Differential information economies: contract based approach

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Differential information economies are investigated from the viewpoint of contract based approach, developed in series of author's papers. The key idea of the approach is to change the study of a "final" allocation of commodities with (different forms) stable webs of barter contracts which implement the allocation. This way main DIE equilibrium notions are clarified and the most significant kinds of core and domination are extracted and their conformity with Walrasian Expectations Equilibrium (WEE) and Rational Expectations Equilibrium (REE) is provided. The series of new concepts of core and domination and appropriate equilibrium notions are introduced: core and equilibrium with differentiated agents (well correspond to conditional expected utilities), interim core and equilibrium. The last one is closely related with REE-equilibrium and clarifies that is an appropriate core and kind of stability. There are proven new theorems on existence of core and quasi-equilibria.

Keywords and Phrases: differentiated information, contractual approach, WEE — Walrasian Expectations Equilibrium, REE — Rational Expectations Equilibrium.

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CONTENTS

NON-TECHNICAL SUMMARY

This study is purely theoretical, concerning our fundamental ideas about markets and their functioning when information is asymmetrically distributed among agents and about what kinds of stable regimes (core, equilibrium) do exist and which of them are the most important ones. Analysis is provided due to a “contractual” approach — a new original method correctly describing economic behavior in disequilibrium situations.

Clearly, the distribution of information plays an important role for effective functioning of economic system and in economic analysis. Specifically informational component plays an important role for commodity allocations among economic agents. However modern economics still has not formed fully valid theoretical model in which economic agents have to make their decisions, given a shortage of information, and, at the same time, they are asymmetrically informed on states of the world. At the present time the theory has too high variety of approaches and solutions, competiting notions of core (stability relative to coalitional threats) and equilibrium. The fact that in real economic life every commodity allocation is implemented via a set of exchange dealings going among economic agents is commonly missed. Moreover, not every exchange can be implemented in real economy — there are many reasons for this, one of the most important is informational one. In this context an information means the ability of agents to distinguish future nature events that is modeled via partitions of space of elementary events. For competiting approaches the ability of agents to distinguish events (indistinguishable events belong to the same element of partition) is taken into account in different ways and moreover, like a real life information in the model can be transformed: informational exchange is possible or it can be lost, forgotten... We apply contractual approach to qualify the case; this approach permits only barter contracts that are measurable relative to individual informations, i.e., they can be only dealings where every side clearly understands that it has to do and takes into account informational possibilities to realize commitments. The sum of concluded contracts from a web implements final resource allocation. The stability of the allocation for contractual approach means the stability of web of contracts: nobody wants to break (as a whole or partially etc.) written contracts and no group is interested to sign new contracts. The analysis of this kind stable allocations is realized according to informational possibilities and it leads us to the extraction of valid notions of core and equilibrium. This way some new concepts are introduced.

Applying contractual approach one can take into account the dynamics of informational changes. Indeed, entering in contractual relations economic agents are able to change information that can be accumulated and agents can be learned. This is modeled via informational rule which is introduced into model and is working so that for every current informational distribution (and commodity allocation) the rule puts into correspondence a new informational distribution. Information rules can vary other agents coalitions and being repeatedly applied the rules yield so called limit information — this is information that cannot be improved in further acts of contractual process. This allows to provide a more realistic view for core and equilibrium: one needs to apply limit information.

There are also investigated known in literature theoretical notions of Walrasian Expectations Equilibrium (WEE) and Rational Expectations Equilibrium (REE). These notions were introduced by Radner at the end of 60’s and 70’s of last century and they are still main theoretical equilibrium concepts. WEE-equilibrium is an ordinary competitive equilibrium considered relative to preferences determined by expected utilities. REE is a competitive equilibrium

defined relative to conditional expected utilities, conditional relative every possible world state realization. Thus there is multiplicity of budget constrains in this concept, different constrains correspond to various states of the world. Moreover this concept assumes that agents are able to extract information from the prices distribution — price signal defines an event that is understandable by every agent, i.e. this is a common knowledge. However it is not clear how these prices were generated — no purchases, sales or other economic actions were realized outside of equilibrium but the Supreme Being presented prices for us... This and other problems can be treated when one applies contractual approach. Specifically, individuals interact each other in contractual process and, being encountered the case when new beneficial contract is possible only after informational exchange individuals are forced to realize this exchange and then to sign a new contract. However now contractual process continues to develop relative to new informational structure and so on, up to the moment when there is no way to sign new contract even after informational exchange. It is shown that REE-equilibrium can be implemented via this way: this is so when agents are able to freely break contracts even when "tomorrow morning" is already happened, before contract realization. If it is not true, i.e. if contracts signed at the final stage of yesterday trade cannot be broken at the moment "tomorrow morning", then so called private equilibrium is realized that is almost equivalent to WEE. Thus the application of contractual dynamical views allows essentially to clarify equilibrium notions approaching their presentations and clarifying their stability types. Moreover this way allows us to propose an universal concept of core and equilibria, they are two-stage notions that integrate the advantages of both views on equilibrium notions.

1. INTRODUCTION

The information plays a crucial role in the economic analysis. This refers not only to the individual decision-making but also to the functioning of the economy as a whole. In particular, the information plays an important role in resources allocation realized through the system of markets. However, in a classical economic modeling the information problem was not considered in a proper way. For example, in the context of the Arrow – Debreu – McKenzie model it is implicitly supposed that economic life proceeds in a separately taken time period and agents possess enough amount of information on economic variables to make rational decisions, and transactions are carried out during infinitesimal time periods, etc.[1] (e.g., see Aliprantis et al. (1995)). However, in the real world, individuals are forced to make decisions under uncertainty and a shortage of information and, moreover, they are asymmetrically informed on states of the world.[2] Unfortunately classical economic theory does not pay enough attention to the issue of information asymmetry.

Originally, model structures and solution concepts (equilibria, core etc.) adequately considering incompleteness and asymmetry in the individual information have been developed in the 70's of the last century. The information is modeled as a partition of (future) states of nature that defines the space of contingent commodities — goods that are consumed in different states of the world are considered as different ones in spite of the fact that they are identical according to their physical characteristics. The economic life is going on in this space — trade and exchanges of commodities are carried out, and actual consumption of contingent goods is realized. Individual information imposes restriction on the structure of possible contracts — contracts as mappings should be measurable according to the individual information partitions of elementary states of the world. Some new specific equilibrium concepts appear: Walrasian expectations, rational expectations equilibrium, etc. Also, new concepts of the core were elaborated, depending on the hypotheses that are accepted for the informational interchange and for making of collective decisions. The most known among others are the coarse core, the fine core and the private core. See the next sections for more details on the description of an economy with the differential information and for the solution concepts studied in its context. The literature review can be found in section 2.

In the study I investigate DIE-economies by means of the contract based approach developed by the author in Marakulin (2003, 2006).[3] In the framework of the ordinary pure exchange model every contract is an *permissible* exchange of commodities among consumers. Contracts may be added one to another and an allocation of resources may be put into correspondence to every (finite) set of contracts — as a result of the summation of contracts and the initial endowments allocation. It is presumed that every feasible set of (permissible) contracts — let us call it '*a web of contracts*' — may be changed during economic life. Each consumer or a coalition of consumers can *break contracts* in which he/she participates, and a coalition can also *sign a new one*. A stable web of contracts is studied in the theory and it may be stability of different forms

[1] There are known the expanded treatments of Arrow – Debreu model admitting asymmetrically informed agents (e.g., see Radner (1982)), however they also are not quite satisfactory.

[2] They are uncertainly defined factors influencing economic indicators and agents' welfare.

[3] In Marakulin (2003) contractual approach was applied to introduce a correct core concept for incomplete markets; in Marakulin (2006) this approach was applied to describe the none-equilibrium functioning of economy with the subsequent progress to an equilibrium.

that depends on admissible ways of breaking contracts. It may be *total breaking* of a contract only, *partial* or even breaking of contracts from an *equivalent system* of contracts, etc.; mixed variants are possible, too. The formal rules of operating with sets of contracts correspond to different forms of the web stability and therefore to the stability of allocation implemented by a web. The kinds of these 'stabilities', together with the property of contracts to be permissible, reflect the different behavioral, physical and institutional principles, formally given in a game-theoretical form, which one can find in real life and in the neoclassical economic theory. So, different kinds of web stabilities correspond to different kinds of contractual allocations, as well as its modifications, that may relax or strengthen the property of allocation to be stable. Depending on the structure of permissible contracts, one can describe well known economic theoretical notions in terms of a stable web of contracts. In a standard exchange model (perfect market) they are the core, competitive equilibria, the Pareto frontier, etc.; in the incomplete market setting, a new (and correct, according to the author) concept of the core was introduced in Marakulin (2003).

The idea to extend the contract-based approach to the DIE-economies seems quite natural. Indeed, Radner, one of founders of the theory of DIE-economies, already used the term "contract" (in the description of the equilibrium concepts) approximately in the same manner as it was done by the author of the paper, see Radner (1982). Certainly, the requirements for the contracts' *admissibility* for a DIE-economy should include their measurability relative to suitable partition of elementary states space. However, this partition differs in different core and equilibrium concepts. Moreover, the DIE-economies theory so far does not demonstrate appropriate conformity between the (different) core and equilibrium concepts,[4] i.e. the situation differs from the case of a complete market. However, the contractual approach provides a clear-cut possibility to establish this conformity. One can reason as in the standard case (see Marakulin (2003)): for the webs realizing core allocations, in addition to their common properties, it is enough to require their stability relative to the partial contracts' breaking. Further, one have to suggest a price characterization for the obtained allocations. This approach may be useful in discovering new concepts of equilibrium[5] and perhaps the most correct one can be found among others.

2. LITERATURE REVIEW

First of all, we indicate Radner's paper (Radner, 1968), where a model of an economy with differential information is formulated and an appropriate generalization of Walrasian equilibrium is introduced (Walrasian expectations equilibrium — WEE). Moreover, the existence theorem of this equilibrium is also proved in the paper. Further, we note Wilson's seminal paper (Wilson, 1978), which introduces specific concepts of coarse and fine core, studies several interesting specific examples, and also offers original concepts of equilibrium (they do not receive names in

[4] Moreover, the widely known WEE and REE equilibrium concepts are criticized in literature, e.g. see *Preface* in DIE-economies (2004).

[5] In DIE-economies (2004) at the end of *Preface* it is referred to the paper *Tourky, R., Yannelis, N.C.: Private expectations equilibrium, 2003* that realized an attempt to offer an equilibrium notion adequate to private core. However, this paper has not been published yet and I could not find and read it: even on personal cites it is marked as being under work.

the modern literature, see Glycopantis, Yannelis (2004)). Wilson's paper attracted the attention of economists and had a profound impact on the development of economic theory in the context of models with differentiated information. It is necessary to mention another work of Radner (Radner, 1978), which supposes that individuals are capable to extract the information from prices distribution and introduces the important concept of rational expectations equilibrium (REE). This concept is specific, and postulates that consumers maximize *conditional* expected utility[6] with the account of initial and additional information provided by prices. In the subsequent research by many authors (see e.g. collections DIE-economies (2004), Economic Theory (2001)) the theory of information-differentiated economy was developed in many directions. Different core concepts were specified and studied (see section below), the existence and relationships of different equilibrium concepts were studied, important issues of incentive compatibility were analyzed[7] and of implementation of core allocations as an equilibrium in some strategic game (specified via initial model), and also a number of others directions were considered; the most complete surveys of the literature are presented in DIE-economies (2004), Schwable (1999).

In mentioned Wilson's paper (Wilson, 1978), a concept of communication system (non-formalized) appears — as a tool transferring the information from one agent to another. In the subsequent works of Allen (Allen, 1991a) – (Allen, 1994) this idea is generalized and formalized in the form of an information rule: this is a mapping which transforms the information of members of a coalition in the form that can be applied to yield a coalition allocation that can dominate current allocation of the economy. Thus for the economy with the information asymmetry Allen introduces a general method to define a core — applying different rules in a context of the same model (and, therefore, different measurability requirements for dominating allocation) one can obtain almost all concepts known in the literature (coarse, fine, private and so on). Schwalbe (Schwable, 1999) develops Allen's approach further and introduces a concept of *maximal information* — the largest information that an economic agent can receive being a member of all possible coalitions he/she enters with the initial information. Exactly the measurability concerning the maximum of information is required for an allocation to be feasible in the economy as a whole. At the same time, coalitions can dominate, as Allen assumed, only through intra-coalition allocation measurable concerning the information received by a rule from the initial one. However, for both authors it is not clear that actually occurs with the information — participating in different coalitions and, extracting a new information and applying it to form a total allocation, agents forget everything and try to dominate it using transformed initial information again... There is something defective in this view on (intramodel) economic life. According to the author, this approach is not quite satisfactory. However, can one offer something constructive instead of this? In the economic theory, the basic analysis concentrates on the study of general allocation of resources that possesses equilibrium properties. In fact, in real life, and from the viewpoint of the *contractual approach*, this allocation is a result of exchange transactions between economic agents. However, in doing so, an information interchange is going on also, and not every transaction (exchange) can be realized, one of the reasons is the information shortcoming.

[6] This basically differs the concept from the WEE-equilibrium introduced in Radner (1968) that also is called Radner equilibrium (not to confuse with REE), see Radner (1968).

[7] This means the absence of the revealed motives for agents to misinform other agents about states of the world — potentially there is such possibility, because an agent can know a state precisely (after its realization) but others are not capable to distinguish it from (some) other states.

The idea of the barter exchange (contract) is by no means new in theoretical economics and seemingly goes back to classical Edgeworth results (1881), but it usually appeared as an interpretation, in the form of net trades in a formal model. The contract as a barter exchange of commodities is appeared then in the works of other authors (though the theory of contracts was not elaborated in a proper way), including Russian ones: Polterovich (1970) and Makarov (1980, 1982). Then in Kozyrev (1982, 1981) there was suggested to partially break contracts and there were obtained some preliminary positive results. Partially breaking contracts being incorporated in stability notion of webs of contracts allow for complete markets to describe Walrasian equilibrium alternatively. Finally, due to the author's results Marakulin (2003, 2006) a basis of contracts theory was appeared that can be considered as replenishment of the classical views on market economy functioning.

According to the author's view, contractual approach is a rather convenient tool for the modeling of situations with asymmetrically informed agents. Clearly, the correct description of real transactions requires at least a clarification of the kinds of contracts that can be signed, i.e. one needs to define permissible contracts. The sum of the concluded contracts and the initial endowments realizes an allocation of resources that can or cannot possess different properties of stability: the theory of (barter) contracts studies so called contractual, proper contractual, perfect contractual allocations and so on, see Marakulin (2003), and also the contractual processes that converge to them, see Marakulin (2006). The notion of the admissible contract plays an important role in the contractual approach. For the model with asymmetric information, the admissibility should include the requirement of measurability determined by the distribution of the information among economic agents. Besides, contractual approach assumes certain dynamics of exchange processes, during which an information interchange can occur. It is natural to suppose that information distribution in the economic environment is non-uniform, non-static and changes eventually during economic interactions between agents. However, what may result from information exchanges? In our opinion, it may be so-called *limit information*, introduced by the author in Marakulin (2009).

The limit information is a result of an information exchange realized by a chain of economic interactions between individuals in the framework of an intra-coalition (barter) commodity exchange and the ongoing information interchange (formal definition is presented below). As examples show, generally different "chains" can lead to different distributions of the information that cannot be improved in subsequent exchange operations. It is proved in Marakulin (2009) that for the *monotonic* rule of *information sharing* the limit information is unique and this means that in this case the concept becomes correct and can be effectively applied to study DIE-economies.

3. ECONOMIES WITH ASYMMETRIC INFORMATION

In this section, the simplest model of an economy with differential (asymmetric) information is described. In the model framework, it is possible to consider some key issues of the economic theory: the concept and the existence of equilibria, various definitions of the core and their implementation as a Bayesian Nash sub-game perfect equilibrium.

In the economic literature, the information was considered from two points. On the one hand, the information has some properties of the goods and thereby it is similar to other market goods: it can be bought, sold or exchanged on the information markets. On the other hand, the information can be individualized and considered as the characteristic of the agent similar to his/her initial endowments and preference relation. Our model takes into account both aspects of the information. The model description applies the terminology of Schwable (1999) and DIE-economies (2004), where the modern views are presented.

3.1. Agents and their information.

Let us consider an exchange economy with a finite set of agents $\mathcal{I} = \{1, 2, \ldots, n\}$. Specific feature of the economy is explicitly introduced information on the states of the world (nature), that agents have. Generally different agents possess different information and their information can vary during agents' economic activities.

The information is modeled as follows. Let us consider measurable space $(\Omega, P(\Omega))$ of the nature events and suppose for simplicity that Ω is a *finite* set. The elements $\omega \in \Omega$ are called states of the nature (the world) or elementary events. Here $P(\Omega)$ is a partition[8] of Ω (in general case one has to consider on Ω an algebra of subsets-events). Partition P_i is associated with the ith agent information. If the information element of partition consists of several states of the nature then it means that the agent *is not able to distinguish* these states. This is a way to describe the ability of agent to distinguish events.

Let $P_i(\omega)$ denote an element of P_i that includes the state $\omega \in \Omega$. It is said that the information P is *finer than (better of)* information P' if each element P is a subset of some element of P', i.e., $P(\omega) \subseteq P'(\omega), \forall\, \omega \in \Omega$. So, one information is finer than another one if it is capable better distinguishes elementary events of the nature. On the set of all partitions of Ω the relation "*finer*" is a partial ordering and it is denoted as

$$P \succeq P' \iff P \ \textit{\textbf{finer than}} \ P' \iff P' \ \textit{\textbf{coarser than}} \ P.$$

On the set of all partitions $\succeq$ defines the lattice structure, i.e., any finite set of partitions has supremum and infimum.

The ordered set (train) of the individualized partitions $\mathbb{P} = (P_i)_{i \in \mathcal{I}}$ is called ***information structure of the economy***.

The relation $\succeq$ for information partitions induces a partial ordering on set of all information structures, that is defined by a rule:

$$\mathbb{P} \succeq \mathbb{P}' \iff P_i \succeq P_i', \quad \forall\, i \in \mathcal{I}.$$

The relation $\succeq$ is applied also for the coalition information structures — for comparison different information provisions of some coalition.

Let's recall some known definitions (e.g., see Schwable (1999)):

*An information is called **perfect** or **complete** if it includes only one-element subsets of Ω. An information structure of the economy is called **perfect** if the information of each agent is perfect.*

*An information structure $(P_i)_{i \in \mathcal{I}}$ is called **asymmetric**, if $P_i \neq P_j$ for some $i, j \in \mathcal{I}$, $i \neq j$.*

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[8] The *partition* of a set is a set of pairwise disjoint non-empty subsets such that their union gives the whole set.

3.2. Commodities, agents and consumption plans

Let there be l physically different commodities (goods) in the economy. Thus the space of the physical commodities is l-dimensional Euclidian space $E = \mathbb{R}^l$. When different events are realized then individuals can potentially consume different bundles of physical (contingent) commodities. The space of contingent commodities is $L = (\mathbb{R}^l)^\Omega = \mathbf{Map}(\Omega, \mathbb{R}^l)$, the set of all maps from the space of elementary events Ω in $\mathbb{R}^l$.

Let P be some partition of Ω. Function f with domain Ω is called *P-measurable*, if it is a constant[9] on the elements of P. For a partition P specify the set

$$\mathbf{Map}_P(\Omega, \mathbb{R}^l) := \{f : \Omega \to \mathbb{R}^l \mid f|_{P(\omega)} = const\},$$

this is a subspace of $\mathbf{Map}(\Omega, \mathbb{R}^l)$ of all P-measurable functions. Space $\mathbf{Map}_P(\Omega, \mathbb{R}^l)$ for $P = P_i$ is the set of *information admissible* consumption bundles of the individual i and this is a basic feature of economy with differentiated information.

An information may be considered as a component of consumption bundle. Thus uncertainty is modeled in the following way: the goods, besides a place and time of availability, are specified also by a state of the world. As soon as the physical goods are considered in the interrelation with the states of the world, it is natural to postulate that space of admissible consumption bundles depends on the information. Thereupon *the generalized space of the goods* $\mathbf{Map}(\Omega, \mathbb{R}^l) \times P^*$ is arisen (P^* denotes the set of all partitions). In this space the goods are presented ordered pairs, consisting of a state contingent commodities and an information. Thus the information becomes a part of goods definition.

So, each individual $i \in \mathcal{I}$ is characterized by initial endowments $(e_i, P_i^0) \in \mathbf{Map}(\Omega, \mathbb{R}^l) \times P^*$, where $e_i \in \mathbf{Map}(\Omega, \mathbb{R}^l)$ denotes the endowments of goods of the agent i, and P_i^0 is his/her initial stock of the information (partition of Ω). For simplicity let's assume, that individuals are able to consume only non-negative quantities of physical goods and that $e_i \geq 0$, $\forall\, i \in \mathcal{I}$.

The consumption set of the agent $i \in \mathcal{I}$ is defined by the following:

$$\mathcal{X}_i := \{(x, P) \in \mathbf{Map}(\Omega, \mathbb{R}^l_+) \times P^* \mid x - e_i \in \mathbf{Map}_P(\Omega, \mathbb{R}^l)\}.$$

Obviously that $(e_i, P_i^0) \in \mathcal{X}_i$ since the contract $e_i - e_i = 0$ is compatible with any information.

Thus consumption plan of the agent i is represented by the ordered couple (x_i, P_i), where the first component $x_i \in \mathbf{Map}(\Omega, \mathbb{R}^l_+)$ is a bundle of contingent commodities and the second one is an information of the agent. Consumption set of the agent consists of all consumption plans that are realized by a web of contracts compatible with the information P_i. Once again notice that agents cannot consume the goods independently of their information. That is if an agent does not distinguish two states of the world and if his/her consumption plan realized by a contract is such that it is different in these states of the nature then it is not placed in the agent's consumption set.

A train consisting of consumption plans of agents $((x_i, P_i))_{i \in \mathcal{I}}$ where $(x_i, P_i) \in \mathcal{X}_i$ is called an ***allocation***.

[9] For a finite algebra of events this definition is equivalent to the standard one.

3.3. Coalitions and information rules

Nonempty subsets of $\mathcal{I}$ are called *coalitions*. Let us presume that all coalitions are permissible and let $C := 2^{\mathcal{I}} \setminus \{\emptyset\}$. It is supposed, that members of a coalition are able to exchange own experience, that is to change their individual information. Redistributions of agents' initial endowments are carried out on the basis of this modified information. Let's consider the process of *informational exchange* among the agents entered in a coalition.

In the most general form an informational interchange is described by the information rule that is one of the model parameters. For every coalition and its set of (private) information the information rule puts into correspondence a new set of information, formally this is defined as follows:

Information rule for a coalition *$S \subseteq \mathcal{I}$ is a mapping $k_S : (P^*)^S \to (P^*)^S$ satisfying $k_{\{i\}}(P) = P$, $\forall\, i \in \mathcal{I}$, $P \in P^*$.*

Information ***rule for an economy*** *is $(2^{|\mathcal{I}|} - 1)$-tuple of mappings $k = (k_S)_{S\in C}$ where to each coalition S there corresponds the proper rule k_S.*

Substantially it is possible to express in such a manner. A member of a coalition $i \in S$ with the information P_i after joining to the coalition S has an access to the information $P_i' = k_S^i((P_j)_{j\in S})$, this is the ith component (projection) of mapping k_S calculated at the "point" $P_S = (P_i)_{i\in S}$. Exactly this information the individual i can use within the limits of coalition activity. For singleton coalitions information change does not occur.

So, the result of an information rule application is a new information which agents can use for the redistributions of initial endowments within the limits of the coalition.

We call $k = (k_S)_{S\in C}$ a rule of ***information sharing*** *if $k_S((P_i)_{i\in S}) \succeq (P_i)_{i\in S}$ for all $(P_i)_{i\in\mathcal{I}}$, $\forall\, S \in C$.*

Certainly, particular cases of an information rule are the rule where information interchange does not occur at all (this is *private* rule) and zero rule where irrespective to the initial information the information $P_i = \{\Omega\}$ is attributed to each member of a coalition.

Finally I'd like to note that informational exchange can certainly depend on not only individual information of coalition members but also a current allocation, i.e. current bundles of consumed contingent commodities. Moreover it seems natural to think that the only incentive to share information is the possibility (hope) to sign a new mutually beneficial contract for new information endowments. To formally describe the extended treatment of informational rule it is enough to change *domain* for mappings k_S, $S \in C$ and think they are defined on the sets $\mathcal{X}_S = \prod_{i\in S} \mathcal{X}_i$ instead of $P_S^* = (P^*)^S$, i.e. one has to assume

$$k_S : \mathcal{X}_S \to (P^*)^S, \quad \forall\, S \in C.$$

This extended treatment of informational rule will be applied in further analysis.

3.4. Feasible allocations

Definition of allocation has a great value for the consideration of the core of economy. Certainly, feasible allocations of an economy should be at least physically admissible. However, if the economy with the asymmetric information is considered then admissibility definition should take into account also the agents' initial information and the information rule that operates in the model.

In the literature there are two the most known definitions of feasible allocation. In the first one (applied by Yannelis et al.) an allocation is called feasible if it is compatible with the initial information of individuals. Thus, by Yannelis the ability of an information exchange between agents is not accepted into account. The second one is appeared in works of Allen (see, e.g., Allen (1991a,b)) where the ability of an information exchange are taken into consideration. By Allen allocations measurable relative to the information accessible to all members of economy are called feasible, i.e., it is the measurability relative to coarsest information. These definitions are not free from shortcomings. Yannelis' definition is poorly motivated from substantial party: it is intuitively clear that an feasibility of allocation should somehow depend on the information rule, the informational world is not static. Definition by Allen is excessively rigid, that can entail emptiness of the core and also it can happen that agents cannot use any information at all and only initial endowments become (single) feasible allocation. Being motivated these reasons, Schwalbe (Schwable, 1999) introduces the following concept of *maximal information* that is applied him further to define feasible allocation.

Maximal information *of agent i relatively an informational rule $k = (k_S)_{S\in C}$ is the information* $P_i^{max} := \bigvee_{S\in C} k_S^i((P_j{}^0)_{j\in S})$.

Thus, the maximal information is such an information that would be received by the agent if he/she would be able to join to all possible coalitions simultaneously. However each individual enters coalition activity with the initial information. Another point of view was suggested in Marakulin (2009) where concept of limit information was introduced.

An information is called ***limit information*** *if a sequence of coalitions does exist such that the information formed at last stage of informational exchange cannot further be changed (unimprovable for each agent).* Formally: $\mathbb{P}^{lim} = (P_i^{lim})_{i\in\mathcal{I}}$ is defined by a (finite) chain of coalitions $S_1, S_2, \ldots, S_m \subseteq \mathcal{I}$, so that:

(i) $\quad P_i^{\xi} = k_{S_\xi}^i((P_j^{\xi-1})_{j\in S_\xi})$, $i \in S_\xi$, $P_i^{\xi} = P_i^{\xi-1}$, $i \in \mathcal{I}\setminus S_\xi$, $\xi = 1, 2, \ldots, m$;

(ii) $\quad \forall\, S \subseteq \mathcal{I}$, $\forall\, i \in S$, $k_S^i((P_j^m)_{j\in S}) = P_i^m = P_i^{lim}$.

In general limit information can depend on its implementing coalition chain. However for a *monotonic* information *sharing* rule it is uniquely defined (see Marakulin (2009)).

Definition 1. *An allocation $((x_i, P_i))_{i\in\mathcal{I}}$ is called* ***feasible*** *if the maps $x_i : \Omega \to \mathbb{R}_+^l$ and the partitions P_i, $i \in \mathcal{I}$ obey:*

(i) $\sum_{i\in\mathcal{I}} x_i = \sum_{i\in\mathcal{I}} e_i$,

(ii) $(x_i - e_i) : \Omega \to \mathbb{R}^l$ *is measurable relative to* P_i^{eco} $\forall\, i \in \mathcal{I}$.

Here the first condition reflects a physical feasibility of the allocation, while the second one — its informational feasibility since it requires the measurability of gross contract (net trade) realizing the allocation. Notice that it should be specially stipulated about what information P_i^{eco} in item (**ii**) is applied. In particular, P_i^{eco} can coincide with P_i^{max} (Schwalbe), P_i^0 (Yannelis), $\bigwedge_{j\in\mathcal{I}} P_j$ (Allen) or even P_i^{lim} (Marakulin).

3.5. Preferences, equilibria and core

Let's assume for simplicity, that preferences are defined via utility functions $u_i : \mathcal{X}_i \to \mathbb{R}$, $i \in \mathcal{I}$, where consumption sets $\mathcal{X}_i$ have been specified above. It is supposed usually that utilities are invariant relative to information:

$$u_i(x, P) = u_i(x, P'), \quad \forall\,(x, P), (x, P') \in \mathcal{X}_i,$$

that can be interpreted as the fact that the information itself does not generate any utility.[10] This assumption means that there is an utility $u_i(x) = u_i(x, \Omega^*)$, defined on E_+^Ω such that by definition $u_i(x, P) = u_i(x, \Omega^*)$, $\forall\,(x, P) \in \mathcal{X}_i$. Thus if a partition P is *fixed* then the set of all P-measurable functions forms a subspace in E^Ω and formally utility function $u_i(\cdot, P)$ can be defined as a reduction of known function $u_i(\cdot, \Omega^*)$. In this sense utility does not depend on information.

Sometimes it is supposed, that utility is defined via so called *randomized* utility function $u_i : \Omega \times \mathbb{R}_+^l \to \mathbb{R}$ where $u_i(\omega, x_i(\omega))$ is a "value" of utility in a state ω for the consumption plan $x_i(\omega)$. In that case the expected utility can be used (ex ante expected utility), calculated by the formula

$$u_i(x_i) = \sum_{\omega\in\Omega} u_i(\omega, x_i(\omega)) q_i(\omega),$$

where $q_i(\omega)$ is defined in the model an (individualized) prior,[11] i.e. a priori defined probability of a state ω realization. Further let us consider equilibrium concepts.

• WEE — *Radner or Walrasian expectations equilibrium.* For the model it is assumed in addition the existence of a specific partition of Ω[12] denoted as $\mathcal{F}$ such that $\mathcal{F} \succeq \bigvee_{i\in\mathcal{I}} P_i$, i.e. this partition is finer than any individual information partition. Further, the $\mathcal{F}$-measurable *system of the prices* $p : \Omega \to \mathbb{R}_+^l$ is considered that defines budget sets:

$$B_i(p) = \{(x, P) \in \mathcal{X}_i \mid P = P_i^0 \;\&\; \sum_{\omega\in\Omega} x_i(\omega) p(\omega) \le \sum_{\omega\in\Omega} e_i(\omega) p(\omega)\}, \quad i \in \mathcal{I}.$$

*A couple (p, x) such that $x = (x_i)_{\mathcal{I}}$, $x_i \in B_i(p)$, $i \in \mathcal{I}$ is called **Walrasian expectations or Radner equilibrium** if the following conditions are satisfied:*

(i) $\forall\, i \in \mathcal{I}$ *the plan x_i maximizes on $B_i(p)$ expected utility* $\sum_{\omega\in\Omega} u_i(\omega, x_i(\omega)) q_i(\omega)$;

(ii) $\sum_{i\in\mathcal{I}} x_i \le \sum_{i\in\mathcal{I}} e_i \;\;\&\;\; \sum_{\omega\in\Omega} p(\omega) \sum_{i\in\mathcal{I}} x_i(\omega) = \sum_{\omega\in\Omega} p(\omega) \sum_{i\in\mathcal{I}} e_i(\omega)$.

Notice, that this is an free disposal equilibrium concept formed on stage before the realization of any state of the nature and in the absence of any kind of an additional information about the realized event.

• REE — *rational expectations equilibrium.* As well as for WEE-equilibrium there is a partition $\mathcal{F}$ and $\mathcal{F}$-measurable *system of the prices* $p : \Omega \to \mathbb{R}_+^l$. It is supposed that having received price signals, agents are able to extract from them an information and then they use it to make own

[10] It is important to remember that the contract $x - e_i$ has to be measurable relative to both partitions P and P'.

[11] In general $q_i(\cdot)$ is the density of a priori probability distribution (prior).

[12] In general settings this is σ-algebra of events.

rational decisions. Formally, function $p(\cdot)$ generates a partition $\sigma(p)$ of Ω, this is the coarsest partition concerning which $p(\cdot)$ is measurable. Having learnt $\sigma(p)$, the agent i has the access to the information $\sigma(p) \vee P_i = \mathcal{G}_i$.

*A couple (p, x) such that $x = (x_i)_{\mathcal{I}}$, $x_i \in E_+^{\Omega}$, $i \in \mathcal{I}$ is called **rational expectations equilibrium**, if the following conditions are satisfied:*

(i) $\forall\, i \in \mathcal{I}$ *the plan* $x_i : \Omega \to \mathbb{R}_+^l$ *is* $\mathcal{G}_i$*-measurable;*

(ii) $\forall\, i \in \mathcal{I}$ *and* $\forall\, \omega \in \Omega$ *the plan* x_i *maximizes the conditional expected utility (interim expected utility)* $\sum_{\omega' \in \Omega} u_i(\omega', x_i(\omega')) q_i(\omega' | \mathcal{G}_i)(\omega)$[13] *under budget constraint*[14] $p(\omega)x_i(\omega) \leq p(\omega)e_i(\omega)$*;*

(iii) $\sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} e_i$.

Notice, that the concept of REE-equilibrium is intermediate (interim) one since all individual decisions are made not before but after a price signal. Another difference with WEE-equilibrium consists in that now there is a set of budget constraints — one for each event that the agent is able to understand.

• *Core.* Core concept in economy with the differentiable information has own specificity and, in difference with the complete market, is generally not unequivocally defined. This is so because besides a physical feasibility of an allocation it is necessary to consider its information feasibility, i.e., its measurability concerning the information partition(s). Thus, it is necessary to define what kind of measurability is applied for a final allocation in an economy as a whole and what measurability requirements are applied for the intra-coalition allocations to dominate current distribution of resources. Available in literature (not all!) and other possible variants of these requirements are presented in the following table:

econ.\coal.	$\wedge_S k_S^i(\mathbb{P})$	$\wedge_S P_i$	P_i	$\{k_S^i(\mathbb{P})\}$	P_i^{lim}	$\vee_S P_i$	$\vee_S k_S^i(\mathbb{P})$	author
$\wedge_{\mathcal{I}} P_i$		strong coarse	α			β		Yannelis
P_i		coarse	private			fine		Yannelis
$\vee_{\mathcal{I}} P_i$		γ	δ			weak fine		Yannelis
$\forall$ f-allocation		coarse				fine		Wilson
$\wedge_{\mathcal{I}} k_{\mathcal{I}}^i(\mathbb{P})$	coarse							Allen
$\{k_{\mathcal{I}}^i(\mathbb{P})\}$							fine	Allen
$\{P_i^{max}\}$				k-core				Schwalbe
$\{P_i^{lim}\}$				k-limit	limit			Marakulin

Here the first column presents requirements to allocation in economy and the first row to intra-coalition allocations. The intersection of a column and a row presents the name of core concept, where in the last row two new concepts are presented and these notions will be studied in the project. Resuming this table one can notice that the approach using an information rule is the most general: commonly it is enough to specified a rule.

[13] Here $q_i(\omega'|\mathcal{G}_i)(\omega)$ is conditional probability distribution when the element $\mathcal{G}_i(\omega)$ of partition $\mathcal{G}_i$ is realized.

[14] It is useful to remember that in view of $\mathcal{G}_i$-measurability of all functions from the inequality and therefore that all of them are constant on $\mathcal{G}_i(\omega)$, because of homogeneity of budget inequality, this inequality is true on the whole $\mathcal{G}_i(\omega)$ (according to a content) and not only for the state ω.

4. CONTRACT BASED APPROACH AND DIE-ECONOMY

It is a firm author's opinion that in the investigation of DIE-economies it is possible to apply effectively contractual approach developed in Marakulin (2003, 2006). Further let us recall briefly its substantive provisions.

Formally any reallocation of commodities $v = (v_i)_{i\in\mathcal{I}}$, where v_i is an element of commodity space L, $i \in \mathcal{I}$, i.e., any vector v satisfying $\sum_{i\in\mathcal{I}} v_i = 0$ is called a (barter) *contract*. In contractual economy model there is exogenously specified a set W of *permissible* contracts. The *finite*[15] set V of elements from $\mathcal{W}$ is called *a web of contracts* relative to $y \in L^n$ if $\forall\, U \subset V$ an allocation $y + \sum_{v\in U} v$ is feasible, i.e., any subset of contracts can be broken without damage for a feasibility of an allocation realized. Let $x(V) = e + \sum_{v\in V} v$ denote an allocation realized via web V relative to initial endowments $e = (e_i)_{i\in\mathcal{I}}$. It is supposed, that any contract v from the web V can be *broken off* by any agent from $\mathrm{supp}\,(v) = \{i \in \mathcal{I} \mid v_i \neq 0\}$ since he/she simply may not keep his/her contractual obligations. Also a non-empty group (coalition) of consumers can *sign* any number of new contracts. Being applied jointly, i.e., as a simultaneous procedure, these operations allow coalition $T \subseteq \mathcal{I}$ to yield new webs of contracts. The set of all such webs is denoted by $F(V, T)$. It is required formally that each element $U \in F(V, T)$ has to satisfy the following properties:
(i) $v \in V \setminus U \;\Rightarrow\; S(v) \cap T \neq \emptyset$ (only members of T are able to break off contracts from V),
(ii) $v \in U \setminus V \;\Rightarrow\; S(v) \subset T$ (only members of T can sign new contracts).
In contract-based approach the notion of domination via a coalition is extended onto webs of contracts. This property of domination via coalition $T \subseteq \mathcal{I}$, being written as $U \underset{T}{\succ} V$ (U dominates V via coalition T), means that
(i) $U \in F(V, T)$,
(ii) $x_i(U) \underset{i}{\succ} x_i(V)$ for all $i \in T$.

A *web* of contracts V is called *stable* if there is no web U and no coalition $T \subseteq \mathcal{I}$, $T \neq \emptyset$ such that $U \underset{T}{\succ} V$.

A *web* of contracts V is called *lower stable* if there is no web U and no coalition $T \subseteq \mathcal{I}$, $T \neq \emptyset$ such that $U \underset{T}{\succ} V$ and $U \subseteq V$.

A *web* of contracts V is called *upper stable* if there is no web U and no coalition $T \subseteq \mathcal{I}$, $T \neq \emptyset$ such that $U \underset{T}{\succ} V$ and $V \subseteq U$.

An allocation x is called *contractual* (lower, upper contractual) if $x = x(V)$ for a stable (lower, upper stable) web V.

In any standard market every core allocation allows an alternative description as contractual one; accordingly, Pareto optimal allocations correspond to upper contractual ones, and individual rational allocations are lower contractual ones etc. The concept of proper contractual allocation is also important, this concept realizes (assumptions: interior point, smooth preferences) an alternative description of equilibria.

The notion of proper contractual allocation is introduced due to the following construction.

[15] It is assumed for simplicity. In general it is possible to consider infinite families of contracts, especially for a contractual trajectories where derivative at a time moment t corresponds to a momentary contract and current allocation is presented as an initial allocation plus an integral of momentary contracts over time period $[0, t]$.

Given a web of contracts V and scalars α_v, $0 \le \alpha_v \le 1$, $v \in V$ define $\alpha V = \{\alpha_v \cdot v\}_{v \in V}$ and consider the web $U = (\alpha V) \cup ((\mathbf{1} - \alpha)V)$, where $\mathbf{1} = (1, 1, \ldots, 1)$. Thus contracts from U are produced from the contracts from V due to partition into several contracts (decomposition in a sum) under condition of preservation of exchange proportions and volumes of exchanged commodities.

• *An allocation x is called proper contractual if there exits a web V such that $x = x(V)$ and for every vector $\mathbf{0} \le \alpha \le \mathbf{1}$ and the web $U = (\alpha V) \cup ((\mathbf{1} - \alpha)V)$ the allocation $x = x(U)$ is contractual.*

The economic meaning of proper contractual stability of an allocation is, that we allow the agents not only to sign new contracts but also to partially break contracts if exchange proportions remain constant. This extends agents' operating potentialities and approaches contractual processes to market processes under perfect competition conditions. In Kozyrev (1982, 1981) there were stated (see also Marakulin (2003)) that for an economy with perfect information under some technical assumptions proper contractual allocations are competitive equilibria. This fact takes place only for differentiable utilities and when the allocation is an interior point of consumption sets. In a context of this study this result is quite important one and let us consider it in more details at least for an example in Edgeworth box.

Let there be a two agents and two commodities exchange economy and an proper contractual allocation (x, y), $x + y = e^1 + e^2 = \bar{e}$, where $e^1, e^2 \in \mathbb{R}^2_+$, $x, y \in \mathrm{int}\mathbb{R}^2_+$ and $x = (x_1, x_2)$ denotes 1st agent consumption bundle, $y = (y_1, y_2)$ is applied for second one and e^1, e^2 denote the agents' initial endowments. Now notice that every proper contractual allocation has to be Pareto optimal and in view of stability relative to partial breakings $e^1 + \alpha(x - e^1) \precsim_1 x$, $e^2 + \alpha(y - e^2) \precsim_2 y$ take place for every $0 \le \alpha \le 1$. In Edgeworth box and in geometrical terms this means that the sets $\mathcal{P}_1(x)$, $\{\bar{e}\} - \mathcal{P}_2(\bar{e} - x)$ and $\mathrm{conv}\{x, e^1\}$ have pairwise empty intersections;[16] here $\mathcal{P}_i(z)$ denotes the set of all bundles strictly preferred by agent i to $z \in \mathbb{R}^2_+$. So, this property unambiguously characterizes proper contractual allocations and moreover for smooth preferences and the interior allocation the linear segment $\mathrm{conv}\{x, e^1\}$ can be unambiguously extended to a hyperplane separating two preferred points sets. It is easy to see that a normal vector to the hyperplane can be taken as a vector of equilibrium prices. The following example illustrates the fact and shows that when assumptions are invalid then not every proper contractual allocation is equilibrium.

Let $u_1(x_1, x_2) = 2\sqrt{x_1 x_2} + x_1 + x_2$, $u_2(y_1, y_2) = 2\sqrt{y_1 y_2} + y_1 + y_2 + \min\{y_1, y_2\}$, where a lower index denotes a number of commodity. Initial endowments: $e_1 = (1, 0)$, $e_2 = (0, 1)$, $\bar{e} = e_1 + e_2 = (1, 1)$. Now one obtains the following Figure 1 (Edgeworth box). One can see that the only equilibrium point is $x = (\frac{1}{2}, \frac{1}{2})$ but every point of the linear segment $\mathrm{conv}\{(\frac{1}{2}, \frac{1}{2}), (\frac{3}{5}, \frac{3}{5})\}$ corresponds to proper contractual allocation. A detailed study of the example one can find in Marakulin (2003).

In context of contractual approach there is another alternative possibility to describe equilibria in specific contractual terms: it can be done even under weaker model assumptions. This is realized due to concept of *fuzzy contractual* allocation that is described below.

The notion of properly contractual allocation presumes that agents are able to partially break contracts in such a way that every contract may be divided in several contracts with equal

[16] The first one corresponds to Pareto optimality and the other ones to the stability of allocation relative to partial breakings. If A and B are subsets of a vector space then $A - B = \{a - b \mid a \in A, b \in B\}$.

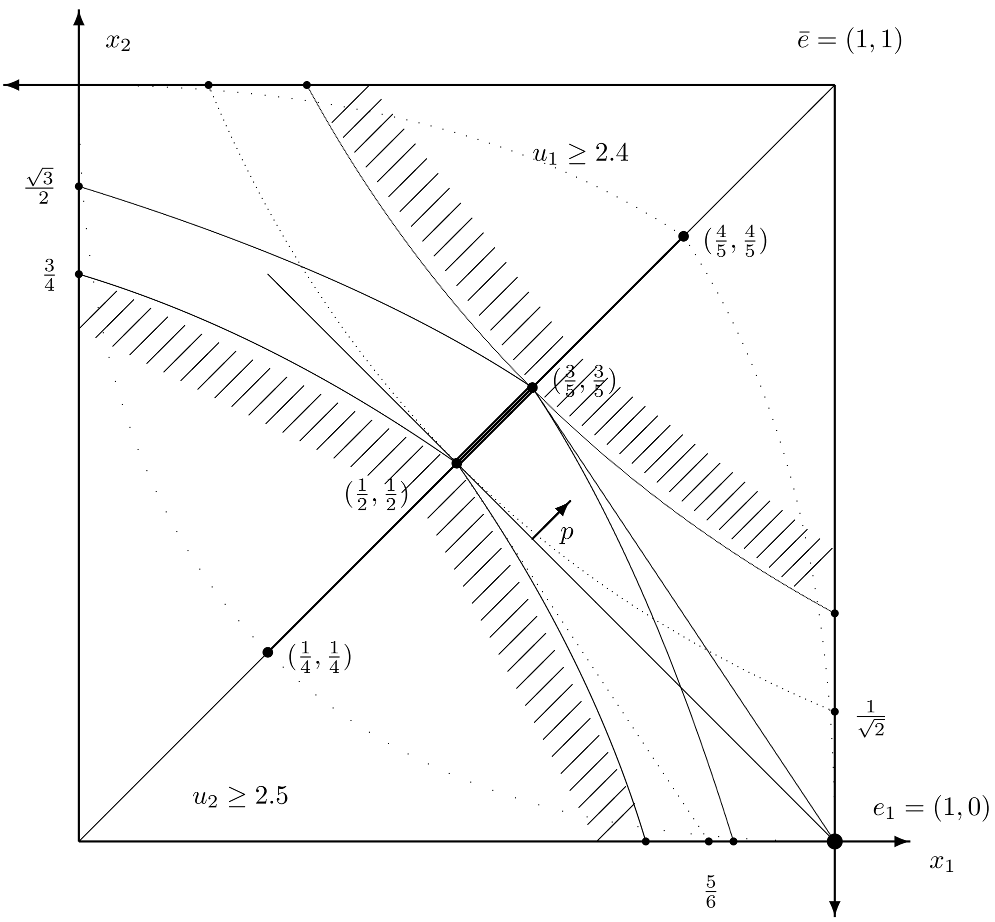

Fig. 1. Proper contractual allocations for non-smooth preferences.

exchange proportions and some of these contracts may be broken, i.e., instead of contract $v \in V$ the agents may deal with a finite family of contracts $\{u_\xi\}$, such that $\sum u_\xi = v$ and $u_\xi = \lambda_\xi v$ for some real $\lambda_\xi \geq 0$ for all ξ. Thus for partial breaking of contract v the members of coalition $S = \text{supp}(v)$ have to coordinate their actions. Relaxing this coordination requirement for fuzzy contractual allocation we allow the agents to break contracts *asymmetrically* and together with $\sum u_\xi = v$ to require $(u_\xi)_i = \lambda_{\xi i} v_i$ for a real $\lambda_{\xi i} \geq 0$ for all ξ and i. Notice that now vectors u_ξ *may not be contracts* at all, since $\sum_{i \in \mathcal{I}} u_{\xi i} = 0$ may not hold.

Let V be a web of contracts. For every $v \in V$ consider and put into correspondence a n-dimension vector

$$t^v = (t_1^v, t_2^v, \dots, t_n^v), \quad 0 \leq t_i^v \leq 1, \quad \forall\, i \in \mathcal{I},$$

and let

$$v^t = (t_1^v v_1, t_2^v v_2, \dots, t_n^v v_n)$$

be the vector of commodity bundles formed from contract $v = (v_i)_{i \in \mathcal{I}}$ when all agents "break" individual bundles (fragments) of this contract in shares $(1 - t_i^v)_{i \in \mathcal{I}}$. Denote $T(V) = T = \{t^v \mid v \in V\}$ and introduce

$$V^T = \{v^t \mid v \in V,\ t^v \in T\}, \quad \Delta(V^T) = \sum_{v^t \in V^T} v^t. \tag{1}$$

Definition 2. *An allocation x is called fuzzy contractual if there exists a proper*[17] *web V such that $x = x(V)$ and for every $T(V)$ the allocation $x^T = e + \Delta(V^T)$ is upper contractual.*

In economic terms this notion can be explained in the following way. During recontracting agents may make mistakes, coordination among coalition members may work imperfectly and so on. As a result an agent i can (erroneously) think that after partial breaking of current contracts he/she will have a commodity bundle x_i^T and that commodities from x_i^T may be exchanged in a new (mutually beneficial) contract. If allocation $x(V)$ is not fuzzy contractual then the last may (potentially) destroy agreements and allocation will be changed. Thus fuzzy contractual allocations are protected from this kind of agreements destructions.

We start from a preliminary result describing mathematical properties of fuzzy contractual allocations that is of interest in its own right.

Proposition 4.1. *An allocation x is fuzzy contractual if and only if*

$$\text{co}\{x_i, e_i\} \cap \mathcal{P}_i(x_i) = \emptyset, \quad \forall\, i \in \mathcal{I}, \tag{2}$$

$$\prod_{\mathcal{I}} [(\mathcal{P}_i(x_i) + \text{co}\{0, e_i - x_i\}) \cup \{e_i\}] \bigcap \{(z_1, \dots, z_n) \in E^{\mathcal{I}} \mid \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \omega_i\} = \{e\} \tag{3}$$

are true.

Notice that in this proposition $\mathcal{P}_i(x_i) = \emptyset$ is possible for some $i \in \mathcal{I}$: by definition $\emptyset + A = \emptyset$ for any A. Relation (2) is equivalent to the allocation x is lower properly contractual one, i.e. this is so that nobody wants to partially break contracts. Proposition 4.1 allows to conclude that fuzzy contractual allocations are the elements of fuzzy core, see Marakulin (2003) and

[17] This is a web stable relative to partial contracts breakings.

section 5.3 below (Proposition 5.2 and Corollary 5.4). At the same time equilibrium properties of fuzzy core allocations (they present quasiequilibria) are well known in literature. Proof of Proposition 4.1 is presented in section 9.

In a context of DIE-economy it seems possible to apply every known concept of contractual approach and the concepts of proper and fuzzy contractual allocations are the most important among others because of their relationships with equilibrium. First of all one has to specify a set $\mathcal{W}$ of all permissible contracts — at least one has to require the appropriate measurability of contracts. Then it will necessary to present contractual characterization of equilibria and cores of different forms (fuzzy core too) applied for DIE-economies. Finally, for equilibria which are still described as proper contractual allocations (or even as fuzzy contractual), one needs to give a price characterization. This way may produce some new equilibrium concepts and clarify some properties of known ones.

5. CONTRACTUAL VIEWS ON EQUILIBRIUM AND CORE OF INFORMATION DIFFERENTIAL ECONOMIES

5.1. The refinement and assumptions in DIE-economies

The subject of the study is an economic exchange model with asymmetrically informed agents (differentiated information), that can be shortly written in the following way:

$$\mathcal{E}^{di} = \langle \mathcal{I}, \mathbb{R}^l, \Omega, (k_S(\cdot))_{S\in C}, (X_i, \mathcal{P}_i, P_i, e_i)_{i\in\mathcal{I}} \rangle.$$

Here:

$\mathcal{I} = \{1, \dots, n\}$ is a (finite) set of agents (traders or consumers);

$\mathbb{R}^l$ denotes a *space* of (physical) *commodities* and $l \in \mathbb{N}$ is their number;

Ω is a *finite* set of elementary events (or states of the world);

$L = (\mathbb{R}^l)^\Omega$ is a *space* of contingent *commodities.*

For each $i \in \mathcal{I}$ there are specified:

$X_i = L_+$ consumption set of agent i;

$\mathcal{P}_i : X_i \Rightarrow X_i$ a point-to-set mapping, defines *preference relation*, where $\mathcal{P}_i(x_i)$ is a set of all consumption bundles strictly preferred by i to bundle $x_i \in X_i$. It is also applied $y_i \succ_i x_i \iff y_i \in \mathcal{P}_i(x_i)$;

P_i is a partition of Ω, characterizes agent i's initial information;

$e_i \in X_i$ initial endowments of contingent commodities.

In the model context it can be presented also an information *rule of economy* $k = (k_S)_{S\in C}$, where $k_S : (P^*)^S \to (P^*)^S$ is an informational rule of coalition $S \in C = 2^{\mathcal{I}} \setminus \{\emptyset\}$, such that $k_{\{i\}}(P) = P$, $\forall\, i \in \mathcal{I}$ and P^* is a set of all partitions Ω.

One will assume without lost of generality that

$$\bigcap_{\mathcal{I}} P_i(\omega) = \{\omega\},\ \forall\, \omega \in \Omega \iff \bigvee_{\mathcal{I}} P_i = \Omega^*, \qquad (4)$$

where $P_i(\omega)$ is an element of partition P_i containing the state $\omega \in \Omega$, and Ω^* is the partition into one-element sets (perfect information).

Let $e = (e_i)_{i\in\mathcal{I}}$ denote a vector of initial endowments of all traders of model under study, define $X = \prod_{i\in\mathcal{I}} X_i$ and put

$$\mathcal{A}(X) = \{x \in X \mid \sum_{i\in\mathcal{I}} x_i = \sum_{i\in\mathcal{I}} e_i\} \; -$$

this is the set of all *physically feasible allocations* in the model $\mathcal{E}^{di}$. Further I shall always assume that model $\mathcal{E}^{di}$ obeys the following assumptions:

(A) *For each* $i \in \mathcal{I}$: $e_i \in X_i$ *and every* $x_i \in X_i$ *there is an open convex* $G_i \subset L$ *such that* $\mathcal{P}_i(x_i) = G_i \cap X_i$ *and* $x_i \in \overline{\mathcal{P}_i(x_i)} \setminus \mathcal{P}_i(x_i)$.[18]

If preferences can be represented by means of the differentiable utility functions one will say about smooth model: The economy $\mathcal{E}^{di}$ is called ***smooth*** if for every $i \in \mathcal{I}$

$$\mathcal{P}_i(x_i) = \{y \in X_i \mid u_i(y) > u_i(x_i)\} \quad \forall\, x_i \in X_i$$

for some *differentiable* functions u_i, defined on (some) open neighbourhood of set X_i, and $\nabla u_i(x_i) \neq 0\ \forall\, x_i \in X_i$.

The structure of contractual economy can be introduced for the model $\mathcal{E}^{di}$ by means of the set $\mathcal{W} \subset L^{\mathcal{I}}$ of *admissible* (permissible) contracts. It is postulated in general case that $\mathcal{W}$ is *star-shaped* at zero[19] in $L^{\mathcal{I}}$. Moreover, in contractual context there is no essential need to consider a generalized space of information admissible commodities, and all subjects connected with the informational admissibility of a consumption bundle are possible to lower on level of concluded contracts and the specification of the set $\mathcal{W}$.

First one important note concerning preferences. In the known literature there is widely used the expected utilities calculated through randomized utility functions $u_i(\omega, x_i(\omega))$, $x_i : \Omega \to \mathbb{R}^l_+$, $i \in \mathcal{I}$. I think that it has been happened because of the need to apply conditional expectation of utility: for introduction and studies of such concepts as an equilibrium in rational expectations (REE) and incentive compatibility — stability relative to potentially misreported states of the world. However it seems obviously, that one possible to manage without this (expected) specification of preferences. The point is that for a realized (or not) event E preferences should not be directly connected with preferences for $\omega \notin E$. Here I speak about preferences as ordinal category, instead of utility, which of course will depend on consumption in $\omega \notin E$. This means that preferences of the individual can be defined, for example, by means of an generalized-separable function having the following functional form:

$$u_i(x_i(\cdot)) = \psi_i(\varphi^i_1(x_i(\omega_1)), \varphi^i_2(x_i(\omega_2)), \ldots, \varphi^i_k(x_i(\omega_k))), \quad \Omega = \{\omega_1, \omega_2, \ldots, \omega_k\}. \tag{5}$$

For a correct presentation it is necessary to assume, that function $\psi_i(\cdot)$ is continuous and strictly monotonically increasing by each of arguments, and $\varphi^i_\omega(\cdot)$ are ordinary quasi-concave functions setting preferences.[20] It is obvious, that for any fixed consumption outside of any

[18] The symbol $\overline{A}$ denotes the closure of A and $\setminus$ is set for the set-theoretical difference.

[19] It means: $v \in \mathcal{W} \ \Rightarrow\ \lambda v \in \mathcal{W}\ \ \forall\, 0 \le \lambda \le 1$.

[20] These are randomized utilities mentioned above, then they are curtailed by a means of nonlinear function ψ_i.

(only one!) elementary event $\omega \in \Omega$ (or for an event $E \in P_i$ any function of presented form induces the identical preferences on the space contingent commodities reduced to this state ω. This corresponds to the notion of *weak* separable utility.

Strongly separable utility has the following functional form:

$$u_i(x_i(\cdot)) = \psi_i(\varphi_1^i(x_i(\omega_1)) + \varphi_2^i(x_i(\omega_2)) + \cdots + \varphi_k^i(x_i(\omega_k))), \quad \Omega = \{\omega_1, \omega_2, \ldots, \omega_k\}. \tag{6}$$

Essential difference is that for every *set* $E \subset \Omega$ of elementary events for contingent commodities related with states from E, there is inducted a preference relation that does not depend on current consumption in external for E states $\omega \in \Omega \setminus E$. For the proof of this result see Barten, Bohm (1982), Debreu (1960). Moreover, the requirements for a general case are clear also: it is necessary, that consumption in events exterior (distinct from) to any set of elementary events do not influence preferences within the limits of the given event. Formally it should be in general case of strong separability

$$\mathrm{Pr_E}\left[\mathcal{P}(x_{|E}, y_{|\Omega\setminus E}) \bigcap ((x_{|E}, y_{|\Omega\setminus E}) + \mathcal{L}^E)\right] = \mathrm{Pr_E}\left[\mathcal{P}(x_{|E}, z_{|\Omega\setminus E}) \bigcap ((x_{|E}, z_{|\Omega\setminus E}) + \mathcal{L}^E)\right] \tag{7}$$

$$\forall \, measurable \; y_{|\Omega\setminus E}, z_{|\Omega\setminus E} : \Omega \setminus E \to \mathbb{R}^l_+,$$

where $\mathrm{Pr_E}[\cdot]$ is a projecting operator onto the subspace of functions defined on $E \subset \Omega$. This can be defined as a multiplication of a function from L on characteristic function χ^E of set $E \subseteq \Omega$.[21] Further I shall assume these preferences properties or if it is necessary the functional form (5). Moreover preferences are assumed to be non-satiated relative to every elementary event, i.e., in terms of (5) it has to be:

$\forall\, \omega \in \Omega$, $\forall\, i \in \mathcal{I}$ functions $\varphi^i_\omega : \mathbb{R}^l_+ \to \mathbb{R}$ are quasi-concave and locally non-satiated on $\mathbb{R}^l_+$.

Further let us turn to the contractual analysis of private core and a related equilibrium concept.

5.2. WEE-equilibria and private core, WEE-lim-equilibria and limit core

Before to formulate and prove different mathematical statements, let us attempt to understand that contractual approach can add in our understanding of functioning of the economy with asymmetrically informed agents. Let's imagine that is "today" and "tomorrow" and that ***today*** we need to plan our tomorrow consumption. We have not exact information what will happen tomorrow, but at least for ourselves we know that exactly we *will be able* to understand tomorrow, i.e. when this "tomorrow" will start to be realized. For the model those events which we will be able to understand tomorrow, form a ***partition*** of the set Ω of all elementary events of tomorrow. For different agents these partitions are different ones and this is informational asymmetry exhibited. Planning the tomorrow's consumption the individual should agree with other agents on what commodity changes will be made tomorrow. However a satisfaction of the individual from the consumption tomorrow commodities essentially depends on what events will happen tomorrow. For example, if I plan tomorrow a trip to the nature, but there will be a rain, I will get wet and by the evening my temperature will rise then it will be required febrifugal and other medicines. If the rain will not be realized, then for me the value of an umbrella and medical products will be insignificant. However already today I should provide

[21] It is defined as $\chi^E(\omega) = 1$ for $\omega \in E$ and zero otherwise.

a possibility of a tomorrow rain and conclude agreements on the change of some commodities (money?) on an umbrella and medicines. Moreover the change on an umbrella will be desirable for making only when it will be precisely known, that the rain will happen, and concerning medicines — when the temperature has already risen (or it is precisely known, that will rise). Here it is important, that the individual first of all ***should understand event*** which specifies the exchange. It, of course, concerns each side in the exchange agreement that we intend to conclude already today; the contract concluded in this agreement should be began from the text: "If there will be a rain, then..." and/or from the text: "If there will be a rain and my temperature will rise..." The mathematical design of this situations can be the following one.

A (barter) contract $v = (v_i)_{\mathcal{I}}$ is a tuple maps $v_i : \Omega \to \mathbb{R}^l$, $i \in \mathcal{I}$, satisfying to measurability

$$\forall\, i \in \mathcal{I}, \quad v_i(\cdot) \text{ measurable subject to } P_i, \tag{8}$$

and also to a standard balancing assumption:

$$\forall\, \omega \in \Omega, \quad \sum_{i \in \mathcal{I}} v_i(\omega) = 0. \tag{9}$$

The first of these requirements says that if an individual is not able to distinguish one elementary event from another one. For example if a and b from Ω are indistinguishable for i, it means that a and b have placed in a common element of an informational partition, then obligations of an individual under the contract should be identical, i.e. it should be $v_i(a) = v_i(b)$ and the function as a whole should be constant on each of elements of an informational partition. In this context condition (8) restricts the area $\mathcal{W}$ of permissible contracts.

If there are no other restrictions imposed and regarding contracts breaking one supposes only a possibility of their full break, then as a contractual allocation we will receive in accuracy allocations from the private core which definition is the following one.

Definition 3. *Private core $\mathcal{C}^{pr}(\mathcal{E}^{di})$ for an economy $\mathcal{E}^{di}$ with asymmetrically informed agents consist of allocations $x = (x_i)_{\mathcal{I}} \in X$ such that:*

(*i*) $\sum\limits_{i \in \mathcal{I}} x_i = \sum\limits_{i \in \mathcal{I}} e_i$,

(*ii*) $(x_i - e_i) : \Omega \to \mathbb{R}^l$ *is P_i-measurable for all $i \in \mathcal{I}$,*

(*iii*) $\nexists S \subseteq \mathcal{I} : \exists y^S = (y_i)_S \mid \forall\, i \in S,\ y_i \in X_i$ *is such that $(y_i - e_i)$ is P_i-measurable,* $y_i \succ_i x_i$ & $\sum_S (y_i - e_i) = 0$.

Here condition (*iii*) presents usual coalitions non-domination requirement considered with the account of measurability of dominating allocation relative to the private information. Clearly one can take $V = \{x - e\}$ as a web of contracts implementing allocation x from the core $\mathcal{C}^{pr}(\mathcal{E}^{di})$. The coincidence of concepts (initial and contractual) is checked directly just one needs take into account the differences in notation and terminology. Further let us turn to equilibrium notion.

According to the general methodology of the contractual approach an allocation implemented by a web of admissible contracts ***cannot*** be an equilibrium if the web is ***upper unstable*** or it is ***unstable*** relative to ***partial breakings*** of contracts. May be it will not enough to suggest a correct definition of equilibrium in the most general setting, but these properties should be fulfilled with necessity. So, to give a possible equilibrium treatment one needs at least to characterize upper stable webs of contracts.

Let's recall that the web of contracts V is called upper stable, if the allocation implemented by this web $x = e + \sum_{v \in V} v$, there are no coalition that has a mutually beneficial contract:

$$\nexists\, v = (v_i)_{\mathcal{I}} \in \mathcal{W}: \quad x_i + v_i \succ_i x_i \quad \forall\, i \in \text{supp}\,(v).$$

Further we shall need the following

Lemma 5.1 (ABOUT MUTUALLY BENEFICIAL CONTRACT). *Let $S \subseteq \mathcal{I}$, $S \neq \emptyset$ be a coalition and $A \subseteq \Omega$ be an event understandable by every coalition S member.*[22] *Then if* ***there are no*** *mutually beneficial exchange of contingent commodities for coalition S members then there does exist a vector $p \in (\mathbb{R}^l)^A$, $p \neq 0$, and vectors $q_i \in (\mathbb{R}^l)^A$, $i \in S$, such that*

$$\forall\, i \in S \;\; \forall\, E \in P_i,\; E \subseteq A \quad \sum_{\omega \in E} q_i(\omega) = 0 \tag{10}$$

and

$$\forall\, i \in S \quad p + q_i \neq 0 \quad \& \quad \langle \mathcal{P}_i(x_i), p + q_i \rangle \geq \langle x_i, p + q_i \rangle \tag{11}$$

holds.

Inverse: let there be vectors satisfying (10), (11) and in (11) at least one inequality is strict.[23] *Then for a coalition S* ***there is no*** *mutually beneficial contract.*

The lemma implies two important for further considerations corollaries.

Corollary 5.1. *Let in Lemma 5.1 conditions preferences be described via differentiable utility functions and let $x = (x_i)_{\mathcal{I}}$ be an interior relative to A and S allocation.*[24] *Then for the coalition S* ***there is no*** *mutually beneficial contract if and only if when there exists a vector $p \in (\mathbb{R}^l)^A$, $p \neq 0$ and $\lambda_i > 0$, $i \in S$, such that*

$$\forall\, E \in P_i,\; E \subseteq A \quad \lambda_i \sum_{\omega \in E} \nabla_\omega u_i(x_i) = \sum_{\omega \in E} p(\omega) \quad \forall\, i \in S.$$

The requirements in the last relations can be considered as a (linear) system of equations subject to $\lambda_i > 0$, $i \in \mathcal{I}$ and $p(\omega) \in \mathbb{R}^l$, $\omega \in A$. If a solution does exist then contract is impossible. The following corollary presents the most simply verified criterium of mutually beneficial contract existence.

Corollary 5.2. *Let in Lemma 5.1 and its Corollary 5.1 conditions an event $E \in P_i$ for each $i \in S$, i.e. all coalition S members do not differentiate states from E. Then mutually beneficial contract does exists if and only if $h_i = \sum_{\omega \in A} \nabla_\omega u_i(x_i(\cdot))$, $i \in S$ form a non-collinear system of vectors.*

[22] That is $\forall\, i \in S$ set A is P_i-measurable.

[23] That is $\exists i : \langle \mathcal{P}_i(x_i), p + q_i \rangle > \langle x_i, p + q_i \rangle \iff \langle y, p + q_i \rangle > \langle x_i, p + q_i \rangle \;\forall\, y \in \mathcal{P}_i(x_i)$.

[24] Therefore $x_i(\omega) \in \text{int}\mathbb{R}^l_+$, $\omega \in A$, $i \in S$.

So as soon as by definition of upper stable proper contactual allocation in the case $A = \Omega$ and coalition $S = \mathcal{I}$ mutually beneficial contract is impossible then one can apply Lemma 5.1 conclusion. This implies the existence of map $p : \Omega \to \mathbb{R}^l$, that satisfies (11) and it can be taken as a *price* map.

Further to derive an equilibrium notion one has also reveal stability relative to partial contracts breakings. For differentiable utilities this is equivalent to requirement

$$\langle \nabla u_i(x_i), x_i - e_i \rangle \geq 0, \quad \forall\, i \in \mathcal{I},$$

that for interior points yields

$$\langle p + q_i, x_i - e_i \rangle \geq 0, \quad \forall\, i \in \mathcal{I}.$$

Clear that all inequalities can be realized here only in the form of strict equalities. Notice also that one has considered only the case of single-element web of contracts $V = \{x - e\}$. Further, in view of (8), (10) for every contract $v = (v_i)_{\mathcal{I}} \in \mathcal{W}$ one has $\langle q_i, v_i \rangle = 0$, $\forall\, i \in \mathcal{I}$, that under presented assumptions means that previous relations are equivalent to $\langle p, (\mathcal{P}_i(x_i) - e_i) \cap \mathcal{L}_i \rangle > 0$, $i \in \mathcal{I}$, where

$$\mathcal{L}_i = \mathbf{Map}_{P_i}(\Omega, \mathbb{R}^l) = \{f \mid f : \Omega \to \mathbb{R}^l \ is\ P_i - measurable\}, \quad i \in \mathcal{I}.$$

As a result I am going to the following equilibrium concept that according to contractual point correctly corresponds to the notion of private core.

Definition 4. *A couple (x, p), $x = (x_i)_{\mathcal{I}} \in X$, $p : \Omega \to L'$, $p \neq 0$ is called ex ante private* ***quasi-equilibrium*** *if it satisfies:*

- *(i)* $(x_i - e_i) : \Omega \to \mathbb{R}^l$ *is P_i-measurable for all $i \in \mathcal{I}$,*
- *(ii)* $0 \not\equiv \langle p, (\mathcal{P}_i(x_i) - e_i) \cap \mathcal{L}_i \rangle \geq 0$, $i \in \mathcal{I}$,
- *(iii)* $\sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} e_i$.

If in (ii) all inequalities are ***strict****, then the pair (x, p) is called* ***equilibrium****.*

The direct comparison of definitions shows that this concept coincides with WEE-equilibrium for a more general form of preferences and when feasibility conditions are presented in the form of equality.

5.3. Private core and equilibria in perfect competition conditions

In the previous section there were considered notions of contractual allocation and private equilibrium closely related with the known in literature notions of private core and WEE-equilibrium. However it was not clearly identified the relationship between contractual allocation and private equilibrium: it is necessary to clarify perfect competition case when being correctly defined core and equilibrium have to coincide. While we are looking for the answer this quest we also clarify conditions under which ex ante private equilibria do exist.

Below there are applied standard analytic methods based on the study of replicated economies. Further I first recall known definitions adapting them to our context. First of all I briefly study the existence of core for model $\mathcal{E}^{di}$.

For a model of differential information economy one can put into correspondence some cooperative game with non-transferable utility (to be short, a NTU-game). Recall that formally a NTU-game (for details, see Moulin (1988), for example) is a couple, $(\mathcal{I}, (V(S))_{S\subseteq\mathcal{I}})$, described by the set of players (agents) $\mathcal{I} = \{1, \dots, n\}$, $(n \geq 2)$ and the sets of permissible vector-payoffs $V(S) \subseteq \mathbb{R}^S$ for every (nonempty) coalition $S \subset \mathcal{I}$, which have to satisfy the following properties:

- $V(S)$ is the nonempty closed subset in $\mathbb{R}^S$,
- $V(S)$ is comprehensive from below, i.e., $x \in V(S)$ and $y \leq x$ imply $y \in V(S)$,
- the set of all individual-rational vector-payoffs from $V(S)$, is by definition the set
$$Q(S) := \{v \in V(S) \ \mid \ v_i \geq V(\{i\}) \ \forall\, i \in S\}, \tag{12}$$
which is nonempty and bounded from above in $\mathbb{R}^S$.

Now let us consider a game construction corresponding to the concept of ex ante private core in an economic model and where informational partitions P_i^S, $i \in S$, $S \subseteq \mathcal{I}$ are applied in an appropriate way. Partition P_i^S describes information that individual $i \in S$ can apply in the finding of a dominating allocation via coalition S. In this case the set of all permissible vector-payoffs for coalition S is determined by formula

$$V(S) = \{(v_i)_{i\in S} \leq (u_i(x_i))_{i\in S} \ \mid \ (x_i)_{i\in S} \in \mathcal{A}(S) = \mathcal{A}(\mathbb{P}^S)\},$$
$$\mathcal{A}(\mathbb{P}^S) = \{y^S = (y_i)_S \mid \sum_S y_i = \sum_S e_i \ \ \& \ \ (y_i - e_i)(\cdot) \ \textit{is} \ P_i^S \ \textit{mesurable}, \ \ y_i \in (\mathbb{R}_+^l)^\Omega, \ i \in S\}.$$

Clearly that the sets $V(S)$ satisfy all the above described necessary conditions, it can be checked easily due to the compactness of set of feasible allocations and via the continuity of utilities in the initial economic model.

Recall that the family $\mathcal{B}$ of subsets in $\mathcal{I}$ is said to be *balanced*, if for every $S \in \mathcal{B}$ there is a real $\lambda_S \geq 0$, such that

$$\sum_{S\in\mathcal{B}: i\in S} \lambda_S = 1 \ \ \forall\, i \in \mathcal{I}$$

holds or, in an equivalent form,

$$\sum_{S\in\mathcal{B}} \lambda_S \mathbf{1}_S = \mathbf{1}_{\mathcal{I}}$$

takes place where, by definition, $\mathbf{1}_S \in \mathbb{R}^{\mathcal{I}}$ is such a vector that $(\mathbf{1}_S)_i = 1$ for $i \in S$ and $(\mathbf{1}_S)_i = 0$ if $i \notin S$, i.e., this is the indicator-function of the set S.

A game $(\mathcal{I}, (V(S))_{S\subset\mathcal{I}})$ is said to be *balanced* if for every balanced family $\mathcal{B}$ of coalitions

$$\bigcap_{S\in\mathcal{B}} \mathrm{pr}_{|S}^{-1}(V(S)) \subseteq V(\mathcal{I}).$$

Here $\mathrm{pr}_{|S}(\cdot)$ is the projection map of space $\mathbb{R}^{\mathcal{I}}$ onto $\mathbb{R}^S$.

The famous Scarf's theorem states that the core of a balanced game $(\mathcal{I}, (V(S))_{S\subseteq\mathcal{I}})$ is nonempty. Applying this theorem and using standard arguments, one can prove the following

Proposition 5.1. *Let $P_i^S \preceq P_i^{\mathcal{I}}$ for every (admissible) coalition $S \subseteq \mathcal{I}$ and each $i \in S$ and let agents' preferences be defined via concave continuous utility functions.*[25] *Then $\mathcal{C}(\mathcal{E}^{di}) \neq \emptyset$.*

[25] It is enough to have the compactness of $\mathcal{A}(\mathbb{P}^{\mathcal{I}})$ the set of all feasible for grand coalition allocations. Now we have it by definition of the model.

Proof of Proposition 5.1. The only thing that really needs to be accurately considered is the checking a game constructed via a model of economy is balanced one. Let $\mathcal{B}$ be any balanced family and let $(\lambda_S)_{\mathcal{B}}$ be a corresponding family of balancing coefficients. Let $v = (v_i)_{\mathcal{I}} \in \mathbb{R}^{\mathcal{I}}$ be a vector such that $\forall\, S \in \mathcal{B}$

$$\exists y^S = (y_i^S)_S \in \mathcal{A}(S) \mid \;\; v_i \leq u_i(y_i^S),\;\; \forall\, i \in S$$

is true. Now by definition of balanced family for any i we have $\lambda_S v_i \leq \lambda_S u_i(y_i^S)$, that being summed by coalitions from $\mathcal{B}$ including i, due to $u_i(\cdot)$ is concave function, yields

$$v_i = \sum_{S\in\mathcal{B}:i\in S} \lambda_S v_i \leq \sum_{S\in\mathcal{B}:i\in S} \lambda_S u_i(y_i^S) \leq u_i(\sum_{S\in\mathcal{B}:i\in S} \lambda_S y_i^S).$$

Moreover

$$(y_i^S - e_i)\ is\ P_i^S - mesurable,\ \forall\, S \in \mathcal{B} : i \in S\ \Rightarrow \sum_{S\in\mathcal{B}:i\in S} \lambda_S (y_i^S - e_i)\ is\ \bigvee_{S\in\mathcal{B}} P_i^S - mesurable,$$

and in view of assumption $\bigvee_{S\in\mathcal{B}} P_i^S \preceq P_i^{\mathcal{I}}$, one concludes

$$x_i^{\mathcal{B}} = \sum_{S\in\mathcal{B}:i\in S} \lambda_S y_i^S\ is\ P_i^{\mathcal{I}} - mesurable,\ \forall\, i \in \mathcal{I}\ \ \Rightarrow\ \ (x_i^{\mathcal{B}})_{\mathcal{I}} \in \mathcal{A}(\mathcal{I}).$$

As soon as we had $v_i \leq u_i(x_i^{\mathcal{B}})$, $i \in \mathcal{I}$, then now by game's definition we conclude $v \in V(\mathcal{I})$ as wanted to prove. ■

As a corollary to the proposition one can conclude core is non-empty in an economy such that for grand coalition allocations measurable relative to limit information are considered and for a coalitional domination — relative to information achieved via any fixed sequence of coalitions information sharing, the sequence is fixed for the definition. In this context we have got the definitions:

Definition 5. *Let $\alpha = \{S_1, S_2, \ldots, S_m\}$ be an arbitrary (possibly empty) sequence of coalitions and let $\mathbb{P} = \{P_i\}_{i\in\mathcal{I}}$ be an information structure. Define $\mathbb{P}^\alpha = \{P_i^\alpha\}_{i\in\mathcal{I}}$, where $P_i^\alpha = k_{S_m}^i(k_{S_{m-1}}^i(\ldots k_{S_1}^i(\mathbb{P})))$, $i \in \mathcal{I}$.*

***Limit core** $\mathcal{C}_{k\alpha}^{lim}(\mathcal{E}^{di})$ of **$k\alpha$-type** consist of allocations $x = (x_i)_{\mathcal{I}} \in X$ such that:*

(i) $\sum_{i\in\mathcal{I}} x_i = \sum_{i\in\mathcal{I}} e_i$,

(ii) $(x_i - e_i) : \Omega \longrightarrow \mathbb{R}^l$ *is* P_i^{lim}*-measurable for all* $i \in \mathcal{I}$.

(iii) $\nexists S \subseteq \mathcal{I} : \exists y^S = (y_i)_S \mid \forall\, i \in S,\ y_i \in X_i$ *is such that* $(y_i - e_i)$ *is* $k_S^i(\mathbb{P}^\alpha)$*-measurable,* $y_i \succ_i x_i\ \&\ \sum_S (y_i - e_i) = 0$.

*Limit core of $k\alpha$-type for $\alpha = \emptyset$ is called **limit k-core**.*

***Limit core** $\mathcal{C}^{lim}(\mathcal{E}^{di})$ consist of allocations, which belong to every $k\alpha$-core for all sequences α, i.e.*

$$\mathcal{C}^{lim}(\mathcal{E}^{di}) = \bigcap_{\alpha} \mathcal{C}_{k\alpha}^{lim}(\mathcal{E}^{di}).$$

Applying Proposition 5.1 and due to limit information is unique (Marakulin, 2009) one concludes

Corollary 5.3. *Let consumers' preferences be defined via concave continuous utility functions and information rule $k = (k_S)_{S\in C}$ be a monotonous rule of information sharing. Then limit core and limit k-core are non-empty.*

Let us turn now to replicated models. A differential information economy replica of volume $r \in \mathbb{N}$ is called the economy $\mathcal{E}^{di}_r$, in which r exact copies of each consumer from initial model $\mathcal{E}^{di}$ is put into correspondence in $\mathcal{E}^{di}_r$. The agents from $\mathcal{E}^{di}_r$ are numbered by double index (i, m), $i \in \mathcal{I}$, $m = 1, \dots, r$, and it is put $X_{im} = X_i$, $\omega_{im} = \omega_i$. Agents' preferences are defined and take values in X_{im} due to identification $\mathcal{P}_{im} = \mathcal{P}_i$. An information that agents have in a replica exactly repeats the initial one and is the same for agents of the same types, i.e., $P_{im} = P_i$, $\forall\, i, m$. To an initial economy $\mathcal{E}^{di}$ allocation $x = (x_i)_\mathcal{I}$, we can put into correspondence the replicated economy allocation $x^r = (x^r_{im})$ by the rule $x_{im} = x_i$, $\forall\, i, m$.

Definition 6. *An allocation $x = (x_i)_\mathcal{I}$ is called ex ante private Edgeworth equilibrium for model $\mathcal{E}^{di}$ if $x^r \in \mathcal{C}(\mathcal{E}^{di}_r)$ for every natural $r = 1, 2, \dots$ The set of all ex ante Edgeworth equilibria is denoted as $\mathcal{C}^e(\mathcal{E}^{di})$.*

Now let us consider the most characteristic properties of Edgeworth equilibria: their existence and relationships with ex ante private equilibria.

Theorem 5.1. *Let every agent's preference be determined via continuous and concave utility function on $(\mathbb{R}^l_+)^\Omega$. Then ex ante private Edgeworth equilibria do exist.*

Remark 5.1. Probably it is not a strongest possible existence result,[26] however this is still important and non-trivial result that is simply formulated and clearly proven.

The next stage of analysis is to put into correspondence for Edgeworth equilibria the elements of fuzzy core and then to characterize them in value units. Let us do it taking into account non-symmetrically distributed information.

Recall that any vector

$$t = (t_1, \dots, t_n) \neq 0, \quad 0 \le t_i \le 1 \ \ \forall\, i \in \mathcal{I}$$

may be identified with a fuzzy coalition, where the real number t_i being interpreted as the measure of agent i in the coalition. A coalition t is said to dominate (block) an allocation $x \in \mathcal{A}(X)$ if there exists $y^t \in \prod_\mathcal{I} X_i$ such that

$$y^t_i - e_i \in \mathcal{L}_i,\ \forall\, i \in \mathcal{I} \ \ \& \ \ \sum_{i\in\mathcal{I}} t_i y^t_i = \sum_{i\in\mathcal{I}} t_i e_i \iff \sum_{i\in\mathcal{I}} t_i (y^t_i - e_i) = 0 \qquad (13)$$

[26] One needs only the continuity and convexity of preferences (i.e. convex upper level setts) but the existence of utility functions does not play special role although it is applied in presented proof. It is also important that the set of all feasible allocations is a compact but the fact that commodity space is finite dimensional is not essential factor.

and

$$y_i^t \succ_i x_i \quad \forall\, i \in \mathrm{supp}(t) = \{i \in \mathcal{I} \mid t_i > 0\}. \tag{14}$$

For non-satiated on $\mathcal{L}_i$ preferences conditions (13), (14) can be equivalently rewritten in the form[27]

$$0 \in \sum_{i \in \mathcal{I}} t_i[(\mathcal{P}_i(x_i) - e_i) \cap \mathcal{L}_i].$$

The set of all feasible allocations which cannot be dominated by fuzzy coalitions is denoted by $\mathcal{C}^f(\mathcal{E}^{di})$ and is called the *fuzzy core*. If preferences are convex and non-satiated then it can be characterized in the following way:

$$x \in \mathcal{C}^f(\mathcal{E}^{di}) \quad \Longleftrightarrow \quad 0 \notin \mathrm{co} \bigcup_{i \in \mathcal{I}} [(\mathcal{P}_i(x_i) - e_i) \cap \mathcal{L}_i]. \tag{15}$$

Theorem 5.2. *Let $\mathcal{P}_i(x_i)$ be convex and open in X_i for each agent $i \in \mathcal{I}$ and $\forall\, x = (x_i)_{\mathcal{I}} \in \mathcal{A}(X)$. Then fuzzy core coincides with the set of all ex ante private Edgeworth equilibria, i.e.*

$$\mathcal{C}^e(\mathcal{E}^{di}) = \mathcal{C}^f(\mathcal{E}^{di}).$$

Proof of Theorem 5.2. To state the theorem it is enough to show that $\mathcal{C}^e(\mathcal{E}^{di}) \subseteq \mathcal{C}^f(\mathcal{E}^{di})$. Assume contrary and find $x \in \mathcal{C}^e(\mathcal{E}^{di})$, which is dominated by a fuzzy coalition $t \neq 0$. Due to definition there is $y^t \in \prod_{\mathcal{I}} X_i$, such that (13), (14) hold. Further for $t_i > 0$ define

$$z_i = (t_i/s_i)y_i^t + (1 - t_i/s_i)e_i \quad \Longleftrightarrow \quad s_i(z_i - e_i) = t_i(y_i^t - e_i),$$

where *rational* s_i are such that $t_i \leq s_i \leq 1$ is true and for $t_i = 0$ take $s_i = 0$. By construction $z_s = (z_i)_{\mathcal{I}} \in \prod_{\mathcal{I}} X_i$ and for each individuals the map $z_i - e_i$ is measurable relative to the same partition (algebra) as is measurable $y_i^t - e_i$ and

$$\sum_{i \in \mathcal{I}} s_i(z_i - e_i) = 0$$

holds. Since $\mathcal{P}_i(x)$ are convex and relatively open by assumptions then numbers s_i can be chosen so that $z_i \in \mathcal{P}_i(x)$ is true for all i satisfying $s_i > 0$. However this contradicts to the choice of $x \in \mathcal{C}^e(\mathcal{E}^{di})$. Theorem 5.2 is proven. ■

In a supplement to considered above (15) I'll need below also in an additional characterization of fuzzy core. Let us consider the sets

$$\Omega_i(x_i) = \mathrm{co}(\mathcal{P}_i(x_i) \cup \{e_i\}), \quad i \in \mathcal{I}.$$

In Marakulin (2003) these sets were applied to give a characterization of fuzzy core for a model with symmetrically distributed information. Being adopted to asymmetrical case it gives the result:

[27] Admitting inaccuracy here and below we will sometimes identify a vector with a single element set, including it.

Proposition 5.2. *An allocation $x \in \mathcal{A}(X)$ is the element of fuzzy core if and only if relation*

$$\prod_{\mathcal{I}} \Omega_i(x_i) \bigcap \{(z_1, \ldots, z_n) \mid \sum_{i\in\mathcal{I}} z_i = \sum_{i\in\mathcal{I}} e_i\} \bigcap (\prod_{\mathcal{I}} \mathcal{L}_i + (e_1, \ldots, e_n)) = \{(e_1, \ldots, e_n)\} \tag{16}$$

is true.

Notice that characterization (16) is also valid for satiated preferences.

Applying (16) one can prove the following corollary convenient in various applications. This will allow us to state the main result of this section.

Corollary 5.4. *Let $x \in \mathcal{A}(X)$ and $\mathcal{P}_i(x_i) \neq \emptyset$ for all $i \in \mathcal{I}$. Then $x \in \mathcal{C}^f(\mathcal{E}^{di})$ implies*:

$$\prod_{\mathcal{I}} (\mathcal{P}_i(x) + \text{co}\{0, e_i - x_i\}) \bigcap \{(z_1, \ldots, z_n) \mid \sum_{i\in\mathcal{I}} z_i = \sum_{i\in\mathcal{I}} e_i\} \bigcap (\prod_{\mathcal{I}} \mathcal{L}_i + (e_1, \ldots, e_n)) = \emptyset. \tag{17}$$

Theorem 5.3. *Every ex ante private Edgeworth equilibrium is a quasi-equilibrium.*

This theorem together with theorem 5.1 presents the main result of the section and says that ex ante private core shrinks to equilibria of the same type and defined by Definition 4. However this statement to have an perfect mathematical formalization it is necessary the theorem conditions to complete with requirements that guarantee every quasi-equilibrium is equilibrium in fact. Notice also that an attempt to prove this theorem starting from characterization (15) encounters with serious mathematical difficulties because the sets $(\mathcal{P}_i(x_i) - e_i) \cap \mathcal{L}_i$ have *empty* interior and therefore one cannot guarantee that functional separating a set from the right hand side of (15) and point '0' is not zero on $\mathcal{L}_i$, that contradicts to condition (*ii*) by quasi-equilibrium Definition 4.

There is another and purely contractual presentation of private equilibrium and, therefore, WEE-equilibrium. This presentation is based on the notion of fuzzy contractual allocation, see Definition 2 on page 19. I think in general this is meaningful substantial notion that is more natural then Edgeworth equilibrium. Moreover for economy with asymmetrically distributed information the concept of fuzzy contractual allocation is so efficient as for symmetrical case: the only difference with Definition 2 is an additional measurement requirement for consumption flows received from a contract, this is measurement relative to individual (private) information. For the model $\mathcal{E}^{di}$ this way produced notion has the following characteristic property.

Proposition 5.3. *An allocation x is fuzzy contractual if and only if*

$$\text{co}\{x_i, e_i\} \cap \mathcal{P}_i(x_i) = \emptyset, \quad \forall\, i \in \mathcal{I}, \tag{18}$$

$$\prod_{\mathcal{I}} [(\mathcal{P}_i(x_i) + \text{co}\{0, e_i - x_i\}) \cup \{e_i\}] \bigcap \{(z_1, \ldots, z_n) \mid \sum_{i\in\mathcal{I}} z_i = \sum_{i\in\mathcal{I}} e_i\} \bigcap$$

$$\bigcap (\prod_{\mathcal{I}} \mathcal{L}_i + (e_1, \ldots, e_n)) = \{(e_1, \ldots, e_n)\}. \tag{19}$$

One sees the only difference with symmetric case characterization that is in formula (19) where a new element is appeared in intersection, this thing defines aggregated contract $z-e$ is measured. The proof of Proposition 5.3 follows the arguments of Proposition 4.1. From

$$\Omega_i(x_i) \subset (\mathcal{P}_i(x_i) + \mathrm{co}\{0, e_i - x_i\}) \cup \{e_i\}, \quad \forall\, i \in \mathcal{I}$$

and Proposition 5.2 one can immediately concludes that fuzzy contractual allocations are the elements of fuzzy core. On the other hand Corollary 5.4 shows that (compare (17) with (19)), the difference is negligible...

6. REE-EQUILIBRIUM AND CONTRACTUAL INTERIM CORE

In the subsequent analysis it will be useful to have the following temporary submissions on various stages of contractual processes in an economy with the differentiated information.

F Today afternoon: Consumption of contracts	$\Rightarrow$	A Today after afternoon: New contracting	$\Rightarrow$	$\cdots$ Night	$\Rightarrow$	B Tomorrow before afternoon: Recontracting	$\Rightarrow$	F Tomorrow afternoon: Consumption of contracts

Timing of contractual processes

6.1. Preliminary analysis

Before the passing to the description of formal constructions let's try to clarify the REE-equilibrium concept (in rational expectations). What is going on?

Somebody (the Lord, God) in some intermediate moment between today and tomorrow (on the diagram at night) presents for agents reviewing the prices or exacter a function of the prices $p : \Omega \to \mathbb{R}^l$, on which tomorrow trade will be carried out. Having realized function $p(\cdot)$, agents extract from it the information as follows. They know, that when tomorrow will occur, and they will see (on screen monitors, in shop) the realized price $\bar{p}$ and one can conclude that (in the morning till a dinner) an elementary event from the set

$$p^{-1}(\bar{p}) = \{\omega \in \Omega \mid p(\omega) = \bar{p}\}$$

is realized. The agent i has also an own information P_i (a partition of Ω) on events of next day. This is why he/she knows today that tomorrow he/she will be able to distinguish what event from $p^{-1}(\bar{p}) \cap P_i(\omega) \neq \emptyset$, $\omega \in \Omega$ is realized. However it means that having received information about function $p : \Omega \to \mathbb{R}^l$ individual i is able to form plans of purchases and sellings, i.e. that for every $E = E(\bar{p}, \omega) = p^{-1}(\bar{p}) \cap P_i(\omega) \neq \emptyset$ he/she has to find a vector $v_i(E) \in \mathbb{R}^l$ maximizing agent's utility under condition an event E is realized. In this case the individual consumption bundle is a bundle $e_i(\omega) + v_i(E)$, $\omega \in E$. Moreover an equilibrium situation is so that there is the balance of demand and supply (of purchases and sellings) for every elementary event that can be realized tomorrow.

Let's imagine, that the God has transmitted the prices that explicitly distinguish all world states... Then in an intermediate stage when price signal is obtained, each individual will receive a possibility to understand (distinguish) every state of the world and, therefore, the own market for every of the future elementary events should be developed for which it is possible to consider competitive equilibrium and the core as in routine Arrow–Debreu model. Certainly, those prices that were transmitted by the God should be the prices of an equilibrium for each of the markets determined by states of the world. The said can be easily checked from the definition of equilibrium in rational expectations. However what is about core? Undoubtedly in this case one has to take the Cartesian product of cores — in this way all known theoretical results can be obtained and generalized. Thus

$$\sigma(p) = \Omega^* \quad \Rightarrow \quad \mathcal{C}(\mathcal{E}^{di}) = \prod_{\omega\in\Omega} \mathcal{C}(\mathcal{E}^{di}(\omega)). \tag{20}$$

Before the consideration general case and the finding an adequate contractual presentation for core and equilibrium in the rational expectations let us consider the following modeling example.

Assume $\Omega = \{a, b, c\}$ and there are three agents $\mathcal{I} = \{1, 2, 3\}$ with initial information

$$P_1 = \{\{a\}, \{b, c\}\}, \quad P_2 = P_3 = \{\{a, b\}, \{c\}\}.$$

Let's assume that the nature has transmitted a price signal such that $p' = p(a) = p(b) \neq p(c) = p''$. What has happened? The agents have been informed not only on different variants of tomorrow prices, but also they have got a toll to distinguish the future events $E' = \{a, b\}$ and $E'' = \{c\}$. Certainly, in the example it useful only for the first individual who has now the perfect information. However, the partition $\{E', E''\}$ is the information which each individual possesses now and everybody knows, that other ones also know (the common knowledge). Thus, already today there can be developed and start to function the markets of the future purchases and sellings, specified by events E' or E''. Here for E' every deal starts with a preamble: "if E' has happened then..."; similar for E''. And what does happen according to REE-equilibrium concept? The following treatment is possible:

For $E' = \{a, b\}$:

$i = 1,\ \omega = a$**,** bundle $x_1(a)$ solves the problem

$$u_1(\varphi^1_a(y_1(a)), *, *) \to \max, \ \ s.t.\ \ y_1(a) \geq 0 \ \ \& \ \ \langle p', y_1(a) - e_1(a)\rangle \leq 0;$$

$i = 1,\ \omega = b$**,** bundle $x_1(b)$ solves the problem

$$u_1(*, \varphi^1_b(y_1(b)), *) \to \max, \ \ s.t.\ \ y_1(b) \geq 0 \ \ \& \ \ \langle p', y_1(b) - e_1(b)\rangle \leq 0;$$

$i = 2,\ E = \{a, b\}$**,** bundle $x_2 = x_2(a, b)$ solves the problem (accounting that $e_2(a) = e_2(b)$)

$$u_2(\varphi^2_a(y_2(a)), \varphi^2_b(y_2(b)), *) \to \max, \ \ s.t.\ \ y_2(a) = y_2(b) \geq 0 \ \ \& \ \ \langle p', y_2 - e_2(a)\rangle \leq 0.$$

$i = 3,\ E = \{a, b\}$**,** bundle $x_3 = x_3(a, b)$ solves the problem (accounting that $e_3(a) = e_3(b)$)

$$u_3(\varphi^3_a(y_3(a)), \varphi^3_b(y_3(b)), *) \to \max, \ \ s.t.\ \ y_3(a) = y_3(b) \geq 0 \ \ \& \ \ \langle p', y_3 - e_3(a)\rangle \leq 0;$$

Balance: $x_1(a) + x_2(a) + x_3(a) = e_1(a) + e_2(a) + e_3(a)$ &
$x_1(b) + x_2(b) + x_3(b) = e_1(b) + e_2(b) + e_3(b)$.

For $E'' = \{c\}$:

$i = 1,\ \omega = c,$ bundle $x_1(c)$ solves the problem

$$u_1(*, *, \varphi_c^1(y_1(c))) \to \max,\ \ s.t.\ \ y_1(c) \geq 0\ \ \&\ \ \langle p'', y_1(c) - e_1(c)\rangle \leq 0;$$

$i = 2,\ \omega = c,$ bundle $x_2(c)$ solves the problem

$$u_2(*, *, \varphi_c^2(y_2(c))) \to \max,\ \ s.t.\ \ y_2(c) \geq 0\ \ \&\ \ \langle p'', y_2(c) - e_2(c)\rangle \leq 0;$$

$i = 3,\ \omega = c,$ bundle $x_3(c)$ solves the problem

$$u_3(*, *, \varphi_c^3(y_3(c))) \to \max,\ \ s.t.\ \ y_3(c) \geq 0\ \ \&\ \ \langle p'', y_3(c) - e_3(c)\rangle \leq 0;$$

Balance: $x_1(c) + x_2(c) + x_3(c) = e_1(c) + e_2(c) + e_3(c)$.

One can see from the presented relations that events E' and E'' in the model context generate rather independent economic structures, submodels, for which reduced on the event REE-allocation has specific equilibrium properties. The model relative to $E' = \{a, b\}$ has two characteristic features:

(i) ***prices*** for elementary events a and b ***coincide***;

(ii) ***1st individual***, which is able to distinguish events a and b, in the submodel is ***presented by two*** different consumer ***problems*** (only prices p' are common thing) with different utilities: $\varphi_a^1(\cdot)$ and $\varphi_b^1(\cdot)$. One may state that the ***individual is 'duplicated'***: one his face works on the market defined by a, another one by b.

One important relationship between items (i) and (ii) has to be noted: if (i) were broken, i.e. if prices in states a and b were different, then all agents were 'duplicated' and the market were disintegrated in two parts... Namely due to (i) and that some individuals cannot to distinguish events their common market is developed. Further I try to analysis the case in more details. Once again contractual approach will be applied for the analysis.

6.2. Contractual approach and differentiated agents

According to general contractual settings, to apply approach one has to define a set W of all permissible contracts correctly reflecting model context. As it was postulated above a contract is tuple of maps $v_i : \Omega \to \mathbb{R}^l$, $i \in \mathcal{I}$, which obeys measurability (8) and balances (9) requirements; recall them:

$$\forall\, i \in \mathcal{I},\ \ v_i(\cdot)\ is\ P_i - \text{measurable},\ \ \forall\, \omega \in \Omega\ \ \sum_{i \in \mathcal{I}} v_i(\omega) = 0. \tag{21}$$

However does it well corresponds to REE-concept in the above example context? The main point is how in the contract and contractual interaction of individuals the possibility of agent decomposition in a pair, as in an example considered above. In general an agent has to be decomposed into several artificial agents each of them corresponds to the element of informational partition. Apparently it is necessary to postulate specially.

For this purpose let us form a new set of individuals, which are indexed by couples (i, E) where the first component is a number (name) of the individual, and $E \in P_i$ is an element of his/her informational partitions. Here (i, E) can be treated as the individual in one of possible variants of *tomorrow's* implementations of the agent i and it corresponds to a knowledge of this agent. So, suppose

$$\Im = \{(i, E) \mid i \in \mathcal{I},\ E \in P_i\}.$$

Further one will specify contracts which 'duplicated agent' is able to conclude. According to construction such agent can live and function only if the event E there will be realized, therefore for $(i,E) \in \Im$ one will specify

$$v_i^E : \Omega \to \mathbb{R}^l : \quad v_i^E(\omega) = v_i^E(\omega'),\ \forall\, \omega, \omega' \in E \ \ \& \ \ v_i^E(\omega) = 0,\ \forall\, \omega'' \in \Omega \setminus E \iff \tag{22}$$

$$v_i^E(\cdot) \ \ P_i - \text{measurable} \ \ \& \ \ \operatorname{supp}(v_i^E) = E \in P_i.$$[28]

Certainly a tuple $(v_i^E)_{\Im}$ can be considered as a contract only if the balance restriction

$$\sum_{\Im} v_i^E = 0 \tag{23}$$

is satisfied. For the individual $(i,E) \in \Im$ to function as a real economic agent one has to endow him/her with initial commodity bundle and preferences. Put

$$e_i^E = e_i \cdot \chi^E, \quad E \in P_i \ \ \& \ \ (i,E) \in \Im,$$

where $\chi_E(\cdot)$ is a characteristic function of the set $E \subseteq \Omega$ and let

$$\mathcal{L}_i^E = \{y \cdot \chi^E(\cdot) \in L \mid y \in \mathbb{R}^l\}$$

be a subspace corresponded to individual $(i,E) \in \Im$ (its dimension is l) in the space of contingent commodities $L = (\mathbb{R}^l)^\Omega$. Further let us define preferences and consumption sets. Put

$$X_i^E = (\mathcal{L}^E + e_i^E) \cap X_i$$

and define on X_i^E relation $\succ_i^E$ by formula

$$y_i^E \succ_i^E x_i^E \iff \exists z_i^{\Omega\setminus E} : \Omega \setminus E \to \mathbb{R}^l \mid \ (y_i^E, z_i^{\Omega\setminus E}) \succ_i (x_i^E, z_i^{\Omega\setminus E}),$$

that can also be rewritten in an equivalent form as

$$\mathcal{P}_i^E(x_i^E) = [\mathcal{P}_i(x_i) \cap (x_i + \mathcal{L}^E)] \cdot \chi^E, \quad for \ \ x_i^E = x_i \cdot \chi^E.$$

Due to assumptions (5) – (7) this is a correct way to present preferences for an economy with differentiated information. Notice that

$$\sum_{E \in P_i} \mathcal{P}_i^E(x_i^E) \subset \mathcal{P}_i(x_i)$$

is always true but the reverse inclusion is false. As a result of presented constructions one comes to an economic model $\mathcal{E}^{\Im}$.

So, I see two possibilities to apply contractual approach in economies with asymmetrically informed individuals:

(i) The set of agents is the same as in the initial model of economy and contracts are completely determined by conditions (21).

(ii) The set of agents varies on $\Im$ to which contract specifications (22)–(23) are applied.

[28] Notice that any map $v_i(\cdot)$ applied in contract definition (9) and measurable by P_i can be decomposed into direct sum of maps v_i^E, $E \in P_i$.

These possibilities lead to different concepts of the contractual allocations and corresponding concepts of the core and an equilibrium. In the first case this a priori private core and an equilibrium (analogue WEE), in the second one *new*concepts of core and an equilibrium are introduced; I would name them as core and equilibrium with *differentiated agents*.[29] The variant (i) was analyzed in previous sections, I continue further considering the second variant which eventually deduces us to REE-equilibrium.

First let's consider the simplest possibility to introduce a core by analogue with a standard case, but now already in described above model with duplicated agents; I have named it as the D-core.

Definition 7. *D-core $\mathcal{C}^d(\mathcal{E}^{di})$ of economy $\mathcal{E}^{di}$ with asymmetrically informed agents consist of allocations $x = (x_i)_\mathcal{I} \in X$ such that:*

(i) $\sum\limits_{i\in\mathcal{I}} x_i = \sum\limits_{i\in\mathcal{I}} e_i$,

(ii) $(x_i - e_i) : \Omega \to \mathbb{R}^l$ *is P_i-measurable for all $i \in \mathcal{I}$.*

(iii) $\nexists S \subseteq \Im : \exists y^S = (y_i^E)_{(i,E)\in S} \mid \forall\,(i,E) \in S\ y_i^E \in X_i^E$ *is such that* $y_i^E \succ_i^E x_i^E$ & $\sum_{(i,E)\in S}(y_i^E - e_i^E) = 0$.

One can see that the only but basic difference of the D-core from private one consists in that for a domination coalition are formed from duplicated agents of the economy. In so doing in the core of this type the allocations are placed that can be by a stable system of contracts, stable in an intermediate stage of uncertainty implementation, i. e. at a moment when each agent is able to understand every state of the nature $\omega \in \Omega$ (it still is not realized!) while only in a form $P_i(\omega)$. Here coalitions of duplicated agents can be formed because uncertainty still not definitively resolved. Later, probably, individuals learn more about state ω but when it will happen all contracts will already be realized. Moreover, it should be clear that there are no possibilities for a deceit here, since if more informed agent will report a false state — ω' instead of true ω — when it cannot bring a damage to the less informed individuals because in both cases in a gross contract $(v_i)_\mathcal{I}$ these individuals have the same vector of mutual deliveries, because by (8) one has $v_i(\omega') = v_i(\omega)$ for $\omega', \omega \in P_i(\omega)$.

Regarding the existence of D-core it is enough to note that Theorem 5.1 is applicable here with more or less simple modifications be conditioned by reproduction of agents — game of economy is still balanced that is necessary for core existence.

Further let us turn to an equilibrium concept that correctly corresponds to the D-core and contractual approach point of view. I start analysis deriving conditions that characterizes mutually beneficial exchange for the economy with differentiated agents: this is once more lemma about mutually beneficial contract.

Lemma 6.1 (ABOUT MUTUALLY BENEFICIAL CONTRACT FOR D-AGENTS). *Let $S \subseteq \mathcal{I}$, $S \neq \emptyset$ be a coalition and $A \subseteq \Omega$ be an event understandable by every coalition S member and let*

$$S(\Im) = \{(i,E) \mid i \in S,\ E \in P_i,\ E \subseteq A\}.$$

[29] Differentiated information about the future induced in the present differentiation and 'reproduction' of agents in the future: depending on the information different agents 'are mirrored' in different ways.

Then if ***there are no*** *mutually beneficial exchange of contingent commodities for coalition* $S(\Im)$ *members then there does exist a vector* $p \in (\mathbb{R}^l)^A$, $p \neq 0$, *and vectors* $q_i \in (\mathbb{R}^l)^A$, $i \in S$, *such that*

$$\forall\, i \in S\ \forall\, E \in P_i,\ E \subseteq A \quad \sum_{\omega \in E} q_i(\omega) = 0 \tag{24}$$

and

$$\forall\, i \in S \quad p + q_i \neq 0 \ \ \&\ \ \langle \mathcal{P}_i^E(x_i^E), p + q_i \rangle \geq \langle x_i^E, p + q_i \rangle \ \ \forall\, E \in P_i,\ E \subseteq A \tag{25}$$

holds.

Inverse: let there be vectors satisfying (24), (25). Then for a coalition S ***there is no*** *mutually beneficial contract in which an agent* $(i, E) \in S(\Im)$ *is non-trivially involved for which inequality (25) has a strict form*[30].

The proof of Lemma 6.1 is realized by method similar applied in Lemma 5.1 proof. The only modification is due to D-agent specification: in an appropriate intersection instead of $\mathcal{P}_i(x_i)$ one has take the set $\sum_{E \in P_i,\ E \subseteq A} \mathcal{P}_i^E(x_i^E)$; let me omit other details. Notice also the coincidence between (24) with an analogue requirement (10) applied in Lemma 5.1.

Similarly to Lemma 5.1, Lemma 6.1 has an important for below analysis corollary, that I formulate with the aim to present full argumentation and to better specify contractual approach for differentiated agents.

Corollary 6.1. *Let in Lemma 6.1 conditions preferences be described via differentiable utility functions and let* $x = (x_i)_{\mathcal{I}}$ *be an interior relative to A and S allocation.*[31] *Then* ***there is no*** *mutually beneficial contract for the coalition* $S(\Im)$ *if and only if when there exists a vector* $p \in (\mathbb{R}^l)^A$, $p \neq 0$ *and* $\lambda_{i,E} \geq 0$, $(i, E) \in S(\Im)$ *non-zero for some* $E \in P_i$ $\forall\, i \in S$, *such that*

$$\forall\, E \in P_i,\ E \subseteq A \quad \lambda_{i,E} \sum_{\omega \in E} \nabla_\omega u_i(x_i) = \sum_{\omega \in E} p(\omega) \quad \forall\, i \in S.$$

Notice some distinctions with Corollary 5.1: factors $\lambda_{i,E}$ have doubled indexes and, in general, some of them can be zeros.

Now for the concept of D-equilibrium to be done it will be enough to add budget constrains to the latter lemma conclusion.

Definition 8. *A couple* (x, p), $x = (x_i)_{\mathcal{I}} \in X$, $p : \Omega \to \mathbb{R}^l$, *is called* ***private D-quasi-equilibrium*** *if it satisfies:*

(i) $(x_i - e_i) : \Omega \to \mathbb{R}^l$ *is* P_i*-measurable for all* $i \in \mathcal{I}$.

(ii) $0 \not\equiv \langle p^E, (\mathcal{P}_i^E(x_i^E) - e_i^E) \rangle \geq 0$, $\langle p^E, x_i^E - e_i^E \rangle = 0$, $\forall\, (i, E) \in \Im$,

(iii) $\sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} e_i$.

If in item (ii) all inequalities are ***strict****, then a pair* (x, p) *is called* ***D-equilibrium****.*

[30] See footnote above.

[31] See Corollary 5.1.

Notice that due to the definition $p(\cdot) \neq 0$ but there is also possible $p^E = p \cdot \chi^E \equiv 0$ for some $E \in P_i$, $i \in \mathcal{I}$. Notice also an alternative form of condition (ii) presentation: $\forall\,(i, E) \in \Im$,

$$\sum_{\omega \in E} p(\omega) x_i(\omega) = \sum_{\omega \in E} p(\omega) e_i(\omega) \quad \&$$

$$\langle \sum_{\omega \in E} p(\omega), y \rangle \geq 0 \;\forall\, y \in \mathbb{R}^l : (y \cdot \chi_E(\cdot) + x_i) \in \mathcal{P}_i(x_i).$$

Thus individual (i, E) takes prices in aggregated form, aggregated relative to states from E, that can be treated as a form of expected prices and profit under condition of event E. Notice that if one adds in the definition the requirement

$$p(\omega) = p(\omega'), \quad \forall\, \omega, \omega' \in E \in P_i, \quad \forall\, i \in \mathcal{I},$$

then one yields the notion generalizing REE-equilibrium to the case of general preferences.

The existence of D-quasi-equilibrium can be proven applying the same methods that were used above for ex ante private equilibrium: due to replicated models passing to fuzzy core for which elements price characterization in the form of quasi-equilibria can be given.

Finishing the section let us consider the concepts of D-core and D-equilibrium, and also of private core and equilibrium in an example of DIE-economy, that admits a demonstration on Edgeworth box (in spite of the fact the space of contingent commodities is 4-dimensional one).

Example 6.1. Let us consider a standard pure exchange economy with asymmetrically informed agents. Let there be 2 types of physically different goods, 2 agents, 2 states of the nature and assume that 2nd agent is able to distinguish them but 1st is not. Thus we have:

$i = 1, 2$, $\Omega = \{a, b\}$, $P_1 = \{\Omega\}$, $P_2 = \{\{a\}, \{b\}\}$,

$\mathbb{R}^2$ is the commodity space,

$L = \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$ is the space of contingent commodities. Let $X_1 = X_2 = \mathbb{R}^4_+$ and consider the following utility functions and initial endowments:

$u_x = u_1(x(a), x(b)) = \ln(x_1(a)) + \ln(x_2(a)) + \ln(x_1(b)) + \ln(x_2(b))$,

$e_x = e_1 = ((3\frac{1}{2}, \frac{1}{2}), (3\frac{1}{2}, \frac{1}{2}))$;

$u_y = u_2(y(a), y(b)) = 2\ln(y_1(a)) + \ln(y_2(a)) + \ln(y_1(b)) + 2\ln(y_2(b))$,

$e_y = e_2 = ((1\frac{1}{2}, 2\frac{1}{2}), (\frac{1}{2}, 3\frac{1}{2}))$.

Thus total endowments in the economy are presented by the vector

$\bar{e} = (\bar{e}(a), \bar{e}(b)), \quad \bar{e}(a) = (5, 3), \quad \bar{e}(b) = (4, 4)$.

Further assume that between agents there is ***no informational exchange***. Now the measurability of contract $(v, -v)$, $v = (v(a), v(b)) \in L$ implies that $v(a) = v(b) = w \in \mathbb{R}^2$ and, therefore,

$$x(a) = e_x(a) + v(a) = e_x(b) + v(b) = x(b) = (3\tfrac{1}{2}, \tfrac{1}{2}) + (w_1, w_2).$$

In general for the states of future 2nd individual may have different consumption plans but now they also can be expressed in the terms of 2-dimension contract (w_1, w_2):

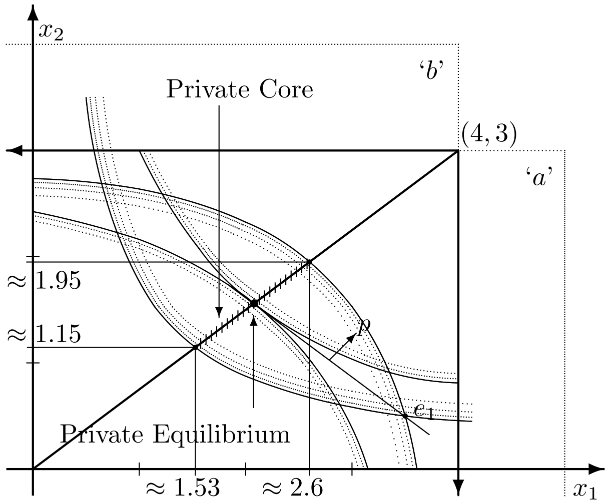

Fig. 2. *Private core and equilibrium*

$$y(a) = e_y(a) - v(a) = (1\tfrac{1}{2}, 2\tfrac{1}{2}) - (w_1, w_2), \quad y(b) = e_y(b) - v(b) = (\tfrac{1}{2}, 3\tfrac{1}{2}) - (w_1, w_2).$$

As soon as 2nd agent is not able to consume negative quantities then

$$y(a) = (1\tfrac{1}{2}, 2\tfrac{1}{2}) - (w_1, w_2) \geq 0, \quad y(b) = (\tfrac{1}{2}, 3\tfrac{1}{2}) - (w_1, w_2) \geq 0 \quad \Rightarrow \quad (w_1, w_2) \leq (\tfrac{1}{2}, 2\tfrac{1}{2})$$

and we are coming to 2-dimensional Edgeworth box presented in 1st agent consumption plans:

$$E = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq (x_1, x_2) \leq (4, 3)\}.$$

Now Pareto boundary can be found applying (for example) 2nd Welfare Theorem that allows to conclude that gradients have to be collinear ones. So, this boundary is described by the following system of equations:

$$\begin{cases} \frac{2}{x_1} = \lambda(\frac{2}{5-x_1} + \frac{1}{4-x_1}), \\ \frac{2}{x_2} = \lambda(\frac{2}{4-x_2} + \frac{1}{3-x_2}), \quad \lambda > 0. \end{cases}$$

Eliminating $\lambda > 0$ from the system one obtains 3rd order equation that determines Pareto boundary in the interior points of box. Sufficiently good its approximation is a curve defined by explicit linear equation $x_2 = \frac{3}{4}x_1$, i.e. this is the diagonal of the box. Further, if one adds 1st agent budget constrain (equality) to the equation of Pareto boundary then one comes to the system of equations that describes private equilibrium (2nd budget constrain is fulfilled automatically). As soon as for interior point first order conditions imply prices are collinear to the gradients of utilities, one finds

$$\langle \nabla u_1(x), x \rangle = \langle \nabla u_1(x), e_1(a) \rangle \quad \Rightarrow \quad 4 = \frac{7}{x_1} + \frac{1}{x_2} \quad \Rightarrow \quad x_1 \approx \frac{25}{12} \approx 2.08, \quad x_2 \approx \frac{25}{16} \approx 1.56.$$

The results of above analysis are graphically presented in Figure 2.

Further let us consider D-core and D-equilibrium. With this in mind let us find Pareto boundaries in the models reduced to the states a, b. For 'a' one obtains the following system of equations:

$$\begin{cases} \frac{2}{x_1} = \lambda\frac{2}{5-x_1}, \\ \frac{2}{x_2} = \lambda\frac{1}{3-x_2}, \quad \lambda > 0 \end{cases} \iff x_2 = \frac{6x_1}{x_1 + 5}.$$

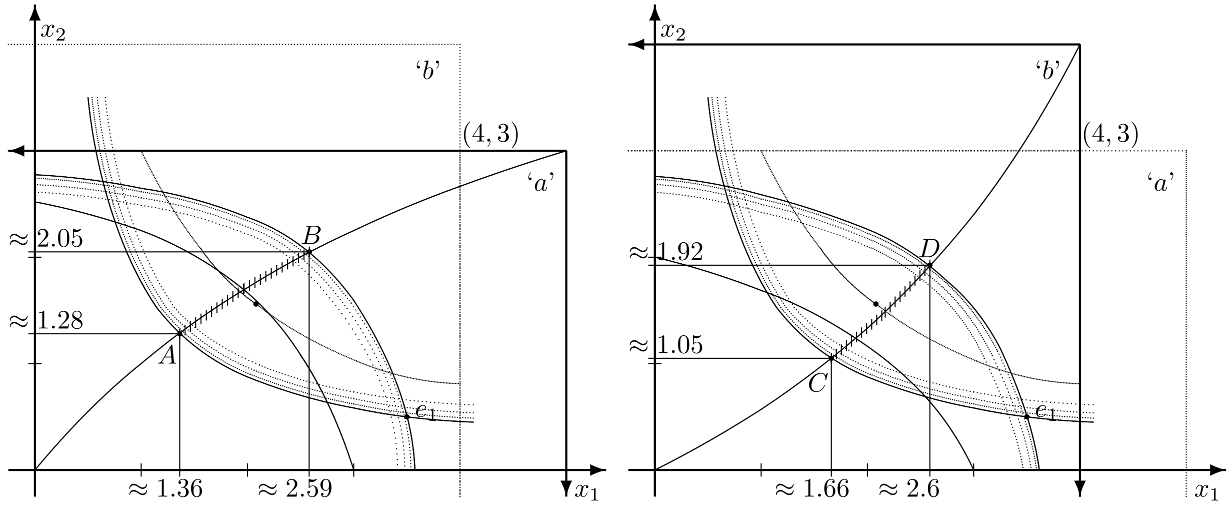

Fig. 3. *D-core: analysis for events 'a' and 'b'*

Here a "core" is presented by a curved segment on the Pareto curve, placed between two indifference curves passed through the initial endowments point; here the endpoints of the segment are $A \approx (1.36, 1.28)$ and $B \approx (2.59, 2.05)$.

For 'b' the similar system is the following:

$$\begin{cases} \frac{2}{x_1} = \lambda \frac{1}{4-x_1}, \\ \frac{2}{x_2} = \lambda \frac{2}{4-x_2}, \quad \lambda > 0 \end{cases} \iff \quad x_2 = \frac{4x_1}{8-x_1}. \tag{26}$$

A "core" corresponds with a curved segment of Pareto curve, that is placed between points $C \approx (1.66, 1.05)$ and $D \approx (2.6, 1.92)$. Graphical construction for the Edgeworth box is provided in Figure 3. Further let us reduce provided constructions into joint picture having this way obtained properly D-core.

Really for example presented only coalition of all players can be efficient, because other coalitions can sign only zero-contract. This is why the elements of D-core are completely described as being Pareto optimal and individually rational ones (i.e. they cannot be dominated by one-element coalitions). In our context Pareto optimality of an allocation $(\bar{x}, \bar{x}, \bar{e}(a) - \bar{x}, \bar{e}(b) - \bar{x}) \in \mathbb{R}^8_+$ means the intersection of the following three sets is empty:

$$\{(x_1, x_2) \in \mathbb{R}^2 \mid u_1(x(a), x(b)) > u_1(\bar{x}, \bar{x}), x(a) = x(b) = (x_1, x_2)\} \bigcap$$

$$\bigcap \{(x_1, x_2) \in \mathbb{R}^2 \mid u_2^a(\bar{e}(a) - (x_1, x_2)) > u_2^a(\bar{e}(a) - (\bar{x}_1, \bar{x}_2))\} \bigcap$$

$$\bigcap \{(x_1, x_2) \in \mathbb{R}^2 \mid u_2^b(\bar{e}(b) - (x_1, x_2)) > u_2^b(\bar{e}(b) - (\bar{x}_1, \bar{x}_2))\} = \emptyset.$$

Here the first one is the set of all strictly preferred consumption bundles for the 1st agent, that is written according to his/her impossibility to differentiate elementary events a and b. Two other sets are the sets of all strictly preferred consumption bundles for the duplicated agents for the 2nd individual: his/her possible implementations in the future events a and b. All sets are written in variables associated with the consumption of the 1st individual. Graphical and numerical construction for the example is provided in Figure 4, where a shaded set of points $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is represented as a curved area $ACDE$ which boundaries are the fragments of cores for events 'a' and 'b', and also fragments of indifference curves of 1st agent and the duplicate $(2,b)$. Notice that here a part of core for 'a', that includes endpoint B, is intercepted by the indifference curve of $(2,b)$ and as a result curved segment AE is considered where $E \approx (2.47, 2.5)$.

Further in the context of example let us consider the notion of D-equilibrium. Briefly speaking, D-quasiequilibria are the allocations in which equilibria in elementary events 'a' and 'b' are implemented. In so doing for ***"complementary"*** elementary event ***prices are equal to zero***; for example for D-equilibrium induced by event 'b' ***prices for event 'a' are zeros***. Calculate these equilibria. According to analysis provided above it can be done adding to Pareto boundary equation the budget equation (equality) for the 1st agent being calculated with respect to prices $p=(p_1,p_2)=\nabla_x u_1(x,x)=(\frac{2}{x_1},\frac{2}{x_2}) \Rightarrow \langle p,x\rangle = 4 = \frac{7}{x_1}+\frac{1}{x_2} = \langle p, e_1(a)\rangle$. As a result for '$a$' one obtains the following system of equations which solution gives equilibrium consumption bundle for the 1st agent and equilibrium prices:

$$\begin{cases} x_2 = \frac{6x_1}{x_1+5} \\ \frac{7}{x_1}+\frac{1}{x_2}=4 \end{cases} \quad \Rightarrow \quad 6(4x_1-7)=x_1+5 \quad \Rightarrow \quad x_1=\frac{47}{23},\ x_2=\frac{47}{27} \quad \Rightarrow \quad p=(23,27).$$

Similarly one can find equilibrium inducted by event 'b'. One obtains the following system of equations for the finding of the 1st agent equilibrium consumption bundle:

$$\begin{cases} x_2 = \frac{4x_1}{8-x_1} \\ \frac{7}{x_1}+\frac{1}{x_2}=4 \end{cases} \quad \Rightarrow \quad 4(4x_1-7)=8-x_1 \quad \Rightarrow \quad x_1=\frac{36}{17},\ x_2=\frac{36}{25} \quad \Rightarrow \quad p=(17,25).$$

So one finds the following D-equilibria:

$$p_a=(23,27),\ p_b=0,\ x(a)=x(b)=\left(\frac{47}{23},\frac{47}{27}\right),\ y(a)=\left(\frac{68}{23},\frac{34}{27}\right),\ y(b)=\left(\frac{45}{23},\frac{61}{27}\right),$$

$$p_a=0,\ p_b=(17,25),\ x(a)=x(b)=\left(\frac{36}{17},\frac{36}{25}\right),\ y(a)=\left(\frac{49}{17},\frac{39}{25}\right),\ y(b)=\left(\frac{32}{17},\frac{64}{25}\right).$$

Here in both cases vector y was found by formulae $y(a)=\bar{e}(a)-x(a)$ and $y(b)=\bar{e}(b)-x(b)$.

Now I would rise an interested and important for contractual approach question: can all presented (quasi)equilibrium allocations be implemented in real economic life and that of them have better chances for a long-run existence? An answer is presented due to contractual approach: long-run living allocation has to be at least lower stable, i.e. stable relative to partial breaking of contracts implementing the allocation. However equilibrium induced by the state 'a' does not obey this criterium. In order to assure this one needs to calculate directional derivative in direction to initial endowments: all of them have to be non-positive. However non-zero derivative can has only duplicated agent $(2,b)$. Being provided the calculations one finds:

$$\nabla_y u_2^b(y)=(\tfrac{1}{y_1},\tfrac{2}{y_2}),\quad y(b)=(\tfrac{45}{23},\tfrac{61}{27}) \quad \Rightarrow \quad \nabla_y u_2^b(y(b))=(\tfrac{23}{45},\tfrac{54}{61}) \quad \Rightarrow$$

$$e_2(b)-y(b)=(\tfrac{1}{2},\tfrac{7}{2})-(\tfrac{45}{23},\tfrac{61}{27})=(\tfrac{-67}{23\cdot 2},\tfrac{67}{54}) \quad \Rightarrow$$

$$\langle \nabla_y u_2^b(y), e_2(b)-y(b)\rangle = \langle (\tfrac{23}{45},\tfrac{54}{61}),(\tfrac{-67}{23\cdot 2},\tfrac{67}{54})\rangle = \tfrac{67}{61}-\tfrac{67}{90}>0.$$

Thus directional derivative in direction $e_2(b)-y(b)$ is strictly more zero that implies agent-duplicate $(2,b)$ will partially break contracts... Another quasiequilibrium is induced by the state 'b' and is lower stable in fact that can be checked from the calculation:

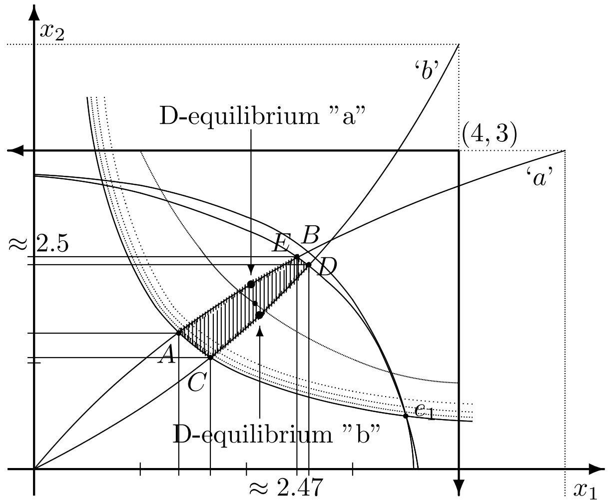

Fig. 4. *D-core: shaded curved area ACDE*

$$\nabla_y u_2^a(y) = (\tfrac{2}{y_1}, \tfrac{1}{y_2}) = (\tfrac{34}{49}, \tfrac{25}{39}) \;\Rightarrow\; \langle \nabla_y u_2^a(y), e_2(a) - y(a)\rangle = \langle (\tfrac{34}{49}, \tfrac{25}{39}), (\tfrac{-47}{34}, \tfrac{47}{50})\rangle = \tfrac{47}{78} - \tfrac{47}{49} < 0.$$

Thus this quasiequilibrium have more chances to exist than the previous one and therefore the state 'b' which specifies viable quasiequilibrium has in contractual process an advantage in comparison with the state 'a'.

In conclusion let us show that there are no other D-(quasi)equilibria in the example and therefore there are no real equilibria also at all. It can be understood coming to a contradiction. Suppose $p_a \neq 0$ and $p_b \neq 0$ are equilibrium prices. Then according to item (ii) of Definition 8 and necessary conditions for an extremum one will has: $\exists\ \alpha > 0,\ \beta > 0,\ \gamma > 0$ such that $\alpha \nabla_x u_1(x, x) = p_a + p_b$, $\beta \nabla_y u_2(y) = p_a$, $\gamma \nabla_z u_2(z) = p_b$ for $y = \bar{e}(a) - x$, $z = \bar{e}(b) - x$. However for the current train of consumption bundles it has to be true budget equalities in addition:

$$(p_a + p_b)x = (p_a + p_b)e_1(a) \;\Rightarrow\; (p_a + p_b)\bot(x - e_1(a)) = 0,\; w = (x - e_1(a)) = (x - e_1(b));$$

$$p_a y = p_a e_2(a) \;\;\Rightarrow\;\; p_a \bot (y - e_2(a)) = 0,\; -w = (y - e_2(a));$$

$$p_b z = p_b e_2(b) \;\;\Rightarrow\;\; p_b \bot (z - e_2(b)) = 0,\; -w = (z - e_2(b)).$$

This implies that each vector from p_a, p_b and $p_a + p_b$ is orthogonal to the vector $w \neq 0$. Therefore, because the space is two-dimensional, all three gradients utilities are *pairwise collinear* one to another. Hence couple (x, y) has to be Pareto optimal for 'a', but the couple (x, z) is optimal for 'b'. Thus the vector $x = (x_1, x_2)$ has to satisfy to each of two obtained above equations of Pareto boundaries, but there are no such solutions. ∎

6.3. Interim core and REE-equilibria

To discuss more the concept of the D-core and an equilibrium introduced above let us consider the following example of an asymmetric information.

Let there are 2 agents, 1st knows $\{\{a\}, \{b\}\}$, second — $\{\{a, b\}\}$. What will happen in case of D-core? The specificity is that in coalitions $\{(1, a), (2, ab)\}$ and $\{(1, b), (2, ab)\}$ by informational reasons the exchange is impossible and therefore their possibilities to dominate will be equal to one-element coalitions (even less). The only efficient coalition will be coalition of all duplicated players (notice, that it not the same, that in case of private core). However does it ensure the necessary stability?

Really if, for example, elementary event 'a' is realized "tomorrow morning", then 1st agent can suggest to 2nd one to break old contract $v = (v_1^a, v_1^b, v_2^{ab})$, $v_1^a = v_1^b = -v_2^{ab}$ and to sign a new w, if there is such (w_1^a, w_1^b, w_2^{ab}), $w_1^a = w_1^b = -w_2^{ab}$ that

$$\varphi_a^1(w_1^a + e_1^a) > \varphi_a^1(v_1^a + e_1^a) \;\; \& \;\; \psi^2(w_2^{ab} + e_2^a, w_2^{ab} + e_2^b) > \psi^2(v_2^{ab} + e_2^a, v_2^{ab} + e_2^b).$$

It is important that for the first agent a significant role plays only a duplicate 'a', and the second duplicate 'b' is a dummy: it is so because event 'a' is already realized and 1st agent knows it for certain. The second agent is not able to receive the new information, but to conclude the good deal, 1st individual accompanies him, promising identical deliveries in case of any of possible events. Clearly, when event 'b' is realized the similar things are happened only function $\varphi_a^1(\cdot)$ is replaced on $\varphi_b^1(\cdot)$.

The main question for the example: whether the contract from D-core is stable relative to described threats? There are no the obvious foundations to state it. For this reason I suggest one more concept of the core which form contracts stable relative to attempts to open new recontracting process in a state "tomorrow morning".

Definition 9. *Interim-core $\mathcal{C}^{int}(\mathcal{E}^{di})$ for the economy $\mathcal{E}^{di}$ with asymmetrically informed agents consist of allocations $x = (x_i)_{\mathcal{I}} \in X$ such that:*

(*i*) $\sum\limits_{i\in\mathcal{I}} x_i = \sum\limits_{i\in\mathcal{I}} e_i$,

(*ii*) $(x_i - e_i) : \Omega \to \mathbb{R}^l$ *is P_i-measurable for all $i \in \mathcal{I}$,*

(*iii*) $\forall\, \omega \in \Omega \; \nexists\, S(\omega) = S \subseteq \mathcal{I} : \exists y^S = (y_i^E)_{(i,E)\in\Im : i\in S}$, $y_i^E \in \mathcal{L}^E$, $(i, E) \in \Im \;|$

(a) $\forall\, i \in S$ *for* $F = P_i(\omega)$ $\; y_i^F \in X_i^F \;\&\; y_i^F \succ_i^F x_i^F$ *and*

(b) $\sum_{(i,E)\in\Im : i\in S} (y_i^E - e_i^E) = 0.$

One can notice again that due to item (*iii*)(**a**) preference in consumption is required only for duplicates $(i, F) \in \Im$ such that $F = P_i(\omega)$ under supposition the nature has realized event ω. Comparing definitions one concludes that interim core is a subset of D-core, i.e.

$$\mathcal{C}^{int}(\mathcal{E}^{di}) \subseteq \mathcal{C}^{d}(\mathcal{E}^{di})$$

is always true. However example 6.1 analyzed above shows that interim core may be empty and moreover it is typical case for asymmetric information. Our analysis is continued by

Example 6.2. Consider a pure exchange economy with asymmetrically informed agents from the previous section, Example 6.1. One can note from item (*iii*) of interim core definition that consumption bundle corresponding to a current elementary event $\omega \in \Omega$ has to be an allocation from the core of economy reduced onto the event. Moreover it has to be true for ***every*** elementary event. These cores corresponding the events 'a' and 'b' were specified during example 6.1 analysis, they are represented in Figure 3, and Figure 4 presents a summarized picture. It shows that for the area bounded by indifference curves, Pareto boundaries being separately specified for these events are not intersected, i.e. presented ***requirements are inconsistent***. Moreover the situation cannot be changed in general for any small variation of utilities and will become even worse if one will try to increase commodity space dimension: this is so because dimension of Pareto boundaries will be the same (number of agents minus 1), i.e. it will be an one-dimensional curve in 3 and more dimensional space and (it possible to say) the probability of these curves have nonempty intersection is zero. Notice that asymmetry of informational distribution plays the key role in this conclusion.

One can conclude from example presented that in order to have non-empty interim core one has to require the coincidence of individual preferences, associated with different states of the world and induced to commodity space (space of contracts!). In other words to obtain non-empty intersection of cores and, therefore, non-empty interim core one needs to require the person that is able to distinguish events has equal preferences (at least locally) for the states of the world that cannot be distinguished by another agent. However namely when interim core is empty for a current allocation realized via a web of contracts, the agents have incentives to open information: as a result of their informational sharing they discover new possibilities to conclude new mutually beneficial contract.

It is obvious, that there are informational structures, when the core is not empty under rather weak other assumptions. However I believe that generically it can be only for the symmetric information and for asymmetrically informed agents interim core generally is empty. ■

The following statement says that if in D-equilibrium by Definition 8 to require in addition that if for any two states of the world that are indiscernible at least for one individual then prices have identical proportions in these states, then this allocation belongs to interim core.

Proposition 6.1. *Let (x,p) be a D-equilibrium by Definition 8 such that*

$$\forall\, i \in \mathcal{I} \;\; \forall\, \omega, \omega' \in E \in P_i, \;\; \exists\, \lambda_{\omega'} \geq 0 : \;\; \lambda_{\omega'} p(\omega) = p(\omega'). \tag{27}$$

Then $x \in \mathcal{C}^{int}(\mathcal{E}^{di})$, i.e. this is an interim-core allocation.

Proof of Proposition 6.1. By contradiction: assume there is $\omega \in \Omega$ such that a coalition S dominates allocation x:

$$\exists y^S = (y_i^E)_{(i,E)\in\Im : i\in S} \mid \sum_{(i,E)\in\Im : i\in S} (y_i^E - e_i^E) = 0 \quad \& \quad [y_i^F \succ_i^F x_i^F, \; F = P_i(\omega)] \;\; \forall\, i \in S.$$

By (ii) in Definition 8 conclude

$$\langle p^F, y_i^F \rangle > \langle p^F, x_i^F \rangle = \langle p^F, e_i^F \rangle, \;\; F = P_i(\omega) \quad \forall\, i \in S,$$

that by $y_i(\omega') - e_i(\omega') = y_i(\omega) - e_i(\omega)$ for all $\omega' \in P_i(\omega)$, $i \in S$ and (27) yields

$$0 < \langle p^F, y_i^F - e_i^F \rangle = \langle p(\omega), y_i(\omega) - e_i(\omega) \rangle \sum_{\omega' \in F} \lambda_{\omega'}, \;\; \forall\, i \in S \;\; \Rightarrow \;\; \langle p(\omega), \sum_S (y_i(\omega) - e_i(\omega)) \rangle > 0,$$

that implies $\sum_S (y_i(\omega) - e_i(\omega)) \neq 0$ and therefore contradicts to $(iii)(\mathbf{b})$ of Definition 9. ■

Proposition 6.1 characterizes D-equilibria that are the elements of interim core, however it would be necessary to present or clarify their existence conditions. The following proposition is important for understanding this problem: there are analyzed D-equilibrium allocations, realized as fuzzy contractual ones, such that there are no individuals having incentives to share information, because it is impossible to sign new beneficial contract.

Proposition 6.2. *Let $x = (x_i)_{i\in\mathcal{I}}$ be a fuzzy contractual allocation with differentiated agents. Then there exist prices $p : \Omega \to \mathbb{R}^l$, for each $i \in \mathcal{I}$ and $E \in P_i$ satisfying*

$$\sum_{\omega\in E} p(\omega)x_i(\omega) = \sum_{\omega\in E} p(\omega)e_i(\omega)$$

and

$$\langle \sum_{\omega\in E} p(\omega), z\rangle \geq 0,\ \forall\, z \in \mathbb{R}^l :\quad x_i + z\cdot\chi^E \in \mathcal{P}_i(x_i). \tag{28}$$

Thus fuzzy contractual allocation is D-quasiequilibrium .

Let, in addition, the allocation $x = (x_i)_{i\in\mathcal{I}}$ be such that there is ***no*** *possibility to sign a new mutually beneficial* ***contract for any kind of informational sharing****. Then (28) is replaced by a more qualified requirement*

$$\sum_{\omega\in E} p(\omega)z(\omega) \geq 0,\ \forall\, z : \Omega \to \mathbb{R}^l :\quad x_i + z\cdot\chi^E \in \mathcal{P}_i(x_i). \tag{29}$$

It can be specially noted the difference in formulae (28) and (29): a *vector* $z \in \mathbb{R}^l$ is applied in the first case, but a *function* $z : \Omega \to \mathbb{R}^l$ in second one.

For contractual process the meaning of the proposition is that if (29) is invalid for fuzzy contractual allocation with D-agents, then there may happen informational sharing that will imply new acts in contractual process.

The last proposition reveal some specific features of equilibrium in the case when agents have no incentives to open information. However the arising question is to estimate how typical is this case and can one qualify it as a generical for D-equilibria to obey (29) in addition. With this in mind let us comparatively analyze this for D-equilibria. Now compare the number of independent constrains and the number of applied variables in the model situation considered above where 1st agent is able to distinguish events a and b, but 2nd *is not able* to do it. Calculate the balance between variables and constrains (variables minus constrains). So for D-equilibria one has:

Direct variables applied for allocation:

$$l \text{ (1st for `a')} + l \text{ (1st for `b')} + l \text{ (2nd for `ab')} - 2l \text{ (number of balance equations)} = l.$$

Price variables and budget constrains[32] yield:

$$l - 1\ (p(a) \text{ normalized}) + l \text{ (for } p(b)) - 2 \text{ (number of independent budget equations)} = 2l - 3.$$

First order conditions (collinearity of prices and gradients):

$$1 - l \text{ (1st for `a')} + 1 - l \text{ (1st for `b')} + 1 - l \text{ (2nd for `ab')} = 3 - 3l.$$

As a result, summing right hand parts, one finds

$$l + 2l - 3 + 3 - 3l = 0,$$

[32] Standardly: if material balance is true then one (any) of budget equalities followed by other ones. This is why number of independent constrains is a unit less of total quantity.

i.e. in generic situation for twice differentiable utility functions one can expect a *finite* number of D-equilibria.

What are the changes in the calculations for allocation satisfying (29)? Only the final addend is changed, namely in first order conditions for 2nd individual there are appeared l additional constrains[33], i.e. the total balance will be negative one and one can expect that generically there are no these allocations at all... For more detailed analysis one can turn to the consideration of example 6.1.

However what does happen in general case and what concept of an equilibrium does correspond to the interim core? To answer the question correctly I need the following technical result.

Lemma 6.2. *Let $\mathcal{E}^{di}$ be a smooth economy and $x = (x_i)_{\mathcal{I}}$ be an* ***interior*** *allocation from interim core, $x \in \mathcal{C}^{int}(\mathcal{E}^{di})$. Then there are prices $p : \Omega \to \mathbb{R}^l$, $p(\omega) \neq 0 \;\forall\, \omega \in \Omega$ and real $\lambda_{i,E} > 0$, $(i, E) \in \Im$ such that*

$$\lambda_{i,E} \sum_{\omega \in E} \nabla_\omega u_i(x_i) = p(\omega) = p(\omega') \quad \forall\, \omega, \omega' \in E, \quad \forall\, (i, E) \in \Im. \tag{30}$$

Therefore, for these prices one also has

$$\langle \sum_{\omega \in E} p(\omega), z \rangle > 0, \;\forall\, z \in \mathbb{R}^l : \quad x_i + z \cdot \chi^E \in \mathcal{P}_i(x_i). \tag{31}$$

Notice the relationship of (31) with relation (28), that characterizes fuzzy contractual allocation with D-agents and that corresponds to (*ii*) of D-(quasi)equilibrium definition (Definition 8): a difference is only the strict form of inequality.

Proof of Lemma 6.2. Analyzing Definition 9 one finds that for every $\omega \in \Omega$ allocation $x(\omega) = (x_i(\omega))_{\mathcal{I}}$ is Pareto optimal in the model reduced to the event ω, where agents' utilities are defined on $X_i(\omega) = \mathbb{R}^l_+$ via $\vartheta_i(y_i) = u_i(z_i(\cdot))$ as follows

$$z_i(\omega') = \begin{cases} y_i, & \omega' \in P_i(\omega), \\ x_i(\omega'), & \omega' \notin P_i(\omega). \end{cases}$$

This standardly implies the existence of vector $p(\omega) \in \mathbb{R}^l$, $p(\omega) \neq 0$ and real $\mu_{i,\omega} > 0$ such that $\mu_{i,\omega} \nabla \vartheta_i(x_i) = p(\omega)$. Thus for normalized $p(\omega)$ one can conclude (because $p(\omega) = \mu_{i,\omega} \nabla \vartheta_i(x_i) = p(\omega')$), that

$$p(\omega) = p(\omega') \quad for \;\; \omega, \omega' \in E, \quad (i, E) \in \Im.$$

This proves (30) in view of $\sum_{\omega \in E} \nabla_\omega u_i(x_i) = \nabla \vartheta_i(y_i)$ for initial utilities. Now relation (31) follows directly. Lemma has proven. ■

Now one can formulate an equilibrium notion well corresponding to interim core concept. This definition is based on Lemma 6.2 result the only that we need is to add to relations (30), (31) a budget feasibility (inequality realized as equality)

[33] Instead of $\nabla_a u_2(x_2^a, x_2^b) + \nabla_b u_2(x_2^a, x_2^b) = \lambda(p^a + p^b)$ for $x_2^a - e_2^a = x_2^b - e_2^b$ one has to use $\nabla_{ab} u_2(x_2^a, x_2^b) = \lambda(p^a, p^b)$.

Definition 10. *A couple (x, p) is called interim equilibrium if it is a D-equilibrium according to Definition 8 and in addition:*

$$\forall\, i \in \mathcal{I}, \quad \forall\, \omega \in \Omega \;\; p(\omega) = p(\omega') \neq 0 \;\; \forall\, \omega' \in P_i(\omega).$$

holds.

Notice that this definition almost exactly corresponds to the notion of REE-equilibrium, see page 15. The only difference is the specific probability context applied for REE and also the possibility to consider finer informational structures, enriched via price informational channel. Notice also that interim equilibrium being an element of interim core (due to Proposition 6.1) may not exist; the existence of it can be provided for a specific informational structures but to achieve it informational exchange has to be realized.

What is the dynamics of contractual process that drives economy to interim equilibrium? Clearly this process has to include an endogenous process of informational sharing. I shall continue the analysis of considered above example, in which $\Omega = \{a, b\}$, $P_1 = \{\{a\}, \{b\}\}$, $P_2 = \{\Omega\}$.

What will be in an equilibrium? In the variant that corresponds to private equilibrium, 1st agent should reach a maximum of utility under constrain

$$p(a)x(a) + p(b)x(b) \leq p(a)e_1(a) + p(b)e_1(b).$$

On the other hand in variant corresponding to D-equilibrium this agent has to solve two problems:

$$\varphi_a^1(x(a)) \rightarrow \max \;\; s.t.\; p(a)x(a) \leq p(a)e_1(a) \;\; \& \;\; \varphi_b^1(x(b)) \rightarrow \max \;\; s.t.\; p(b)x(b) \leq p(b)e_1(b).$$

Second agent in both cases solves a problem under an additional constrain:

$$[p(a) + p(b)]y(a, b) \leq p(a)e_2(a) + p(b)e_2(b), \;\; y(a, b) = y(a) = y(b).$$

In equilibrium solutions of these problems has to be feasible.

The question is raised: will contractual process be completed in this (equilibrium) allocation or it can continue developing? Probably the first agent will wish to share the information with 2nd and to transmit him the ability to distinguish a and b? He/she can do it only if he/she will receive real advantage (profit). This advantage can be realized as a new favourable contract — having learnt a difference between states, 2nd agent will agree to sign contract beneficial for the first agent. Let us write the conditions of this agreement for differentiated agents:

$$\exists\; (v(a), v(b)) : \;\; \langle p(a), v(a)\rangle > 0, \quad \langle p(b), v(b)\rangle > 0 \;\; -$$

these are payoffs of 1st in every of future states and

$$\langle p(a) + q, -v(a)\rangle > 0, \quad \langle p(b) - q, -v(b)\rangle > 0$$

are payoffs of second agent (possibly not all > 0, one zero can be); here q is a vector which exists via condition that *there is no* mutually beneficial contract under *fixed* information (Pareto optimality), see Lemma 6.1. As soon as for $q \neq \lambda p(a)$ vectors $p(a)$ and $-[p(a) + q]$ are non-collinear, then there exists $v(a)$ satisfying

$$\langle p(a), v(a)\rangle > 0 \;\; \& \;\; \langle p(a) + q, -v(a)\rangle > 0.$$

Thus there is the following: 1st says to 2nd: "I can learn you to distinguish 'a' and 'b', and you will learn me to distinguish event $\{a, b\}$ (from 2nd partition) and then we will be able to find mutually beneficial exchange..."

When this cannot be happened in general (let the allocation be interior)? Lemmas 5.1, 6.1, their Corollaries 5.1, 5.2, 6.1 and Proposition 6.2 give answers. It will not be happened if and only if for event $a \in \Omega$

$$\forall\, z \in \mathbb{R}^l \;\; \langle p(a), z\rangle > 0 \;\Rightarrow\; \langle p(a) + q_i(a), z\rangle > 0 \;\forall\, i \in \mathcal{I},$$

is true and similar requirement for the event $b \in \Omega$:

$$\forall\, z \in \mathbb{R}^l \;\; \langle p(b), z\rangle > 0 \;\Rightarrow\; \langle p(b) + q_i(b), z\rangle > 0 \;\forall\, i \in \mathcal{I}.$$

In view of Farkas's lemma 1st of these conditions is equivalent to the vectors $p(a)$ and $p(a)+q_i(a)$ are collinear for all $i \in \mathcal{I}$; the same thing for a second condition but now for event b. Certainly, speaking about prices $p(a)$ and $p(a) + q_i(a)$ I mean gradients of utilities related with prices via conditions $\exists\, \lambda_{1a} > 0$: $\nabla_a u_1(x_1) = \lambda_{1a} p(a)$ and $\exists\, \lambda_{2ab} > 0$: $\nabla_a u_2(x_2) = \lambda_{2ab}(p(a) + q_2)$ $\nabla_b u_2(x_2) = \lambda_{2ab}(p(b) - q_2)$ and similar relations for other individuals and for the state 'b'. However the collinearity of $p(a)$ and $p(a) + q_i(a)$ implies collinearity of $p(a)$ and $q_i(a) \neq 0$ and analogously the collinearity of $p(b)$ and $q_i(b) = -q_i(a)$ that implies $p(a)$ and $p(b)$ are collinear vectors.

Thus if informed individual $i = 1$ has non-collinear gradients $\nabla_a u_1(x_1)$ and $\nabla_b u_1(x_1)$ then mutually beneficial ***contract*** with individual $i = 2$ will ***not be found*** only if $(\nabla_a u_2(x_2), \nabla_b u_2(x_2)) = (\alpha \nabla_a u_1(x_1), \beta \nabla_b u_1(x_1))$ for some real $\alpha > 0$ and $\beta > 0$ (for this case[34] $q = 0$). Moreover, due to Lemma 6.2 for non-collinear gradients individual $i = 1$ understands clearly that if one of events 'a' or 'b' will be realized 'tomorrow morning' then he/she or other agents will renew contractual process, i.e. current allocation is not stable in fact. Both instances motivate $i = 1$ to open (share) today information to $i = 2$ and then contractual process can be proceeding under a new informational allocation. This will proceed till a new contractual allocation corresponding to D-equilibrium with proportional prices (equal being renormalized!) at indistinguishable states of the world, i.e. then an interim equilibrium is realized.

Further one describes a possible procedure of mutually beneficial contract searching during contractual process ongoing with sharing of information. Let for an event $a \in \Omega$ vectors $p(a) + q_i(a)$ and $p(a) + q_j(a)$ are non-collinear. Now consider event

$$E_{ij}(a) = P_i(a) \cap P_j(a).$$

Individuals i, j can be learnt to distinguish this event if they share information about $P_i(a)$ and $P_j(a)$. Further if vectors

$$\sum_{\omega \in E_{ij}(a)} (p(\omega) + q_i(\omega)) \;\;\&\;\; \sum_{\omega \in E_{ij}(a)} (p(\omega) + q_j(\omega)) \qquad (32)$$

are non-collinear then due to Corollaries 5.1, 6.1 to lemmas on mutually beneficial contract these individuals will be able to find a mutually beneficial exchange for the event $E_{ij}(a)$; otherwise if

[34] Notice that in this case there is no necessity to share information at all and if nevertheless 1st will teach 2nd agent to distinguish 'a' and 'b', then nothing will be happened and current allocation will not change. However after informational sharing the allocation will obtain an additional stability.

vectors (32) are collinear then a third participant can be invited to joint to coalition, let his/her number be k, and such that

$$E_{ijk}(a) = P_i(a) \cap P_j(a) \cap P_k(a) \neq E_{ij}(a),$$

i.e. k-th individual is able to improve information about event a. Now if vectors

$$\sum_{\omega \in E_{ijk}(a)} (p(\omega) + q_i(\omega)) \quad \& \quad \sum_{\omega \in E_{ijk}(a)} (p(\omega) + q_j(\omega)) \quad \& \quad \sum_{\omega \in E_{ijk}(a)} (p(\omega) + q_k(\omega))$$

are non-collinear then coalition $\{i, j, k\}$ is able to find a new mutually beneficial contract; otherwise one needs to involve in a coalition one more individual and so on. This process has to finish at some time because by (4)

$$\bigcap_{\mathcal{I}} P_i(a) = \{a\},$$

and among vectors $p(a) + q_i(a)$, $i \in \mathcal{I}$ there exists at least one non-collinear pair.

If the moment for a possibility of a mutually beneficial exchanging has occurred, then information interchange is realized and ordinary contractual process is going up to approach to a new dead-end situation: there is no possibility for an interchanging without new information division.

So, what it is as a result? The system as a whole can come (if it is not cycled) to allocation such that even after the information sharing a mutually beneficial exchange is impossible. Besides, it should be stable contractual allocation. However what is the case?

(i) This way and also applying other methods that can be outside the model framework (for example via a price channel) there was attained a new information distribution that is finer than the previous one but in general is coarse than supremal information: $\tilde{\mathbb{P}} = (\tilde{P}_i)_{\mathcal{I}} \succeq (P_i)_{\mathcal{I}} = \mathbb{P}$, $\tilde{P}_i \preceq \bigvee_{j \in \mathcal{I}} P_j$, $i \in \mathcal{I}$.

(ii) For the attained level of informational distribution a new mutually beneficial contract is impossible even after additional information sharing, formally (via Lemma 6.1): $\exists p : \Omega \to \mathbb{R}^l$, $p \neq 0$ and $\lambda_{i,E} > 0$, $(i, E) \in \Im$, such that

$$\forall\, E \in \tilde{P}_i, \quad \lambda_{i,E} \sum_{\omega \in E} \nabla_\omega u_i(x_i) = \sum_{\omega \in E} p(\omega) \quad \forall\, i \in \mathcal{I}. \tag{33}$$

Moreover for stability of allocation at the moment of ‘tomorrow morning’, condition (27) of prices are proportional at all states of the world that are indistinguishable for an individual; this can be written as for every $\forall\, \bar{\omega} \in \Omega$ the family of vectors $\{\sum_{\omega \in P_i(\bar{\omega})} \nabla_\omega u_i(x_i)_{i \in \mathcal{I}}\}$ is collinear one. At the same time the requirement for a system of gradients $\{\nabla_{\bar{\omega}} u_i(x_i)\}_{i \in \mathcal{I}}$ be collinear for $\forall\, \bar{\omega} \in \Omega$, that formally implies impossibility to sign new mutually beneficial contract after informational sharing, seems to be excessive one. This is so because the informed agents are not motivated enough to share information and because the ability to distinguish or not distinguish something is a private property of an individual and in general this is unknown for contractual partner. Here the disproportion in gradients (prices, exchange proportions) can be considered as a signal to share information: there is no exchange without this signal. Thus one has described a contractual process delivering a fixed-point allocation that corresponds to interim equilibrium but it is happened now for enriched informational structure.

And what is about core? It has to be a core in the model with agents' "duplicates" constructed for an information distribution $\tilde{\mathbb{P}} = (\tilde{P}_i)_{\mathcal{I}}$, realized at the last stage of information sharing:

$$\mathcal{C}(\mathcal{E}^{di}, \Im, \tilde{\mathbb{P}}), \quad \Im(\tilde{\mathbb{P}}) = \{(i, E) \mid i \in \mathcal{I},\ E \in \tilde{P}_i\}.$$

For a further convenient and short usage let us call this core as ***D-interim-core***. Now one always has by construction

$$\forall\, i \in \mathcal{I} \quad \tilde{P}_i \preceq \bigvee_{j \in \mathcal{I}} P_j.$$

Moreover there are reasons to think that in generic case (for almost all economies) the described process will lead to supremal information, i.e. it will be

$$\forall\, i \in \mathcal{I} \quad \tilde{P}_i = \bigvee_{j \in \mathcal{I}} P_j = P_\vee.$$

For this case one will have

$$\mathcal{C}(\mathcal{E}^{di}, \Im, \tilde{\mathbb{P}}) = \mathcal{C}(\mathcal{E}^{di}, \Im, P_\vee) = \prod_{E \in P_\vee} \mathcal{C}(\mathcal{E}^{di}(E)),$$

where $\mathcal{C}(\mathcal{E}^{di}(E))$ is a core in the model $\mathcal{E}^{di}(E)$ with agents as in initial model but commodity space is changes on

$$\{z : E \to \mathbb{R}^l \mid \quad z(\omega) = z(\omega'),\ \forall\, \omega, \omega' \in E\}$$

and with endowments preferences of initial model are inducted on positive cone of this space (this is applied as a consumption set). In particular if (4) is true, then

$$P_\vee = \Omega^* \quad \Rightarrow \quad \mathcal{C}(\mathcal{E}^{di}) = \prod_{\omega \in \Omega} \mathcal{C}(\mathcal{E}^{di}(\omega)).$$

The final remark. The presented core can be considered also as a core, that generically corresponds to the concept of REE-equilibrium. In general case one needs to take the core of economy with differentiated agents: in this model the individual information is specified so as it is presented in REE-concept: supremum of the initial information and an information received via the price channel. Here one can feel inconformity between the concept of core (stability on the basis of the barter trade) and the presence of prices and trade under these prices... However REE-equilibrium concept was not invented by me.

6.4. Universal equilibrium and core notions in DIE-economies

In the previous sections there were proposed and analyzed different possibilities to apply contractual approach to the model with differentiated information. I think that contractual approach turned out as an efficient tool that allows us to investigate not only private core and equilibrium but also to introduce a sequence of new concepts (D-core and equilibrium, interim concepts etc.), driving us to a contractual analogue of REE-equilibrium notion. In this section I want to propose one more equilibrium notion based on a complex-contractual treatment: under some circumstances this implements private equilibrium, under other ones REE-equilibrium but often and often it will present something intermediate one.

Let us deeper think about that is the fundamental difference between two equilibrium concepts, private equilibrium (WEE) and equilibrium in rational expectations (REE), between private equilibrium in contractual sense and interim equilibrium (when nobody wants to reveal information). I believe that the main difference consists in:

(i) For the first (private) case contracts concluded today at the end of negotiation stage will be *certainly realized tomorrow*, nevertheless that tomorrow some individuals will wish somehow to change contracts, i.e. the fact that concluded contracts are implemented confirming the agent's reputation as a reliable partner is more important than some temporarily gains.

(ii) For the second case, this is the case of interim equilibrium and core, all contracts are initially less stable and, according to the game rules, at the moment "tomorrow morning" *contractual process can be renewed*, individuals will break existing and sign new contracts; they will also to reveal information.

Further I would like to note first that if one will not take into account possibilities of informational exchange then statistically (in expectations) the first noted variant is more preferable for individuals since it leads to higher utility values. The weak point in this approach is stability and implementation... Does it possible presented approaches to be combined in a form of united concept? Certainly it is if one applies contractual point of view. Namely, one can specify for every coalition a subspace of strongly stable contracts, contracts of this kind will certainly be implemented tomorrow; this is a subspace of space of contracts where the distribution of information is already taken into account. Yet, it is not forbidden for individuals to conclude all other contracts but their tomorrow implementations are not guaranteed: tomorrow morning a new contractual process can be initiated. Agreements realized among a narrow group of responsible individuals (reliable gentlemen, they are always better than their word) can serve the example of strong kind contracts. Another example presents the trade of future contracts: the deal being concluded today is realized tomorrow (a month or half year later etc). Contracts of second kind look like preliminary agreements implementing a preliminary stage of final contract search that in view of informational constrains can be found only "tomorrow morning" when some information will appear on a true state of the world.

How in general has contractual process to go? There are two stages, during the first one only high-stable contracts are concluded, and all other contracts are signed in second stage, where in difference with the first one informational exchange can also be going being an integrated part of contractual process that drives economy to an interim equilibrium. Further one considers a formalized mathematical presentations (unfortunally rather cumbersome ones).

Let us consider model $\mathcal{E}^{di}$ of pure exchange economy with differentiated information described on page 20. Let us add to the model a new element $\mathcal{W} = \bigcup_{S\subseteq\mathcal{I}} \mathcal{W}^S$, where $\mathcal{W}^S \subseteq L^{\mathcal{I}}$ is a space of strong stable contracts that can be signed by coalition $S \subseteq \mathcal{I}$ members, they are contracts that after finish of contractual stage "today" will certainly implemented tomorrow[35]. Let $\mathcal{W}_i$ be a ***projection*** $\mathcal{W}$ on i-th actor in the space of allocations $L^{\mathcal{I}}$. Further let us consider two notions, a core and equilibrium, that well correspond to contractual point of view and differential information.

Definition 11. *An allocation $x \in \prod_{\mathcal{I}} X_i$ is an element of contractual core if there exists a web of contracts $V = \{v^S\}_{S\subseteq\mathcal{I}}$, $v^S \in W^S$, $S \subseteq \mathcal{I}$ such that $y = e + \sum_{v^S\in V} v^S \in \prod_{\mathcal{I}} X_i$ and the allocations x and y obey conditions: $y = (y_i)_{\mathcal{I}}$ is an element of private core with respect to $\mathcal{W}$, i.e.*

(i) $\sum\limits_{i\in\mathcal{I}} v_i = 0,$

[35] Here $v = (v_i)_{\mathcal{I}} \in \mathcal{W}^S \Rightarrow v_i = 0$ for $i \notin S$.

(ii) $v_i : \Omega \to \mathbb{R}^l$ *is* P_i*-measurable for all* $i \in \mathcal{I}$*,*

(iii) $\nexists S \subseteq \mathcal{I} : \exists z^S = (z_i)_S, \quad (z^S - e^S) \in \mathcal{W}^S \mid \forall\, i \in S,\ z_i \in X_i$ *is such that*

$(z_i - e_i)$ *is* P_i*-measurable,* $z_i \succ_i y_i$ & $\sum_S (z_i - e_i) = 0$*;*

and the allocation x *is an element of interim core relative to endowments* $y = (y_i)_{\mathcal{I}}$ *and (some) informational structure* $\tilde{\mathbb{P}} = (\tilde{P}_i)_{i \in \mathcal{I}}$*, obtained as a result of contractual process of informational sharing, i.e.* x *obeys in addition*

(iv) $\sum\limits_{i \in \mathcal{I}} x_i = \sum\limits_{i \in \mathcal{I}} e_i$*,*

(v) $(x_i - y_i) : \Omega \to \mathbb{R}^l$ *is* $\tilde{P}_i$*-measurable for all* $i \in \mathcal{I}$*,*

(vi) $\forall\, \omega \in \Omega \ \nexists\, S(\omega) = S \subseteq \mathcal{I} : \exists z^S = (z_i)_{i \in S} \in (\mathbb{R}^l_+)^S,\ \sum_{i \in S} (z_i - y_i(\omega)) = 0$ &

$z_i \cdot \chi_F \succ_i^F x_i^F,\ \forall\, i \in S$ *for* $F = \tilde{P}_i(\omega)$

Notice a small technical difference in the presentation of *(vi)* from the latter definition and *(iii)* from Definition 9. On the existence of contractual core one can note that the existence of private core with respect to $\mathcal{W}$ can be stated without problems applying technique presented in the proof of Theorem 9.1. Further one can take any allocation from the core and activate contractual process that leads us to a fixed point from the D-core (the existence of this can be stated directly similarly as in Theorem 9.1). Now if this point is not in interim core then an informational sharing is realized and a new contractual process is initiated that again leads to a point from D-core, but now already relative to enriched informational structure, and so on.

Further let us consider a two stages contractual concept of equilibrium.

Definition 12. *A couple of allocations* $(y, x) \in X \times X$ *and a couple of price mappings* (q, p)*,* $q, p : \Omega \to \mathbb{R}^l$ *is called contractual equilibrium if the couple* (y, q) *is a private equilibrium with respect to* $\mathcal{W}$*, i.e., first the following is true:*

(i) $(y_i - e_i) : \Omega \to \mathbb{R}^l$ *is* P_i*-measurable for all* $i \in \mathcal{I}$*,*

(ii) $0 \not\equiv \langle p, (\mathcal{P}_i(y_i) - e_i) \cap \mathcal{L}_i \cap \mathcal{W}_i \rangle \geq 0$[36]*,* $i \in \mathcal{I}$*,*

(iii) $\sum_{i \in \mathcal{I}} y_i = \sum_{i \in \mathcal{I}} e_i$*.*

Second requirement is that pair (x, p) *is an interim equilibrium with respect to endowments* $y = (y_i)_{\mathcal{I}}$ *and an informational structure* $\tilde{P}_i = P_i \vee \sigma(p)$*, where* $\sigma(p)$ *is the algebra of events (field), generated by the price map* $p(\cdot)$[37]*:*

(iv) $(x_i - y_i) : \Omega \to \mathbb{R}^l$ *is* $\tilde{P}_i$*-measurable for all* $i \in \mathcal{I}$*,*

(v) $\langle p^E, (\mathcal{P}_i^E(x_i^E) - y_i^E) \rangle > 0, \ \ \langle p^E, x_i^E - y_i^E \rangle = 0, \quad \forall\, i,\ E = \tilde{P}_i(\omega),\ \forall\, \omega \in \Omega,$

(vi) $\sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} e_i = \sum_{i \in \mathcal{I}} y_i$*.*

On the existence of equilibria defined via the above conception one can only note that everything that is known for the core one needs to apply to find fuzzy contractual elements that corresponds

[36] Recall that $\mathcal{L}_i$ is, in the space of contingent commodities, a space of maps measurable with respect to i^{th} agent informational partition.

[37] The coarsest σ-algebra, such that $p(\cdot)$ is measurable map.

to equilibria. Notice also that an attempt to consider an equilibrium with a common prices for both stages instead of two-stages $q(\cdot)$, $p(\cdot)$ leads us to the notion whose existence is unclear. Moreover, it is also unclear how it can be interpreted, since in this case contractual process has to be treated as going simultaneously for strong and weak stable contracts.

7. COMPUTER SIMULATION OF CONTRACTUAL PROCESSES IN INFORMATION DIFFERENTIAL ECONOMIES

In this section there are presented some results of computer simulation for contractual processes that drives economy to the contractual allocations of different kinds: private core and equilibrium, D-core and D-equilibrium, and interim concepts too.

Contractual processes were suggested and studied in my paper (Marakulin, 2006), where a standard pure exchange economy with *symmetrically* informed agents was considered as a basic model; there were suggested several types of processes, the most major among others is proper contractual one. For processes of this kind partial breaking of the contracts is allowed in the context of several basic hypotheses, determining the character of contracts' breaking process, are formulated in a general kind and for major particular cases. They are the following:

(IB) — instantaneous breaking of the contracts;

(UB) — uniform breaking of all contracts;

(CUB) — uniform breaking of gross within-coalitional contracts.

Combinations of these hypotheses result to proper-contractual trajectories of a different kind of a generality. Under **(IB)** and **(UB)** contractual trajectory turns out *aggregated*, under **(IB)** and **(CUB)** — *coalitional-contractual*; there are given formal and mathematically reasonable definitions. Besides, concept of *trade rule* is introduced; this is a map, unequivocally determining mutually beneficial contract for the current consumption plans, having some additional good mathematical properties. By use of a trade rule a contractual trajectory of each mentioned kinds is unequivocally determined. The special type of *benevolent* rules of trade is stood out as rules which determine a new contract allowing the break of gross barter contract only if being realized *every* new mutually beneficial contract involves contracts' breaking. Just for this class of benevolent processes the basic positive results about convergence were received. In general case contractual processes may be cycled However their steady points always have specific equilibrium properties (i.e. it is an element of core or quasiequilibrium, depending on the kind of process) that attracts an interest.

In several student works provided under my scientific supervision there were considered and modeled in computer simulation contractual processes where a new mutually beneficial contract and, therefore, trade rule as a whole, is determined according to one of known in bargaining theory kinds of solution. In this context there were considered Nash's and Kalai–Smorodinsky solutions, see Rubinstein, Osborne (1990), Thomson (1994), that are the most interested ones from theoretical and practical point of views. In this paper there were modeled processes that applies so called "local Nash rule".

There is considered the following model of pure exchange economy with asymmetrically informed agents. There are 2 kinds of physically different commodities, 2 individuals and 3 states of the world. Preferences are defined by Cobb–Douglas utility functions $u_1(x)$, $x \in \mathbb{R}^6$ and $u_2(y)$, $y \in \mathbb{R}^6$ presented in the logarithmic form:

$$u_1(x) = \alpha_{a_1} \ln(x_{a_1}) + \alpha_{a_2} \ln(x_{a_2}) + \alpha_{b_1} \ln(x_{b_1}) + \alpha_{b_2} \ln(x_{b_2}) + \alpha_{c_1} \ln(x_{c_1}) + \alpha_{c_2} \ln(x_{c_2}), \quad x \gg 0,$$

$$u_2(y) = \beta_{a_1} \ln(y_{a_1}) + \beta_{a_2} \ln(y_{a_2}) + \beta_{b_1} \ln(y_{b_1}) + \beta_{b_2} \ln(y_{b_2}) + \beta_{c_1} \ln(y_{c_1}) + \beta_{c_2} \ln(y_{c_2}), \quad y \gg 0.$$

The coefficients of these functions are defined at the start of modeling program:

$$\alpha = (\alpha_{a_1}, \alpha_{a_2}, \alpha_{b_1}, \alpha_{b_2}, \alpha_{c_1}, \alpha_{c_2}), \quad \beta = (\beta_{a_1}, \beta_{a_2}, \beta_{b_1}, \beta_{b_2}, \beta_{c_1}, \beta_{c_2}).$$

Initial endowments are e=(e_1, e_2), $e_1 = (e_1^1, e_1^2, e_1^3)$, $e_2 = (e_2^1, e_2^2, e_2^3)$ and information is P_1, P_2 that is presented as partitions of $\Omega = \{a, b, c\}$, it is the set of all states of the world. In the program this information is defined as an ordered array of natural numbers; here the state a has index 1, b — 2, c — 3. Moreover if for one of agents some elements of array concise then it means this agent is not able to distinguish states corresponding to the elements' indexes, i.e. the states are placed in a common element of partition. Besides there are defined parameters: *Step* > 0 for volume of contract and *Accuracy* > 0 for the accuracy of equilibrium calculation. Program starts from the initial endowments $(x, y)^{(0)} = (e_1, e_2)$.

Further there is described a programmed procedure applied for the search of private equilibrium under endogenously going process of informational sharing.

Now I first describe trade rule applied for our process. This rule was called "local Nash rule" because the levels of marginal profit from contracts are defined via Nash solution. One can show that for a pair agents coalition this rule corresponds to the principle of "moving along the bisectrix" of angle formed by gradients of utility functions (formally for the second one needs to take antigradient): it means equal contractual profit (it is calculated as an inner product of gradient and the vector of contractual flow). Let $w_1^{(n)} = \mathrm{Pr}_W(\nabla u_1(x^{(n)}))$ and $w_2^{(n)} = \mathrm{Pr}_W(\nabla u_2(y^{(n)}))$, where $\mathrm{Pr}_W(\cdot)$ denotes the projection on the space $W \subseteq \mathbb{R}^6$ of *permissible contracts* (parameterized in the 1st agent consumption). Now we have:

$$v^{(n)} = (v^{(n)}, -v^{(n)}), \ v^{(n)} := \frac{w_1^{(n)}}{|w_1^{(n)}|} - \frac{w_2^{(n)}}{|w_2^{(n)}|}.$$

Thus gradients are projected on the space of *permissible contracts* (one needs measurability relative to information of first and second agent) and then they are normalized. The volume of contract (length of vector) $v^{(n)} = (v_1^{(n)}, v_2^{(n)})$ is defined equal to *step*, i.e. $v^{(n)} := \mathit{step} * v^{(n)}/|v^{(n)}|$[38]. Further the volume of contract becomes less again (it is divided on 1.1) until contract be mutually beneficial or while its volume will not become less of *accuracy* ($|v^{(n)}| < \mathit{accuracy}$). In the first case contract $v^{(n)}$ is signed and system transits to allocation $(x, y)^{(n)} + v^{(n)}$ that can be further partially broken off. In the second case we think that programme has found an equilibrium under presented information and $(x, y)^{(n)}$ is written in the file. When an equilibrium relative to a fixed information is already found, the next step realizing information sharing can be done.

Contracts breaking procedure is specified as follows. After every time when new contract is concluded the first (or second) individual partially break gross contract in the volume $\frac{|v^{(n)}|}{10}$, and if necessary this breaking is repeated while it is profitable for the agent. Let $\lambda^{(n)}$ denote

[38] It is a little bit incorrect to apply a common notation for initial and normalized contract.

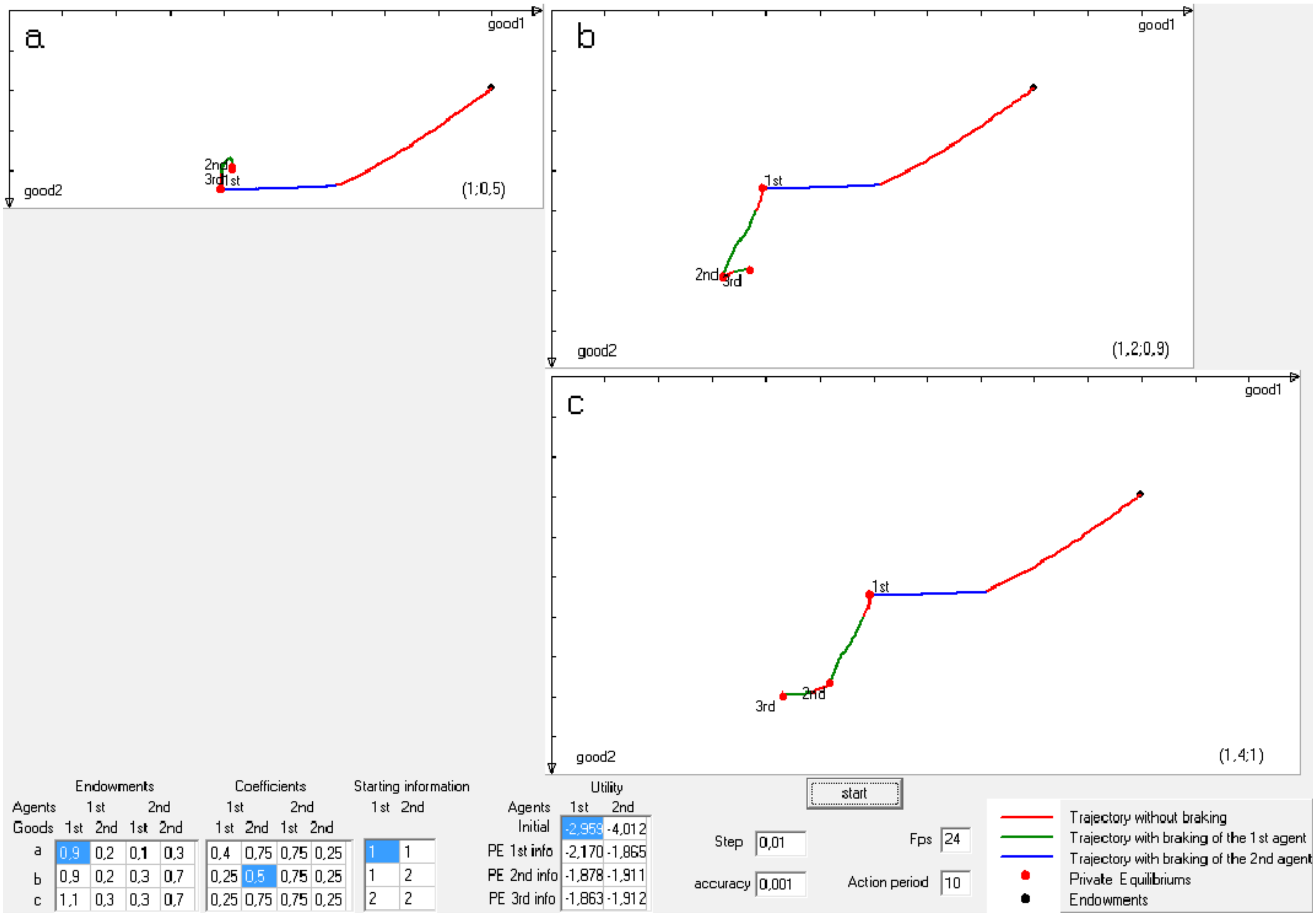

Fig. 5. Dynamics of private contractual process with an informational sharing

a number of these breaking iterations for a current situation. Then a resulting transition is realized by formula:

$$(x,y)^{(n+1)} = (x,y)^{(n)} + v^{(n)} + \lambda^{(n)} * \frac{|v^{(n)}|}{10} * (e - (x,y)^{(n)} - v^{(n)}).$$

Information sharing: at every time when equilibrium relative to presented information has found it can be profitable for agents to change information (to share). An agent has incentives to share information (wants to inform another agent how he/she can distinguish an element of partition from another one), if projection of his utility gradient on the space of contracts differs with the gradient (i.e. it is nontrivial: gradient does not belong the space[39]). Further, supposing that trial informational sharing individuals find a new mutually beneficial contract according to the rule described above, but now already for an updated informational structure. Now if the volume of new contract is greater of *accuracy*, then the agent really shares the information (trial sharing becomes real one) and the stage is finished. Otherwise the agent thinks informational sharing inefficient however he/she can also try to use another method of informational sharing (e.g. in sharing procedure one can apply another element of informational partition). Finally if both agents get nowhere then contractual process is finished and programme ends the work.

The results of programme work are presented in a picture, see Figure 5. Here *three* Edgeworth boxes for the states of the world $\{a, b, c\}$ (left upper angle in a box) are presented. Contractual process starts from initial endowments (small black filled circles) and its developing is presented

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[39] Really programme inspects both gradients, that a little bit extends possibilities for informational exchange.

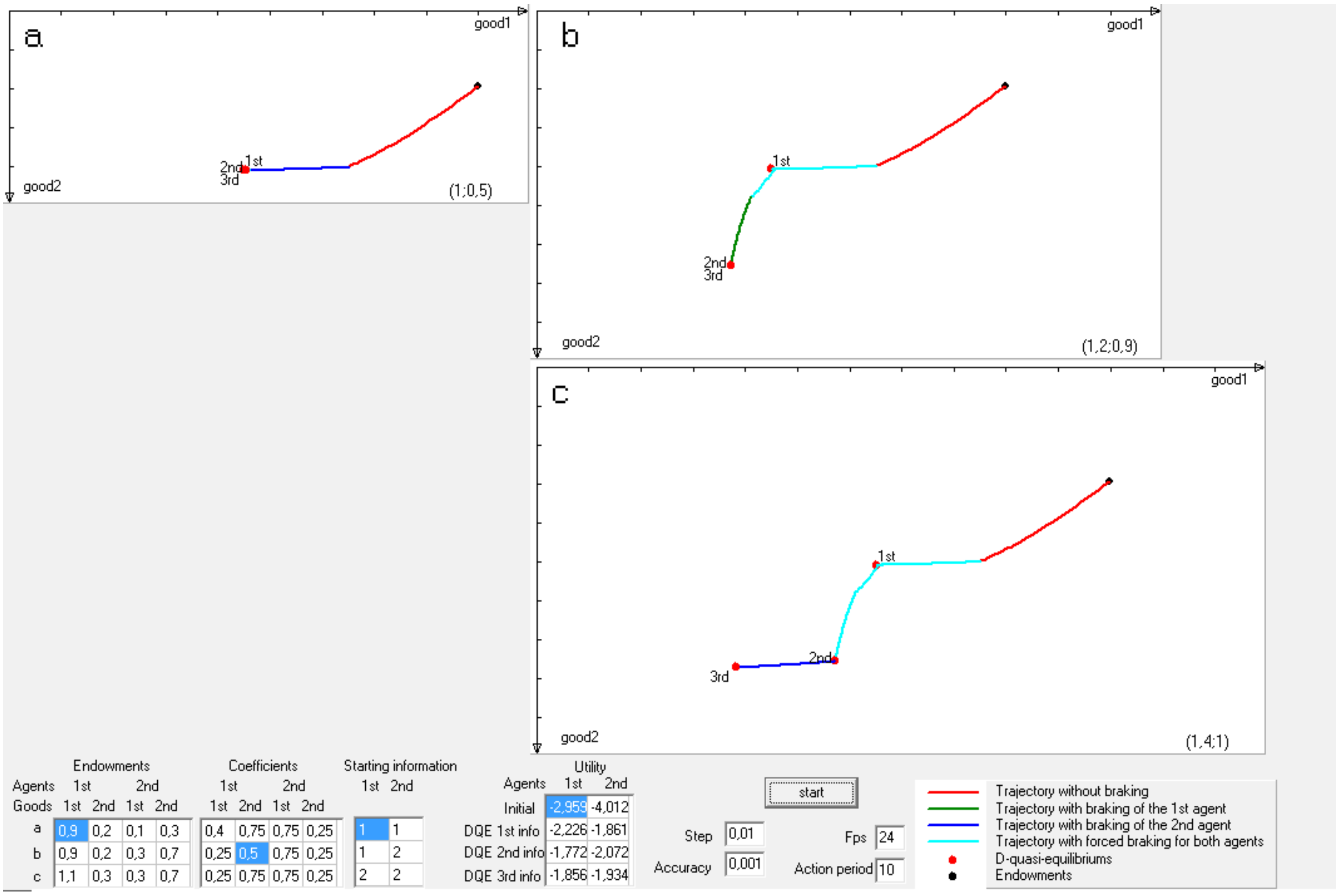

Fig. 6. Dynamics of contractual process with differentiated agents and an informational sharing

by curves where red color denotes no breaking of contracts, green one — 1^{st} breaks, blue — 2^{nd}. Red small filled circles denote equilibria that correspond to information of $1st$, $2nd$ and $3rd$ types: these structures were consequently improved during sharing process. Here $1st$ corresponds to initial information, $2nd$ was obtained after sharing initiated by 2^{nd} agent and then $3rd$ was obtained via 1^{st} agent sharing. Initial model parameters can be changed and they are presented in the bottom panel of screen, from the left to right: 1st table presents initial endowments data (3×4 matrix); 2nd presents the factors of utility functions; 3rd is an initial information; 4th presents the dynamics of utility changes over the equilibrium allocations relative to different information; further there are presented *step*, *accuracy*, number of iterations etc. Analyzing the obtained results one can see rather exotic character of utility and allocation changes. In particular one can see that at the state 'a' after two stages of informational sharing equilibrium allocation comes back to allocation implemented relative to initial information; another funny fact is that after first stage of informational sharing 2^{nd} agent has lost because his/her utility decreased in equilibrium allocations, nevertheless he/she initiated the sharing. Clearly at the stage of sharing the agent expected that utility will increase and locally it is so, but then it decreases: this is because of specific properties of contractual process where rational decisions are taken only in bounded sense, not in global one.

Similar computer simulation was conducted for contractual processes driving to D-quasiequilibrium. The results are presented in Figure 6. It is curiously that after the fist act of informational sharing when 2^{nd} teaches 1^{st} to distinguish 'a' and 'b' resulting market for 'a' is separated, but nevertheless contractual process does not develop and allocation for 'a' is the same. The similar thing for 'b' and 'c', where after when 2^{nd} studied to distinguish them, the situation for 'b' does not change and a dynamic is appeared only in the market for 'c'. In so doing the states of the world can be ordered in a natural way. It is happened because for the initial information the state 'a' is a bottleneck, for new information with 'b' and 'c' it is 'b'; notice that exactly these events implement D-quasiequilibria as an equilibrium relative to the presented state of the world, see Example 6.1. It is also curious that for differentiated agents and some initial data directly after informational sharing contractual process can develop in such a way that first the partial breaking of signed contracts is realized and new contracts are signed only later (for data presented in the picture it is not so). This fact has a simple explanation: after informational sharing in the model with for differentiated agents some new agents are appeared (for an agent utility function is disintegrated into several parts), and for some of them it can be profitable to break off partially gross contract. It never can be true for a private equilibrium model.

8. CONCLUSION

During realization of the project the analysis of the various concepts of core and equilibrium has been carried out from the point of view of the contractual approach: there were analyzed concepts of ex ante private core and equilibrium concerning also the limit information; there were introduced concepts of D-core and D-equilibrium that then were developed up to interim core and interim equilibrium. The notion of an ex ante private equilibrium is very closely related (develops) with the notion of Walrasian Equilibrium in Expectations (WEE), and interim equilibrium develops the concept of Equilibrium in Rational in Expectations (REE). The concept of interim core is also new and through interim equilibrium it corresponds to REE.

Obtained mathematical results: there were proven existence theorems for ex ante private equilibrium and its coincidence with ex ante private core in perfect competition conditions under, I think so, quite comprehensible model assumptions. Similar results were designated for D-core and D-equilibrium. There were described example illustrated investigated notions. This example shows that usually (generically) interim core and equilibria do not exist. However this is the case when one can reveal information and this allow agents to sign a new contract. When interim equilibrium (element of core) is implemented then contractual process is finished.

It is important, that in all offered and studied concepts individuals cannot lie with an advantage for themselves and, thereby, all concepts are "incentive compatible." Certainly, it happens due to the measurability of contracts concerning the private information. However, to realize WEE-equilibrium and allocations from the private core at the moment when tomorrow has already occurred, one needs already today to ensure the mechanism of their implementation: legally accomplished agreements and so on. It is so because if it is not an allocation from interim core then there always will be found a group of agents wishing to break off contracts "concluded at yesterday evening" and to conclude the new "at today morning", i.e. when uncertainty is resolved (cleared) at least only partially. Moreover, if there is not stability relative to partial breakings (therefore, it not equilibrium with differentiated agents) there will be persons interested in partial break of contracts... In the absence of an effective mechanism controlling the implementation of agreements, it will mean short-deliveries by the concluded contracts and, as a consequence, will imply a disbalance of economy as a whole. These reasons leaded me to the idea on the most "correct" concept of an equilibrium and core described in section 6.4. In words: it was assumed, that in the space of all admissible contracts measurable in the necessary measure, the special subspace is selected and contracts from it have the special exogenously given form of stability: if a contract of this type is concluded today, then by no means it will be realized tomorrow. Other contracts have not so strong stability but agents can conclude their also: they are just preliminary agreements. Thus, these contracts are the specific form of the optional agreement: tomorrow morning individuals can change the mind and break off the contract without any problems for itself: this is under game rules. In such a way one comes to that is named, according to terminology from Marakulin (2003), complex contractual allocations. These allocations are implemented by stable webs of contracts accounting the indicated specificity and different possibilities to break contracts: only as a whole or it is possible to do partially. In such a way specific concepts of contractual core and an equilibrium were introduced.

Introduced notions can be illustrated by the conceptual example. Assume, that there is a group responsible for the own word agents, these people always keep the promises given by them even if it happens to be unprofitable at present and they can have damage in comparison with a case of a break of concluded earlier (yesterday) contracts. This contract was mutually beneficial within the limits of known yesterday information and according to presented uncertainty. Today there is appeared a fresh information though uncertainty still was resolved only partially. And now one of parties to an agreement realizes that in new facts it would be beneficial to break off the contract may be only partially, however he/she does not do it... Though it seems unreasonable, however it can be very reasonable for strategic long-term plans, because the expected utility is applied in the latter case and a current utility in the former one. However not all individuals behave yourself so in these circumstances and even if you are a responsible person in the sense of contractual obligations, it still does not mean that you are already the club member: it is necessary that other agents have known and believed in it... So, the contracts concluded among club members of responsible agents will be realized tomorrow for certain and everybody knows this while other contracts — depending on a situation, and nobody knows in advance what exactly will be realized...

9. PROOFS

Proof of Proposition 4.1. Let x be a fuzzy contractual allocation implemented by a proper web V, i.e., $x = x(V)$ for some web V, satisfying Definition 2. Suppose that (3) is false and therefore does exist $y = (y_i)_\mathcal{I} \neq e$ which belongs to the left part of equality (3). Consider coalition $S = \{i \in \mathcal{I} \mid y_i \neq e_i\}$. Notice $\mathcal{P}_i(x_i) \neq \emptyset$, $i \in S$ and find $z_i \in \mathcal{P}_i(x_i)$, $i \in S$ such that $y_i = z_i + \lambda_i(e_i - x_i)$, for some real $0 \leq \lambda_i \leq 1$, $i \in S$ and $y_i = e_i$, $i \notin S$. Determine $u_i = y_i - e_i$, $i \in \mathcal{I}$. Since $\sum_{i\in\mathcal{I}} y_i = \sum_{i\in\mathcal{I}} e_i$ then $\sum_\mathcal{I} u_i = 0$ and therefore $u = (u_i)_{i\in\mathcal{I}}$ is a contract with $\operatorname{supp}(u) = S \neq \emptyset$. One can write

$$z_i = y_i - e_i + \lambda_i(x_i - e_i) + e_i = u_i + \lambda_i \sum_{v\in V} v_i + e_i, \quad i \in S.$$

Now for all $v \in V$ put $t_i = t_i^v = \lambda_i$, $i \in S$, and $t_i = t_i^v \in [0, 1]$, $i \notin S$ and apply $T(V) = \{t^v\}_{v\in V}$ for allocation $x = x(V)$. We have $x^T = e + \Delta(V^T)$, where by construction $x_i^T = e_i + t_i(x_i - e_i)$, $\forall\, i \in \mathcal{I}$. Therefore by construction

$$u_i + x_i^T = z_i \in \mathcal{P}_i(x_i), \quad \forall\, i \in S.$$

Thus x^T is not upper contractual and this contradicts the fact that allocation x is fuzzy contractual.

Show that if a lower contractual allocation x satisfies (3) then it is fuzzy contractual relative to web $V = \{x - e\}$. Assume contrary and find $T = \{t\}$ and a contract $u = (u_i)_{\mathcal{I}}$, $\text{supp}\,(u) = S \neq \emptyset$, such that

$$u_i + t_i(x_i - e_i) + e_i \in \mathcal{P}_i(x_i), \quad \forall\, i \in S \iff z_i = u_i + e_i \in \mathcal{P}_i(x_i) + t_i(e_i - x_i), \quad \forall\, i \in S.$$

Let us determine $z_i = e_i$ for $i \notin S$. Now due to contract's definition conclude $\sum_{i\in\mathcal{I}} z_i = \sum_{i\in\mathcal{I}} e_i$ that implies the allocation $z \neq e$ belongs to the left part of (3) and this is a contradiction. ■

Further first let us present some definitions. I start from the affine space $\mathcal{A}(\mathbb{P})$ of all balanced allocations measurable relative to an information structure $\mathbb{P} = \{P_i\}_{\mathcal{I}}$, where $L = (\mathbb{R}^l)^\Omega$ is a space of contingent commodities

$$\mathcal{A}(\mathbb{P}) = \{(x_i)_{i\in\mathcal{I}} \mid \sum_{i\in\mathcal{I}} x_i = \sum_{i\in\mathcal{I}} e_i \;\;\&\;\; \forall\, i \in \mathcal{I} \;\; x_i : \Omega \to \mathbb{R}^l \;\; P_i - \text{measurable}\}.$$

Let us write the requirement of measurability of maps $x_i(\cdot)$ in detailed form, one has $\iff$
$x_i(\cdot)$ is a constant on the elements of information partition $P_i \iff$
$x_i(\cdot) \in \mathbf{Map}_{P_i}(\Omega, \mathbb{R}^l) \iff$

$$\forall\, \omega \in \Omega, \quad x_i(\omega') = x_i(\omega) \;\; for\ all\ \omega' \in P_i(\omega). \tag{34}$$

Lemma 9.1. *Let P be a partition of Ω. Then a space orthogonal to subspace $\mathbf{Map}_P(\Omega, \mathbb{R}^l)$ of all P-measurable, with values in $\mathbb{R}^l$ functions is described as:*

$$[\mathbf{Map}_P(\Omega, \mathbb{R}^l)]^\perp = \{q : \Omega \to \mathbb{R}^l \mid \forall\, \omega \in \Omega, \sum_{\omega' \in P(\omega)} q(\omega') = 0\}. \tag{35}$$

Thus the lemma states that functional $q_i(\cdot)$ vanishes on the elements satisfying to (34) if and only if when for every element $P_i(\omega)$ of partition P_i the sum of vectors $q_i(\omega')$ taking over $\omega' \in P_i(\omega)$ is equal to zero.

Proof of Lemma 9.1. Let $z \in \mathbf{Map}_P(\Omega, \mathbb{R}^l)$ and q be from a dual space. Let $E_1, E_2, \ldots, E_k$ be the elements of partition P. Calculate inner product:

$$\langle z, q\rangle = \sum_{\omega\in E_1} \langle z(\omega), q(\omega)\rangle + \sum_{\omega\in E_2} \langle z(\omega), q(\omega)\rangle + \cdots + \sum_{\omega\in E_k} \langle z(\omega), q(\omega)\rangle =$$

$$= \langle z(\omega_1), \sum_{\omega\in E_1} q(\omega)\rangle + \langle z(\omega_2), \sum_{\omega\in E_2} q(\omega)\rangle + \cdots + \langle z(\omega_k), \sum_{\omega\in E_k} q(\omega)\rangle,$$

where $\omega_j \in E_j$, $j = 1, 2, \ldots, k$ are chosen in an arbitrary way. Now since the vectors $z(w_j) \in \mathbb{R}^l$ can be arbitrary ones then for the last summation to be zero it is necessary and sufficient $\sum_{\omega\in E_j} q(\omega) = 0$, $j = 1, 2, \ldots, k$. The proof is complete. ■

Proof of Lemma 5.1 (about mutually beneficial contract). Consider the case when there is no mutually beneficial contract. This can be written in the following way: let

$$\mathcal{V}_S = \{v = (v_i)_S \in L^S \mid v_i : \Omega \to \mathbb{R}^l \ is\ P_i - measurable,\ v_i(\omega) = 0,\ \omega \notin A,\ i \in S, \sum_S v_i = 0\},$$

be the space of all possible contracts for coalition S and such that an exchange is realized in the limits of event A. Then

$$\nexists\, v \in \mathcal{V}_S :\ (x_i + v_i)_S \in \prod_S \mathcal{P}_i(x_i) \quad \Longleftrightarrow \quad \prod_S (\mathcal{P}_i(x_i) - \{x_i\}) \bigcap \mathcal{V}_S = \emptyset.$$

In the latter formula two convex sets are intersected and the first of them has non-empty interior. Therefore separation theorem can be applied and one can conclude the existence of a vector (functional) $f = (f_i)_{i\in S} \in (L^S)' = L^S$, $f \neq 0$, such that

$$\langle f, \prod_S (\mathcal{P}_i(x_i) - \{x_i\})\rangle \geq \langle f, \mathcal{V}_S\rangle. \tag{36}$$

As soon as $\mathcal{V}_S$ is a subspace then $\langle f, \mathcal{V}_S\rangle = 0$ and moreover via

$$\mathcal{V}_S = \prod_{i\in S} \mathcal{V}_A^i \bigcap \{v = (v_i)_{i\in S} \in L^S \mid \sum_S v_i = 0\},$$

where $\mathcal{V}_A^i = \{v_i \in L \mid v_i : \Omega \to \mathbb{R}^l,\ P_i - measurable,\ v_i(\omega) = 0, \forall\, \omega \notin A\}$, then f is decomposed into a sum of two functionals $f = q + p$ such that $q = (q_i)_S$ is zero on the first of intersected sets and $p = (p_i)_S$ on the second ones. Now applying Lemma 9.1 one can concludes:

$$\langle q, \prod_{i\in S} \mathcal{V}_A^i\rangle = 0 \ \Rightarrow\ \langle q_i, \mathcal{V}_A^i\rangle = 0\ \forall\, i \in S \ \Rightarrow\ \forall\, E \in P_i,\ E \subseteq A \sum_{\omega\in E} q_i(\omega) = 0 \ \ \forall\, i \in S; \tag{37}$$

the second part of previous conclusion yields

$$\langle p, \{v = (v_i)_{i\in S} \in L^S \mid \sum_S v_i = 0\}\rangle = 0 \quad \Rightarrow \quad p_i = p_j = p,\ \forall\, i \neq j,\ i, j \in S.$$

As a result one has:

$$\exists p \in (\mathbb{R}^l)^\Omega :\ \ \forall\, i \in S\ \ \exists q_i \in (\mathbb{R}^l)^\Omega \ \mid\ f_i = p + q_i \ \ \&\ \ \forall\, E \in P_i,\ E \subseteq A \ \sum_{\omega\in E} q_i(\omega) = 0.$$

On the other hand the last one together with (36) implies $\langle f, \prod_S (\mathcal{P}_i(x_i) - \{x_i\})\rangle \geq 0$, that in view of $0 \in \mathrm{cl}(\mathcal{P}_i(x_i) - \{x_i\})$, $i \in \mathcal{I}$ (local non-satiation of preferences) allows to conclude

$$\langle f_i, \mathcal{P}_i(x_i)\rangle \geq \langle f_i, x_i\rangle,\ \ i \in S.$$

Further let us show that $p \neq 0$ and $p + q_i \neq 0$ for all i.

Increasing consumption of each individual for a unit of every commodity in every state of the world from A, one can construct an allocation strictly preferred by each agent and it is from *interior* of $(\mathbb{R}^l)^A \cap \prod_S \mathcal{P}_i(x_i)$. This implies that a value of f on this allocation has to be strictly more of its value on $x = (x_i)_S$, since otherwise inequality (36) will imply that for a neighborhood of origin functional f vanishes and therefore is equal to zero. However this contradicts to the functional's choice via separation theorem. However in view of (37) the difference in functional values will equal to the value $\sum_{\omega \in A}(p(\omega) \cdot \mathbf{1}) \neq 0$, that in particular proves that $p_{|_A} \neq 0$.

Further assume $p + q_i = 0$ for some i. Then $\sum_{\omega \in A}(p(\omega) + q_i(\omega)) \cdot \mathbf{1} = 0$, that in view of (37) implies $\sum_{\omega \in A}(p(\omega) \cdot \mathbf{1}) = -\sum_{\omega \in A}(q_i(\omega) \cdot \mathbf{1} = 0$, due to the previous one it is impossible.

The second part of Lemma conclusion can be proven in a standard way. Assume that in lemma conditions there exists a mutually beneficial contract $v = (v_i)_S$, i.e. there is a vector having properties: $x_i + v_i \in \mathcal{P}_i(x_i)\ \forall\, i \in S$ & $\sum_S v_i = 0$. Applying (11) and summing inequalities one finds $\sum_S p(x_i + v_i) > \sum_S px_i$ that is impossible. ■

Proof of Corollary 5.1 (to Lemma 5.1 about mutually beneficial contract). In view of (11) for differentiated preferences in an interior point vector $f_i = p + q_i \neq 0$ has to be proportional to the gradient of utility function. Moreover, since $f_i \neq 0$ for each $i \in S$, then the proportionality coefficient has to be *strictly* more than zero. Therefore one can conclude:

$$\forall\, i \in S\ \exists\, \lambda_i > 0 :\ \lambda_i \nabla u_i(x_i) = (p + q_i),$$

that being summed over $\omega \in E \in P_i$, $E \subseteq A$ due to (10) yields

$$\forall\, E \in P_i,\ E \subseteq A\ \ \lambda_i \sum_{\omega \in E} \nabla_\omega u_i(x_i) = \sum_{\omega \in E} p(\omega)\ \ \forall\, i \in S,$$

as we wanted to prove. Easy to see that presented arguments are reversible ones. ■

Let us consider a cooperative game with fractional coalitions. For $r \in \mathbb{N}$ a fractional coalition is defined by a vector

$$d = (d_1, d_2, \ldots, d_n) :\quad rd_i \in \{0, 1, \ldots, r\}$$

and by a set of feasible payoff-vectors for agents non-trivially entering in the coalitions:

$$V(d) \subset \mathbb{R}^{\operatorname{supp}(d)},\ \ \operatorname{supp}(d) = \{i \in \mathcal{I} \mid d_i > 0\}.$$

Let D denote a set of all non-zero fractional coalitions. Like ordinary ones fractional coalitions can be applied to dominate a current payoff vector; doing so we are going to the notion of fractional core, this is a set

$$\mathcal{C}_q(V) = \{x \in V(\mathcal{I}) \mid \nexists d \in D :\ \exists\, y \in V(d) \text{ such that } y_i > x_i\ \forall\, i \in \operatorname{supp}(d)\}.$$

Further I consider a generalization of Scarf's theorem to the fractional coalitions case. To do this I need notion of balanced family of coalitions to be extended to fractional coalitions case.

The family of *fractional* coalitions $\mathcal{B} \subseteq D$ is called *balanced*, if for each $d \in \mathcal{B}$ there is real $\lambda_d \geq 0$ such that

$$\sum_{d \in \mathcal{B}} \lambda_d d_i = 1\ \ \forall\, i \in \mathcal{I}$$

holds, or in an equivalent form

$$\sum_{d \in \mathcal{B}} \lambda_d d = \mathbf{1}_{\mathcal{I}}. \tag{38}$$

A game (D, V) is called *balanced*, if for every balanced family of coalitions $\mathcal{B}$

$$\bigcap_{d \in \mathcal{B}} \operatorname{pr}^{-1}_{|_{S(d)}}(V(d)) \subseteq V(\mathcal{I})$$

is true, where $\mathrm{pr}_{|_{S(d)}}(\cdot)$ is a projecting map for the space $\mathbb{R}^{\mathcal{I}}$ onto $\mathbb{R}^{S(d)}$, $S(d) = \mathrm{supp}\,(d)$.

Theorem 9.1. *Let (D, V) be a cooperative game with fractional coalitions and let ordinary requirements are satisfied for fractional coalitions (closeness, comprehensiveness from below, (12) and etc.). Then if (D, V) is balanced then $\mathcal{C}_q(V) \neq \emptyset$.*

Proof of Theorem 9.1. The proof is based on the application of Kakutani's fixed point theorem to a point-to-set mapping constructed in an appropriate way. An idea of the proof is similar to Danilov's proof (see lecture 22 from Danilov (2002)), which first suggested a short and elegant proof of Scarf's theorem based on Kakutani's fixed point theorem. This construction is presented below.

One can think without lost of generality that one-element coalitions are able to earn zero and not more, i.e. $V(\mathbf{1}_{\{i\}}) = (-\infty, 0]\ \forall\, i \in \mathcal{I}$. Further consider sets

$$\widetilde{V}(d) = V(d) \cap \mathbb{R}_+^{\mathrm{supp}\,(d)} \neq \emptyset, \quad d \in D.$$

By assumption (12) all these sets are non-empty compacts. Therefore there exists a real $c > 0$ such that a cube with the side $2c$ centered in origin includes in its interior every of these sets, i.e.

$$\widetilde{V}(d) \subset \mathrm{int} G \cap \mathbb{R}^{\mathrm{supp}\,(d)}, \quad G = \{z \in \mathbb{R}^{\mathcal{I}} \mid -c(1,1,\ldots,1) \leq z \leq c(1,1,\ldots,1)\}, \quad \forall\, d \in D$$

holds.

On the cube G define the following point-to-set mappings:[40]

$$\chi_d(z) = \begin{cases} \{2\}, & \text{if} \quad z_d \in \mathrm{int}(\widetilde{V}(d) - \mathbb{R}_+^{\mathrm{supp}\,(d)}), \\ \{0\}, & \text{if} \quad z_d \notin (\widetilde{V}(d) - \mathbb{R}_+^{\mathrm{supp}\,(d)}), \\ [0,2], & \quad \text{otherwise.} \end{cases}$$

So, by definition $\chi_d(z)$ takes value $\{2\}$ if $z_d = (z_i)_{i \in \mathrm{supp}\,(d)}$ is in interior of "corrected" set of coalition abilities $\widetilde{V}(d) - \mathbb{R}_+^{\mathrm{supp}\,(d)}$, is equal to the segment $[0, 2]$ on its boundary and coincides with $\{0\}$ behind its limits.

Further on the cube $[0, 2]^D$ define a mapping $\varphi(\cdot)$ with values in G by formula:

$$\varphi(\Lambda) = \operatorname*{argmax}_{z \in G} \langle z, \sum_{d \in D} \lambda_d d - \mathbf{1}_{\mathcal{I}} \rangle, \quad \Lambda = (\lambda_d)_{d \in D} \in [0, 2]^D. \tag{39}$$

Finally define a map of compact $G \times [0, 2]^D$ into itself by formula

$$\psi(z, \Lambda) = \varphi(\Lambda) \times \prod_D \chi_d(z) \subset G \times [0, 2]^D, \quad (z, \Lambda) \in G \times [0, 2]^D.$$

One can easily conclude via construction that the map ψ has closed graph and non-empty convex values. Thus Kakutani's fixed point theorem conditions are satisfied and one concludes the existence of couple $(\bar{z}, \bar{\Lambda}) \in G \times [0, 2]^D$, such that

$$(\bar{z}, \bar{\Lambda}) \in \psi(\bar{z}, \bar{\Lambda}).$$

[40] They are constructed by closing of characteristic set function graph and taking convex hulls of images.

Further let us show that $\bar{z} \in \mathcal{C}_q(V)$. To do it first show that in the fixed point

$$\sum_{d \in D} \bar{\lambda}_d d = \mathbf{1}_{\mathcal{I}}, \quad \bar{\lambda}_d \in \chi_d(\bar{z}),\ d \in D \tag{40}$$

holds, i.e. $\bar{\Lambda}$ is a balanced family. Assuming contrary one has:

$$\sum_{d \in D} \bar{\lambda}_d d - \mathbf{1}_{\mathcal{I}} \neq 0 \ \Rightarrow\ \exists\, i \in \mathcal{I} : \sum_{d \in D} \bar{\lambda}_d d_i - 1 \neq 0.$$

Consider the first possibility: let $\sum_{d \in D} \bar{\lambda}_d d_i - 1 > 0$ be true for an i. Now by (39) one has $\bar{z}_i \in \varphi_i(\bar{\Lambda}) = \{c\}$, that due to the choice of c and construction gives $\chi_d(\bar{z}) = \{0\}\ \forall\, d \in D$ that implies $\sum_{d \in D} \bar{\lambda}_d d_i = 0$. This contradicts the assumption.

Further assume $\sum_{d \in D} \bar{\lambda}_d d_i - 1 < 0$ for some i. Now by (39) one has $\bar{z}_i \in \varphi_i(\Lambda) = \{-c\} < 0$, that due to construction gives $\bar{\lambda}_{\mathbf{1}_{\{i\}}} = 2 \Rightarrow \sum_{d \in D} \bar{\lambda}_d d_i \geq 2$. One has again a contradiction. So (40) is true because other possibilities cannot be realized.

Condition (40) means that the bundle $\bar{\Lambda} = (\bar{\lambda}_d)_{d \in D}$ is balanced and by $\chi_d(z)$ construction for $\bar{\lambda}_d > 0$, $\bar{\lambda}_d \in \chi_d(\bar{z})$ it has to be $\bar{z}_d \in V(d)$, that due to the game is balanced implies $\bar{z} \in V(\mathcal{I})$. Finally if it would occur a coalition $d \in D$ dominating $\bar{z}$, then it would mean that $\bar{\lambda}_d = 2 \in \chi_d(\bar{z})$ is true for d, this again contradicts to (40). So one has found a non-dominated via coalitions payoff vector $\bar{z}$ from $V(\mathcal{I})$ and therefore $\mathcal{C}_q(V) \neq \emptyset$. ■

Proof of Theorem 5.1. The result will be stated if one shows that symmetric part of core

$$\{x = (x_{im})_{m=1, i \in \mathcal{I}}^{m=r} \in \mathcal{C}(\mathcal{E}_r^{di}) \mid x_{im} = x_{im'}\ \forall\, m, m' = 1, \ldots, r,\ \forall\, i \in \mathcal{I}\} = \mathbb{S}(\mathcal{C}(\mathcal{E}_r^{di}))$$

is non-empty for each r-replica of studied model. To do it let us put into correspondence to economy $\mathcal{E}_r^{di}$ a cooperative game of n-persons with the fractional coalitions. For a coalition $d = (d_1, d_2, \ldots, d_n)$, $rd_i \in \{0, 1, \ldots, r\}\ \forall\, i \in \mathcal{I}$ define the set

$$\mathcal{A}(d) = \{y^d = (y_i)_{\mathcal{I}} \mid \sum_{\mathcal{I}} d_i y_i = \sum_{\mathcal{I}} d_i e_i\ \&\ (y_i - e_i)(\cdot)\ \textit{is}\ P_i^0\ \textit{mesurable},\ y_i \in (\mathbb{R}_+^l)^{\Omega},\ i \in \mathcal{I}\}.$$

This set differs from the set for ordinary coalitions in the part of inter-coalitional allocations that correspond to allocation for a coalition of replica economy. Further define

$$V(d) = \{(v_i)_{i \in \operatorname{supp}(d)} \leq (u_i(x_i^d))_{i \in \operatorname{supp}(d)} \ \mid\ x^d \in \mathcal{A}(d)\}.$$

Now to apply Theorem 9.1 it will enough to check the constructed game (D, V) is balanced one. Let us do it.

Let $\mathcal{B} \subseteq D$ be a *balanced* family of fractional coalitions and let $\lambda_d \geq 0$, $d \in \mathcal{B}$ be scalar coefficients system satisfying to (38). Let $v \in \mathbb{R}^{\mathcal{I}}$ be such that $\forall\, d \in D$

$$(v_i)_{i \in \operatorname{supp}(d)} \in V(d) \quad \Longleftrightarrow \quad \exists x^d \in \mathcal{A}(d) :\ (v_i)_{i \in \operatorname{supp}(d)} \leq (u_i(x_i^d))_{i \in \operatorname{supp}(d)}.$$

Consider an allocation for grand coalition defined by formulas:

$$z_i = \sum_{d \in \mathcal{B}} \lambda_d d_i (x_i^d - e_i), \quad i \in \mathcal{I}.$$

By construction $z_i(\cdot)$ is P_i-measurable and

$$\sum_{i\in\mathcal{I}} z_i = \sum_{i\in\mathcal{I}}[\sum_{d\in\mathcal{B}} \lambda_d d_i(x_i^d - e_i)] = \sum_{d\in\mathcal{B}} \lambda_d \sum_{i\in\mathcal{I}} d_i(x_i^d - e_i) = 0$$

takes place. Therefore $z = (z_i)_{i\in\mathcal{I}}$ is a permissible contract and in view of $\sum_{d\in\mathcal{B}} \lambda_d d_i = 1$

$$z_i + e_i = \sum_{d\in\mathcal{B}} \lambda_d d_i x_i^d \in (\mathbb{R}^l_+)^\Omega, \quad i \in \mathcal{I}.$$

holds. Further multiply inequality $v_i \leq u_i(x_i^d)$ on $\lambda_d d_i \geq 0$ and sum the products over $d \in \mathcal{B}$ to obtain

$$v_i = (\sum_{d\in\mathcal{B}} \lambda_d d_i)v_i \leq \sum_{d\in\mathcal{B}} \lambda_d d_i u_i(x_i^d) \leq u_i(\sum_{d\in\mathcal{B}} \lambda_d d_i x_i^d) = u_i(z_i + e_i), \quad \forall\, i \in \mathcal{I}.$$

By definition of $V(\mathcal{I})$ this means $v \in V(\mathcal{I})$ and therefore the game is balanced. Now by Theorem 9.1 there is some $\bar{v} \in \mathcal{C}_q(V)$ such that

$$\exists\, x \in \mathcal{A}(\mathcal{I}) : \bar{v} = u(x) \;\;\&\;\; \nexists\, d \in D \mid \exists y^d \in \mathcal{A}(d) : \;\; u_i(x_i^d) < u_i(y_i^d), \quad i \in \operatorname{supp}(d). \tag{41}$$

Now let us show that $x^r = (x, x, \ldots, x) \in \mathbb{S}(\mathcal{C}(\mathcal{E}_r^{di}))$. Assume contrary and let $T \subseteq \mathcal{I} \times \{1, \ldots, r\}$ be a dominating coalition, i.e.

$$\exists z^T = (z_{im})_{(i,m)\in T} : \quad u_i(z_{im}) > u_i(x_i)\; \forall\, (i, m) \in T.$$

Further define $T(i) = \{m \in \{1, \ldots, r\} \mid (i, m) \in T\}$ where $|T(i)| = \operatorname{card}(T(i))$ is a number of its elements and let $d_i = \frac{|T(i)|}{r}$, $i \in \mathcal{I}$ define a fractional coalition $d = (d_1, d_2, \ldots, d_n)$. Further define allocation $y^d \in \mathrm{A}(d)$ putting

$$y_i^d = \frac{1}{|T(i)|} \sum_{m\in T(i)} z_{im}, \quad i \in \operatorname{supp}(d).$$

Now in view of the convexity of preferences (concave utilities) and since T dominates x^r conclude

$$y_i^d \succ_i x_i \;\; \forall\, i \in \operatorname{supp}(d),$$

that contradicts to (41). This finishes the proof that the symmetric part of every replicated economy is non-empty. Projecting this set onto the space of allocations one concludes that for every $r \in \mathbb{N}$ sets

$$\{x \in \mathcal{A}(\mathcal{I}) \mid x^r \in \mathcal{C}(\mathcal{E}_r^{di})\}$$

are non-empty and compacts. Now since these sets are included one to another decreasing when r is risen, then by the lemma on included compacts their intersection is non-empty — as we wanted to prove. ■

Proof of Corollary 5.4. The argument in the proving of this result is based on Propositions 5.2 and relation (16), which characterizes fuzzy core elements. We need to show that (16) implies (17).

Assume that x satisfies (16) and suppose that (17) is false. This implies that there is a vector $t = (t_1, \ldots, t_n)$, $0 \le t_i \le 1$ and bundles $z_i \succ_i x_i$, $i \in \mathcal{I}$ such that

$$\sum_{\mathcal{I}} z_i + \sum_{\mathcal{I}} t_i(e_i - x_i) = \sum_{\mathcal{I}} e_i, \quad z_i - e_i \in \mathcal{L}_i, \quad i \in \mathcal{I}, \tag{42}$$

holds. Now for a real $0 < \beta \le \frac{1}{2}$ consider the vector $y = y(\beta) = (y_i)_{i\in\mathcal{I}}$, where

$$y_i(\beta) = \beta[z_i + t_i(e_i - x_i)] + (1-\beta)x_i, \quad i \in \mathcal{I}.$$

In view of (42) and $x \in \mathcal{A}(X)$ we have $\sum_{\mathcal{I}} y_i(\beta) = \sum_{\mathcal{I}} e_i$ and $y_i(\beta) - e_i \in \mathcal{L}_i$, $i \in \mathcal{I}$ for every β. Now vectors $y_i(\beta)$ can be presented in the form

$$y_i(\beta) = (1-\beta t_i)x_i + \beta t_i e_i + (1-\beta t_i)\frac{\beta}{1-\beta t_i}(z_i - x_i), \quad i \in \mathcal{I},$$

where by the choice of β we have $\mu_i = \frac{\beta}{1-\beta t_i} \le 1$. This due to preferences assumptions for $i \in \mathcal{I}$ implies

$$\mu_i(z_i - x_i) \in \mathcal{P}_i(x) - x_i \;\Rightarrow\; \exists\, \eta_i \in \mathcal{P}_i(x) : \; \mu_i(z_i - x_i) = \eta_i - x_i.$$

Therefore the previous formula gives

$$y_i = (1-\beta t_i)\eta_i + \beta t_i e_i,$$

that implies $y_i \in \Omega_i(x_i)$, $i \in \mathcal{I}$. This allows us to apply relation (16), concluding $y = y(\beta) = (e_1, e_2, \ldots, e_n)$ for *all* real $0 < \beta \le \frac{1}{2}$. Write this equality componentwise and due to $y_i(\beta)$ specification find

$$\beta[z_i + t_i(e_i - x_i)] + (1-\beta)x_i = e_i \;\Rightarrow\; z_i + t_i(e_i - x_i) = x_i + \frac{e_i - x_i}{\beta}$$

that has to be true for all $i \in \mathcal{I}$ and *all* $0 < \beta \le \frac{1}{2}$. However these equalities (consider *different* β) can be true only if $x_i = e_i = z_i$, $i \in \mathcal{I}$, that due to the choice of z_i implies $x_i \succ_i x_i$ and contradicts to assumptions on preferences. Proof is completed. ■

Proof of Theorem 5.3. Take $x \in \mathcal{C}^e(\mathcal{E}^{di})$. Apply Corollary 5.4 and formula (17). Now separation theorem can be applied to find a functional (vector) $f = (f_1, f_2, \ldots, f_n) \in L^{\mathcal{I}}$, $f \ne 0$ separating the left hand side set in (17) from the intersection of others. One has:

$$\langle f, \prod_{\mathcal{I}}(\mathcal{P}_i(x) + \mathrm{co}\{0, e_i - x_i\})\rangle \ge \langle f, \{(z_1, \ldots, z_n) \mid \sum_{i\in\mathcal{I}} z_i = \sum_{i\in\mathcal{I}} e_i\} \bigcap (\prod_{\mathcal{I}} \mathcal{L}_i + (e_1, \ldots, e_n))\rangle. \tag{43}$$

As soon as in the right hand side of the inequality there are functional values presented on non-trivial affine subspace, then the functional $f(\cdot)$ has to be constant on the subspace that is equivalent to

$$\langle f, \prod_{\mathcal{I}} \mathcal{L}_i \bigcap \{(z_1, \ldots, z_n) \in L^{\mathcal{I}} \mid \sum_{i\in\mathcal{I}} z_i = 0\}\rangle = 0.$$

This implies that f is decomposed into sum of two functionals $f = q + p$ such that $q = (q_i)_{\mathcal{I}}$ is zero on the first of intersected in last formula sets and $p = (p_i)_{\mathcal{I}}$ is zero on the second one. This in view of Lemma 9.1 gives:

$$\langle q, \prod_{i\in\mathcal{I}} \mathcal{L}_i\rangle = 0 \;\;\Rightarrow\;\; \langle q_i, \mathcal{L}_i\rangle = 0 \;\; \forall\, i \in \mathcal{I} \;\;\Rightarrow\;\; \forall\, E \in P_i, \;\; \sum_{\omega\in E} q_i(\omega) = 0 \;\; \forall\, i \in \mathcal{I}; \tag{44}$$

the second part of conclusion yields

$$\langle p, \{v = (v_i)_{i\in\mathcal{I}} \in L^{\mathcal{I}} \mid \sum_{\mathcal{I}} v_i = 0\}\rangle = 0 \quad \Rightarrow \quad p_i = p_j = p,\ \forall\, i, j \in \mathcal{I}.$$

As a result one has:

$$\exists p \in (\mathbb{R}^l)^{\Omega}: \quad \forall\, i \in \mathcal{I} \ \ \exists q_i \in (\mathbb{R}^l)^{\Omega} \ \mid \ f_i = p + q_i \ \ \& \ \ \forall\, E \in P_i, \ \sum_{\omega\in E} q_i(\omega) = 0.$$

Further one can realize standard argumentation as earlier in Lemma 5.1 (about mutually beneficial contract) and state an analog of formula (11). Really, computing the right hand part in (43) accounting the obtained conclusions one finds $\sum_{j\in\mathcal{I}}\langle p + q_j, e_j\rangle$. On the other hand take $e_j = x_j + (e_j - x_j) \in \mathrm{cl}(\mathcal{P}_j(x) + \mathrm{co}\{0, e_j - x_j\})$ for $j \neq i$ and $z_i + (e_i - x_i) \in (\mathcal{P}_i(x) + \mathrm{co}\{0, e_i - x_i\})$ and substitute this to the left hand side of (43), one obtains $\langle p + q_i, z_i - x_i\rangle + \sum_{j\in\mathcal{I}}\langle p + q_j, e_j\rangle$. Reducing identical terms and taking into account the choice of $z_i \in \mathcal{P}_i(x)$ is arbitrary, one concludes

$$\langle p + q_i, \mathcal{P}_i(x_i)\rangle \geq \langle p + q_i, x_i\rangle \ \ \& \ \ p + q_i \neq 0 \ \ \forall\, i \in \mathcal{I},$$

where the latter inequality and $p \neq 0$ can be proven as it was done in Lemma 5.1. Finally, to prove budget equalities in the last argumentation instead of $(e_i - x_i) \in \mathrm{co}\{0, e_i - x_i\}$ take $0 \in \mathrm{co}\{0, e_i - x_i\}$ concluding then

$$\langle p + q_i, \mathcal{P}_i(x_i)\rangle \geq \langle p + q_i, e_i\rangle \ \ \forall\, i \in \mathcal{I}.$$

Now since $x_i \in \mathrm{cl}\mathcal{P}_i(x)$ then $\langle p + q_i, x_i\rangle \geq \langle p + q_i, e_i\rangle \Rightarrow \langle p + q_i, x_i - e_i\rangle \geq 0$ that in view of (44) and $x_i - e_i \in \mathcal{L}_i$ yields $\langle p, x_i - e_i\rangle \geq 0$, $i \in \mathcal{I}$. However due to allocation is feasible the last one is possible only then $\langle p, x_i\rangle = \langle p, e_i\rangle$, $\forall\, i \in \mathcal{I}$. Theorem 5.3 is proven. ■

Proof of Proposition 6.2. State the first part of proposition. First of all its conditions have to be presented in the convenient for mathematical analysis form. To do it let us reformulate relation (19) for the case of differentiated agents. Define

$$L(E) = \{z : \Omega \to \mathbb{R}^l \mid z(\omega) = 0,\ \omega \notin E\} = \chi^E \cdot L.$$

Then for fuzzy contractual allocation with fixed information one has

$$\prod_{i\in\mathcal{I}} \sum_{E\in P_i} [((\chi^E \cdot \mathcal{P}_i(x_i)) \cap (x_i^E + L(E)) + \mathrm{co}\{0, e_i^E - x_i^E\}) \cup \{e_i^E\}] \bigcap$$

$$\bigcap \{z = (z_i)_{\mathcal{I}} \mid z_i = \sum_{E\in P_i} z_i^E,\ \forall\, i \in \mathcal{I} \ \ \& \ \ \sum_{i\in\mathcal{I}} z_i = \sum_{i\in\mathcal{I}} e_i \ \&$$

$$(z_i^E - e_i^E)_{|E} = const,\ \&\ (z_i^E - e_i^E)_{|\Omega\setminus E} = 0,\ \forall\, (i, E) \in \Im\} = \{(e_1, \ldots, e_n)\}. \tag{45}$$

Notice that in difference with (19) there is no necessity to include a third element in the latter intersection; this element was applied in (19) in order to specify contracts measurable relative to individual information. Similarly instead of $\mathcal{P}_i^E(x_i)$ in the first set under intersection there were applied sets $(\chi^E \cdot \mathcal{P}_i(x_i)) \cap (x_i^E + L(E))$, where $(\chi^E \cdot \mathcal{P}_i(x_i))$ is algebraic product of characteristic set function with the set of all preferred bundles.

Further, in order to present price characteristics for the allocation let us apply to intersection (45) separation theorem and consider a liner functional (vector) $\pi = (p_i)_{i\in\mathcal{I}} \neq 0$ separating these sets. This functional has the following specific structure, that can be checked arguing in similar manner to the proof of Lemma 5.1 (about mutually beneficial contract, see page 59):

$$\exists p : \Omega \to \mathbb{R}^l \ \ \& \ \ q_i : \Omega \to \mathbb{R}^l, p + q_i \neq 0, \ \forall\, i \in \mathcal{I}$$

such that $\forall\, i \in \mathcal{I}\ \forall\, E \in P_i$

$$\sum_{\omega\in E} q_i(\omega) = 0 \ \ \&$$

$$\langle(\chi^E \cdot \mathcal{P}_i(x_i)) \cap (x_i^E + L(E)), p^E + q_i^E\rangle = \langle \mathcal{P}_i(x_i) \cap (x_i + L(E)), p^E + q_i^E\rangle \geq \langle x_i^E, p^E + q_i^E\rangle = \langle e_i^E, p^E + q_i^E\rangle. \tag{46}$$

Notice that here functional $p^E + q_i^E$ supports set $\mathcal{P}_i(x_i) \cap (x_i + L(E))$, where in difference with $\mathcal{P}_i^E(x_i)$, for different states of nature from E different commodity flows can be consumed, i.e. $(z_i - e_i)_{|E} \neq \mathit{const}$ is possible for $z_i \in \mathcal{P}_i(x_i) \cap (x_i + L(E))$. Moreover notice that now one cannot guaranty that $p^E + q_i^E \neq 0$ for all $E \in P_i$; it known only that at least one of them is not zero (since $p + q_i \neq 0$). Further due to $\langle \chi^E, q_i^E\rangle = 0$ one can avoid q_i from relations (46) and for $x_i + z \cdot \chi^E \in \mathcal{P}_i(x_i)$, $z \in \mathbb{R}^l$ rewrite it in an equivalent form: $\forall\,(i, E) \in \Im$,

$$\langle \sum_{\omega\in E} p(\omega), x_i^E - e_i^E\rangle = 0 \ \ \& \ \ \langle \sum_{\omega\in E} p(\omega), z\rangle \geq 0, \ \forall\, z \in \mathbb{R}^l : \ x_i + z \cdot \chi^E \in \mathcal{P}_i(x_i),$$

that proves (28) and the first part of Lemma. Notice that this exactly thing required by definition of D-quasiequilibrium.

Further, the fact that after every possible informational sharing there are no mutually beneficial contracts is equivalent to that there are no beneficial contracts relative to complete information. However one has to remember that current allocation is still realized via web of contracts obtained for differentiated agents and only these contracts can be broken: partially, fuzzy or as a whole. This means that in order to all conditions of second part be satisfied it is sufficient the second set from intersection change on similar one but without measurement requirements (one needs to study the case when all distinguished sets have only one element, i.e. when $E = \{\omega\}$, $\omega \in \Omega$). As a result we have

$$\prod_{i\in\mathcal{I}} \sum_{E\in P_i} [(\chi^E \cdot \mathcal{P}_i(x_i)) \cap (x_i^E + L(E)) + \mathrm{co}\{0, e_i^E - x_i^E\}) \cup \{e_i^E\}] \bigcap \{z = (z_i)_{i\in\mathcal{I}} \mid \sum_{i\in\mathcal{I}} z_i = \sum_{i\in\mathcal{I}} e_i\}.$$

Further standardly applying separation theorem one concludes the existence of “price” map $p : \Omega \to \mathbb{R}^l$ such that $\forall\,(i, E) \in \Im$,

$$\sum_{\omega\in E} p(\omega)x_i(\omega) = \sum_{\omega\in E} p(\omega)e_i(\omega) \ \& \ \sum_{\omega\in E} p(\omega)z(\omega) \geq 0, \ \forall\, z : \Omega \to \mathbb{R}^l : x_i + z \cdot \chi^E \in \mathcal{P}_i(x_i) \ -$$

as we wanted to prove. ■

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