_{\sigma=1}/\rho^2_{\sigma=1} = \gamma\rho^1_{\sigma=2}/\rho^2_{\sigma=2} \quad \Longleftrightarrow \quad \lambda = \gamma\frac{\rho^2_{\sigma=1}\rho^1_{\sigma=2}}{\rho^1_{\sigma=1}\rho^2_{\sigma=2}}. \tag{2.27}$$

Thus if parameters $\lambda > 0$, $\gamma > 0$ satisfy the last relation and simultaneously *do not satisfy* (2.26), the corresponding allocation belongs to the incomplete core.

Further let us consider properly *Hart's example*, which corresponds to our model with two assets under an additional condition:

$$\rho^1_\sigma = \rho^2_\sigma = \rho_\sigma,\ \sigma = 1, 2 \quad \& \quad \alpha_{\sigma=1} = \alpha_{\sigma=2} = \alpha,\ \beta_{\sigma=1} = \beta_{\sigma=2} = \beta.$$

The left hand side of this requirement and (2.27) implies $\lambda = \gamma$, i.e., an allocation is optimal by Pareto iff $\lambda = \gamma$. Initial endowments for Hart's example are determined as

$$\omega_1^{\sigma=1} = (1-\varepsilon, 1-\varepsilon),\ \ \omega_1^{\sigma=2} = (\varepsilon, \varepsilon),\ \ \omega_1^\sigma + \omega_2^\sigma = (1, 1),\ \sigma = 1, 2,$$

where real $0 < \varepsilon < 1$.

Let us show that for Hart's example the set of allocations from the incomplete core, which corresponds to the *non-degenerated* matrix of financial returns, forms the empty set. In fact if $(x, y) \in \mathcal{C}(\mathcal{E}^{in})$ and $\det(\mathrm{P}_1\mathrm{A}) \neq 0$, then (x, y) is Pareto optimal and $\lambda = \gamma$. However then from $\alpha_\sigma = \alpha$, $\beta_\sigma = \beta$, $\sigma = 1, 2$ one can conclude the coincidence of matrix $\mathrm{P}_1\mathrm{A}$ rows and therefore $\det(\mathrm{P}_1\mathrm{A}) = 0$ — contradiction.

Next consider the second possibility: $(x, y) \in \mathcal{C}(\mathcal{E}^{in})$ and $\det(\mathrm{P}_1\mathrm{A}) = 0$. It may be realized only if system (2.25) is solvable. The last one for this case is equivalent to the solvability of system

$$\begin{pmatrix} \alpha + \lambda\beta \\ \alpha + \gamma\beta \end{pmatrix} z = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} (\alpha + \lambda\beta, 1 - \alpha + \lambda(1-\beta))\omega_1^{\sigma=1} \\ (\alpha + \gamma\beta, 1 - \alpha + \gamma(1-\beta))\omega_1^{\sigma=2} \end{pmatrix};$$

substituting for the value of initial endowments, and realizing some elementary transformations we find

$$\begin{pmatrix} \alpha + \lambda\beta \\ \alpha + \gamma\beta \end{pmatrix} z = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} (1-\varepsilon)(1+\lambda) \\ \varepsilon(1+\gamma) \end{pmatrix}. \tag{2.28}$$

Next, substituting $\alpha_\sigma = \alpha$, $\beta_\sigma = \beta$, $\sigma = 1, 2$ in (2.26) and doing transformations, we obtain

$$\det(\mathrm{P}_1\mathrm{A}) = 0 \quad \Longleftrightarrow \quad (\gamma - \lambda)(\alpha - \beta) = 0.$$

Thus the matrix of financial returns is degenerated only if $\gamma = \lambda$ (as seen above) or when $\alpha = \beta$. In the first case (2.28) may be fulfilled only if $\varepsilon = 1/2$. In the second case, (2.28) is reduced to

$$\alpha \begin{pmatrix} 1 + \lambda \\ 1 + \gamma \end{pmatrix} z = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} (1-\varepsilon)(1+\lambda) \\ \varepsilon(1+\gamma) \end{pmatrix} = \begin{pmatrix} \lambda(\varepsilon - 1) + \varepsilon \\ 1 + \varepsilon + \varepsilon\gamma \end{pmatrix}.$$

This system is solvable only if

$$\frac{\lambda(\varepsilon - 1) + \varepsilon}{1 + \varepsilon + \varepsilon\gamma} = \frac{1 + \lambda}{1 + \gamma} \quad \Longleftrightarrow \quad 1 + 2\lambda + \lambda\gamma = 0.$$

However, the last equation cannot be solved for $\lambda > 0$ and $\gamma > 0$.

Let us resume our analysis. For Hart's example the core of an incomplete market is a *non-empty* set *only* for $\varepsilon = 1/2$, and for this case the incomplete core coincides with the classical market core (since then the solvability of (2.28) is equivalent to an allocation is Pareto optimal). For $\varepsilon \neq 1/2$, the *core is empty*, which can be explained via the specific features of given model parameters: preferences and real assets. This peculiarity is such that contracting each other at every nature event and applying real assets (in a given structure) in the present, the agents are not able to arrive at Pareto optimal allocation, regardless of the fact that there is potentially enough assets (so much as there are many future states of the world). In other words any feasible net of contracts is unstable in the sense that coalition $\{1, 2\}$ of all market operators is able to find an opportunity to sign a new exchange contract, taking into account the ability to break some of the given contracts (remember that for future events one can break virtual contracts). Figure 2.2 illustrates the case. In this figure the abilities $V(1, 2) = u[\mathcal{A}(\mathcal{X})]$ of coalition $\{1, 2\}$ are described in the criterial space of "utilities" in a standard manner. The abilities of singleton coalitions are presented via vector $u(\omega) = (u_1(\omega_1), u_2(\omega_2))$. Curve AB, representing a part of the

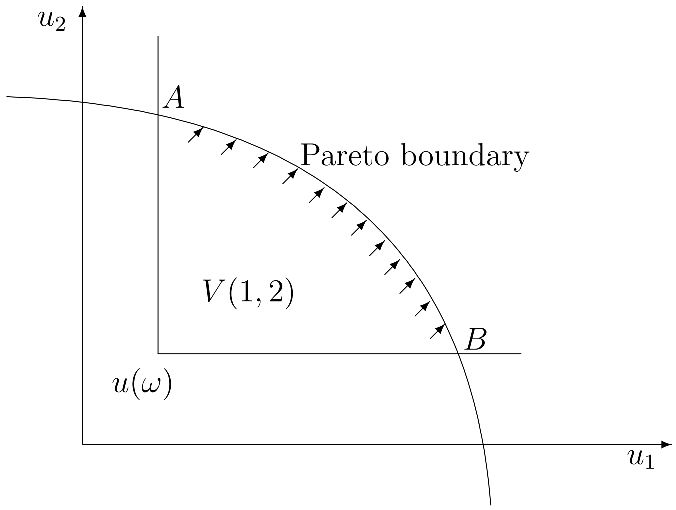

Figure 2.2: Classical and incomplete core in Hart's example

Pareto boundary, corresponds to the standard core of the market. For an incomplete market the points of this curve are not available for consumers since a bundle of utilities from the curve is realized via an allocation which is not $p^{\mathbf{1}}$-feasible. Notice that one can infinitesimally closely approach the points of this curve via allocations which are *partially* Pareto optimal and $p^{\mathbf{1}}$-feasible relative to partial Pareto prices. In fact, partial Pareto boundary is completely parameterized by couples $(\lambda, \gamma) \gg 0$, and in doing so a point belongs to the (classical) Pareto boundary only if $\lambda = \gamma$. Moreover, the matrix of financial returns is also degenerated only if $\lambda = \gamma$ (let for simplicity $\alpha \neq \beta$). Thus for every point $(\lambda, \gamma) \gg 0$, $\lambda \neq \gamma$ a partially optimal and simultaneously $p^{\mathbf{1}}$-feasible allocation may be put into correspondence, which for λ/γ being near enough to 1 realizes a utility vector, which is close enough to the Pareto optimal one. In this economy Pareto optimal allocations may be attained *only in a limit*, and a *sequence of contracts*, reflecting the exchange of assets in the $p^{\mathbf{1}}$-feasibility condition in this passing to a limit, which is an *unbounded* one. The last observation is rather important, and I are going to discuss it more detailed below.

A contract of this kind for some $(\lambda, \gamma) \gg 0$, $\lambda \neq \gamma$ can be calculated as a solution of system (2.25), that for $\det(\mathrm{P_1A}) \neq 0$ (true for $\lambda \neq \gamma$) yields

$$z = [\mathrm{P_1A}]^{-1} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \mathrm{P_1}\omega_1 \right].$$

However for $\gamma \to \lambda$ we have $\det(\mathrm{P_1A}) \to 0$ and, since $\det([\mathrm{P_1A}]^{-1}) = 1/\det(\mathrm{P_1A})$, then $\det([\mathrm{P_1A}]^{-1}) \to \infty$. Therefore, $||[\mathrm{P_1A}]^{-1}||$ (the norm of operator $[\mathrm{P_1A}]^{-1}$, considered as a function of parameters λ, γ), is unbounded for $\gamma \to \lambda > 0$. Moreover, one can show that exactly for the vectors of form $y^{\lambda\gamma} = (1,1) - \mathrm{P_1}\omega_1$ (prices depend on λ, γ), operator $[\mathrm{P_1A}]^{-1}$ values are norm-unbounded for $\gamma \to \lambda > 0$. It seems to be true that exactly this fact is the main cause of the potential non-stability of financial market. Speaking substantially in terms of contracts, one may see that during the contracting and recontracting process, market operators may realize “a race to infinity”, i.e., the total volume of contracts for one of the agents may rise with no limit. The problem can be solved if one imposes some constraints on the total volume of contracts from the asset market for each agent; it is enough to restrict only the volumes of sales or purchase, but with the same style for all agents. Notice that choosing some finite but big enough constraints, one can realize allocations that are near enough to the allocations (or to the utilities bundles) of a classical core. □

2.3 Proofs

Proof of Lemma 2.1. The first part of item (i) Lemma 2.1 is obvious. To check the second part, recall that for every contractual $x = (x_i)_{i\in\mathcal{I}},\ x_i = (x_i^0, \ldots, x_i^s)$, the representation (2.2) or equivalent relation(2.3) takes place. Further, as soon as $(\bar{x}_i^\sigma)_{i\in\mathcal{I}}$ is an equilibrium allocation for reduced onto $\sigma = 1, \ldots, s$ economy $\mathcal{E}^\sigma$ equipped with endowments $(\omega_i^\sigma + A_\sigma \Delta z_i(W))$ (it was noted above and follows from Theorem 1.2), then

$$p_\sigma \bar{x}_i^\sigma = p_\sigma \omega_i^\sigma + p_\sigma A_\sigma \Delta z_i, \tag{2.29}$$

and $\bar{x}_i \in \mathcal{H}_i$ for all i. Moreover, the equilibrium properties of $\bar{x}^\sigma$ and Theorem 1.2 imply $p_\sigma v_i^\sigma = 0,\ v^\sigma \in V^\sigma$ for all i and $\sigma \geq 1$. Now if one breaks a part of contract $v^\sigma \in V^\sigma$, then the new allocation $(\hat{x}_i^\sigma)_{i\in\mathcal{I}}$ satisfies the system (2.29). Therefore $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n) \in H$. Note that breaking of a share of contracts in the present touches the exchanges of assets and the agents just realize the new allocation, for which the condition to be in H is realized for a new $\Delta\tilde{z}_i$. The last one ends with the checking of (i).

To see that statement (ii) is true, let us presume that some semi-perfect contractual x is Pareto-dominated by allocation $y \in H$. Now since $x, y \in H$, then there are such z, z', that the following equalities are true:

$$p_\sigma x_i^\sigma - p_\sigma \omega_i^\sigma = p_\sigma A_\sigma z_i, \quad \sigma = 1, \ldots, s,\ i \in \mathcal{I},$$

$$p_\sigma y_i^\sigma - p_\sigma \omega_i^\sigma = p_\sigma A_\sigma z_i', \quad \sigma = 1, \ldots, s,\ i \in \mathcal{I}.$$

As soon as all contracts from V^σ, $\sigma \geq 1$ in the web $V = \bigcup_{\sigma=1}^{s} V^\sigma \cup W$, which realizes $x = x(V)$, are perfect, due to perfect contract definition one may change contracts related to the future states of the world and realize x^σ by the (proper) web, which consists of two contracts — v'^σ and v''^σ, defined by formulas:

$$v'^\sigma_i = y_i^\sigma - \omega_i^\sigma - A_\sigma z_i', \quad i \in \mathcal{I},$$

$$v''^\sigma_i = x_i^\sigma - y_i^\sigma - A_\sigma(z_i - z_i'), \quad i \in \mathcal{I},$$

for all $\sigma = 1, \ldots, s$, and saving in the present "old" contracts. This is a web due to the fact that the consumption sets are rectangular. Now due to Proposition 1.1 and its corollaries, to check the properness of v'^σ and v''^σ it is enough to verify that $v_i'^\sigma p_\sigma = 0$ and $v_i''^\sigma p_\sigma = 0$ for all i, which we already have. In view of Definition 2.3, the new web of contracts has to be stable relative to the *simultaneous* procedure of contracts breaking and signing a new contract in the *"present."* Now one can break (as a whole) the contracts of the second type and all contracts in the present and sign the new contract $\hat{w}^0 = (y^0 - \omega^0, z')$ for $\sigma = 0$. In so doing the agents can realize allocation y, which contradicts the definition of semi-perfect (first) and proper-perfect (second) contractual allocations.

Item (*iii*) follows from the definition of semi-perfect contractual allocation and from Theorem 1.2. $\square$

Proof of Lemma 2.2. Let us write in matrix form the conditions, that define the allocations from H. In fact for *feasible* $x \in H$, there are $z_i \in \mathbb{R}^k,\ i \in \mathcal{I}$ such that

$$\begin{cases} \sum\limits_{i=1}^{n} x_i = \sum\limits_{i=1}^{n} \omega_i; \\ \sum\limits_{i=1}^{n} z_i = 0; \\ p_\sigma x_i^\sigma - p_\sigma A_\sigma z_i = p_\sigma \omega_i^\sigma,\ \sigma = 1, \ldots, s \end{cases}$$

holds. Notice that if balance relations and budget constraints $p_\sigma x_i^\sigma - p_\sigma A_\sigma z_i = p_\sigma \omega_i^\sigma$ are satisfied for some fixed $\sigma \geq 1$ and all $i \in \mathcal{I} \setminus \{i_0\}$, then the last budget constraint is also true automatically. This is why all agent i_0's budget constraints, being linear dependent, may be removed from the system of linear equations defining the space H. One may think without loss of generality that $i_0 = n$. Denote by B the matrix

$$\begin{pmatrix}
E_l & 0 & \ldots & 0 & . & E_l & 0 & \ldots & 0 & E_l & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
0 & E_l & . & 0 & \cdots & 0 & E_l & . & 0 & 0 & E_l & . & 0 & 0 & \ldots & 0 & 0 \\
\vdots & . & . & . & . & \vdots & . & . & & . & . & . & . & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & . & E_l & . & 0 & 0 & . & E_l & 0 & 0 & . & E_l & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & E_k & \ldots & E_k & E_k \\
0 & p_1 & . & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & -p_1 A_1 & . & 0 & 0 \\
\vdots & . & . & . & . & \vdots & \vdots & . & \vdots & \vdots & \vdots & . & \vdots & \vdots & . & \vdots & \vdots \\
0 & 0 & . & p_s & . & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & -p_s A_s & . & 0 & 0 \\
\vdots & \vdots & . & . & . & \vdots & . & . & \vdots & \vdots & \vdots & . & \vdots & . & . & . & \vdots \\
0 & 0 & \ldots & 0 & \ldots & 0 & p_1 & . & 0 & 0 & 0 & \ldots & 0 & 0 & . & -p_1 A_1 & 0 \\
\vdots & \vdots & . & \vdots & . & \vdots & . & . & . & \vdots & \vdots & . & \vdots & \vdots & . & \vdots & \vdots \\
0 & 0 & \ldots & 0 & \ldots & 0 & 0 & . & p_s & 0 & 0 & \ldots & 0 & 0 & . & -p_s A_s & 0
\end{pmatrix}$$

Here in standard manner E_l and E_k denote the unit matrices of an appropriate size and p_1, p_s, $p_1 A_1$ and $p_s A_s$ are row-vectors. Clearly we have the following equivalence: *feasible* $x \in H \iff$ *there exists* $z = (z_1, \ldots, z_n)$ *such that*

$$\mathrm{B} \quad * \quad \begin{pmatrix} x_1^0 \\ \vdots \\ x_1^s \\ \vdots \\ x_n^0 \\ \vdots \\ x_n^s \\ z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} \bar{\omega}^0 \\ \vdots \\ \bar{\omega}^s \\ \mathbf{0}_k \\ p_1 \omega_1^1 \\ \vdots \\ p_s \omega_1^s \\ \vdots \\ p_1 \omega_{n-1}^1 \\ \vdots \\ p_s \omega_{n-1}^s \end{pmatrix}.$$

Further let us consider the subspace

$$H^Z = \{((x_1, z_1), ., (x_n, z_n)) \in (\mathbb{R}^{l(s+1)} \times \mathbb{R}^k)^{\mathcal{I}} \mid B(x_1, \ldots, x_n, z_1, \ldots, z_n) = 0\};$$

this is the *kernel* of operator B(.), in which the order of components is changed for the convenience of the below considerations. Due to Lemma 2.1, the allocation $\bar{x} \in H$ and is Pareto-optimal relative to H. Therefore

$$\prod_{\mathcal{I}}[(\mathcal{P}_i(\bar{x}_i) - \bar{x}_i) \times \mathbb{R}^k] \cap H^Z = \emptyset$$

takes place (note that via the second part of (**A**) each of these sets is nonempty). Now by the separation theorem we may find such linear functional $f = (f_1, \ldots, f_n) \neq 0,\ f_i = (f_i^x, f_i^z) \in \mathbb{R}^{l(s+1)+k}$ that

$$\langle f, \prod_{\mathcal{I}}[(\mathcal{P}_i(\bar{x}) - \bar{x}_i) \times \mathbb{R}^k]\rangle \geq \langle f, H^Z\rangle$$

holds. Notice the functional f is constant (and hence is equal to zero) onto subspace H^Z, since the right-hand side of the last inequality is bounded. Therefore

$$\langle f, \prod_{\mathcal{I}}[(P_i(\bar{x}) - \bar{x}_i) \times \mathbb{R}^k]\rangle \geq 0 \tag{2.30}$$

is true. Let us show further that $f_i^z = 0,\ i \in \mathcal{I}$, i.e.,

$$f_i = (f_i^0, \ldots, f_i^s, \underbrace{0, \ldots, 0}_{k})$$

holds for every $i \in \mathcal{I}$. In fact, consider fixed $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_I),\ \hat{x}_i \in (\mathcal{P}_i(\bar{x}) - \bar{x}_i)$, $i \in \mathcal{I},\ \hat{u}_j \in \mathbb{R}^k,\ j \neq i_0$ for some $i_0 \in \mathcal{I}$. In view of (2.30), for any $u \in \mathbb{R}^k$ we have

$$\sum_{j \neq i_0}\langle f_j, (\hat{x}_j, \hat{u}_j)\rangle + \langle f_{i_0}, (\hat{x}_{i_0}, u)\rangle \geq 0,$$

which is possible only if $f_{i_0}^z = 0$ and in view of the arbitrariness of i_0, for all $i_0 \in \mathcal{I}$. For the convenience of the below notations I will identify the functional f_i with f_i^x, i.e., by convention let us put $f_i = (f_i^x, 0) = (f_i, 0)$.

Let us show further that

$$\langle f_i, (\mathcal{P}_i(\bar{x}) - \bar{x}_i)\rangle \geq 0,\ i \in \mathcal{I}, \tag{2.31}$$

and moreover, if $f_i \neq 0$ and $\bar{x}_i \in \operatorname{int} X_i$, then the inequality is strict. Presuming the contrary one can find consumer j_0 such that

$$\langle f_{j_0}, (y_{j_0} - \bar{x}_{j_0})\rangle = -\varepsilon$$

for some $\varepsilon > 0,\ y_{j_0} \in \mathcal{P}_{j_0}(\bar{x})$. Now due to the local-nonsatiation of preferences, we can conclude

$$\forall\, i \neq j_0 \quad \forall\, \delta > 0 \quad \exists\, y_i \in \mathcal{P}_i(\bar{x}) \ :\ \| y_i - \bar{x}_i \| \leq \delta.$$

Now placing vector $y = (y_1, \ldots, y_n)$ in (2.30), we obtain

$$\langle f, (y - \bar{x})\rangle = \sum_{i \neq j_0} \langle f_i, (y_i - \bar{x}_i)\rangle - \varepsilon \leq \delta \sum_{i \neq j_0} \| f_i \| - \varepsilon.$$

However we have $\sum_{i \neq j_0} \| f_i \| \neq 0$ (otherwise (2.30) is false) and may choose $\delta < \varepsilon / \sum_{i \neq j_0} \| f_i \|$, and find $\langle f, (y - \bar{x})\rangle < 0$, which contradicts (2.30). Thus (2.31) is true in the non-strict form of inequalities. Now assumption $\bar{x} \in \text{int} X$ and (**A**) for $f_i \neq 0$ standardly imply the strictness of these inequalities.

Further, the fact that functional $f = (f_1, \ldots, f_n)$ is constant onto subspace H^Z implies that this functional can be represented as a linear combination of the vector-rows of matrix B. Now using the structure of matrix B, one can conclude the existence of such real λ_i^σ, $\sigma \geq 1$ $i \in \mathcal{I}$, $i \neq n$ and such vectors $q \in \mathbb{R}^k$, $\bar{p} = (\bar{p}_0, \ldots, \bar{p}_s) \in E'$, that for all $i \neq n$ the following system of linear equations is true:

$$\begin{cases} f_i^0 = \bar{p}_0, \\ f_i^\sigma = \bar{p}_\sigma + \lambda_i^\sigma p_\sigma, \qquad \sigma \geq 1, \\ f_i^z = 0 = -q + \sum\limits_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma A_\sigma, \end{cases} \tag{2.32}$$

and for $i = n$ we have $f_n^z = 0 = -q + 0$ and $f_n^x = \bar{p}$. Putting $\lambda_n^\sigma = 0$ for all $\sigma \geq 1$, one may think (2.32) is true for all $i \in \mathcal{I}$. Moreover system (2.32) implies that $q = \sum\limits_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma A_\sigma = 0$ for all i. Note also that assumption $\bar{p}_0 = 0$ contradicts (**S**), the local-nonsatiation in each spot market (and therefore for $\sigma = 0$[8]). Therefore $f_i \neq 0$ and on the right-hand side of (2.31) we have a *strict inequality for all* i. Moreover as soon as $f_n = \bar{p} \neq 0$, due to the same arguments — from the local-nonsatiation agent n in each future spot market — we conclude that $\bar{p}_\sigma \neq 0$ for all i and σ. It is also clear that due to (2.31) and (**S**), we have $f_i^\sigma \neq 0$ for all i and σ.

Now let us show that $\bar{p}$ satisfies the other requirements of Lemma 2.2. Having this in mind, first note that by subspaces $\mathcal{H}_i$ specification for every $x_i \in \mathcal{H}_i$ we have

$$p_\sigma(x_i^\sigma - \omega_i^\sigma) = p_\sigma A_\sigma z_i, \ \sigma = 1, \ldots, s$$

for some $z_i \in \mathbb{R}^k$. Now multiplying equalities on λ_i^σ and then summing them by $\sigma = 1, \ldots, s$, one obtains

$$\sum_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma (x_i^\sigma - \omega_i^\sigma) = \sum_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma A_\sigma z_i = (\sum_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma A_\sigma) z_i = q z_i. \tag{2.33}$$

Next let us recall that due to (2.31) and the above considerations we also have

$$\langle f_i, (\mathcal{P}_i(\bar{x}) - \bar{x}_i)\rangle > 0 \Longrightarrow \langle f_i, (y_i - \bar{x}_i)\rangle > 0 \ \ \forall y_i \in \mathcal{P}_i(\bar{x}) \cap \mathcal{H}_i \neq \emptyset \tag{2.34}$$

for all $i \in \mathcal{I}$. Now substituting the representation of f_i from (2.32), we obtain

$$\bar{p}_0(y_i^0 - \bar{x}_i^0) + \sum_{\sigma=1}^{s} \langle (\bar{p}_\sigma + \lambda_i^\sigma p_\sigma), (y_i^\sigma - \bar{x}_i^\sigma)\rangle > 0 \ \ \forall y_i \in \mathcal{P}_i(\bar{x}) \cap \mathcal{H}_i.$$

[8] Since due to $\mathcal{H}_i$ specification, if $y_i \in \mathcal{H}_i$ then for $\bar{y}_i = (\bar{y}_i^\sigma)_{\sigma=0}^{\sigma=s}$, where $\bar{y}_i^\sigma = y_i^\sigma$ for $\sigma \geq 1$, we have $\bar{y}_i \in \mathcal{H}_i$ for every $\bar{y}_i^0$, and therefore $\mathcal{P}_i(\bar{x}) \cap \mathcal{H}_i \neq \emptyset$ *for all* i.

Subtracting from the left and right-hand sides of the inequality the value $\sum_{\sigma=1}^{s} \lambda_i^\sigma (p_\sigma \omega_i^\sigma)$, after transformations we obtain

$$\bar{p}_0 y_i^0 + \sum_{\sigma=1}^{s} \bar{p}_\sigma y_i^\sigma + \sum_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma (y_i^\sigma - \omega_i^\sigma) > \bar{p}_0 \bar{x}_i^0 + \sum_{\sigma=1}^{s} \bar{p}_\sigma \bar{x}_i^\sigma + \sum_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma (\bar{x}_i^\sigma - \omega_i^\sigma).$$

Since $\bar{x}_i, y_i \in \mathcal{H}_i$, there are such $\bar{z}_i = \bar{z}_i(\bar{x}_i)$ $z_i = z_i(y_i)$ that relations (2.33) are true. Now due to the previous inequality we get

$$\langle \bar{p}, y_i \rangle + q z_i > \langle \bar{p}, \bar{x}_i \rangle + q \bar{z}_i.$$

However from the last equation of (2.32) we have $q = f_n^z = 0$, which gives

$$\langle \bar{p}, y_i \rangle > \langle \bar{p}, \bar{x}_i \rangle \;\; \forall\, y_i \in \mathcal{P}_i(\bar{x}) \cap \mathcal{H}_i \iff \langle \bar{p}, ((P_i(\bar{x}) \cap \mathcal{H}_i) - \bar{x}_i) \rangle > 0. \qquad (2.35)$$

The sufficiency of relations (2.9) and (2.10) for an allocation $\bar{x} \in \mathcal{A}(\mathcal{X}) \cap H(p^{\mathbf{1}})$ to be Pareto $H(p^{\mathbf{1}})$-optimal is stated quite standardly. In fact let there be $y = (y_i)_{\mathcal{I}} \in \mathcal{A}(\mathcal{X}) \cap H(p^{\mathbf{1}})$ such that $y_i \succ_i \bar{x}_i$ is true for all i. Then as soon as the right-hand side in (2.9) is fulfilled for $y = (y_i)_{\mathcal{I}} \in H(p^{\mathbf{1}})$ due to $H(p^{\mathbf{1}})$ determination, via the left-hand part of (2.9), we can conclude $\bar{p} y_i > \bar{p} \bar{x}_i$ for all i. Now, summing inequalities over i one finds $\bar{p} \sum_{i \in \mathcal{I}} y_i > \bar{p} \sum_{i \in \mathcal{I}} \bar{x}_i$. Since $\sum_{i \in \mathcal{I}} y_i = \sum_{i \in \mathcal{I}} \bar{x}_i = \sum_{i \in \mathcal{I}} \omega_i$, we are coming to a contradiction. Lemma 2.2 is proved. □

Proof of Corollary 2.1. We have to consider the smooth case in the context of Lemma 2.2. On the necessary side, for the existence of values $\alpha_i > 0$ and $\lambda_i^\sigma \; \forall\, \sigma \geq 1$, one can state it directly from relations (2.9) and (2.10), applying separation theorem (or simply from the necessary conditions of the convex programming problem). However the easiest way to see it may be found from condition $\bar{x} \in \mathrm{int} X$ and relations (2.31), stated in the proof of Lemma 2.2. From this we conclude in a standard way the existence of such $\alpha_i > 0$, that $\mathrm{grad}\, u_i(\bar{x}_i) = \alpha_i f_i$ ($\alpha_i \neq 0$ due to $f_i \neq 0$ and $\mathrm{grad}\, u_i(\bar{x}_i) \neq 0$). Finally one needs to apply (2.32).

To state the sufficiency, let us show that relations (2.9) and (2.10) are true. For some i and $y_i \in \mathcal{P}_i(\bar{x}_i)$, assume $\exists\, z_i \in \mathbb{R}^k \; : \; p_\sigma (y_i^\sigma - \omega_i^\sigma) = p_\sigma A_\sigma z_i \; \forall\, \sigma \geq 1$. Due to gradient's properties for interior points we have

$$\langle \mathrm{grad}\, u_i(\bar{x}_i), y_i \rangle > \langle \mathrm{grad}\, u_i(\bar{x}_i), \bar{x}_i \rangle \quad \forall\, i \in \mathcal{I}.$$

Now substituting the gradient presentation given in the corollary conditions, one can conclude

$$\alpha_i \bar{p} y_i + \sum_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma y_i^\sigma > \alpha_i \bar{p} \bar{x}_i + \sum_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma \bar{x}_i^\sigma.$$

However there are $z_i, \bar{z}_i \in \mathbb{R}^k$, such that $p_\sigma y_i^\sigma = p_\sigma \omega_i^\sigma + p_\sigma A_\sigma z_i$ & $p_\sigma \bar{x}_i^\sigma = p_\sigma \omega_i^\sigma + p_\sigma A_\sigma \bar{z}_i \; \forall\, \sigma \geq 1$. Substituting these expressions in the former formula, one can find

$$\alpha_i \bar{p} y_i + \sum_{\sigma=1}^{s} p_\sigma \omega_i^\sigma + (\sum_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma A_\sigma) z_i > \alpha_i \bar{p} \bar{x}_i + \sum_{\sigma=1}^{s} p_\sigma \omega_i^\sigma + (\sum_{\sigma=1}^{s} \lambda_i^\sigma p_\sigma A_\sigma) \bar{z}_i,$$

that due to $\sum_{\sigma=1}^{s}\lambda_i^\sigma p_\sigma A_\sigma = 0$ and $\alpha_i > 0$ gives the result. □

Proof of Theorem 2.1. To check the inclusion

$$W(\mathcal{E}^{in}) \cap \mathrm{int}X \subset \mathrm{int}X \cap \mathcal{D}^{cp}(\mathcal{E}^{in}),$$

take some $x \in W(\mathcal{E}^{in}) \cap \mathrm{int}X$ and consider the web $V = \{v^\sigma\}_{\sigma=0}^{\sigma=s}$, where

$$v^0 = (x^0 - \omega^0, z), \quad v^\sigma = (x^\sigma - \omega^\sigma - A_\sigma z), \ \ \sigma = 1, \ldots, s$$

and by convention $A_\sigma z = (A_\sigma z_i)_{i\in\mathcal{I}}$ for the appropriate portfolios z_i existing due to the GEI-definition. Due to Theorem 1.2 and the GEI-equilibrium specification, it is easy to see that v^0 is proper and v^σ are perfect contracts for all $\sigma \geq 1$. Also due to the equilibrium specification one can easy to see that this web satisfies the condition of contracts' common stability by Definition 2.4.

Let us check the inverse inclusion. Let $\bar{x} \in \mathrm{int}X \cap \mathcal{D}^{cp}(\mathcal{E}^{in})$. By definition there is a weak stable web of contracts $V = \bigcup_{\sigma=1}^{s} V^\sigma \bigcup W$, realizing the allocation $\bar{x} = x(V)$, so that all contracts from V^σ are perfect, all contracts from W are proper, and the web is stable relative to the simultaneous procedure of breaking (corresponding to the type of contract) and signing new ones in the "present".

Since we always have $\mathcal{D}^{cp}(\mathcal{E}^{in}) \subset \mathcal{D}^{sp}(\mathcal{E}^{in})$, then due to Lemma 2.1 and condition $\bar{x} \in \mathrm{int}X \cap \mathcal{D}^{cp}(\mathcal{E}^{in})$, one can conclude that the conditions of Lemma 2.2 and its Corollary 2.2 are satisfied. Now let $\bar{p} = (\bar{p}_0, \ldots, \bar{p}_s)$ be a vector, which due to Corollary 2.2 corresponds (uniquely) to the allocation $\bar{x}$ and satisfies (2.11). We have to show that there exists such $\bar{z}$ that $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$, where $\bar{q} = \sum_{\sigma=1}^{\sigma=s}\bar{p}_\sigma A_\sigma$, is the GEI-equilibrium allocation of model $\mathcal{E}^{in}$. Having this in mind first let us state that $\bar{x} \in \mathrm{int}X \cap \mathcal{D}^{cp}(\mathcal{E}^{in})$ implies

$$\langle \bar{p}, \bar{x}_i\rangle = \langle \bar{p}, \omega_i\rangle, \ i \in \mathcal{I}.$$

Assuming the contrary, suppose there is consumer i_0 such that $\bar{p}\,\omega_{i_0} > \bar{p}\,\bar{x}_{i_0}$. Then via the smoothness of preferences, we obtain

$$\bar{x}_{i_0} + \mu(\omega_{i_0} - \bar{x}_{i_0}) \succ_{i_0} \bar{x}_{i_0}$$

for some real $\mu > 0$ small enough. However now, using (2.3), one can write

$$\bar{x}_{i_0}^0 + \mu(\omega_{i_0}^0 - \bar{x}_{i_0}^0) = \omega_{i_0}^0 + \Delta_{i_0}^0(V) - \mu\Delta_{i_0}^0(V) = \omega_{i_0}^0 + (1-\mu)\Delta_{i_0}^0(W),$$

$$\bar{x}_{i_0}^\sigma + \mu(\omega_{i_0}^\sigma - \bar{x}_{i_0}^\sigma) = \omega_{i_0}^\sigma + \Delta_{i_0}(V^\sigma) + A_\sigma\Delta z_{i_0} - \mu\Delta_{i_0}(V^\sigma) - \mu A_\sigma \Delta z_{i_0} =$$
$$= \omega_{i_0}^\sigma + (1-\mu)\Delta_{i_0}(V^\sigma) + (1-\mu)A_\sigma\Delta z_{i_0}, \ \sigma = 1, \ldots, s.$$

Clearly by the choice of μ one can think $0 \leq (1-\mu) < 1, \quad \mu > 0$, and the participant i_0 can partially break all contracts in a share μ, increasing utility, which contradicts the lower stability of the proper contractual allocation $\bar{x}$. Therefore $\langle \bar{p}, \bar{x}_i\rangle \leq \langle \bar{p}, \omega_i\rangle$ for all $i \in \mathcal{I}$. Now via the feasibility of $\bar{x}$, one obtains the result.

Further, from Lemma 2.1, Lemma 2.2 and its Corollary 2.2 we have $\bar{x} \in H(\bar{p}^{\mathbf{1}})$, that means

$$\bar{p}_\sigma \bar{x}_i^\sigma = \bar{p}_\sigma \omega_i^\sigma + \bar{p}_\sigma A_\sigma \bar{z}_i \ \ \forall\, \sigma \geq 1$$

for some $\bar{z}_i \in \mathbb{R}^k$ and all $i \in \mathcal{I}$, and also $\sum_{i\in\mathcal{I}} \bar{z}_i = 0$. Let us take this $\bar{z} = (\bar{z}_i)_{i\in\mathcal{I}}$ as a net trade portfolio for allocation $\bar{x}$. From the above relations, using

$$\bar{p}\bar{x}_i = \sum_{\sigma=0}^{s} \bar{p}_\sigma \bar{x}_i^\sigma = \bar{p}_0\bar{x}_i^0 + \sum_{\sigma=1}^{s} \bar{p}_\sigma \omega_i^\sigma + \sum_{\sigma=1}^{s} \bar{p}_\sigma A_\sigma \bar{z}_i = \sum_{\sigma=0}^{s} \bar{p}_\sigma \omega_i^\sigma$$

one can easily conclude that $\bar{p}_0\bar{x}_i^0 = \bar{p}_0\omega_i^0 - \bar{q}\bar{z}_i$ takes place for $\bar{q} = \sum_{\sigma=1}^{\sigma=s} \bar{p}_\sigma A_\sigma$. Now applying Corollary 2.2 and (2.11), one can conclude that for each i the vector $\bar{x}_i$ is the maximal element of $\succ_i$ on the set $\mathcal{B}_i(\bar{p}, \bar{q})$ of all $x_i \in X_i$, satisfying the conditions

$$\exists\, z_i \in \mathbb{R}^k \;:\; \bar{p}_0 x_i^0 = \bar{p}\omega_i^0 - \bar{q}z_i \;\;\&\;\; \bar{p}_\sigma x_i^\sigma = \bar{p}\omega_i^\sigma + \bar{p}_\sigma A_\sigma z_i \;\;\forall\, \sigma \geq 1.$$

However, using (**S**), local nonsatiation in each of the spot markets, and following along standard line of augmentation, one can state that if $\succ_i$ attains a maximal point[9] on the set of all $x_i \in X_i$ such that

$$\exists\, z_i \in \mathbb{R}^k \;:\; \bar{p}_0 x_i^0 \leq \bar{p}\omega_i^0 - \bar{q}z_i \;\;\&\;\; \bar{p}_\sigma x_i^\sigma \leq \bar{p}\omega_i^\sigma + \bar{p}_\sigma A_\sigma z_i \;\;\forall\, \sigma \geq 1$$

is true (in fact equal to i's budget set for incomplete market), then this point undoubtedly has to belong to the set $\mathcal{B}_i(\bar{p}, \bar{q})$ (i.e., for this point all inequalities are realized in the form of an equality). So we have proven condition (i) of Definition 2.1 for$(\bar{x}, \bar{z}, \bar{p}, \bar{q})$.

As the requirement (ii) of this definition is also obviously true, Theorem 2.1 is proved. □

Proof of Proposition 2.1. To apply Scarf's theorem we need to show that the game $(\mathcal{I}, V_p)$, determined via an incomplete market under fixed prices $p^{\mathbf{1}}$, is balanced. With this in mind let us consider a balanced family of coalitions $\mathcal{B}$. It needs to be shown that for every utility-vector v, corresponding to some $p^{\mathbf{1}}$-feasible allocation, the following

$$[\forall S \in \mathcal{B} \;\; \mathrm{pr}_{|S}(v) \in V_p(S)] \Rightarrow v_{\mathcal{I}} \in V_p(\mathcal{I})$$

is true. In fact, due to the game definition, for each $S \in \mathcal{B}$ there is $x^S \in \mathcal{A}_p(S)$ such that $v_S = (v_i)_{i\in S} \leq (u_i(x_i^S))_{i\in S} = u_S(x^S)$. Using the balanced family definition, for each $S \in \mathcal{B}$ one can find a real $\lambda_S \geq 0$. Now, multiplying inequalities on λ_S and summing then by S from $\mathcal{B}$, due to the concavity of the utility functions we yield $(v_i)_{i\in\mathcal{I}} \leq (u_i(\bar{x}_i))_{i\in\mathcal{I}}$ for some $\bar{x} \in \mathcal{A}_p(\mathcal{I})$, which proves $v_{\mathcal{I}} \in V_p(\mathcal{I})$.

Finally, if $\bar{v}_{\mathcal{I}}$ is the vector realizing the maximum of $\sum_{i\in\mathcal{I}} v_i$ subject to $v_{\mathcal{I}} = (v_i)_{\mathcal{I}}$ from the core of $(\mathcal{I}, V_p)$ (which is compact), then obviously the allocation corresponding to it is from $C_{\mathbf{p}}(\mathcal{E})$. □

Proof of Lemma 2.3. To state the conclusion of lemma on the side of necessity, let us consider some $x \in \mathcal{C}(\mathcal{E}^{in}) \cap \mathrm{int}X$. Clearly, every allocation from the core of $\mathcal{E}^{in}$ is a relative equilibrium for the markets of future states of the world in reduced economies $\mathcal{E}_\sigma^{in}$ relative to endowments $\omega_i^\sigma + A_\sigma \Delta z_i(W)$, $i \in \mathcal{I}$ and subject to the fixed

[9] To be sure such point does exist, one may assume, in addition, strict monotonicity for at least one agent's preferences and every consumption set is bounded from below. This is so provided that budget sets are compact, which due to the continuity of preferences, gives the result.

consumption in all other markets (since every contract for future events is perfect, then it follows from Theorem 1.2). This in particular proves (i). Next let us prove (ii). Due to $x \in \mathrm{int}X$ and all utilities being differentiable, the vectors of equilibrium prices $\bar{p}_\sigma$, $\sigma \geq 1$ are uniquely determined (up to normalization), and it follows from **(M)** that $\bar{p}_\sigma \gg 0$. Show that $x \in C_{\bar{\mathbf{p}}}(\mathcal{E}^{in})$. The proof parallels the proof of item (ii) from Lemma 2.1. Presuming the contrary, find $y \in \mathcal{A}_{\bar{p}}(S)$ such that $y \succ_S x$ relative to $\bar{p}^{\mathbf{1}}$. Since $x \in \mathcal{A}_{\bar{p}}(\mathcal{I})$, $y \in \mathcal{A}_{\bar{p}}(S)$, there are such z, z' that the equalities

$$\bar{p}_\sigma x_i^\sigma - \bar{p}_\sigma \omega_i^\sigma = \bar{p}_\sigma A_\sigma z_i, \quad \sigma = 1, \dots, s, \; i \in \mathcal{I},$$

$$\bar{p}_\sigma y_i^\sigma - \bar{p}_\sigma \omega_i^\sigma = \bar{p}_\sigma A_\sigma z_i', \quad \sigma = 1, \dots, s, \; i \in S$$

are fulfilled. As soon as all contracts for future states are assumed to be perfect, we can substitute the initial web V^σ, which realizes the allocation $(x_i^\sigma)_{i\in\mathcal{I}}$ by the web consisting of two *proper* contracts v'^σ and v''^σ:

$$v_i'^\sigma = y_i^\sigma - \omega_i^\sigma - A_\sigma z_i', \quad i \in S, \quad v_i'^\sigma = 0, \quad i \in \mathcal{I} \setminus S,$$

$$v_i''^\sigma = (x_i^\sigma - y_i^\sigma) - A_\sigma(z_i - z_i'), \; i \in S, \quad v_i''^\sigma = x_i^\sigma - \omega_i^\sigma - A_\sigma z_i, \; i \in \mathcal{I} \setminus S.$$

The last one can be done for all $\sigma = 1, \dots, s$. At present one can save "old" contracts. Due to consumption sets being rectangular and due to the $\mathcal{A}_{\bar{p}}(S)$ specification, one can easy see that all of them are *contracts* forming a *web*. Recall that to see if these contracts are proper it is enough to check $v_i'^\sigma \bar{p}_\sigma = 0$ $v_i''^\sigma \bar{p}_\sigma = 0$ for all i. In so doing, by definition, the new web has to be stable relative to the *simultaneous* procedure of breaking and signing a new contract in the *"present."* However breaking all contracts of a second kind now and all contracts in the present and signing for $t = \sigma = 0$ the new contract $w = [(y_i^0 - \omega_i^0, z_i')]_{i\in S}$, the members of coalition S are able to realize the allocation $y \in \mathcal{A}_{\bar{p}}(S)$, which contradicts the incomplete market core definition.

Further let us prove the sufficiency of items (i) and (ii) of Lemma 2.3. By assumption, x^σ is Pareto optimal in a σ-reduced model for a fixed $x^{-\sigma}$ relative to endowments $\omega_i^\sigma + A_\sigma z_i$, $\sigma \geq 1$. And moreover, $x \in C_{\mathbf{p}}(\mathcal{E}^{in}) \cap \mathrm{int}X$ for partial Pareto prices $p^{\mathbf{1}} = (p_1, \dots, p_s)$. Therefore there are $z_i \in \mathbb{R}^k$ satisfying the condition

$$\mathrm{P}_1 x_i^{\mathbf{1}} = \mathrm{P}_1 \omega_i^{\mathbf{1}} + \mathrm{P}_1 \mathrm{A} z_i.$$

Now let us consider the web $V = \bigcup_{\sigma\geq 1} \{v^\sigma\} \cup \{w\}$, where

$$w = (x^0 - \omega^0, z), \quad v^\sigma = (x^\sigma - \omega^\sigma - A_\sigma z), \quad \sigma = 1, \dots, s$$

for $A_\sigma z = (A_\sigma z_i)_{i\in\mathcal{I}}$. We have to prove that the allocation $x = x(V)$, realized via this web, belongs to the incomplete market core, i.e., $x = x(V) \in \mathcal{C}(\mathcal{E}^{in})$.

Next we show first that each contract v^σ, $\sigma \geq 1$ is perfect in fact. Really, for fixed $\sigma \geq 1$ due to a specification we have $p_\sigma v_i^\sigma = 0$ for all i. Since the prices $p^{\mathbf{1}}$ are partial Pareto, then for all i we also have

$$p_\sigma y_i^\sigma > p_\sigma x_i^\sigma \quad \forall\, y_i = (y_i^\sigma, x_i^{-\sigma}) \in \mathcal{P}_i(x_i).$$

Now one can apply Proposition 1.1, and using the sufficiency of (1.1) conclude that contract v^σ is coherent. But applying Theorem 1.2 for a Pareto optimal allocation

$x \in \text{int}X$ realized by coherent web (since *(ii)* implies *(iv)*) we can conclude that the web is perfect. Therefore, contract v^σ is perfect and this is true for all $\sigma \geq 1$.

Let a coalition $S \subset \mathcal{I}$, virtual proper webs $V^\sigma \sim \{v^\sigma\}$ and their subwebs $U^\sigma \subset V^\sigma$, which are realized after this coalition breaks a part of its virtual contracts, be given for all $\sigma \geq 1$, and let $\text{supp}(u^\sigma) \subseteq S$ for every $u^\sigma \in U^\sigma$ and each $\sigma \geq 1$, i.e., (2.4) is true.

In the present let the members of S sign a new contract (u^0, z^0), breaking contract w. In such a case, the members of S realize the following allocation $y = (y_i)_{i \in S}$:

$$y_i^0 = \omega_i^0 + u_i^0, \qquad i \in S,$$

$$y_i^\sigma = \omega_i^\sigma + \Delta_i(U^\sigma) + A_\sigma z_i^0, \quad \sigma \geq 1, \ i \in S.$$

Now, since contracts from U^σ are proper and in view of Theorem 1.2, we get $p_\sigma u_i^\sigma = 0$, $u^\sigma \in U^\sigma$, which implies

$$p_\sigma y_i^\sigma = p_\sigma \omega_i^\sigma + p_\sigma A_\sigma z_i^0, \quad \sigma \geq 1, \ i \in S.$$

From (2.4) we have $\sum_S \Delta_i(U^\sigma) = 0$ for all $\sigma \geq 1$, and by contract specification $\sum_S u_i^0 = 0$ and $\sum_S z_i^0 = 0$; this proves $\sum_S y_i = \sum_S \omega_i$. So as a result we have $y \in \mathcal{A}_p(S)$ and see that domination by coalition S is impossible.

So, it is proved that contracts v^σ, $\sigma = 1, \ldots, s$ are perfect and the web is stable by Definition 2.3. Thus x is a semi-perfect contractual allocation, as we wanted to prove. □

Proof of Theorem 2.2. The proof is based on the simple application of Corollary 2.4 of Lemma 2.3, however let us consider this in detailed form.

First of all let us remark that assumption $\overline{\mathcal{P}_i(\omega_i)} \subset \text{int}X_i \ \forall\, i \in \mathcal{I}$, implies that $x \in \text{int}X$ as for $x \in \mathcal{C}(\mathcal{E})$ and also for $x \in \mathcal{C}(\mathcal{E}^{in})$. This is why in below considerations we can always think $x \in \text{int}X$.

Let us show the inclusion $\mathcal{C}(\mathcal{E}) \subseteq \mathcal{C}(\mathcal{E}^{in})$. Let $x \in \mathcal{C}(\mathcal{E})$. Then for each σ the allocation x_σ is partially Pareto optimal, which implies the existence of price-vector $\bar{p}^{\mathbf{1}} \gg 0$[10] such that condition $(\tilde{x}_i^\sigma, x_i^{-\sigma}) \succ_i (x_i^\sigma, x_i^{-\sigma})$ (relative to fixed $x^{-\sigma}$) implies $\bar{p}^\sigma \tilde{x}_i^\sigma > \bar{p}^\sigma x_i^\sigma$ for all i. Now let us represent x as a $\bar{p}^{\mathbf{1}}$-feasible allocation (to apply Corollary 2.4). For this we need to find a feasible trade net of portfolios $(z_i)_{i \in \mathcal{I}}$, satisfying the system of linear equations

$$\mathrm{P}_1 x_i^{\mathbf{1}} = \mathrm{P}_1 \omega_i^{\mathbf{1}} + \mathrm{P}_1 \mathrm{A} z_i. \tag{2.36}$$

Since the market is complete, this system is solvable relative to z_i. Therefore $x \in \mathcal{A}_{\bar{p}}(\mathcal{I})$. Now let us presume that $x \notin \mathcal{C}(\mathcal{E}^{in})$. Then due to Corollary 2.4, we get $x \notin C_{\mathbf{p}}(\mathcal{E}^{in})$ for each $p^{\mathbf{1}} \gg 0$, and hence for given $\bar{p}^{\mathbf{1}} \gg 0$. Thus there is such coalition $S \subset \mathcal{I}$ and $\bar{p}^{\mathbf{1}}$-feasible for S allocation y that

$$y_i \succ_i x_i, \quad \forall\, i \in S.$$

[10] The property $\bar{p}^{\mathbf{1}} \gg 0$ follows from the strict monotonicity of utilities, which is guaranteed due to assumption **(M)**.

However the last one contradicts $x \in \mathcal{C}(\mathcal{E})$, and this ends the check for inclusion. It is easy to see that the web realizing the given complex contractual allocation $x \in \mathcal{C}(\mathcal{E}^{in})$ is the collection

$$V = \{(u, z)\} \bigcup_{\sigma=1}^{s} \{v^\sigma\},$$

where $u = x^0 - \omega^0$ and $z = (z_i)_{i \in \mathcal{I}}$ are such that z_i satisfy the system (2.36),[11] and $v_i^\sigma = x_i^\sigma - \omega_i^\sigma - A^\sigma z_i$ for every $i \in \mathcal{I}$ and $\sigma = 1, \ldots, s$.

Let us prove the inverse inclusion $\mathcal{C}(\mathcal{E}^{in}) \subseteq \mathcal{C}(\mathcal{E})$. Consider $x \in \mathcal{C}(\mathcal{E}^{in})$. Due to Corollary 2.4 there exists a price-vector $\bar{p}^{\mathbf{1}} \gg 0$ such that $x \in C_{\bar{\mathbf{p}}}(\mathcal{E}^{in})$. Note if allocation y dominates via S allocation x (in an ordinary sense), i.e., if there are such $y_i \in X_i$ that $y_i \succ_i x_i$ for all $i \in S$ and $\sum_{i \in S} y_i = \sum_{i \in S} \omega_i$, then in view of market completeness one can find such $\tilde{z}_i \in \mathbb{R}^k$, $i \in S$ that system (2.36) has a solution relative to $\tilde{z}_i$ and for fixed y_i (substitute $\tilde{z}_i$ instead of z_i and y_i instead of x_i). Since these $\tilde{z}_i$ satisfy $\sum_S \tilde{z}_i = 0$, then $y \in \mathcal{A}_{\bar{p}}(S)$, and therefore $x \notin C_{\bar{\mathbf{p}}}(\mathcal{E}^{in})$, which is a contradiction. □

Proof of Lemma 2.4. The proof of lemma is reduced to the application of the separation theorem to a convex set properly constructed,; this set correspond to the ability of fuzzy coalitions to dominate an allocation.

Analogously to formula (2.6), let us determine the subspaces

$$\mathcal{H}_i = \mathcal{H} + \omega_i, \quad \mathcal{H} = \{y \in \mathbb{R}^{l(s+1)} \mid \exists\, z \in \mathbb{R}^k \ : \ p_\sigma y^\sigma = p_\sigma A_\sigma z, \ \forall\, \sigma \geq 1\}.$$

Next let us take any consumer $i_0 \in \mathcal{I}$, let $i_0 = 1$, and determine the following set

$$G = G(x) = \mathrm{co}\left[(\mathcal{P}_1(x_1) - \omega_1) \bigcup \left(\bigcup_{i=2}^{n} [(\mathcal{P}_i(x_i) - \omega_i) \cap \mathcal{H}]\right)\right].$$

Now show that if $0 \in G$, then there is a fuzzy coalition p-dominating given allocation x in an incomplete market. In fact, $0 \in G$ implies the existence of $t = (t_1, \ldots, t_n) \geq 0$, $\sum t_i = 1$ such that for some $y_i \in \mathcal{P}_i(x_i)$

$$\sum_{i \in \mathcal{I}} t_i(y_i - \omega_i) = 0 \iff \sum_{i \in \mathcal{I}} t_i y_i = \sum_{i \in \mathcal{I}} t_i \omega_i \tag{2.37}$$

is true and moreover for $i = 2, \ldots, n$

$$\exists\, z_i \in \mathbb{R}^k \ : \quad \mathrm{P}_1 y_i = \mathrm{P}_1 \omega_i + \mathrm{P}_1 \mathrm{A} z_i$$

takes place. To check the fuzzy domination definition in part (2.15), it is sufficient to state the last relation for $i = 1 \in \mathrm{supp}(t)$. To realize this, multiply (2.37) on matrix P_1, which after transformations due to $t_1 \neq 0$ yields

$$\mathrm{P}_1 y_i = \mathrm{P}_1 \omega_i + \mathrm{P}_1 \mathrm{A}(-\textstyle\sum_{i=2}^{n} \frac{t_i}{t_1} z_i).$$

[11] More exactly, it is the system defined via the square non-degenerated submatrix of $\mathrm{P}_1\mathrm{A}$, whose dimension is equal to s.

Thus one can take $z_1 = -\sum_{i=2}^{n} \frac{t_i}{t_1} z_i$ as the necessary solution (portfolio) for agent 1. As a result we conclude that coalition t can p-dominate the allocation.

Therefore, for every fuzzy incomplete market core element x, it has to be that $0 \notin G$ and since $\operatorname{int} G \neq \emptyset$ (because of $\operatorname{int} X_1 \neq \emptyset$ due to (**A**) and $\operatorname{int} \mathcal{P}_1(x_1) \neq \emptyset$), then one can apply the separation theorem and find such *non-zero* $\bar{p}$ that

$$\langle \bar{p}, G \rangle \geq 0.$$

Since $\mathcal{P}_1(x_1) - \omega_1$ and $(\mathcal{P}_i(x_i) - \omega_i) \cap \mathcal{H}$, $i = 2, \ldots, n$ are the subsets of G, we conclude

$$\langle \bar{p}, \mathcal{P}_1(x_1) \rangle \geq \bar{p}\omega_1 \quad \& \quad \langle \bar{p}, \mathcal{P}_i(x_i) \cap (\omega_i + \mathcal{H}) \rangle \geq \bar{p}\omega_i, \quad \forall\, i = 2, \ldots, n.$$

Moreover the first of these inequalities due to (**S**), $x_1 \in \operatorname{int} X_1$ and $\bar{p} \neq 0$ implies $\bar{p}_\sigma \neq 0$, $\forall\, \sigma$ in a standard way (presuming the contrary, one can conclude $\bar{p} = 0$, that is impossible). Lemma is proved. □

Conclusion

One of the results of this investigation is an development of (exchange) contracts theory, the elements of which were founded in Makarov (1980), (1982) and Kozyrev (1981), (1982). Applying this theory, first developed in an abstract way and then for classical markets, the analysis of incomplete market theory was realized. The main goal of this analysis and this project as a whole is to study the ability to correctly introduce domination via coalitions and subsequently a core concept for incomplete markets. This concept was introduced in this paper, and it is based on our contractual approach. In so doing, it was proved that a suggested incomplete market core satisfies the following (two) requirements:

- If an economy is described as an incomplete market but is mathematically equivalent to a classical pure exchange model (i.e., it is a complete economy in fact), then the core in the context of an incomplete market has to coincide with the classical core of a pure exchange economy.

- In conditions of perfect competition, the core of an incomplete market coincides with the set of equilibrium allocations.

Namely due to these two properties, one can state that the correct concept of core is truely introduced. Then it seems to be true that this concept is a natural generalization of the classical approach save its the most meaningful properties. It is important that the suggested concept being based on the notion of a contract does not address to current allocation value characterizations — attention is concentrated on exchange contracts, their sets and the different kinds of stability properties. Some kind of value characterizations (p-core, fuzzy p-domination) have appeared in the analysis of core allocations, but it is more a technical element of investigation than an element of its foundation. The mathematical generality of the proved theorems seems to be reasonable in the incomplete market context. This is why the investigator of the project think that the main goal of the project has been attained. Further, let us consider the detailed results and conclusions that have been achieved in this investigation.

Main results

Let us start from a conceptual description of the main notions of elaborated contracts theory.

• *The theory of contracts.* A contract is an elementary exchange of commodities among the members of a group (coalition) of economic agents. Not every exchange presents a *permissible* contract; there are a lot of reasons for this (physical, institutional, behavioral etc.). Contracts my be added to each other and with the initial endowments. This way, to any (finite) set of contracts an allocation of commodities among agents may be put into correspondence. Each consumer or their coalition can *break contracts* in which they participate, and a coalition can also *sign a new one.* A finite collection of contracts forms a web, if after breaking any subset of contracts, a feasible allocation is realized. From the set of all webs one extracts *stable* webs. They are webs such that no agents are stimulated to change them; no coalition can break a part of these contracts and sign a new one and still have an advantage for all of its members. The kind of web stability can be differentiated and, in addition, extracted: *lower* stable (stability relative only to breaking given contracts), *upper* stable (nobody wants to sign a new contract without breaking given ones) and *weak stable* (both properties together but in separately applied). Sometimes contracts can be divided (partitioned) into (proportional) parts, and some of these obtained contracts can be broken (partial breaking). Admission of ability to partially break contracts raises the stability of final resource allocation and of its realized web. The webs which are lower stable relative to the partial breaking of contracts (from the web) are called *proper*. An allocation is called proper if a web can be realized such that any another web produced from the given one as its partition is stable (the agents have no incentives to partially break contracts and simultaneously sign a new one). The proper webs are considered to be long-living ones. For the class of proper webs an *equivalence relation* is introduced. This equivalence identifies the webs that realize the same deviations of consumption programs for each agent. A web, which is equivalent to some given one, is called a *virtual* web. This equivalence relation is applied to introduce another important notion, the perfect contract. A contract is called *perfect* if being substituted by any virtual web, nobody can get an advantage from breaking contracts from this web. Namely, perfect contracts play the key role in the introducing the core concept in an incomplete market. A web containing only perfect contracts is called perfect. A structure of permissible contracts for an economy may allow the ability to classify contracts via its characteristic feature (type). In such a case one can require contracts of different types to satisfy different kinds of stability. An allocation that can be realized by a stable web (in one sense) where there is correspondence between the type of contract and the kind of its stability is called *complex contractual*.

- *Perfect market.* For a classical pure exchange economy the following results were obtained. A contract-based economy is put into correspondence to the model, and it is postulated that *all* contracts are *permissible* for this contractual economy. In such a case, some well known notions of economic theory obtain treatments in pure contractual terms. For example, the Pareto optimality is equivalent to the ability to realize an allocation via an upper stable web. An allocation belongs to the core if and only if it can be realized by some stable web of contracts. If a point is interior and the model is smooth, then the allocation is equilibrium one if and only if it may be realized by a web of contracts that is stable under conditions allowing contracts to be partially broken; thus the web is perfect or proper (weaker requirements are also possible). In the classical model, the conditions of *perfect competition*, which mean that the ability of the core allocation to be price decentralized, are (by the fact) equivalent to the stability of the web realizing this allocation under conditions allowing contracts to be *partially broken*.

• *Incomplete market.* The application of contracts theory to incomplete markets yields the following main results. A contract-based model, which is put into correspondence to an incomplete market, is such that only contracts realizing the *exchange* of commodities for some *fixed event* are permissible ones, where exchanges via *real assets* are allowed in the *present*, where the trade of the given initial model collection of fixed standard contracts is realized. Applying only these standard contracts, one can realize the commodity exchange between different states of nature. In so doing, one can also realize an exchange of commodities in the present for the consumption at future events. Let the model be smooth and a point (allocation) be taken from the interior of the consumption sets in all subsequent considerations.

For the allocations of an *incomplete market core*, it is suggested to take complex contractual allocations, which can be realized via a web of permissible contracts that is stable in the following sense. The web as a whole is weak stable and all contracts corresponding to future events are perfect. The web also has to satisfy the additional stability property: there is no coalition for which it is profitable to break all contracts with non-members of the coalition; therewith, for future events they can break contracts from virtual webs, but present contracts may be broken only as a whole, and moreover the coalition can sign a new contract in the present. It has been proved that if the structure of assets in an (incomplete) market is complete, then the incomplete market core coincides with the classical one. Here an *assets structure is complete* if all commodities are desirable (goods) and for all positive prices in future events markets the matrix of financial returns, defined via real assets matrix, has a rank equal to the number of future events. Simply speaking, the latter means that there are enough independent assets — enough to realize any value transfer from one given future event to any other one without loss of value at other non-true events. This means that the considered core concept satisfies the first suggested criteria. It has also been proven that the concept satisfies the second criteria — under perfect competition conditions, the core and equilibria coincides. Moreover this fact is presented in two versions.

The first theorem states the description of GEI-equilibria in pure contractual terms, where a stability relative to the partial breaking of contracts is allowed. More detailed, an incomplete market equilibrium can be describe as a complex contractual allocation of the following kind. There exists a weak stable web of *permissible* contracts such that contracts for future events are perfect and as a whole the web is stable relative to the partial breaking of contracts in the present and to the breaking of virtual contracts for future events, and relative to a new contract in the present that can be signed. In other words for every event, a subweb relative to this event has to be stable therewith for future events in the strong sense (since then contracts are perfect), and the web as a whole has to satisfy the condition of joint stability described above.

The second theorem follows to the classical modelling tradition of perfect competition conditions — being replicated, an allocation can be considered as an allocation of a replicated economy and it has to belong to the core of the replicated model. Then such an allocation may be decentralized.

The facts presented above allow me to assert that the introduced core concept is correct.

The investigation also contains the analysis of *Hart's example* well known in incomplete market theory. Under specific model parameters there is no equilibrium in this example. It was clarified, that the *core* in the described sense *may also not exist*. The cause of this is the specific financial market properties. Namely, if the number of assets is limited, the situation may occur when market operators tend to raise contract volumes for assets with no limit. It seems to be true that this is a degenerated case, which may happened only under a specific relationship between preferences and assets (it is well known in the theory that financial equilibria, which always are in a core, generically do exist). Of course there are more chances for core non-emptiness relative to equilibria. Moreover, the example shows that an incomplete market core may be nonclosed set. In particular, this is why one cannot apply the classical scheme of the equilibrium existence proof, in which (quasi)equilibria allocations are the elements of a symmetric part of core allocations of the replicated model — the intersection of non-empty, bounded, embedded, but nonclosed sets may be empty. In my opinion for the incomplete market core to be non-empty, it is necessary that grand coalition abilities be supplemented by marginal variants of consumption bundles, i.e., one needs in fact to pass to the closure of an appropriate set. The problem of *core emptiness* can be solved if one imposes some institutional constraints on trade volumes for a financial market. It is known that when some constrains of this kind take place, then equilibria exist under rather weak model assumptions. Moreover, relaxing constraints to infinity, one can also consider limit equilibrium allocations (e.g., see Marakulin (1999), Florenzano et al. (1998)). These arguments may be applied to the core concept also, and in such a case one can pass to the consideration of approximating and marginal dominating variants and the incomplete market core with a non-empty core seems possible to introduce. However, these ideas were not elaborated in this investigation.

Resumed conclusions and policy making

In their application, the obtained theoretical results of the investigation can be realized in the following conclusions.

(**i**) The potential instability of the financial assets market is its characteristic feature. This instability can become apparent as an inability to equate supply and demand (no equilibrium) and also as a form of *instability* with respect to *coalition domination*. Formally this instability can be realized via market operators tending to raise the volumes of contracts with no limit.

(**ii**) To stabilize the financial market, the *policy* of institutional *constraints for trade volumes* in the financial market seems to be reasonable. These constraints have to be *provisional* and have to be relaxed to infinity if it is possible. When it is impossible some new assets have to be created and introduced into the financial market in order to stabilize it.

(**iii**) In accordance with item (**ii**), the elements of financial market regulation can be mainly *recommended* to fast *developing national financial markets*, during a transition time period when the number of new assets and services are quickly increasing (they penetrate from abroad and are born in the internal market itself).

It seems to be true that this conclusion may be applied to the case of Russian economy.

Possible subsequent investigations

One possible way to develop the obtained theoretical results of this investigation is to further strengthen them from the formal-mathematical point of view; in particular, the study of the ability to extend results for the non-smooth case is possible. It seems that in a lot of cases, it is possible to do this. However, it may require one to not only continue the study of perfect contracts and webs but also probably a minor modification of this concept. This modification, of course, does not have to change the essence of this for the smooth case. As a whole this is not easy mathematical work, but this requires time and a methodical approach.

I think that the contract-based approach can be applied to suggest a microeconomic foundation of the tâtonnement process, which is the process of price changing. The grounds of this broadly discussed process which is still not proved. An idea is that with every exchange bargain for a coalition, one can associate some prices (possibly not unique). The fact that among these coalition prices there are no equilibrium prices means that a coalition exists that can strike a bargain which is profitable for all its members. In so doing, it is important that in contrast with classical case (as e.g., Smale (1981)), we allow not only a new contract to be signed but also given contracts can be broken. In particular, one can interpret the deviation of current non-equilibrium resource allocation via contractual approach and avoid the standard conjecture that an auctioneer does exist (he/she defines the trade process and the process of price changing to achieve the final allocation). A coalition can realize the same thing via its members and defines itself exchange proportions in the bargain that is realized.

The contract-based approach can be also extended to other more general models in comparison with the studied ones. Among these models there may be sequential markets and markets with transaction costs (for transactions for bargains, see Repullo (1988)), models for which trust among agents is important (e.g., see Gale (1978)), etc. Namely this ability of expansion approach seems the most interesting for the author of this project. For example, it seems very perspective to study economies with information asymmetry of economic agents, the ability indicated by the expert group (I gratefully acknowledge them for this). There is a wide range of literature on this subject, partially presented in our lists of references. In the most complete form, this direction is described in Schwable's monograph (1999). It seems that the contractual approach can be fruitfully applied to the models of this kind. The key point to realize is the correct definition of the set of all permissible contracts, which essentially has to be based on economic agents' information abilities. A generally accepted method to model individual information about the future events of the world is that some sigma-algebra of events is added to the description of an agent; this algebra distinguishes the events which an agent can identify (differentiate). The case becomes rather clear if there is no informational exchange among the members of the dominating coalition. For this case, to define a permissible contract notion, one can postulate the measurability of the consumption bundles vector (mapping), determining the deviation of an agent's current consumption in a contract, i.e., a contract as a whole has to be measurable relative to an appropriate product of sigma-algebras. In this context, one can also consider the finest (weakest) algebra of events, which is coarser (finer) for each agent algebra from a coalition going to sign a contract. In so doing the members of a coalition "emulate" the less (most) informed agents. The admission of informational exchange leads to the enlargement of a model with the rule of information sharing. This case is rather non-trivial, and I am not ready to answer what is the most correct way to model the case via the contract-based approach (also there is no unified opinion in the literature on how one has to postulate the core concept or simply to model this situation). However we think that this also has to be described via the requirement of contract permissibility, expressed in terms of information abilities of a coalition; once again it has to be the measurability relative to some sigma-algebra.

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