# Economies with non-convex production and public goods: contract based approach*

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**Abstract**

The paper applies and elaborates contractual approach to study economies with a production sector. Economies with convex, non-convex production and with public goods are considered. The notions and terms of barter contractual approach developed in Marakulin (2003a) for exchange economies are now modified and extended to the production economy; there are introduced notions of production contract and adopted known earlier notions, they are: a web of contracts, coalitional domination for webs, a partial breaking of contracts and so on. This way a new notion of marginal contractual allocation is introduced for the model with non-convex production. Then it is applied to describe marginal cost pricing equilibrium, the notion used in theory for non-convex case instead of Walrasian equilibrium. Also a specific notions of properly contractual and fuzzy contractual allocations for economy with public goods are introduced and their equivalence with Lindahl equilibrium are stated. In general depending on the type of production economy there are three results on equivalence of equilibria and specific (properly, marginal, fuzzy) contractual allocation, implemented via web of contacts stable relative to a partial break. These theorems can be interpreted as a specific form of perfect competition implemented via contractual approach.

**Keywords and Phrases:** production economy, public goods, contract, contractual allocation, Walrasian equilibrium, MCP-equilibrium, Lindahl equilibrium, Foley core.
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# Contents

# Introduction

One of main objectives of the economic theory and its basic part — general equilibrium theory — consists in the description of resources allocation implemented via the system of the markets. In classical Arrow–Debreu model resulting allocation is arrived as Walrasian (competitive) equilibrium that is the basic object of the theoretical analysis, see Arrow, Debreu (1954), McKenzie (1954), Mas-Colell, et al. (1995), Aliprantis et al. (1989) etc. Arrow–Debreu model was developed and generalized in different directions, one of them being the study of models with non-convex technological sets and with the public goods.

For classical Arrow–Debreu model the convexity of technological sets (and sets of the preferred consumption bundles) is very important assumption, otherwise equilibria may not exist. However non-convexity in technologies is a characteristic for many industrial spheres and it relays with increasing returns from scale (for example, for private municipal enterprizes). Therefore the case of non-convex technological sets is very important theoretical problem. The non-convexity in technologies leads to the known concept of equilibrium with pricing by marginal costs, so called MCP-equilibrium. For the first time the existence of MCP-equilibrium for monopolistic economy with one firm has been established in Mantel (1979). Further in paper of Beato and Mas-Colell (Beato, Mas-Colell, 1985) existence of equilibrium with marginal costs pricing has been proved for several firms with non-convex technologies, most general results have been obtained in Bonnisseau, Cornet (1988, 1990); a survey of literature one can find in Brown (1991). So general equilibrium theory started to develop for non-convex technological sets. Notice in addition, that equilibrium with marginal costs pricing implements only necessary condition for Pareto optimality of current production allocation (and profit maximization of every producer, however it is also sufficient in a convex case).

Modern views on the theory of financing of the public goods were generated by Samuelson papers and of some other authors, see survey Milleron (1972) and Ruys (1974). A public good is a product of collateral consumption of all economic agents. The Pareto efficient mechanism of cost regulation of the public goods is based on individual cost estimations calculated as product of the individual price and total amount of consumption.[1] Certainly, in economy there can be usual commodities, which processes of an exchange and reproduction are carried out by usual market rules. An appropriate theoretical equilibrium concept is defined and called Lindahl equilibrium and it is a Pareto efficient notion. However there is a problem of practical definition of individual prices and corresponded taxes that is difficult theoretical question that still does not have an undoubted answer in classical theory. One way to solve this problem could be based on the cooperative description of equilibrium,

[1] There are known also non-Pareto efficient equilibrium notions, as based on private provision of public goods, e.g. see Florenzano (2009).

in such a way as it is done in models with purely private goods, via a theorem on the coincidence of core and equilibrium under perfect competition conditions. However, examples show that under an ordinary replication[2] of an economy Foley core does not shrinks to equilibrium (see Muench (1972), Milleron (1972) and other papers and an example below borrowed from Buchholz, Peters (2007)). If one does not take into account such extremal results as Conley (1994), in the modern literature the problem finds its resolution only through the transformation of the concept of coalitional domination. Really, instead of the familiar public goods so-called semi-public goods are introduced, as it was done in Vasil'ev et al. (1995) (see also Weber, Wiesmeth (1991), Vasil'ev (1996) and Florenzano, Mercato (2006)). The difference is that now the pleasure of consumed goods depends of the total number of its customers.[3] One can say that consumers are interested in the average level of consumption of public goods. In theory there are appeared such concepts as "returns to group size", see (Roberts, 1974), a congestion of good etc.; in Vasil'ev et al. (1995) there were proved subtle theorems on the coincidence of the core and equilibria and also that it can be obtained only in the case of constant returns to the size of the coalition. In this paper the theorems on the coincidence of the core and Lindahl equilibrium are stated within the contractual approach and under similar assumptions that were done in the papers mentioned above. However, here we are not talking about the size of the coalition or a measure of congestion in public goods: in the model individuals are engaged in ordinary economic activity, they sign and break (partially and asymmetrically) production and barter contracts, thereby producing a stable regime of functioning, which corresponds to a Lindahl equilibrium allocation.

So, the paper is carrying out studies of the mathematical models of economy with production and consumption sectors. Structurally they are Arrow–Debreu models, however in production sector increasing returns from scale are possible and/or production of the public goods is carried out. The analysis is based on the author's approach developed in the series of papers of this decade in context of exchange economies of different kinds and generality (Marakulin (2002, 2003a) etc.). The idea of the barter exchange (contract) is by no means new in theoretical economics and seemingly goes back to classical Edgeworth results (1881), but it usually appeared as an interpretation, in the form of net trades in a formal model. Contract as a barter exchange of commodities is appeared then in the works of other authors (though the theory of contracts was not elaborated in a proper way), including Russian ones:

[2]This is one of possible and the most popular way to model perfect competition conditions, which goes back to Debreu and Scarf. There are other methods, e.g. Aumann's approach, applying a non-atomic measure space of economic agents.

[3]Inversely proportional to the number of users of the same type; not only the total amount, but first of all the density of vehicles and traffic congestion affect the satisfaction of such a good as "transport infrastructure".

Polterovich (1970) and Makarov (1982). Then in Kozyrev (1981) there was suggested the partial breaking of contracts and there were obtained some preliminary positive results. Partially breaking contracts being incorporated in stability notion of webs allow for complete markets to describe alternatively Walrasian equilibrium. Finally, due to the author's results (Marakulin, 2002, 2003a) a basis of contracts theory was constructed that can be considered as cooperative replenishment of the classical views on perfect competition conditions of market economy functioning.

In the consumption sector every contract is an elementary *permissible* exchange of commodities among consumers (barter): the members of a coalition implement the exchange of commodities. Contracts may be added one to another and an allocation of resources may be put into correspondence to every (finite) set of contracts — as a result of the summation of contracts and the initial endowments allocation. The presence of production essentially affects contract definition; now it is not only barter exchange of commodities, but the contract, in which agents bear the material costs defined in a field of production of private and public goods. It is presumed that every feasible set of (permissible) contracts — let us call it '*a web of contracts*' — may be changed during economic life. Each consumer or a coalition of consumers can *break contracts* in which he/she participates, and a coalition can also *sign a new one*. Sometimes the partial break of contracts is also permissible. Stable web of contracts are studied in the theory and it may be stability of different forms that depends on admissible ways of contracts breaking. It may be *total breaking* of a contract, *partial* or even breaking of contracts from an *equivalent system* of contracts, etc.; mixed variants are possible, too. The formal rules of operating with sets of contracts correspond to different forms of the web stability and therefore to the stability of allocation implemented by a web. The kinds of these 'stabilities', together with the property of contracts to be permissible, reflect the different behavioral, physical and institutional principles, formally given in a game-theoretical form. A specific property of our contractual approach is that all processes of production and exchange are going without any kinds of value parameters. So, applying contractual approach we are interested in the stable web of contracts and this stability can be variable.

In previous studies for exchange economies it was shown that competitive equilibrium can be described in terms of stable webs where partial break of contracts is permissible, see Marakulin (2003a), Kozyrev (1981). This paper presents similar description of equilibria for production economies. Moreover here there are studied not only economies with convex production of private goods but also with non-convex production (increasing returns to scale) and with public goods. The paper presents contractual description of Walrasian equilibrium (convex case) and MCP–equilibrium (marginal cost pricing equilibrium) for non-convex production. MCP–equilibria are described as (a new introduced notion) marginally contractual allocations. Similar result of Lindahl equilibrium description is presented for economies with public goods: if aggregated production is presented by a convex cone (constant returns to scale); in general stable webs of studied kinds describe a wider set of Pareto optimal allocations.

The paper is organized as follows. In the first section there is defined Arrow–Debreu–McKenzie model and applied in its context competitive (Walrasian) equilibrium and equilibrium with marginal cost pricing (MCP–equilibrium). Mantel's theorem as being the first existence result on the existence of MCP–equilibrium is also presented here.

The second section is devoted to the description of the main settings of contractual approach and its application to the convex production economy. There are presented model and main definitions; theorem on equivalence of properly contractual allocation and competitive equilibrium is also proved here.

The third section studies contractual approach for economy with non-convex production. Here there is introduced a new notion of marginal contractual allocation and there is proved the theorem on its equivalence with MCP–equilibrium.

The forth section studies contractual approach for economy with public goods — a new specific notion of production contract is introduced here. The equivalence theorems of Lindahl equilibrium and specified properly and fuzzy contractual allocation present the main results of this section. Conclusion ends the paper.

# 1 Economy of Arrow–Debreu type

The formal economy of Arrow–Debreu type in its shortest form is presented by the following bundle of parameters:

$$\mathcal{E}^{AD} = \langle \mathcal{I}, \mathcal{J}, \mathbb{R}^l, \{X_i, \mathcal{P}_i(\cdot), \omega_i, \{\theta_i\}\}_{i\in\mathcal{I}}, \{Y_j\}_{j\in\mathcal{J}} \rangle.$$

Here $\mathcal{I} = \{1, \dots, n\}$ is the set of consumers, $\mathcal{J} = \{1, \dots, m\}$ is the set of producers (firms), $l$ is a number of commodities and $\mathbb{R}^l = E$ is the commodity space. Consumption sets are denoted as $X_i \subset \mathbb{R}^l$ and $X = \prod_{i\in\mathcal{I}} X_i$; agents' preferences are presented by point-to-set mappings $\mathcal{P}_i : X_i \rightrightarrows X_i$, $i \in \mathcal{I}$ where $\mathcal{P}_i(x_i) = \{y_i \in X_i \mid y_i \succ_i x_i\}$ is a set of all consumption bundles strictly preferred by the $i$-th agent to the bundle $x_i$. It is also applied the notation $y_i \succ_i x_i$ which is equivalent to $y_i \in \mathcal{P}_i(x_i)$. Consumers have also initial endowments $\omega_i \in X_i$, $i \in \mathcal{I}$. Determine $\omega = (\omega_1, \dots, \omega_n)$ and let $\bar{\omega} = \sum_{i=1}^{n} \omega_i$. A producer $j \in \mathcal{J}$ is described by a technological set $Y_j \subset E$, $Y = \prod_{j\in\mathcal{J}} Y_j$, defined in terms of material flows, i.e. non-negative component of $y_j \in Y_j$ is an output but if it is negative then it is input of commodity in the units of counting. This way production plans defined for a vector of prices $p = (p_1, \dots, p_l) \in E'$ profit $\pi_j(p, y_j)$ for a plan $y_j \in Y_j$ can be calculated in the form of inner product $\pi_j(p, y_j) = \langle p, y_j \rangle = p \cdot y_j$. There are also presented $nm$ scalar values $\theta_i^j \geq 0$ being the components of vectors $\theta_i = (\theta_i^1, \dots, \theta_i^m)$, they present the shares of $i$ in the profits $\pi_j$ of producers $j \in \mathcal{J}$; due to definition $\sum\limits_{i=1}^{n} \theta_i = (1, \ldots, 1)$. Further let us recall the definition of ***competitive (Walrasian)*** equilibrium.

**Definition** 1.1 *A triplet $(x, y, p)$, where $x = (x_i)_{i \in \mathcal{I}} \in X$ is a family of consumption plans, $y = (y_j)_{j \in \mathcal{J}} \in Y$ are production plans and $p = (p_1, \ldots, p_l) \neq 0$, $p \in E'$ is a price vector, is said to be **quasi-equilibrium**, if:*

$$p \cdot y_j \geq \langle p, Y_j \rangle, \ \forall\, j \in \mathcal{J}, \tag{1.1}$$

[4]

$$\langle p, \mathcal{P}_i(x_i) \rangle \geq p \cdot \omega_i + \sum_{j=1}^{m} \theta_i^j p \cdot y_j = p \cdot x_i, \ \forall\, i \in \mathcal{I}, \tag{1.2}$$

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \omega_i + \sum_{j=1}^{m} y_j. \tag{1.3}$$

*If all inequalities in (1.2) have strict sign then the triplet $(x, y, p)$ is called **competitive (Walrasian)** equilibrium.*

Requirements (1.1)–(1.3) have a familiar economic sense. When inequality (1.2) has strict form it means that consumption plan $x_i$ is an optimal choice (demand) for individual $i$ under his/her budget constraint $p \cdot z_i \leq p \cdot \omega_i + \sum_{j=1}^{m} \theta_i^j p \cdot y_j = r_i(p, y)$, $z_i \in X_i$, where the right hand side presents total agent's income from all channels (the sale of commodities $\omega_i$ and the shares in firm's profits $\theta_i^j p y_j$) under prices $p = (p_1, \ldots, p_l) \in E'$. Condition (1.1) says that producers maximize profit and (1.3) is a material balance condition, that usually is presented as the equality of demand and supply.

Conditions guaranteing existence of equilibria in Arrow–Debreu model are well known in literature, e.g. see Mas-Colell, et al. (1995), Aliprantis et al. (1989), Marakulin (2003b). In consumption sector they are the continuity (different versions are applied), open-convex values, irreflexivity and local non-satiation of agents' preferences $\mathcal{P}_i : X_i \Rightarrow X_i, \forall\, i \in \mathcal{I}$. Consumption sets have to be convex and closed; moreover they have to provide a bounded (compact) set of all feasible allocations. These requirements are sufficient for quasi-equilibria or more refined notion of equilibria with non-standard prices do exist in exchange economy (see Marakulin (2003b), Konovalov, Marakulin (2006)). For strict equilibria do exist one needs to require in addition survival assumptions, it may be resource relatedness, irreducibility or something like this one. Further let us turn to production sector and consider it a little bit more detailed.

For production sector it is usually assumed that for all $j \in \mathcal{J}$ technological sets $Y_j$ have the following properties:

[4] $\langle A, B \rangle = \{\langle a, b \rangle = a \cdot b \mid a \in A,\ b \in B\}$ for all $A, B \subset E$; $A \geq b \iff a \geq b\ \forall\, a \in A$.

- $Y_j$ — *convex*, closed sets (i.e. limit technological processes are permissible),
- $Y_j - \mathbb{R}^l_+ \subset Y_j$ — free disposal condition,
- $Y_j \cap \mathbb{R}^l_+ = \{0\}$ — no free lunch, where $\mathbb{R}^l_+$ is positive orthant of commodity space.
- $Y \bigcap (-Y) = 0$ — the irreversibility of production processes.

In spite of the latter three requirements have an own economic sense they are really needed to provide, together with consumption sets properties (boundedness from below), that the set of all feasible (balanced) allocations is bounded one. Nowadays one can often meet a direct requirement for the feasible allocation set to be bounded. The first assumption is for us now the most of interest and as a part of it the convexity of production sets. Without this requirement equilibria may not exist.

In this study there are analyzed the cases of convex and nonconvex production sets. Nonconvexity in production may occur, for example, due to increasing returns to scale (firm revenues are increasing per unit costs). For example, the recording of the CD-ROM is costlier than its replication, overwriting occurs at low cost. However, this possibility (because of technical-mathematical reasons) has not been studied in the classical version of the existence theory. Of course, in order to an equilibrium does exist under non-convex technology, the concept should be appropriately modified. However, this modification should be such that the new concept is resulted in (or at least has chances) the Pareto optimal allocation, as it is in the convex case. Pricing on the basis of average costs does not satisfy this requirement. The key idea of MCP-equilibrium is that profit maximization is replaced by the (necessary) first-order condition (expressed in terms of gradients of the functions that define the production sets), which in the convex case is also sufficient for a plan to be profit maximizer. Thus the concept directly generalizes the usual competitive equilibrium. As the subject of interpretation one is usually talking about the social planner, who has the ability to "evaluate" the price obtained this way according to the principle of marginal cost pricing (MCP), and then force the manufacturers to adhere the specified production plans. Further I consider the simplest version of the model with a nonconvex production sector, described by differentiable functions.

Assume that the production sets $Y_j$ are described via *differentiable functions* $\varphi_j$ by formula

$$Y_j = \{y \in E \mid \varphi_j(y) \leq 0\},\ j \in \mathcal{J}. \tag{1.4}$$

and, moreover, in this case the boundary of production sets can be defined as[5]

$$\partial Y_j = \{y \in E \mid \varphi_j(y) = 0\} \neq Y_j,\ j \in \mathcal{J}. \tag{1.5}$$

[5] Clearly that in general case the topological boundary of the set may be narrower of the set described below.

Define $X = \prod_{i \in \mathcal{I}} X_i$, $Y = \prod_{j \in \mathcal{J}} Y_j$. The following definition MCP-equilibrium can be found in Mantel (1979), Brown (1991).

**Definition** 1.2 ***MCP-equilibrium*** *(marginal cost pricing) is a triplet* $(x, y, p)$*, where* $x = (x_i)_{i \in \mathcal{I}} \in X$ *is a family of consumption plans,* $y = (y_j)_{j \in \mathcal{J}} \in Y$ *are production plans and* $p = (p_1, \ldots, p_l)$ *is a price vector, which is satisfying the conditions:*

$$y \in \prod_{j \in \mathcal{J}} \partial Y_j \quad \& \quad \exists\, \lambda_j > 0 : \; p = \lambda_j \nabla \varphi_j(y_j), \; \forall\, j \in \mathcal{J}, \tag{1.6}$$

$$\langle p, \mathcal{P}_i(x_i) \rangle > p \cdot \omega_i + \sum_{j \in \mathcal{J}} \theta_i^j p \cdot y_j = p \cdot x_i, \; \forall\, i \in \mathcal{I}, \tag{1.7}$$

$$\sum_{i \in \mathcal{I}} x_i = \sum_{j \in \mathcal{J}} y_j + \sum_{i \in \mathcal{I}} \omega_i. \tag{1.8}$$

The requirement (1.6) is the above-mentioned first-order conditions to which the equilibrium production plans must satisfy instead of the condition of profit maximization (1.1). Conditions (1.7), (1.8) present the optimum of consumer preferences under budget and other constraints and the balance of commodity markets.

In the latter definition it is implicitly assumed that production can be unprofitable, but total taxes cover the losses of firms with nonconvex production sets. It is important to note that if all firms have convex technologies, the concept of equilibrium with marginal cost pricing is turn to be the Walrasian equilibrium in the classical Arrow–Debreu model. Further there is stated one of the simplest result on existence of MCP-equilibria known in the literature (see Brown (1991)).

Consider a model with one firm and let assumptions (1.4), (1.5) hold. In addition let us assume $0 \in Y$, $Y - \mathbb{R}^l_+ \subseteq Y$, the set $(Y + \sum_{\mathcal{I}} \omega_i) \cap \mathbb{R}^l_+$ is bounded and if $y + \sum_{\mathcal{I}} \omega_i \in \partial Y \cap \mathbb{R}^l_+$, then $\nabla \varphi(y) \gg 0$.

Let for all $i \in \mathcal{I}$ consumption sets $X_i = \mathbb{R}^l_+$, and preferences are determined via utility functions $u_i : X_i \to \mathbb{R}$, which are continuous, strictly concave and locally non-satiated ones.

**Theorem** 1.1 (Mantel, 1979) *Under presented assumptions an equilibrium with marginal cost pricing does exist.*

Substantially stronger results can be found in Bonnisseau, Cornet (1988, 1990). Here there are considered many firms that may have smooth boundary, as well as a general rule of pricing. In this context, there is considered the mapping $\psi : \prod_{\mathcal{J}} \partial Y_j \to \mathbb{R}^l_+$, which maps a vector of production prices to a set of production plans. Requirements on the map $\psi(\cdot)$ are very general and this way can be realized as the marginal cost pricing as soon as average cost pricing — ACP, and also other variants, see Bonnisseau, Cornet (1988), Brown (1991).

Further we are passing to the main purpose of the paper — the analysis of contractual approach in Arrow–Debreu model with non-convex technologies as well as with public goods. Let us start from the analysis of convex Arrow–Debreu model.

# 2 Contractual approach in convex Arrow–Debreu model

Consider the following model of an economy with production

$$\mathcal{E} = \langle \mathcal{I}, E, (X_i, \omega_i, Y_i, \mathcal{P}_i) \rangle. \tag{2.1}$$

This model differs from the classical Arrow–Debreu model only in the part of production sector, which assumes the existence of individualized production sets $Y_i \subset E$, $i \in \mathcal{I}$. It is easy to see that the classical model can be reduced to a model of this kind: it is sufficient to specify the individual production sets $Y_i$ as $Y_i = \sum_{j=1}^{m} \theta_i^j \cdot \overline{Y}_j$, where $\overline{Y}_j$ are production sets in the model $\mathcal{E}^{AD}$. Therefore, the results obtained for the model $\mathcal{E}$ will be applied to model $\mathcal{E}^{AD}$ and vice versa. In this case, however, theoretical constructions are considerably simplified. For this model budget constraints are applied in the form

$$p \cdot x_i \leq p \cdot \omega_i + p \cdot y_i, \quad i \in \mathcal{I},$$

i.e., consumer $i$ income from production activities is presented by the value $p \cdot y_i$. Further define $Z_i = X_i \times Y_i$, $Z = \prod_{i \in \mathcal{I}} Z_i$ and let $L = E^{\mathcal{I}} \times E^{\mathcal{I}}$ be a space of allocations. Define

$$\mathcal{A}(\mathcal{E}) = \{z = (x, y) \in X \times Y \mid \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \omega_i + \sum_{i \in \mathcal{I}} y_i\} \ -$$

it is the set of all feasible allocations in model $\mathcal{E}$. In the context of $\mathcal{E}$ equilibrium is defied as foollows:

• *A triplet $(x, y, p)$, where $p \neq 0$, $p \in E'$, $(x, y) \in \mathcal{A}(\mathcal{E})$ is said to be **quasi-equilibrium**, if for each $i \in \mathcal{I}$:*

$$\langle p, \mathcal{P}_i(x_i) \rangle \ \geq p \cdot \omega_i + p \cdot y_i = p \cdot x_i, \tag{2.2}$$

$$p \cdot y_i \geq \langle p, Y_i \rangle. \tag{2.3}$$

*holds. If (2.2) has strict sign for each $i$, then triple $(x, y, p)$ is called **competitive equilibrium**.*

Everywhere below we assume that the model $\mathcal{E}$ satisfies the following assumption.

(**A**) *For each $i \in \mathcal{I}$, $X_i$ is a convex solid closed set (non-empty interior), $\omega_i \in X_i$ and for every $x_i \in \mathrm{pr}_{|X_i}\mathcal{A}(\mathcal{E})$ there exists an open convex $G_i \subset E$ such that*

$$\mathcal{P}_i(x_i) = G_i \cap X_i \;\;\&\;\; x_i \in \overline{\mathcal{P}_i(x_i)} \setminus \mathcal{P}_i(x_i).^6$$

For convenience of further exposition, we introduce a specific notion of a smooth economy.

• *Economy $\mathcal{E}$ is called* ***smooth****, if for each $i$: the set $Y_i$ has a boundary presented as a smooth manifold and*

$$\mathcal{P}_i(x_i) = \{x'_i \in X_i \mid u_i(x'_i) > u_i(x_i)\}, \;\forall\, x_i \in X_i$$

*holds for a differentiable concave function $u_i(\cdot)$ that is defined on an open neighborhood of $X_i$.*

*Economy $\mathcal{E}$ has* ***smooth consumption sector****, if only first requirement is true.*

Recall further the conceptual apparatus of the theory of barter contracts, see Marakulin (2002, 2003a), while adapting it to the model with the productive sector.

Any vector $v = (v_i)_{i\in\mathcal{I}} \in E^{\mathcal{I}}$ satisfying $\sum_{i\in\mathcal{I}} v_i = 0$ is called a barter (exchange) *contract*. Such barter contracts are used in pure exchange economies, as well as in the consumption sector in the economy with production. In what follows, we assume that any barter agreement is valid[7]. With every finite collection $V$ of (permissible) contracts it can be associated allocation $x(V) = \omega + \sum_{v\in U} v$, where $\omega = (\omega_1, \ldots, \omega_n) \in X$ is an initial resource allocation. If $\omega + \sum_{v\in U} v \in X \;\forall U \subseteq V$, i.e. if for any part of contracts is broken one can get anyway a feasible allocation, then we call $V$ as a *web* of contracts.

Consider further the operations of breaking of existing and signing of new contracts. For any contract $v \in V$ define

$$S(v) = \mathrm{supp}(v) = \{i \in \mathcal{I} \mid v_i \neq 0\},$$

this is a support of barter contract $v$. It is assumed that any contract $v \in V$ may be *broken* by any trader in $S(v)$, since he/she simply may not keep his/her contractual obligations. Also a non-empty group (coalition) of consumers can *sign* any number of new contracts. Being applied jointly, i.e., as a simultaneous procedure, these operations allow coalition $T \subseteq \mathcal{I}$ to yield new webs of contracts. The set of all such webs is denoted by $F(V,T)$. Formally, we require that each element $U \in F(V,T)$ has to satisfy the following properties:

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[6]The symbol $\overline{A}$ denotes the closure of $A$ and $\setminus$ is set for the set-theoretical difference.

[7]In general, for the model of contractual economy a set $\mathcal{W} \subseteq E^{\mathcal{I}}$ of *admissible* (resolved) contracts is allocated.

(i) $v \in V \setminus U \Rightarrow S(v) \bigcap T \neq \emptyset$,

(ii) $v \in U \setminus V \Rightarrow S(v) \subset T$.

In Marakulin (2002, 2003a) there were considered and analyzed contracts in the exchange economy. In this paper, contractual concepts are modified and adapted to an economy with production. In the latter case contract is a pair $(v, y) \in E^{\mathcal{I}} \times E^{\mathcal{I}}$, where $v$ is an ordinary barter contract but $y = (y_1, \ldots, y_n)$ is a vector which corresponds to production programs $y_i$ for individuals $i \in \mathcal{I}$. If $(v, y) \in E^{\mathcal{I}} \times Y$, i.e. if each production program is feasible, $y_i \in Y_i$, $i \in \mathcal{I}$, then contract $(v, y)$ is permissible. For a finite collection $V$ of contracts in the model (2.1) one can put into correspondence (consumption) allocation $x(V) = \omega + \sum_{(v,y)\in V} y + \sum_{(v,y)\in V} v$. On the other hand clearly that any contract $(v, y)$ can be decomposed into sum of pure exchange contract $(v, 0)$ and production contract (programs) $(0, y)$. In this case, referring to the web of contracts, there is no need to specify a different sets of production programs, it suffices to take a gross contract, obtained by summing them. Moreover, the actual production component of the contract may be without prejudice to endure beyond the concept of a web of contracts. So, one arrives at the following definition:

A finite collection $V$ of permissible contracts is called a web of contracts relative to pair $(x, y) \in X \times Y$, if

$$x + y + \sum_{v \in U} v \in X, \ \forall U \subseteq V \quad \Longleftrightarrow \quad x_i + y_i + \sum_{v \in U} v_i \in X_i, \ \forall i \in \mathcal{I}, \ \forall U \subseteq V,$$

i.e. under the current consumption $x \in X$ and production $y \in Y$ plans, individuals enter into contractual relationships so that they can break any contracts. Here, the vector of current consumption plans $x = (x_1, \ldots, x_n)$ actually plays a role (variable) of initial endowments. In the case where $x = \omega = (\omega_1, \ldots, \omega_n) \in X$, i.e. when there is a web of contracts relative to a pair $(\omega, y) \in X \times Y$, one calls this web the $y$-web. Specify:

$$x(V, y) = \omega + y + \sum_{v \in V} v \in E^{\mathcal{I}}, \quad z(V, y) = (x(V, y), y) \in E^{\mathcal{I}} \times E^{\mathcal{I}}.$$

Let $y_T = (y_i)_{i\in T}$ be a collection of production plans of individuals entering in coalition $T$. Define $\mathcal{I} \setminus T = -T$, then $y_{-T}$ will present the vector consisting of the production plans of all individuals who were not included in the coalition $T$. Now the collection of all production plans $y = (y_i)_{i\in\mathcal{I}}$ can be written in the form $y = (y_T, y_{-T})$.

Being applied jointly, i.e., as a simultaneous procedure, these operations allow coalition $T \subseteq \mathcal{I}$ to yield new webs of contracts. The set of all such webs is denoted by $F(V, T)$. Formally, we require that each element $U \in F(V, T)$ has to satisfy the following properties:

Let us say that $\tilde{y}$-web $U$, $\tilde{y} = (\tilde{y}_T, y_{-T}) \in Y$, dominates $y$-web $V$, $y = (y_T, y_{-T}) \in Y$ via coalition $T$ (notation $U \succ_T V$), if:

(i) $U \in F(V, T)$,

(ii) $x_i(U, \tilde{y}) \succ_i x_i(V, y)$ for all $i \in T$.

Notice that now production plans of supplementary coalition $y_{-T}$ are unchangeable but $y_T$ can be substituted for new plans $\tilde{y}_T \in \prod_{i \in T} Y_i$. In other words, members of the coalition $T$ can change not only the exchange contracts (breaking of the old (any involving) and conclude new within the coalition contracts), but they are also able to modify their production plans.

**Definition** 2.1 *A $y$-web of contracts $V$ is called stable if there is no $\tilde{y}$-web $U$ and no coalition $T \subset \mathcal{I}$ such that $U \succ_T V$. Allocation $z = (x, y) \in \mathcal{A}(\mathcal{E})$ is called contractual if $x = x(V, y)$ for some stable $y$-web $V$.*

In order to introduce the operation of partial breaking of contracts, consider the following partial order on the set of $y$-webs. This ordering is defined by the rule:

$$U \geq V \Longleftrightarrow \exists \text{ a map } \textbf{onto } f : U \to V \text{ such that}$$

(i) $\lambda f(u) = u$ for some $0 \leq \lambda \leq 1$ and for every $u \in U$,

(ii) $\sum_{u \in f^{-1}(v)} u = v$ for every $v \in V$.

By this definition, a web $U$ consists of a finite partition of contracts from $V$ (decomposition into the sum, see (ii)) subject to the exchange ratios are not changed (due to (i)). Minimal elements of the ordering among all $y$-webs are called root webs.

The ordering induces the following equivalence relation on the set of all $y$-webs:

$$U \simeq V \Longleftrightarrow \exists \text{ y-web } W \text{ such that } V \geq W \text{ and } U \geq W. \tag{2.4}$$

Notice that if $U \simeq V$, then $z(U, y) = z(V, y)$, i.e. equivalent webs yield identical allocations.

In terms of equivalent webs an *allocation $z \in \mathcal{A}(\mathcal{E})$ is called properly contractual, if there is y-web $V$ such that $z = z(V, y)$ and for every $U \simeq V$ allocation $z = z(U, y)$ is contractual one.*

This concept of a properly contractual allocation can be reformulated in the following simplified form. For real $\alpha$ define $\alpha V = \{\alpha \cdot v \mid v \in V\}$, i.e. $\alpha V$ is a web, yielded from $V$ by multiplying contracts on $\alpha$. For $0 \leq \alpha \leq 1$ consider web $U = \alpha V \cup (1 - \alpha) V$, which obviously implements the same allocation $z(U, y) = z(V, y)$. The web $U = \alpha V \cup (1 - \alpha) V$ is called $\alpha$-partition of the web $V$. An allocation $z = z(V, y)$ is properly contractual if $\alpha$-partition of $V$ is stable for every $\alpha \in [0, 1]$. For clarity, below we present a narrative definition in substantial terms.

**Definition** 2.2 *A pair* $(x, y) \in X \times Y$ *is called properly contractual allocation if there is a web* $V$ *such that the following conditions are satisfied:*

*(i)* $x = x(V, y) = \sum_V v + \omega + y.$

*(ii) There are no coalition* $S$*, for which it is profitable (in a separate or simultaneous regime):*

- $(\alpha)$ *to partially break barter contracts;*
- $(\beta)$ *to transit from the given production programs* $y = (y_S, y_{-S})$ *to new production programs* $y' = (y'_S, y_{-S})$*, where* $y'_S \in \prod_{i \in S} Y_i$*;*
- $(\gamma)$ *to sign new contract.*

In Kozyrev (1981), Marakulin (2002, 2003a) it was proved that for the smooth pure exchange economies each interior properly contractual allocation is an equilibrium. Below it will be shown that this result holds for the convex Arrow–Debreu model.

The following lemma gives a characterization of Pareto optimal allocations of the convex economy in value terms and, in fact, this is an analogue of the Second Welfare Theorem. Recall that:

• *A feasible allocation* $(x, y)$ *is said to be (weakly) Pareto optimal if there is no a family* $(x'_i, y'_i)_{i \in \mathcal{I}} \in \mathcal{A}(\mathcal{E})$ *such that* $x'_i \succ_i x_i$ *for all* $i \in \mathcal{I}$*.*

**Lemma** 2.1 *Let for all* $i \in \mathcal{I}$ *the sets* $Y_i$ *be convex and let an allocation* $\bar{z} = (\bar{x}_i, \bar{y}_i)_{i \in \mathcal{I}} \in \prod_{i \in \mathcal{I}} X_i \times Y_i$ *be Pareto optimal one. Then there is a vector* $p \neq 0$ *such that:*

$$\langle p, \mathcal{P}_i(\bar{x}_i) \rangle \ \geq p \cdot \bar{x}_i, \tag{2.5}$$

$$p \cdot \bar{y}_i \geq \langle p, Y_i \rangle. \tag{2.6}$$

*holds for each* $i \in \mathcal{I}$*.*

**Remark** 2.1 If $\mathcal{E}$ is an economy with a smooth consumption sector and if $x_i \in \mathrm{int} X_i$ then there is a real $\lambda_i > 0$ such that $p = \lambda_i \cdot \nabla u_i(x_i)$. ■

*Proof of Lemma 2.1.* First note that assumption (**A**) implies that $\bar{x}_i \notin \mathcal{P}_i(\bar{x}_i)$ and $\bar{x}_i \in \overline{\mathcal{P}_i(\bar{x}_i)}$ is true for all $i \in \mathcal{I}$. Next, write the property that allocation $(\bar{x}_i, \bar{y}_i)_{i \in \mathcal{I}}$ is Pareto optimal in an equivalent form:

$$\prod_{i \in \mathcal{I}} (\mathcal{P}_i(\bar{x}_i) \times Y_i) \bigcap \mathcal{A}(L) = \emptyset,$$

$$\mathcal{A}(L) = \{(x,y) \in L \mid \sum_{i\in\mathcal{I}} x_i = \sum_{i\in\mathcal{I}} \omega_i + \sum_{i\in\mathcal{I}} y_i\}.$$

Since intersected sets are convex and nonempty (because of (**A**)), then by separation theorem there exists a linear functional $f \neq 0$ separating these sets, i.e.

$$f((x_i,y_i)_{i\in\mathcal{I}}) \geq f((\check{x}_i,\check{y}_i)_{i\in\mathcal{I}}), \quad \forall (x_i,y_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} (\mathcal{P}_i(\bar{x}_i) \times Y_i), \quad \forall (\check{x}_i,\check{y}_i)_{i\in\mathcal{I}} \in \mathcal{A}(L).$$

Since our commodity space is finite dimensional, it follows that the linear functional can be represented by the vector (through the inner product), i.e., we can assume that $f = (f_1,\dots,f_n) \in L$. Now the latter relation can be written in the form:

$$\sum_{i\in\mathcal{I}} \langle f_i,(x_i,y_i)\rangle \geq \sum_{i\in\mathcal{I}} \langle f_i,(\check{x}_i,\check{y}_i)\rangle. \tag{2.7}$$

Fix $(x_i,y_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}}(\mathcal{P}_i(\bar{x}_i)\times Y_i)$. From (2.7) the value $\sum_{i\in\mathcal{I}}\langle f_i,(\check{x}_i,\check{y}_i)\rangle$ is bounded on $\mathcal{A}(L)$, i.e. $f$ is bounded from above on $\mathcal{A}(L)$. Show that in fact the functional is constant on $\mathcal{A}(L)$.

Assume contrary. The set $\mathcal{A}(L)$ can be written in the form:

$$\mathcal{A}(L) = -(\omega_1,\dots,\omega_n,0,\dots,0) + H = -(\omega,0) + H,$$

where $H = \{(x,y) \in L \mid \sum\limits_{i\in\mathcal{I}} x_i = \sum\limits_{i\in\mathcal{I}} y_i\}$. Notice that $H$ is a linear space such that the summation $f(\omega,0)+f(h)$ is bounded for all $h \in H$. Therefore $f$ is bounded from above on $H$ that is possible only if $f(h) \equiv 0$ on $H$ (hence functional $f$ is constant on $\mathcal{A}(L)$).

Further, one can note that the vectors of the form

$$((0,0)_1,\dots,(0,0)_{i-1},(x,x)_i,(0,0)_{i+1},\dots,(0,0)_n) \in H, \quad \forall x \in E.$$

Therefore, $f_i(x,x) = 0$ for all $x \in E$. Write $f_i = (f_i^x, f_i^y)$ and obtain $f_i^x\cdot x + f_i^y\cdot x = 0$. Now, since $x$ is an arbitrary chosen, one concludes

$$f_i^x = -f_i^y = p_i. \tag{2.8}$$

Further show that $p_i = p_j$ for all $i,j \in \mathcal{I}$.

Due to (2.8) one has $f_i = (p_i,-p_i)$. Further note that for every pair $(x,y) \in L$ the vector of the form

$$((0,0)_1,\dots,(0,0)_{i-1},(x,y)_i,(0,0)_{i+1},\dots,(0,0)_{j-1},-(x,y)_j,(0,0)_{j+1},\dots,(0,0)_n)$$

belongs to $H$. Therefore one has $f_i(x,y) - f_j(x,y) = 0$. The latter equality holds for every pair $(x,y) \in E\times E$, that implies

$$f_i = f_j = (p,-p) \ \forall\, i,j \in \mathcal{I}.$$

Finally, the fact that $f \neq 0$ implies that $p \neq 0$. Now (2.7) can be rewritten in the form

$$\sum_{i \in \mathcal{I}} \langle (p, -p), (x_i, y_i) \rangle \geq \sum_{i \in \mathcal{I}} \langle (p, -p), (\check{x}_i, \check{y}_i) \rangle. \qquad (2.9)$$

To prove inequality (2.5) let us consider inequality (2.9). Consider $y_i = \check{y}_i = \bar{y}_i$ and let $\check{x}_i = \bar{x}_i$ for all $i \in \mathcal{I}$. In view of (2.7) for all $x_i \in \mathcal{P}_i(\bar{x}_i)$, $i \in \mathcal{I}$ one has:

$$\sum_{i \in \mathcal{I}} \langle (p, -p), (x_i - \bar{x}_i, 0) \rangle \geq 0 \quad \Longleftrightarrow \quad \langle p, x_i \rangle + \sum_{j \neq i} \langle p, (x_j - \bar{x}_j) \rangle \geq \langle p, \bar{x}_i \rangle \ \ \forall\, i \in \mathcal{I}.$$

Fix $i$ and consider $j \neq i$. From (**A**) it follows that $\bar{x}_j \in \overline{\mathcal{P}}_j(\bar{x}_j)$ that allows in the addends with numbers $j \neq i$ in the left hand side of the last inequality pass to the limits by $x_j \to \bar{x}_j$ and as a result one gets $\langle p, x_i \rangle \geq \langle p, \bar{x}_i \rangle$ for all $x_i \in \mathcal{P}_i(\bar{x}_i)$ and all $i \in \mathcal{I}$, this proves (2.5).

To prove relation (2.6) consider $y_j = \check{y}_j = \bar{y}_j$ and $\check{x}_j = \bar{x}_j$ for all $j \neq i$. Let $\check{y}_i = \bar{y}_i$ and $\check{x}_i = \bar{x}_i$. In view of (2.9) one obtains

$$\sum_{j \in \mathcal{I}} p(x_j - \bar{x}_j) - p(y_i - \bar{y}_i) \geq 0.$$

Similarly to the previous one, passing $x_j$ to $\bar{x}_j$ for all $j \in \mathcal{I}$ one concludes

$$-p(y_i - \bar{y}_i) \geq 0 \ \ \forall\, y_i \in Y_i \quad \Longleftrightarrow \quad \langle p, \bar{y}_i \rangle \geq \langle p, Y_i \rangle,$$

as we wanted to prove. ■

The following theorem establishes an equivalence between the equilibrium notion and properly contractual allocation in the economy with convex production sector.

**Theorem** 2.1 *Let $\mathcal{E}$ be contractual economy with convex production and smooth consumption sectors. Let $\bar{z} = (\bar{x}, \bar{y}) \in \mathcal{A}(\mathcal{E})$ and $\bar{x}_i \in \mathrm{int} X_i$ for all $i \in \mathcal{I}$. Then pair $(\bar{z}, p)$ is an equilibrium if and only if $\bar{z}$ is a properly contractual allocation.*

*Proof of Theorem 2.1.* Let us start by checking the sufficiency. Suppose that $\bar{z} = (\bar{x}(V, \bar{y}), \bar{y})$ is a properly contractual allocation. This allocation is Pareto optimal since coalition $\mathcal{I}$ is unable to conclude a new contract beneficial for all members of $\mathcal{I}$. From Lemma 2.1 it follows the existence of $p \neq 0$ which satisfies

$$\langle p, \mathcal{P}_i(x_i) \rangle \geq p \cdot x_i, \quad \forall\, i \in \mathcal{I}.$$

Now, to prove that $(\bar{x}, \bar{y}, p)$ is a quasi-equilibrium, it suffices to show the following

$$p \cdot \bar{x}_i = p \cdot \bar{y}_i + p \cdot \omega_i, \quad \forall\, i \in \mathcal{I}. \qquad (2.10)$$

The proof is by contradiction. Assume that not all equalities (2.10) hold. Then, since the allocation $\bar{z}$ is balanced one conludes:

$$\sum_{i\in\mathcal{I}} \bar{x}_i = \sum_{i\in\mathcal{I}} \bar{y}_i + \sum_{i\in\mathcal{I}} \omega_i \;\Rightarrow\; \sum_{i\in\mathcal{I}} p\bar{x}_i = \sum_{i\in\mathcal{I}} p\bar{y}_i + \sum_{i\in\mathcal{I}} p\omega_i.$$

Therefore there is $i \in \mathcal{I}$, for which

$$p \cdot \bar{x}_i - p \cdot \bar{y}_i < p \cdot \omega_i.$$

This inequality can be rewritten in the form

$$p \cdot \bar{x}_i < p \cdot (\omega_i + \bar{y}_i) = p \cdot x_i,$$

where

$$\bar{x}_i = \omega_i + \bar{y}_i + \sum_{v\in V} v_i, \quad x_i = \omega_i + \bar{y}_i + 0.$$

Further, from $p \cdot \bar{x}_i < p \cdot x_i$ one concludes $p \cdot (x_i - \bar{x}_i) > 0$, that due to Remark 2.1 to Lemma 2.1 ($p = \lambda_i \cdot \nabla u_i(\bar{x}_i)$ for some $\lambda_i > 0$) one has

$$\nabla u_i(\bar{x}_i) \cdot (x_i - \bar{x}_i) > 0.$$

However, this inequality means that the function $u_i$ increases in the direction $x_i - \bar{x}_i = -\sum_{v\in V} v_i$, so there is $\mu \in (0, 1)$ such that

$$u_i(\bar{x}_i - \mu \cdot \sum_{v\in V} v_i) > u_i(\bar{x}_i).$$

Thus it is profitable for agent $i$ to partially break all barter contracts in the volume $\mu V$. Now one arrives to a contradiction with the definition of properly contractual allocation. It remains to note that under the theorem conditions ($\bar{x}_i \in \mathrm{int} X_i$, $\forall\, i$) and in view of assumption (**A**) every quasi-equilibrium is an equilibrium in fact.

To prove the necessity define $v = \bar{x} - \bar{y} - \omega$ and put $V = \{v\}$. In view of equilibrium properties of pair $(\bar{z}, p)$ one has $p \cdot v_i = 0$ for all $i \in \mathcal{I}$. Suppose that $\bar{z}$ is not properly contractual allocation. Then there is a coalition $T \subseteq \mathcal{I}$, real $0 \le \lambda_v \le 1$, contract $w = (w_i)_{i\in T}$ and production plans $y_T \in \prod_T Y_i$ such that $\omega_i + y_i + \lambda_v \cdot v_i + w_i \succ_i \bar{x}_i$, $i \in T$. However from equilibrium properties (1.2) one concludes

$$p \cdot (\omega_i + y_i + \lambda_v \cdot v_i + w_i) > p \cdot \bar{x}_i = p \cdot (\omega_i + \bar{y}_i + v_i),\ i \in T.$$

Therefore $p \cdot w_i > p \cdot (\bar{y}_i - y_i)$ for all $i \in T$, that via (1.1) implies $p \cdot w_i > 0$, hence $p \cdot \sum_{i\in T} w_i \neq 0$. One obtains a contradiction with the definition of contract because it has to be $\sum_{i\in T} w_i = 0$. Theorem 2.1 is proved. ■

# 3 Contractual economies with non-convex production

Consider the model of the economy (2.1) in which we assume the presence of non-convex production sets $Y_i$, $i \in \mathcal{I}$ and assuming the other standard assumptions: closeness, free disposal and so on. Applied in this model a specified concept of properly contractual allocation uses the notion of star-shaped set. Recall that a set $A \subseteq E$ is called a star-shaped with respect to $x \in A$ if for all $y \in A$ linear segment$[x, y] \subset A$; here by definition $[x, y] = \{z \in E \mid \exists\lambda \in [0, 1] : z = \lambda x + (1-\lambda)y\}$.

Let $y = (y_i)_{i\in\mathcal{I}} \in Y$ be any given family of production plans. For each $i \in \mathcal{I}$ specify a set $M_i^y$ as a star-shaped relative to $y_i \in Y_i$ subset of $Y_i$ such that it is the maximal by inclusion among other this kind star-shaped sets.

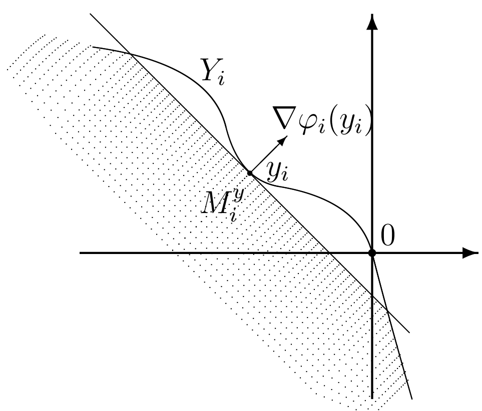


Figure 1: Maximal star-shaped subset $M_i^y$

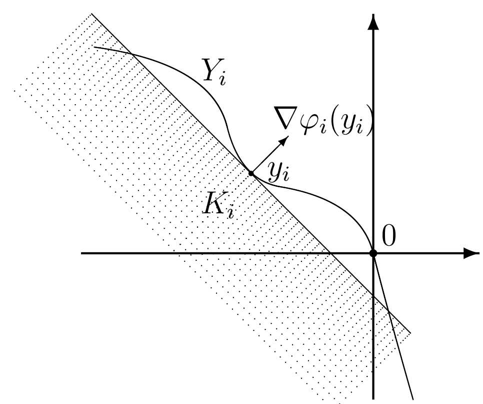


Figure 2: Semispaces $K_i$

Notice that for *convex* case the sets $M_i^y$ and $Y_i$ are *matched* but in the nonconvex variant it can be both ways, see Figure 1. In the case of nonconvex production sets the concept of properly contractual allocation is modified, but it coincides with the above in the convex context.

**Definition** 3.1 *A pair $(x, y) \in X \times Y$ is called* ***marginally contractual*** *allocation if there is a web $V$ such that the following conditions are satisfied:*

*(i)* $x = x(V, y) = \sum_V v + \omega + y.$

*(ii) There are no coalition $S$, for which it is profitable (in a separate or simultaneous regime):*

  ($\alpha$) *to partially break barter contracts;*

($\beta$) *to transit from the given production programs* $y = (y_S, y_{-S})$ *to new production programs* $y' = (y'_S, y_{-S})$*, where* $y'_S \in \prod_{i \in S} M_i^y$*;*

($\gamma$) *to sign new contract.*

Here, in contrast to the convex case, the freedom of the individual in choosing a production plan is limited. It is assumed that the mutual co-operation or some authority establishes joint production plans, and the individuals still have the right to decide: do it or not (a specific form of non-binding agreement). However, the individual deviations from a given production plan is only possible within the sets $M_i^y \subset Y_i$. Again, in the case of convex production $M_i^y$ and $Y_i$ coincide.

Suppose further that the production sets $Y_i$ are described by *differentiable functions* $\varphi_i$ as follows:

$$Y_i = \{y \in E \mid \varphi_i(y) \le 0\},\ i \in \mathcal{I}.$$

Without loss of generality assume also that the boundary of production sets can be defined as[8]

$$\partial Y_i = \{y \in E \mid \varphi_i(y) = 0\} \ne Y_i,\ i \in \mathcal{I}.$$

Next, we need the following

**Lemma** 3.1 *Let* $M_i^y$ *be the maximal star-shaped subset of* $Y_i$ *relative to a point* $y_i \in \partial Y_i$*. Then*

$$M_i^y \subset \{z \in E \mid \varphi_i(z) \le 0\ \&\ \nabla\varphi_i(y_i) \cdot z \le \nabla\varphi_i(y_i) \cdot y_i\}.$$

*Proof of Lemma 3.1.* Take $z \in M_i^y$. By $M_i^y$ definition one has

$$z(\lambda) = (1-\lambda)y_i + \lambda z \in M_i^y \subseteq Y_i,\quad \forall\, \lambda \in [0,1].$$

Therefore, $\varphi_i(z(\lambda)) \le 0$. As soon as $\varphi_i(y_i) = 0$, then

$$\lim_{\lambda \to +0} \frac{\varphi_i(y_i + \lambda(z - y_i)) - 0}{\lambda} = \langle \nabla\varphi_i(y_i), z - y_i \rangle \le 0,$$

that completes the proof of lemma. ■

Relationship between marginal contractual allocation and MCP–equilibrium is established in the following

**Theorem** 3.1 *Let* $\mathcal{E}$ *be smooth economy and let* $x = (x_i)_{i \in \mathcal{I}} \in \mathrm{int}X$ *and* $y = (y_i)_{i \in \mathcal{I}} \in Y$ *be the families of consumption and production plans. Then the triplet* $(x, y, p)$ *is the MCP–equilibrium if and only if* $(x, y)$ *is marginal contractual allocation.*

[8]Note that this implies $\mathrm{int}Y_i \ne \emptyset$.

*Proof of Theorem 3.1.* Necessity. Let $(\bar{x}, \bar{y}, \bar{p})$ be a MCP–equilibrium and let $V = \{v\}$ be a web, where $v = \bar{x} - \bar{y} - \omega$. Show that $(x(V, \bar{y}), \bar{y}) = (\bar{x}, \bar{y})$ is a marginally-contractual allocation. Assume the contrary. Then there are the coalition $S$, the real $\lambda_v \in [0, 1]$, the new contract $u = (u_i)_{i \in \mathcal{I}}$ and new production programs $y' \in \prod_{i \in S} M_i^{\bar{y}}$ such that

$$\lambda_v \cdot v_i + \omega_i + y_i' + u_i = x_i' \succ_i \bar{x}_i, \quad i \in S$$

holds. By equilibrium definition $\bar{p}x_i' > \bar{p}\omega_i + \bar{p}\bar{y}_i$. Since the deviation from the production plan is possible only within $M_i^{\bar{y}}$ then by Lemma 3.1 one obtains

$$\nabla\varphi_i(\bar{y}_i) \cdot y_i' \leq \nabla\varphi_i(\bar{y}_i) \cdot \bar{y}_i.$$

As soon as $\bar{p} = \lambda_i \cdot \nabla\varphi_i(\bar{y}_i)$ for some $\lambda_i > 0$ then $\bar{p} \cdot y_i' \leq \bar{p} \cdot \bar{y}_i$, that for each $i \in S$ yields

$$\bar{p}x_i' > \bar{p}\omega_i + \bar{p}\bar{y}_i \geq \bar{p}\omega_i + \bar{p}y_i' \;\Rightarrow\; \bar{p}x_i' - \bar{p}\omega_i - \bar{p}y_i' > 0.$$

Since $\lambda_v \bar{p} v_i + \bar{p}\omega_i + \bar{p}y_i' + \bar{p}u_i = \bar{p}x_i'$, one has

$$\bar{p}\sum_S u_i = \bar{p}\sum_S x_i' - \bar{p}\sum_S \omega_i - \bar{p}\sum_S y_i' - \lambda_v \bar{p}\sum_S v_i. \tag{3.1}$$

However one already has $\bar{p}\sum_S x_i' - \bar{p}\sum_S \omega_i - \bar{p}\sum_S y_i' > 0$. Now note that the properties of MCP–equilibrium (1.7) imply $\bar{p}v_i = 0$ for all $i \in \mathcal{I}$ that due to (3.1) allows to deduce $\bar{p}\sum_S u_i > 0$, that contradicts to the contract definition.

Now establish the sufficiency. Let $(x(V, y), y)$ be marginally–contractual allocation. Show that there exists a vector $p \neq 0$ such that $(x, y, p)$ is a MCP–equilibrium. Consider the following convex sets (see Fig. 2):

$$K_i = \{y_i' \in E \mid \langle\nabla\varphi_i(y_i), y_i'\rangle < \langle\nabla\varphi_i(y_i), y_i\rangle\} \cup \{y_i\}, \quad i \in \mathcal{I}.$$

Note that

$$\overline{K}_i \supset \{y_i' \in Y_i \mid \langle\nabla\varphi_i(y_i), y_i'\rangle \leq \langle\nabla\varphi_i(y_i), y_i\rangle\} \supset M_i^y, \quad i \in \mathcal{I}.$$

This is a tangent cone to the $Y_i$ going from a point $y_i$.

Put $K = \prod_{i \in \mathcal{I}} K_i$. Next, consider an economy $\mathcal{E}_K$ similar to one under study in which the set $Y_i$ are replaced by $K_i$. Show that the pair $(x, y)$ is Pareto optimal in $\mathcal{E}_K$. Assuming contrary, one finds an attainable allocation $\hat{z} = (\hat{x}, \hat{y}) \in \mathcal{A}(\mathcal{E}_K)$, where $\hat{y} \in \prod_{i \in \mathcal{I}} K_i$ and $\hat{x}_i \succ_i x_i$ for all $i \in \mathcal{I}$. Let

$$y_i(\alpha) = y_i + \alpha(\hat{y}_i - y_i), \quad \alpha \in (0, 1], \quad i \in \mathcal{I}.$$

Since $\hat{y}_i \in K_i$ then for $\hat{y}_i \neq y_i$ one has

$$\langle\nabla\varphi_i(y_i), y_i(\alpha) - y_i\rangle = \alpha\langle\nabla\varphi_i(y_i), \hat{y}_i - y_i\rangle < 0, \quad i \in \mathcal{I}.$$

Recall that $\varphi_i(y_i) = 0$. But then for all sufficiently small $\alpha > 0$ one obtains

$$\varphi_i(y_i(\alpha)) = \varphi_i(y_i) + \langle \nabla\varphi_i(y_i), y_i(\alpha) - y_i\rangle + o(|y_i(\alpha) - y_i|) < 0, \quad i \in \mathcal{I}.$$

Consequently, for some $\varepsilon > 0$ and all $\alpha \in [0, \varepsilon]$, for each $i \in \mathcal{I}$ one has

$$y_i(\alpha) \in M_i^y.$$

Further fix $\alpha \in [0, \varepsilon]$ and put $\tilde{y}_i = y_i(\alpha)$, $\tilde{x}_i = \alpha\hat{x}_i + (1-\alpha)x_i$. Let $u_i = \hat{x}_i - \hat{y}_i - \omega_i$, $i \in \mathcal{I}$. As soon as $\mathcal{P}_i(x_i)$ is convex one has $\tilde{x}_i \succ_i x_i$. On the other hand

$$\tilde{x}_i = \alpha(\hat{y}_i + u_i + \omega_i) + (1-\alpha)(y_i + \sum_{v\in V} v_i + \omega_i) = \omega_i + (1-\alpha)\sum_{v\in V} v_i + \alpha u_i + \tilde{y}_i.$$

Define $W = (1-\alpha)V \cup \{\alpha u\}$. By construction $\alpha u$ is a (new) contract and $\tilde{y}_i \in M_i^{y^*}$ is a new production plan constructed via the initial $\hat{y}_i$. Here new consumption plans $\tilde{x}_i = \tilde{x}_i(W, \tilde{y}_i)$ are strictly preferred to the plans $x_i$ for all $i \in \mathcal{I}$. Now one comes to the contradiction with the definition of marginally–contractual allocation since the allocation $(\tilde{x}, \tilde{y})$ is received from the current one through a partial break of initial contracts in the amount of $\alpha > 0$, the new choice of admissible production plans and the signing of a new barter contract $\alpha u$.

Thus, now we are in the conditions of Lemma 2.1 according to which there is a vector $p \neq 0$ such that for all $i \in \mathcal{I}$

$$\langle p, \mathcal{P}_i(x_i)\rangle \ \geq p \cdot x_i \ \ \& \ \ p \cdot y_i \geq \langle p, K_i\rangle. \tag{3.2}$$

We prove further that $p$ is a price vector for MCP-equilibrium. To this end, we first show that

$$px_i = py_i + p\omega_i, \quad i \in \mathcal{I}.$$

However, this fact is proved in the same way as it was done in the convex case (see Theorem 2.1). As soon as assumptions (**A**) and $x_i \in \text{int}X_i$ for $p \neq 0$ in view of (3.2) together imply

$$\langle p, \mathcal{P}_i(x_i)\rangle > p \cdot x_i, \quad i \in \mathcal{I},$$

then condition (1.7) of Definition 1.2 is proven. Condition (1.6) of Definition 1.2 follows from (3.2) and specification of $K_i$. Finally, (1.8) holds by the definition of contractual allocation. Theorem 3.1 is proved. ■

# 4 Contractual approach in an economy with public goods

An economy with public goods is characterized by the presence of products of special form, which by their physical qualities are the products of public consumption.

Examples of public goods are public television and radio, street lighting, roads, production of "security" (police, national defense, etc.). The list of examples can be continued, but it is clear that always there is a product (good), while consuming a lot of agents. This product is needed to be reproduced (one needs to repair roads, produce and broadcast television programs) that needs to be somehow financed. It is clear that the funding of production of collective consumption commodity should be carried out by all its consumers. In the neoclassical theory of decentralized economy the concept of individual cost estimates is putted as a basis for the mechanism of regulation of the value of public goods, these estimates are calculated as the product of the individual prices of total consumption. Of course, ordinary products may exist in the economy and processes of their allocation and reproduction are carried out under usual market rules. In theory, an appropriate concept of equilibrium (by Lindahl) is defined and studied in such a way that the related allocation is Pareto optimal one.[9] It is a difficult theoretical problem of correct determination of individual prices in practice. Really, in the case of products of private consumption this issue is resolved automatically by the market mechanism, based on a large number of exchange transactions, by the method of price "tatônnement." This method does not work for public goods, because individuals can not in principle to exchange the parts of public goods.[10] From a theoretical point of view individual prices should be proportional to the marginal rate of substitution (exchange), but in terms of utility functions — to a fragment of the gradient, corresponding to public goods. Thus, in order to "evaluate" the individual prices, one must have a purely private information about the preferences of individuals, that is not practicable in real life. Below I suggest a theoretical way to resolve this problem.

## 4.1 Public goods economy and main classical notions

Structure of the economy with public goods is similar to the model of the Arrow–Debreu, the main difference is in the commodity space which represents also public goods. The mechanism of public goods cost regulation is also different and specific. The model has $n$ consumers forming a set $\mathcal{I} = \{1, \dots, n\}$ and $m$ producers (firm) $\mathcal{J} = \{1, \dots, m\}$. In the economy there is represented $l$ types of private goods, their nomenclature is $\{1, \dots, l\}$, and $s$ kinds of public goods, numbered by the indices $\{l + 1, \dots,\ l + s\}$. Thus, the total number of products is $l + s$. Consumers are equipped with individualized *private goods consumption sets* $X_i^p \subset \mathbb{R}^l$ and *common* for all consumers set of *permissible* for consumption *public goods* $X^c \subset \mathbb{R}^s$. So, here

[9]There are also other equilibrium concepts, not necessarily Pareto optimal, see Florenzano (2009).

[10]Therefore, it is recommended in practice to transform the "public goods" into private ones whenever possible, through various specific techniques in order to enable the market mechanism. An example of this is the transition to the counter in water provision.

$\mathbb{R}^{l+s} = E$ is a commodity space. In addition, consumers have initial endowments of private goods $\omega_i \in X_i^p$, $i \in \mathcal{I}$, and economy as a whole has endowments of public goods $\omega^c \in X^c$. In general firms can produce and spend as private and public goods, their production capacities are presented by technological sets $Y_j \subset \mathbb{R}^{l+s}$, $0 \in Y_j$, $j \in \mathcal{J}$. Production plans $y_j \in Y_j$ are written in the form $y_j = (y_j^p, y_j^c)$, where $y_j^p \in \mathbb{R}^l$ is associated with the goods of private consumption and $y_j^c \in \mathbb{R}^s$ — for public ones. A set

$$\mathbb{Z} = \prod_{\mathcal{I}} X_i^p \times X^c \times \prod_{\mathcal{J}} Y_j$$

is identified with the set of all *admissible states* and the space $L = \mathbb{R}^{ln+s+m(l+s)} \supset \mathbb{Z}$ is a *space of allocations*. Consumers' preferences are defined and take values in $X_i^p \times X^c$, i.e. $\mathcal{P}_i : X_i^p \times X^c \Rightarrow X_i^p \times X^c$. The sets $X_i^p \times X^c$ are associated with consumption sets of individuals, they are assumed to be convex. One can see that this model has externalities, concentrated in the area of public goods. In addition, as well as in Arrow–Debreu model there are defined *shares* $\theta_i^j \geq 0$ (the components of vector $\theta_i = (\theta_i^1, \ldots, \theta_i^m)$) of consumer $i$ in a profit of producer $j$. These quantities satisfy $\sum_{i\in\mathcal{I}} \theta_i^j = 1$ for all $j \in \mathcal{J}$ (i.e. profit is completely distributed among all shareholders).

Processes of exchange and reproduction of goods are regulated by the individual prices for public goods $q_i \in \mathbb{R}^s$ and the market price $p \in \mathbb{R}^l$ for the commodities of private consumption. In this case, the budget acceptable consumption plans $(x_i, x^c) \in X_i^p \times X^c$ of individual $i \in \mathcal{I}$ are specified by the constraint

$$\langle x_i, p\rangle + \langle x^c, q_i\rangle \leq \langle \omega_i, p\rangle + \langle \omega^c, q_i\rangle + \sum_{j\in\mathcal{J}} \theta_i^j (p y_j^p + \bar{q} y_j^c),$$

where $y \in \prod_{\mathcal{J}} Y_j$, $q = (q_1, \ldots, q_n) \in [\mathbb{R}^s]^{\mathcal{I}}$, $p \in \mathbb{R}^l$ and $\bar{q} = \sum_{\mathcal{I}} q_i$. So, one can see that an agent's income is formed from three sources: the sale of private endowments $\omega_i$ by market prices $p$, individualized cost value of public goods $\langle q_i, \omega^c\rangle$ and as the "sum of dividends" from the profits of producers. It is also worth to repeat that the producers' profits are determined by "production" prices $(p, \bar{q})$. Now in its shortest form the model under study with public goods can be written as

$$\mathcal{E}^{pg} = \langle \mathcal{I}, \mathcal{J}, \mathbb{R}^l, \mathbb{R}^s, \{X_i^p, \mathcal{P}_i(\cdot), \theta_i, \omega_i\}_{i\in\mathcal{I}}, \{Y_j\}_{j\in\mathcal{J}}, X^c, \omega^c \rangle.$$

In the neoclassical setting Lindahl equilibrium is considered as a main solution concept.

**Definition** 4.1 *An allocation $z = (x, x^c, y) \in \mathbb{Z}$ is said to be **Lindahl equilibrium** with a price bundle $(p, q_1, \ldots, q_n) \in \mathbb{R}^{l+ns}$, if for $\bar{q} = \sum_{\mathcal{I}} q_i$ it obeys:*

$$p y_j^p + \bar{q} y_j^c \geq \langle (p, \bar{q}), Y_j\rangle, \quad j \in \mathcal{J}, \tag{4.1}$$

$$\langle (p, q_i), \mathcal{P}_i(x_i, x^c) \rangle > \omega_i p + \omega^c q_i + \sum_{j \in \mathcal{J}} \theta_i^j (p y_j^p + \bar{q} y_j^c) = x_i p + x^c q_i, \quad i \in \mathcal{I}, \tag{4.2}$$

$$\sum_{\mathcal{I}} x_i = \sum_{\mathcal{I}} \omega_i + \sum_{\mathcal{J}} y_j^p, \tag{4.3}$$

$$x^c = \omega^c + \sum_{\mathcal{J}} y_j^c. \tag{4.4}$$

*In the case of non-strict inequalities in (4.2) it is said to be* ***quasi-equilibrium****.*

Requirements (4.1)–(4.4) have the usual substantial sense. Condition (4.1) implements the principle of producers' profit maximization, (4.2) states that $(x_i, x^c)$ is optimal budget acceptable plan, the condition (4.3) presents the balance of private consumption commodities and (4.4) is the balance of public goods.

Commenting the equilibrium definition, one notes that individuals evaluate the consumption of public goods according to individual prices $q_i$, while in the production there are applied specific industrial prices which are equal to the sum of individual prices, $\bar{q} = \sum_{\mathcal{I}} q_i$. Draw also attention to the specific requirement of the public goods balance: this is so because all individuals consume the same quantity of good — if it is a public one. Further, any allocation from $\mathbb{Z}$ satisfying (4.3) and (4.4), is called *feasible* (valid) one and the set of all such allocations is denoted as $\mathcal{A}(\mathcal{E}^{pg})$.

Without going into details, one notes that for a convex economy Lindahl equilibrium always is Pareto optimal and do exists approximately under the same assumptions as usual Walrasian equilibrium, see e.g. Ruys (1974), Marakulin (2003b) §1.2.3, Florenzano, Mercato (2006), Florenzano (2009). Similarly to usual Arrow–Debreu model in an economy with public goods there is an analog of 2-nd Welfare theorem: every Pareto optimal allocation can be presented in the form of Lindahl equilibrium for a specific redistribution of initial endowments. In other words, for a Pareto optimal allocation one can provide a dual characterization in value categories (an analogue of Lemma 2.1; similar result can be found e.g. in Florenzano (2009)). Insofar as this description and application of mathematical techniques is involved in the subsequent analysis, I present the formalization of an appropriate mathematical result and give its proof. Recall that:

• *A feasible allocation* $(x, y) = ((x_i)_{i \in \mathcal{I}}, (y_j)_{j \in \mathcal{J}}) \in \mathbb{Z}$ *is said to be (weak)* ***Pareto optimal****, if there is no pair* $(x'_i, y'_i)_{i \in \mathcal{I}} \in \mathcal{A}(\mathcal{E}^{pg})$ *such that* $x'_i \succ_i x_i$ *for each* $i \in \mathcal{I}$.

In the following lemma and the subsequent analysis I shall use the assumption[11] (**A**) and apply a specific notion of local *non-satiated* preferences of each individual in a groups of *private* and (separately) *public* goods. The latter means that changing of consumption bundle $(x_i^p, x_i^c)$ only in part of private or (separately) public goods

[11] Here, one has to understand $X_i^p \times X^c$ as a consumption set for individual $i$.

while the consumption from another group of commodities is the same, it is possible to obtain strictly preferred consumption bundle:

$$\mathcal{P}_i(x_i^p, x_i^c) \cap \mathbb{R}^l \times \{x_i^c\} \neq \emptyset \quad \& \quad \mathcal{P}_i(x_i^p, x_i^c) \cap \{x_i^p\} \times \mathbb{R}^s \neq \emptyset, \quad \forall\, i \in \mathcal{I}. \tag{4.5}$$

**Lemma** 4.1 *Let an allocation $\bar{z} = ((\bar{x}_i)_{i\in\mathcal{I}}, \bar{x}^c, (\bar{y}_j)_{j\in\mathcal{J}}) \in \mathcal{A}(\mathcal{E}^{pg})$ be Pareto optimal. Then there is a vector of prices for private goods $p \in \mathbb{R}^l$ and individualized price vectors for public goods $q_i \in \mathbb{R}^s$, $i \in \mathcal{I}$, which* ***not all*** *are equal to* ***zero*** *and such that:*

$$\langle (p, q_i), \mathcal{P}_i(\bar{x}_i, \bar{x}^c)\rangle \ \geq \langle (p, q_i), (\bar{x}_i, \bar{x}^c)\rangle, \ i \in \mathcal{I}, \tag{4.6}$$

$$\langle (p, \sum_{i\in\mathcal{I}} q_i), \bar{y}_j\rangle \geq \langle p, Y_j\rangle, \ j \in \mathcal{J}. \tag{4.7}$$

*If for $\bar{z}$ in addition the consumption of each individual is* ***non-satiated*** *by private and (separately) public goods, and if $(\bar{x}_i, \bar{x}^c) \in \operatorname{int}(X_i^p \times X^c)$ $\forall\, i \in \mathcal{I}$, then all these price vectors* ***are non-zero****, i.e. $p \neq 0$, $q_i \neq 0$ $\forall\, i \in \mathcal{I}$.*

**Remark** 4.1 If $\mathcal{E}^{pg}$ is an economy with a smooth consumption sector and all lemma's assumptions are satisfied then for interior allocation and for every $i$: $\exists\, \lambda_i > 0 \mid (p, q_i) = \lambda_i \cdot \nabla u_i(\bar{x}_i, \bar{x}^c)$. ■

*Proof of Lemma 4.1.* To establish the lemma, one expresses the Pareto optimality of the allocation $\bar{z}$ in a form suitable for the application of the separation theorem. For this purpose, let us define the following affine space:

$$\mathcal{L} = \{(x_1^p, x_1^c, \ldots, x_n^p, x_n^c, y_1^p, y_1^c, \ldots, y_m^p, y_m^c) \in \mathbb{R}^{(l+s)(n+m)} :$$

$$x_1^p + \cdots + x_n^p - (y_1^p + \cdots + y_m^p) = \sum_{\mathcal{I}} \omega_i; \tag{4.8}$$

$$x_2^c - x_1^c = 0, \ x_3^c - x_1^c = 0, \ldots, \ x_n^c - x_1^c = 0; \tag{4.9}$$

$$x_1^c - (y_1^c + \cdots + y_m^c) = \omega^c\}. \tag{4.10}$$

Now, Pareto optimality can be written as

$$\prod_{\mathcal{I}} (\mathcal{P}_i(\bar{x}_i, \bar{x}^c) \times \prod_{\mathcal{J}} Y_j \bigcap \mathcal{L} = \emptyset. \tag{4.11}$$

Since the intersected sets are convex and nonempty (by (**A**)), then by the separation theorem, there exists a linear functional $f \neq 0$, which separates these sets, i.e. if one presents this functional in the form of inner product, then there exists a vector $f = (f_1, \ldots, f_n, g_1, \ldots, g_m)$ such that

$$\langle f, \prod_{\mathcal{I}} (\mathcal{P}_i(\bar{x}_i, \bar{x}^c) \times \prod_{\mathcal{J}} Y_j\rangle \geq \langle f, \mathcal{L}\rangle.$$

Further one reveals the structure of the functional (vector) $f$. Since $f$ is bounded from above on $\mathcal{L}$, then it must be constant on $\mathcal{L}$ and, hence, its representing vector must be located in the orthogonal to $\mathcal{L}$ subspace. However $\mathcal{L}^\perp$ is represented as a linear hull of the normal vectors to the hyperplanes defined by relations (4.8)–(4.10) (it should be remembered that they are equations in vector form). The analysis of these relations gives the following representation: $\exists\, p \in \mathbb{R}^l$, $\exists\, \bar{q}, q_2, \ldots, q_n \in \mathbb{R}^s$ such that for $q_1 = \bar{q} - \sum_{i=2}^n q_i$

$$f = (p, q_1, p, q_2, \ldots, p, q_n, (-p, -\bar{q}), \ldots, (-p, -\bar{q}))$$

holds.

Further, to establish (4.6), consider the value of the functional $f$ on the vector

$$(\bar{x}_1^p, \bar{x}^c, \ldots, \bar{x}_n^p, \bar{x}^c, \bar{y}_1^p, \bar{y}_1^c, \ldots, \bar{y}_m^p, \bar{y}_m^c) \in \mathcal{L} \tag{4.12}$$

and compare it with the value on a similar vector, where on the place of the individual $i$ consumption there is written a vector $(x_i^p, x_i^c) \in \mathcal{P}_i(\bar{x}_i^p, \bar{x}_i^c)$. By (**A**) one has $(\bar{x}_k, \bar{x}^c) \in \overline{\mathcal{P}_k(\bar{x}_k, \bar{x}^c)}$, $\forall\, k \in \mathcal{I}$ and, therefore, constructed the vector belongs to the closure of the set, recorded in the left side of the intersection (4.11). Consequently, the value of the functional on the constructed vector must be not less than its value on the vector (4.12), hence, reducing the common terms in the right and the left hand sides one arrives at (4.6).

Inequalities (4.7) are proved in a similar way: the value of the functional on the vector (4.12) should be compared with the value on a similar vector, where instead of production plan $(\bar{y}_j^p, \bar{y}_j^c)$ there is written (arbitrary chosen) plan $(y_j^p, y_j^c) \in Y_j$. In so doing one arrives at

$$\langle (-p, -\bar{q}), (y_j^p, y_j^c) \rangle \geq \langle (-p, -\bar{q}), (\bar{y}_j^p, \bar{y}_j^c) \rangle, \ \forall\, (y_j^p, y_j^c) \in Y_j \iff$$

$$\langle (p, \bar{q}), \bar{y}_j \rangle \geq \langle (p, \bar{q}), Y_j \rangle,$$

that proves the first part of the lemma. Further one proves the second part.

With this in mind, one considers an inequality of (4.6) such that $(p, q_i) \neq 0$. Suppose, for example, $p = 0$. Now substitute consumption bundle $\bar{x}_i$ by $x_i$ so that $(x_i, \bar{x}^c) \succ_i (\bar{x}_i, \bar{x}^c)$ is true. Next find $x^c$ such that $q_i x^c < q_i \bar{x}^c$ but still $(x_i, x^c) \succ_i (\bar{x}_i, \bar{x}^c)$. Assumptions of lemma (interior point and non-satiation in a separate way, by private and public consumption) allows to do it. However, now one obtains $\langle (p, q_i), (x_i, x^c) \rangle < \langle (p, q_i), (\bar{x}_i, \bar{x}^c) \rangle$, that contradicts (4.6). Therefore $q_i \neq 0$ implies $p \neq 0$. It can be proven similarly that $p \neq 0 \Rightarrow q_i \neq 0$ for each $i$. Lemma 4.1 is proved. ■

In the model $\mathcal{E}^{pg}$ with public goods the concept of Foley core is usually considered in literature and it has a familiar substantial sense: the set of all production allocations, which can be dominated by no coalition, i.e., no group of individuals would benefit to live as a separate economy.

• *An allocation* $z = (x, x^c, y) \in \mathcal{A}(\mathcal{E}^{pg})$ *is said to be dominated (blocked) by coalition* $\emptyset \neq S \subseteq \mathcal{I}$, *if there exist production* $y^S = (y^{Sp}, y^{Sc}) \in Y_S = \sum_{i \in S} \sum_{j \in \mathcal{J}} \theta_i^j Y_j$ *and consumption* $((x_i^S)_S, x^{Sc}) \in \prod_S X_i \times X^c$ *programs such that*

$$\sum_{i \in S} x_i^S = \sum_{i \in S} \omega_i + y^{Sp}, \quad x^{Sc} = \omega^c + y^{Sc} \quad \& \quad (x_i^S, x^{Sc}) \succ_i (x_i, x^c) \ \ \textit{for each} \ \ i \in S.$$

*In an economy with public goods the set of all allocations that are* ***dominated by no coalition*** *is denoted* $\mathcal{C}(\mathcal{E}^{pg})$ *and is called* ***Foley core***.

Foley has introduced this concept (Foley, 1970) and proved, under certain assumptions that Lindahl equilibrium belongs to the core: in the paper this result follows directly from the analogous fact for the fuzzy core, see Proposition 4.1. However, does the Foley core shrink to equilibrium at the model being replicated to infinity? The following example, borrowed from Buchholz, Peters (2007), illustrates the fact well-known in the literature that an infinite replication of the model *does not* imply *Foley core shrinks* to (equal treatment) Lindahl equilibria — in difference with an economy with only private goods. So, how perfect competition has to be presented in public goods economy?

**Example** 4.1 Consider an economy with two individuals $i = 1, 2$ and two goods: one private and another public one. Let $(x_i, x^c) \in \mathbb{R}_+^2$ be permissible consumption plans that agents $i = 1, 2$ can consume in any non-negative quantities. Let consumers have the same Cobb–Douglas utilities specified in logarithmic form as $u_i(x_i, x^c) = \ln(x_i) + \ln(x^c)$ and let the individuals' endowments are $\omega_i = (1, 0)$, $i = 1, 2$ (public goods endowments are equal to 0). Finally, suppose there is a technology set specified as a cone

$$Y = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \leq 0, \quad y_2 \leq -y_1\},$$

i.e. technology is linear and one unit of private good produces a unit of public one. Now let us calculate Lindahl equilibrium according to Definition 4.1.

Due to (4.1) and the fact that public good in equilibrium is not zero, for the (normalized) equilibrium prices, one has $(p, \bar{q}) = (1, 1)$ and the zero profit for producer. From the first order conditions in the consumer's problem, one concludes $\nabla u_i(x_i, x^c) = (\frac{1}{x_i}, \frac{1}{x^c}) = \lambda_i(1, q_i)$, $\lambda_i > 0 \Rightarrow q_i = \frac{x_i}{x^c}$, that from budget equality allows to conclude $x_i + x^c \cdot \frac{x_i}{x^c} = 1 \Rightarrow x_i = \frac{1}{2} \Rightarrow x^c = 2 - x_1 - x_2 = 1$. Thus, in equilibrium one has

$$(x_1, x^c) = (x_2, x^c) = (\tfrac{1}{2}, 1),$$

where a private good price is $p = 1$, and individualized prices are $q_1 = q_2 = \frac{1}{2}$. Further let us find the core.

In an economy with two agents core is a part of the Pareto boundary, where the consumption bundles for each agent are individually rational (utility is not less than in an economy with one agent). Pareto frontier can be found using the above analysis and Lemma 4.1; now the individual rationality constraint is given by $x_i \cdot x^c \geq \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, that allows to describe core structure:

$$\mathcal{C}(\mathcal{E}^{pg}) = \{(x_1, x_2, x^c) \in \mathbb{R}^3_+ \mid x^c = 1,\ x_1 + x_2 = 1,\ x_1 \geq \tfrac{1}{4},\ x_2 \geq \tfrac{1}{4}\}.$$

Using an analysis similar to the above, it is easy to understand that in the $n$-times replicated economy Lindahl equilibrium has the following form:

$$(\bar{x}_{11}, \bar{x}_{21}, \ldots, \bar{x}_{1n}, \bar{x}_{2n}, \bar{x}^c, (y, y^c)) = (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, \tfrac{1}{2}, n, (-n, n)),$$

under prices $p = 1$, $q_{ik} = \frac{1}{2n}$, $i = 1, 2$, $k = 1, \ldots, n$.

Let us consider further the following allocation:

$$z = (x_{11}, x_{21}, \ldots, x_{1n}, x_{2n}, x^c, (y, y^c)) = (\tfrac{5}{8}, \tfrac{3}{8}, \ldots, \tfrac{5}{8}, \tfrac{3}{8}, n, (-n, n)),$$

i.e. agents of the second type are spending for the public good $\frac{5}{8}$ of the private product, and first ones only $\frac{3}{8}$. Now applying Lemma 4.1 one can simply show that this is a Pareto optimal allocation (here similar to above $q_{ik} = \frac{x_{ik}}{x^c}$). Moreover, it belongs to the core. To verify this, consider a coalition $S$, consisting of $l$ 1st type agents and $m$ of 2nd ones. Optimal production of public goods in this group (through inter-coalitional Pareto frontier) is $x^c_S = \frac{l+m}{2} \leq n$. Hence, in order to allow agents of 1st type to reach the level of utility no less than in the allocation $z$, they have to consume private good *more* than $\frac{5}{8}$. But then (by balance) in the group of 2nd type individuals in $S$ there is an individual whose consumption of private good $x^S_{2k}$ is strictly less than $\frac{1}{m}(\frac{l+m}{2} - \frac{5}{8}l) = \frac{1}{2} - \frac{1}{8}\frac{l}{m}$, i.e. $\exists\,(2,k) \in S$: $x^S_{2k} < \frac{1}{2} - \frac{1}{8}\frac{l}{m} \Rightarrow$

$$x^S_{2k} \cdot \tfrac{l+m}{2} < \left(\tfrac{1}{2} - \tfrac{1}{8}\tfrac{l}{m}\right) \cdot \tfrac{l+m}{2} < \left(\tfrac{1}{2} - \tfrac{1}{8}\tfrac{l}{n}\right) \cdot \tfrac{l+n}{2} =$$

$$= \tfrac{1}{16n}(4n - l)(n + l) < \tfrac{1}{16n}(6n^2 - 2(l - n)^2) < \tfrac{6n^2}{16n} = \tfrac{3}{8}n.$$

Taking the logarithm of both sides of the inequality, we conclude that the utility of the individual $(2, k) \in S$ is lower than in the proposed allocation, which contradicts $S$ is blocking. Since the $l$ and $m$ were chosen in an arbitrary way, then the resulting contradiction proves that allocation $z$ belongs to Foley core.

The presented analysis holds for arbitrarily large $n$, which proves that Foley core does not shrinks to Lindahl equilibrium when the model of economy is replicating to infinity. ■

In the next section I introduce and study the concept of the fuzzy core for a model with public goods. Further in the section 4.4 there will be introduced and studied fuzzy contractual allocation. As we shall see, these concepts are closely related to each other, very fruitfully work in the theory of equilibrium and present an adequate solution on the convergence of the core to equilibrium.

## 4.2 Fuzzy core vs. Foley core — what is the best?

Let us start from the definition and analysis of the fuzzy core. Recall that a fuzzy coalition is identified with any vector

$$t = (t_1, t_2, \dots, t_n) \neq 0, \quad 0 \leq t_i \leq 1 \ \ \forall\, i \in \mathcal{I},$$

where real $t_i$ are interpreted as a measure of agent $i$ participation in the coalition activities. The key property that determines the efficiency of fuzzy coalitions is their ability to dominate a current allocation of the economy, and it defines the fuzzy core. In a model with public goods it is an especially peculiar thing.

• *Let a triplet $(x, x^c, y) = z$, where $((x_i)_{i\in\mathcal{I}}, x^c) \in \prod_{\mathcal{I}} X_i^p \times X^c$ are the families of private and public consumption plans, and $y = (y^p, y^c) \in \sum_{\mathcal{J}} Y_j = Y$ is an aggregated production program and let $z$ be a feasible allocation, i.e. $x^c = \omega^c + y^c$ and $\sum_{\mathcal{I}} x_i = \sum_{\mathcal{I}} \omega_i + y^p$ hold. A **fuzzy coalition** $t = (t_1, \dots, t_n)$ **blocks allocation** $z$, if there is a triplet $((\xi_i)_{\mathcal{I}}, \xi^c, (\zeta^p, \zeta^c))$, such that*

$$\sum_{\mathcal{I}} t_i(\xi_i - \omega_i) = \zeta^p, \quad \xi^c = \zeta^c, \quad (\zeta^p, \zeta^c) \in Y \tag{4.13}$$

*and*

$$(\xi_i, \frac{\xi^c}{t_i} + \omega^c) \succ_i (x_i, x^c) \ \ \forall\, i \in \mathcal{I}: \ t_i \neq 0. \tag{4.14}$$

*The set of all allocations that are dominated by no fuzzy coalition is denoted $\mathcal{C}^f(\mathcal{E}^{pg})$ and is called **fuzzy core**.*

Despite the fact that similar constructions already have been appeared in the literature (e.g. see Vasil'ev (1996)), further analysis is quite original.

A meaning of the fuzzy core and the blocking is as follows. Imagine that an index $i \in \mathcal{I}$ specifies only the type of economic agent which is represented by many identical copies (the same number for different types). Then $t_i \in (0, 1]$ is a share of type $i$ individuals, entered in a blocking coalition. Being separated the coalition passes to self-sufficiency of all its needs, that is expressed in the relation (4.13). In this case, however, agents have to improve their situation and reach a more preferred consumption, which is expressed in (4.14), and it contains a key specific of (semi)public goods. Indeed, the agents estimate produced within the coalition public goods as the relative proportion of individuals of this type. In Florenzano, Mercato (2006) a fuzzy core concept is not introduced, however, expressly postulated the appropriate type of domination in a replica of the original model (see Definitions 3.4, 3.5). One can say that not common level of consumption is crucial for agents, but *average* consumption of public goods is really important. Thus, one can speak about semi-public goods, just as it is done in Vasil'ev et al. (1995). Moreover in Vasil'ev et al. (1995) domination similar to describe above is interpreted in the terms of congestion or crowing in its provision (in these terms the results on equivalence of core and equilibria are formulated). For example, (dis)pleasure from the consumption of such good as the opportunity to attend a public skating-rink (park, road infrastructure, etc.) essentially depends on the number of visitors. Below we will see that the domination and the fuzzy core elements can be interpreted in contractual categories, where the coalition has the opportunity to enter into contracts for the production of public goods for an own inter-coalitional consumption. Then the elements of the fuzzy core correspond to a stable set of contracts (webs), subject to approval of the possibility of an asymmetric partial breaking.

Next, I present the first important result of the fuzzy core, proven under the following additional assumptions:

(**P**) *For each $j \in \mathcal{J}$ set $Y_j$ is a convex closed cone with a vertex at zero and let public goods can only be produced and they are not costly factors, i.e. $Y_j \subset \mathbb{R}^l \times \mathbb{R}^s_+$.*

(**M**) *The set of feasible public goods consumption programs $X^c$ is convex, obeys $X^c + \mathbb{R}^s_+ \subseteq X^c$ and that all public goods are desirable*[12] *for each individual, i.e.*

$$(x_i, x^c + z) \succ_i (x_i, x^c) \quad \forall\, (x_i, x^c) \in X_i \times X^c, \ \forall\, z \in \mathrm{int}\mathbb{R}^s_+ \quad \forall\, i \in \mathcal{I}.$$

**Proposition** 4.1 *Let $\mathcal{E}^{pg}$ obey* (**P**), (**M**). *Then Lindahl equilibrium belongs to the fuzzy core.*

*Proof of Proposition 4.1.* Let a triplet $(x, x^c, y) \in \prod_{\mathcal{I}} X_i^p \times X^c \times Y$, $Y = \sum_{\mathcal{J}} Y_j$ satisfy to Definition 4.1. Assume there is a dominating coalition $t = (t_i)_{\mathcal{I}} \neq 0$. Then estimating (4.14) by prices and applying (4.2) find

$$\langle p, \xi_i \rangle + \langle q_i, \frac{\xi^c}{t_i} + \omega^c \rangle > \langle p, x_i \rangle + \langle q_i, x^c \rangle = \langle p, \omega_i \rangle + \langle q_i, \omega^c \rangle + \sum_{j \in \mathcal{J}} \theta_i^j (p y_j^p + \bar{q} y_j^c), \ \ \forall\, i \in \mathcal{I}.$$

Due to (4.1) and assumption (**P**) in the part of technological sets are convex cones with the vertex at zero, conclude that $p y_j^p + \bar{q} y_j^c \geq 0$, $\forall\, i \in \mathcal{I}$. Substituting this in the last formula and multiplying inequalities on $t_i$ one finds:

$$t_i \langle p, \xi_i \rangle + \langle q_i, \xi^c \rangle > t_i \langle p, \omega_i \rangle, \ \ i \in \mathrm{supp}\,(t) \ \Rightarrow \ \langle p, \sum_{\mathrm{supp}\,(t)} t_i(\xi_i - \omega_i) \rangle + \langle \sum_{\mathrm{supp}\,(t)} q_i, \xi^c \rangle > 0.$$

However it follows now from (**P**), (**M**) that $\xi^c \geq 0$ and $q_i \geq 0$ for all $i$, that implies $\langle \sum_{\mathcal{I}} q_i, \xi^c \rangle \geq \langle \sum_{\mathrm{supp}\,(t)} q_i, \xi^c \rangle$ and, therefore,

$$\langle p, \sum_{\mathrm{supp}\,(t)} t_i(\xi_i - \omega_i) \rangle + \langle \sum_{\mathcal{I}} q_i, \xi^c \rangle \geq \langle p, \sum_{\mathrm{supp}\,(t)} t_i(\xi_i - \omega_i) \rangle + \langle \sum_{\mathrm{supp}\,(t)} q_i, \xi^c \rangle > 0.$$

[12] This is the monotonicity of preferences over the group of public goods variables.

Due to (4.13) the last means that by equilibrium prices $(p, \bar{q})$, $\bar{q} = \sum_{\mathcal{I}} q_i$ production plan $(\sum_{\mathrm{supp}\,(t)} t_i(\xi_i - \omega_i), \xi^c) = (\zeta^p, \zeta^c)$ yields a strictly positive profit, that is impossible in view of (4.1) and the right hand side of (4.13) for conical technological sets. ■

In order to better understand the properties of fuzzy core and to establish the exact relationship between the core and equilibrium (equivalence?), I establish the following fuzzy core characterization. Define

$$\Omega_i(x_i, x^c) = \mathrm{co}(\mathcal{P}_i(x_i, x^c) \cup \{(\omega_i, \omega^c)\}), \quad i \in \mathcal{I} \tag{4.15}$$

Due to the convexity of $\mathcal{P}_i(x_i, x^c)$, for $\mathcal{P}_i(x_i, x^c) \neq \emptyset$ (one has it by (**M**)) conclude

$$\mathrm{co}(\mathcal{P}_i(x_i, x^c) \cup \{(\omega_i, \omega^c)\}) = \bigcup_{0 \le \lambda \le 1} [\lambda \mathcal{P}_i(x_i, x^c) + (1 - \lambda)(\omega_i, \omega^c)] =$$

$$= \Omega_i(x_i, x^c) = \bigcup_{0 \le \lambda \le 1} \lambda(\mathcal{P}_i(x_i, x^c) - (\omega_i, \omega^c)) + (\omega_i, \omega^c), \quad i \in \mathcal{I}.$$

Next, consider a set and vector

$$\prod_{\mathcal{I}} \Omega_i(x_i, x^c) \times Y, \quad \tilde{\omega} = ((\omega_1, \omega^c), (\omega_2, \omega^c), \ldots, (\omega_n, \omega^c), 0) \in \mathbb{R}^{(n+1)(l+s)}$$

and condition $z + \tilde{\omega} \in \prod_{\mathcal{I}} \Omega_i(x_i, x^c) \times Y$. In view of the construction and analysis one has a representation

$$z = (\lambda_1(\xi_1 - \omega_1, \xi_1^c - \omega^c), \lambda_2(\xi_2 - \omega_2, \xi_2^c - \omega^c), \ldots, \lambda_n(\xi_n - \omega_n, \xi_n^c - \omega_n^c), \zeta^p, \zeta^c). \tag{4.16}$$

Further, we consider the subspace corresponding to the material balance relations in a model with public goods:

$$\mathcal{L}^{pg} = \{((z_1^p, z_1^c), (z_2^p, z_2^c), \ldots, (z_n^p, z_n^c), (y^p, y^c)) \mid$$

$$z_1^p + z_2^p + \cdots + z_n^p = y^p + \sum_{\mathcal{I}} \omega_i, \quad z_1^c = z_2^c = \cdots = z_n^c = y^c + \omega^c\}. \tag{4.17}$$

Finally, when $z + \tilde{\omega} \in \prod_{\mathcal{I}} \Omega_i(x_i, x^c) \times Y \cap \mathcal{L}^{pg}$ then to constraint (4.16) for $z$ there are added requirements

$$\lambda_1(\xi_1 - \omega_1) + \cdots + \lambda_n(\xi_n - \omega_n) = \zeta^p \quad \& \quad \lambda_1(\xi_1^c - \omega^c) = \cdots = \lambda_n(\xi_n^c - \omega^c) = \zeta^c,$$

that is equivalent to (4.13). For $\xi_i = \frac{\zeta^c}{\lambda_i}$ one has from construction $(\xi_i, \frac{\zeta^c}{\lambda_i} + \omega^c) \in \mathcal{P}_i(x_i, x^c)$, that is equivalent to (4.14), $i \in \mathcal{I}$. Hence, the fuzzy blocking occurs if and only if there exists *nonzero* $z$, satisfying all requirements (i.e. $z + \tilde{\omega}$ belongs to the intersection). This way there was proved characteristic

**Lemma** 4.2 *A feasible allocation* $(x, x^c, y^p, y^c) \in \mathcal{C}^f(\mathcal{E}^{pg})$ *if and only if then*

$$\prod_{\mathcal{I}} \Omega_i(x_i, x^c) \times Y \bigcap \mathcal{L}^{pg} = \{\tilde{\omega}\}. \tag{4.18}$$

As soon as the sets intersected in (4.18) are convex, it allows us to apply the separation theorem for the characterization of the elements of fuzzy core in value categories. One can do this applying the result and the mathematical technique used in Lemma 4.1 to intersection (4.18), and further apply

**Proposition** 4.2 *Let an allocation* $\bar{z} = (\bar{x}, \bar{x}^c, \bar{y}^p, \bar{y}^c) \in \mathcal{C}^f(\mathcal{E}^{pg})$*. Then* $(x, x^c, y^p, y^c)$ *is Pareto optimal and therefore there are prices* $p \in \mathbb{R}^l$*,* $q_i \in \mathbb{R}^s$*,* $i \in \mathcal{I}$*, not all equal to zero such that relations (4.6), (4.7) hold. In addition*

$$\langle (p, q_i), (\bar{x}_i, \bar{x}^c) \rangle = \langle p, \omega_i \rangle + \langle q_i, \omega^c \rangle, \ \ i \in \mathcal{I}, \tag{4.19}$$

*holds, i.e. budget equalities are true and therefore, the allocation is a **quasiequilibrium**. If the second part of Lemma 4.1 conditions is true, then it is a really Lindahl equilibrium.*

Now a theorem on the equivalence of core and equilibrium can be obtained as a consequence of the unconditional approval of the latter in conjunction with the Proposition 4.1 proved above.

**Theorem** 4.1 *Let* $\mathcal{E}^{pg}$ *satisfy* (**P**), (**M**), $(x_i, x^c) \in \text{int}(X_i \times X^c)$*,* $i \in \mathcal{I}$ *and (4.5) is true (separated non-satiation). Then allocation* $z = ((x_i)_{\mathcal{I}}, x^c, y^p, y^c) \in \mathcal{C}^f(\mathcal{E}^{pg})$ *if and only if it is a **Lindahl equilibrium** allocation.*

**Remark** 4.2 In order to avoid such restrictive requirement as an interior point in the consumption of each individual, one can use the *irreducibility* assumption (specified for the public goods) and *nontrivial quasi-equilibrium* property, as in Florenzano, Mercato (2006) and Florenzano (2009). Here and below, I restrict myself to the case of interior point, in order to already surround presentation be not too overburdened. ■

*Proof of Proposition 4.2.* Let $\bar{z} = (\bar{x}, \bar{x}^c, \bar{y}^p, \bar{y}^c) \in \mathcal{C}^f(\mathcal{E}^{pg})$, that due to Lemma 4.2 is equivalent to (4.18). Now applying separation theorem[13], one can find a nonzero linear functional non-strictly separating these sets. Since

$$\prod_{\mathcal{I}} \mathcal{P}_i(x_i, x^c) \times Y \subset \prod_{\mathcal{I}} \Omega_i(x_i, x^c) \times Y,$$

[13]Strictly speaking, to apply the theorem, one has to extract point $\tilde{\omega}$ from the set, located in the left side and notice that it does not imply the loss of convexity. Instead of the set one can also take its interior, that is convex, and clearly has empty intersection with a subspace.

then it is clear that this functional also separates the sets of (4.11), this is an equivalent to Pareto optimality relation (a small difficulty with the fact that $Y = \sum_{\mathcal{J}} Y_j$ can be easily bypassed), and therefore all conclusions regarding to the separating functional from Lemma 4.1 can be applied to our functional. This proves the first part of the statement: there are prices $p \in \mathbb{R}^l$, $q_i \in \mathbb{R}^s$, $i \in \mathcal{I}$, not all zero, such that the relations (4.6), (4.7) are true. Further prove (4.19).

A vector representing functional that separates the sets of (4.18) has the following structure:

$$f = (p, q_1, p, q_2, \dots, p, q_n, (-p, -\bar{q})), \quad \bar{q} = \sum_{\mathcal{I}} q_i.$$

Now from the local nonsatiation conclude $(\bar{x}_i, \bar{x}^c) \in \overline{\mathcal{P}_i(\bar{x}_i, \bar{x}^c)}$, $i \in \mathcal{I}$. Hence $\mathrm{co}\{(\bar{x}_i, \bar{x}^c), (\omega_i, \omega^c)\} \subset \overline{\Omega_i(\bar{x}_i, \bar{x}^c)}$, $i \in \mathcal{I}$ and certainly $\mathrm{co}\{(\bar{y}^p, \bar{y}^c), (0,0)\} \subset Y$. Therefore the set

$$\mathrm{co}\{(\bar{x}_1, \bar{x}^c), (\omega_1, \omega^c)\} \times \cdots \times \mathrm{co}\{(\bar{x}_n, \bar{x}^c), (\omega_n, \omega^c)\} \times \mathrm{co}\{(\bar{y}^p, \bar{y}^c), (0,0)\}$$

has to be separated from affine subspace $\mathcal{L}^{pg} \ni \tilde{\omega}$. Calculating further the value of the functional at the point $\tilde{\omega} = ((\omega_1, \omega^c), \dots, (\omega_n, \omega^c), 0, 0)$ and for an appropriate element of the product, one finds:

$$\sum_{j \neq i} (p\omega_j + q_j \omega^c) + p\bar{x}_i + q_i \bar{x}^c \geq \sum_{j \neq i} (p\omega_j + q_j \omega^c) + p\omega_i + q_i \omega^c \ \ \Rightarrow \ \ p\bar{x}_i + q_i \bar{x}^c \geq p\omega_i + q_i \omega^c.$$

Summing inequalities over $i \in \mathcal{I}$ and taking into account the balance relations $\sum_{\mathcal{I}} \bar{x}_i = \sum_{\mathcal{I}} \omega_i + \bar{y}^p$ and $\bar{x}^c = \omega^c + \bar{y}^c$ find $p\bar{y}^p + \bar{q}\bar{y}^c \geq 0$, that can only be executed in the form of equity (since $Y$ is a convex cone with the vertex at zero). Consequently, each addend inequalities can be fulfilled only in the form of equality. QED. ■

Further I consider the contract based approach.

## 4.3 Contractual approach: partial breaking and proper allocations

The main thing, that is needed to clarify, is what and how a contract is concluded in the production sector and how does it break. The essential difference of the contract associated with the production of public goods with the bundle of production plans is that here there is carried out really a joint production of collective consumption goods. Formally, the contract is

$$(r_1, \dots, r_n, y^c) \in \mathbb{R}^{ln} \times \mathbb{R}^s : \quad (\sum_{\mathcal{I}} r_i, y^c) \in Y = \sum_{\mathcal{J}} Y_j.$$

With every production contract $w = (r, y^c)$, $r = (r_i)_{i\in\mathcal{I}}$ one can associate its support:

$$\mathrm{supp}(w) = \{i \in \mathcal{I} \mid r_i \neq 0\} = S(w),$$

these are the agents who work together to realize the production program $(\sum_{i\in\mathcal{I}} r_i, y^c)$. As it follows from the definition, the main specific of production contract for the model with public goods consists in its cooperative nature. That is, unlike the classical Arrow–Debreu model where production can be individualized, it really is a project of some joint activities concerning to the production of goods $y^c = (y_1^c, y_2^c, \ldots, y_s^c)$ from resources $\bar{r} = \sum_{i\in S(w)} r_i$ obtained from the agents of $S(w)$. Notice also that this is not necessarily here that the private products are only costly factors of production contract: there can be produced not only public goods but private ones too — if the technology allows it... In other words, the vectors $r_i$ and $\bar{r} = \sum_{i\in S(w)} r_i$ can also have positive components. The break of production contract $w$ is possible by any of its members $i \in S(w)$ and it means that all mutual obligations among members of the coalition $S(w)$ are void.

As well as barter, production contracts may form a *web*, i.e. a finite set $W$ of contracts, each subset $U \subseteq W$ of which forms a set of agreements, which correspond to a *feasible* production plan:

$$(\sum_{w\in U}\sum_{i\in\mathcal{I}} r_i^w, \sum_{w\in U} y_w^c) \in \sum_{j\in\mathcal{J}} Y_j.$$

Thus, the specific property of production webs is that the break of a part of contracts have no direct effect on the implementation of other contracts and the corresponding production programs.

Production contracts of a web can be broken not only fully, but also partially — here there is a complete analogy with the case of a pure barter contracts, including the concept of domination by a coalition. Moreover, this analogy is also extended to the specific concepts of webs' stability: there are applied concepts of lower stable (the break), upper (conclusion of a new contract) and just a stable web (simultaneous break and signing of new contracts); in addition, these concepts can be applied to the union of barter (exchange) and production webs. This way *admitting only the total break* of contracts we arrive at the concept of contractual allocation that likes Definition 2.1. A simple analysis of the definitions shows that the contractual allocations of this type in an economy with public goods are exactly the allocations of Foley core, see definition above. Indeed, the fact that a coalition $S \subseteq \mathcal{I}$ dominates (blocks) the current allocation can be expressed in contractual terms as follows: this coalition breaks *all* contracts and sign new inter-coalitional contracts, barter and production, in which only members of the coalition and their technological set $Y_S = \sum_{i\in S}\sum_{j\in\mathcal{J}} \theta_i^j Y_j$ are involved. As a result of these activities, each member of the coalition should improve its position (to increase utility). Finally, the core is the set of feasible allocations implemented by a web of contracts that no coalition can improve upon. Allowing the possibility to break contracts partially one leads to a more qualified type of stability and a number of new concepts, that is reflected in the subsequent definitions.

Next, I introduce a specific notion of properly contractual allocation. Having this in mind, I first extend the relation $\simeq$, see (2.4), to the set of all productional webs. Here $\simeq$ corresponds to the partial breaking of contracts that can be represented so that instead of the web of contract there is considered its partition into two or more webs, so that exchange or production ratios are not changed, but the total volume of exchange and production flows also is saved. Simplistic, but strict enough, this means that the webs $V$, $W$ are replaced by ($\simeq$ equivalent) the webs $\alpha V \cup (1-\alpha)V$, $0 \leq \alpha \leq 1$ and $\beta W \cup (1-\beta)W$, $0 \leq \beta \leq 1$.

Let $V$ be a web of barter contracts and $W$ be a web of production ones in the model $\mathcal{E}^{pg}$. Define $z(V,W) = (x, x^c, y) \in [\mathbb{R}^l]^{\mathcal{I}} \times \mathbb{R}^s \times \mathbb{R}^{(l+s)}$ as an allocation implemented these webs, i.e. for

$$z(V,W) = ((x_i(V,W))_{i\in\mathcal{I}}, x^c(W)), y(V,W)))$$

put

$$x_i(V,W) = \omega_i + \sum_{v\in V} v_i + \sum_{w\in W} r_i^w, \quad i \in \mathcal{I}, \quad y(V,W) = (\sum_{\mathcal{I}} \sum_{w\in W} r_i^w, \sum_{w\in W} y_w^c) \quad (4.20)$$

$$x^c(W) = \omega^c + \sum_{w\in W} y_w^c. \quad (4.21)$$

In terms of equivalent webs an allocation $z \in \mathcal{A}(\mathcal{E}^{pg})$ of the model with public goods is called *properly contractual, if there exist barter $V$ and production $W$ webs such that $z = z(V,W)$ and for every $V' \simeq V$, $W' \simeq W$ allocation $z = z(V', W')$ is contractual one.* Below there is presented a narrative substantial definition.

**Definition** 4.2 *A triplet $(x, x^c, y)$, where $((x_i)_{i\in\mathcal{I}}, x^c) \in \prod_{\mathcal{I}} X_i^p \times X^c$ are the families of private and public consumption plans, and $y \in \sum_{\mathcal{J}} Y_j$ is aggregated production program is called* ***properly contractual allocation****, if there are a barter $V$ and a production $W$ webs such that:*

*(i)* $x_i = x_i(V,W) = \omega_i + \sum_{v\in V} v_i + \sum_{w\in W} r_i^w, \ \ i \in \mathcal{I},$
$x^c = x^c(W) = \omega^c + \sum_{w\in W} y_w^c \ \ \& \ \ y = y(V,W) = (\sum_{\mathcal{I}} \sum_{w\in W} r_i^w, \sum_{w\in W} y_w^c).$

*(ii) There is no coalition $S$, for which it is profitable (in a separate or simultaneous regime):*

- *($\alpha$) to partially break barter and production contracts;*
- *($\beta$) to sign new barter and production contract.*

In other words, the allocation is properly contractual, if it is realized by a pair of stable webs (barter and production) that do not lose stability with respect to any of their partial decomposition.

The main results of this section are the theorems characterizing properly contractual allocations in the value categories that allows to establish their relationship with the equilibria.

**Theorem** 4.2 *Let* $z(V,W) = ((x_i(V,W))_{i\in\mathcal{I}}, x^c(W), y(W))$ *be an allocation implemented by two webs: barter* $V$ *and production* $W$*. Let* $\mathcal{E}^{pg}$ *be a convex model with a smooth consumption sector,* $z(V,W) \in \operatorname{int}(\prod_{\mathcal{I}} X_i^p \times X^c) \times \sum_J Y_j$ *and each individual is* ***non-satiated*** *by private and public goods.*

*Then, if* $z(V,W)$ *is implemented as a properly contractual allocation, then there exist nonzero vectors* $p \in \mathbb{R}^l$*,* $q_i \in \mathbb{R}^s$*,* $i \in \mathcal{I}$ *such that*

$$\langle (p, q_i), \mathcal{P}_i(x_i, x^c)\rangle \ > \langle (p, q_i), (x_i, x^c)\rangle, \ \forall\, i \in \mathcal{I}, \tag{4.22}$$

$$pv_i = 0, \ \forall\, v \in V, \quad \forall\, i \in \mathcal{I}, \tag{4.23}$$

$$pr_i^w + q_i y_w^c \geq 0, \quad \forall\, w \in W, \ \forall\, i \in \mathcal{I}, \tag{4.24}$$

$$\langle (p, \sum_{i\in\mathcal{I}} q_i), y(W)\rangle \geq \langle (p, \sum_{i\in\mathcal{I}} q_i), \sum_{j\in\mathcal{J}} Y_j\rangle. \tag{4.25}$$

The peculiar circulation of Theorem 4.2 result gives the following

**Theorem** 4.3 *In Theorem 4.2 conditions additionally assume* (**P**) *and* (**M**).[14] *Then every contractual allocation satisfying (4.22)–(4.25) is properly contractual one.*

Equalities (4.23) mean that the cost of consumed private goods bundle $x_i$ is equal to the cost of initial endowments plus the cost of private goods flow directed (derived from) in the production sector. While (4.22) means that if the consumption of public goods is valuated through individual prices, then there is no other strictly preferred bundle that is cheaper of this. Inequality (4.24) says that every production contract is individually profitable, and (4.25) shows that the production sector of the economy as a whole operates in a maximum profitable way.

The following corollary of Theorems 4.2, 4.3 actually presents an equivalent description of Lindahl equilibrium in purely contractual categories. Note that the contractual allocation of any type, as well as properly contractual one, do not appeal to the value parameters, their stability has a cooperative nature, expressed in terms of product flows and the contractual obligations among agents. Thus our approach eliminates the main theoretical difficulty of the concept of equilibrium with public goods — the presence of individual prices in the production mechanism of public goods.

[14] Public goods are desirable, can be only produced but cannot be spent and technological set is a convex cone with vertex at zero.

**Corollary** 4.1 *Let conditions of Theorems 4.2, 4.3 be satisfied. Then a feasible allocation* $z = (x, x^c, y) \in \mathbb{Z}$, $(x, x^c) \in \mathrm{int}(\prod_{\mathcal{I}} X_i^p \times X^c)$ *and a bundle of private and individualized prices* $(p, q_1, \ldots, q_n) \in \mathbb{R}^{l+ns}$ *present* ***Lindahl equilibrium if and only if*** *then there are the webs of barter* $V$ *and production* $W$ *contracts, implementing this allocation* $((x_i(V,W))_{i\in\mathcal{I}}, x^c(W), y(W))$ *as a properly contactual one.*

**Remark** 4.3 Note that the only difference in the conditions of Corollary 4.1 with the assumptions of Theorem 4.1 consists in the fact that consumption sector is a smooth one.

*Proof of Corollary 4.1.* Necessity. First one needs having a Lindahl equilibrium to construct two webs of contracts: barter $V$ and production $W$, which implement the equilibrium allocation and satisfy to Definition 4.2. Having this in mind define $\bar{y} = \sum_{\mathcal{J}} y_j = (\bar{r}, \bar{y}^c)$, i.e. one specifies a vector $\bar{r}$ of total production inputs and outputs of private goods as $\bar{r} = \sum_{\mathcal{J}} y_j^p$. Consider a production web $W$, consisting of a single contract $w = (r_1, \ldots, r_n, \bar{y}^c)$, where by definition put $r_i = x_i - \omega_i$. Then as soon as $z$ is balanced one has $\sum_{\mathcal{I}} r_i = \sum_{\mathcal{J}} y_j^p = \bar{r}$ and therefore contract $w$ implements production program $\bar{y} = \sum_{\mathcal{J}} y_j$. Furthermore, by definition $p(x_i - \omega_i - r_i) = 0$, $\forall\, i \in \mathcal{I}$. Next define $v_i = x_i - \omega_i - r_i = 0$ and form a formal web $V = \{v\}$, consisting of a single contract $v = 0$. Further one uses Theorem 4.3 and states (4.22)–(4.25). Really, (4.22) follows from (4.2) and (4.23) is true by construction. Further, let us consider (4.24). Fix $i$ and look at its budget equality (the right hand side of (4.2)):

$$\omega_i p + \omega^c q_i + \sum_{j\in\mathcal{J}} \theta_i^j (p y_j^p + \bar{q} y_j^c) = x_i p + x^c q_i.$$

Now substitute the expressions of consumption bundles for private and public goods $x_i(V,W) = \omega_i + v_i + r_i$, $\forall\, i \in \mathcal{I}$, $x^c = \omega^c + \bar{y}^c$, and find:

$$\omega_i p + \omega^c q_i + \sum_{j\in\mathcal{J}} \theta_i^j (p y_j^p + \bar{q} y_j^c) = (\omega_i + v_i + r_i)p + (\omega^c + \bar{y}^c) q_i.$$

Canceling common terms and in view of $pv_i = 0$ find

$$0 \le \sum_{j\in\mathcal{J}} \theta_i^j (p y_j^p + \bar{q} y_j^c) = p r_i + q_i \bar{y}^c \quad \Rightarrow \quad q_i \bar{y}^c + p r_i \ge 0,$$

that proves (4.24). Condition (4.25) immediately follows from the construction of webs and (4.1). Thus, by construction and Lindahl equilibrium definition (4.22)–(4.25) are satisfied and by Theorem 4.3 the allocation is properly contractual one.

Sufficiency. In the corollary conditions Theorem 4.2 takes place, and hence (4.22)–(4.25) are fulfilled. Since the production sets are cones, then the total profit is zero and by (4.25) the inequalities (4.24) can be performed only in the form of equalities. This provides budget equalities in the right side of (4.2). Other requirements of equilibrium definition are also obviously follows from (4.22)–(4.25). ■

*Proof of Theorem 4.2.* The fact that $z(V, W)$ is a properly contractual allocation implies that allocation $((x_i(V, W))_{i\in\mathcal{I}}, x^c(W), (y^p(W), y^c(W))) = z(V, W)$, where

$$x_i(V, W) = \omega_i + \sum_{v\in V} v_i + \sum_{w\in W} r_i^w, \quad i \in \mathcal{I},$$

$$x^c(W) = \omega^c + \sum_{w\in W} y_w^c$$

and

$$(y^p(W), y^c(W)) = (\sum_{w\in W} \sum_{i\in\mathcal{I}} r_i^w, \sum_{w\in W} y_w^c) \in \sum_{j\in\mathcal{J}} Y_j$$

is feasible and Pareto optimal. Hence, by Lemma 4.1, there exist nonzero vectors $p \in \mathbb{R}^l$, $q_i \in \mathbb{R}^s$, $i \in \mathcal{I}$, satisfying to (4.6), where all inequalities are fulfilled in a strict form and therefore (4.22) is true, and

$$\langle (p, \sum_{i\in\mathcal{I}} q_i), (y^p(W), y^c(W))\rangle \geq \langle (p, \sum_{i\in\mathcal{I}} q_i), \sum_{j\in\mathcal{J}} Y_j\rangle.$$

Next one uses these price vectors and proves (4.23)–(4.25). The requirement (4.25) is true by construction. Let us prove (4.23). It is sufficient to show by definition, that for each individual and every contract $v \in V$ one has $pv_i \geq 0$, $i \in \mathcal{I}$. Assuming contrary one finds contract $v$ and an individual $i$ such that $pv_i < 0$. Due to Remark 4.1, it follows that $\langle \nabla u_i(x_i(V, W), x^c(W)), (-v_i, 0)\rangle > 0$, i.e. the derivative of the utility in the direction $-(v_i, 0)$ is *positive*. Hence, this individual would be advantageous partial break of contract $v$ in a possibly small volume $\alpha > 0$, because in this case locally his/her change of utility is calculated as $\alpha\langle \nabla u_i(x_i(V, W), x^c(W)), (-v_i, 0)\rangle > 0$. This contradicts to the definition of properly contractual allocation.

To prove (4.24), let us consider the value of consumption bundle of the individual $i$ at prices $(p, q_i)$ after a partial break of the production contract $w' \in W$, $i \in \operatorname{supp}(w)$ in a volume $\alpha > 0$. It is easy to see that after the break one has:

$$\langle (p, q_i), (x_i(V, W), x^c(W))\rangle - \alpha\langle (p, q_i), (r_i^{w'}, y_{w'}^c)\rangle.$$

In other words, the value is changed by $-\alpha\langle (p, q_i), (r_i^{w'}, y_{w'}^c)\rangle$, which for $pr_i^{w'} + q_i y_{w'}^c < 0$ is positive. Applying Remark 4.1 to this case, one obtains $\langle \nabla u_i(x_i(V, W), x^c(W)), (-r_i^{w'}, -y_{w'}^c)\rangle > 0$, i.e., derivative of the utility of the individual $i$ along the direction corresponding to the break of the contract $w'$ is strictly greater than zero. Therefore it is beneficial for the individual to break this contract at least in a small volume. Hence, the assumption $pr_i^{w'} + q_i y_{w'}^c < 0$ leads us to a contradiction with the definition of a properly contractual allocation. ■

*Proof of Theorem 4.3.* Let the conditions (4.22)–(4.25) be fulfilled. Arguing by contradiction it is easy to see that (4.22), (4.25) together imply Pareto optimality of allocation implemented by a web of contracts (one comes to a contradiction by comparing the aggregate cost balance of current and dominating allocation).

Without loss of generality, assume that there is only one barter $v$ and one production contract $w$. Suppose there is found the coalition $S \subseteq \mathcal{I}$, interested in a partial break of existing contracts in the amount $1 \geq 1 - \alpha \geq 0$, $1 \geq 1 - \beta \geq 0$ and in the conclusion of new contracts $\tilde{v}$ and $\tilde{w}$. This way obtained consumption bundles should be preferred for the agents $i \in S$. These bundles are

$$\tilde{x}_i = \omega_i + \alpha v_i + \beta r_i + \tilde{v}_i + \tilde{r}_i, \ i \in S, \quad \& \ \ \tilde{x}^c = \omega^c + \beta y^c + \tilde{y}^c,$$

$$x_i = \omega_i + v_i + r_i, \ i \in S, \quad \& \ \ x^c = \omega^c + y^c.$$

Estimating the bundles by prices $(p, q_i)$ and applying (4.22), (4.23), one finds

$$p(\beta r_i + \tilde{v}_i + \tilde{r}_i) + q_i(\beta y^c + \tilde{y}^c) > pr_i + q_i y^c.$$

Summing up these inequalities over $i \in S$, via $\sum_S \tilde{v}_i = 0$ one concludes

$$\beta p \sum_S r_i + p \sum_S \tilde{r}_i + (\sum_S q_i)(\beta y^c + \tilde{y}^c) > p \sum_S r_i + (\sum_S q_i) y^c \ \Rightarrow$$

$$p \sum_S \tilde{r}_i + (\sum_S q_i)\tilde{y}^c > (1 - \beta)[(\sum_S q_i) y^c + p \sum_S r_i]. \tag{4.26}$$

On the other hand, the updated production program must be technologically acceptable, i.e., it has to be

$$(\sum_S \tilde{r}_i + \beta \sum_{\mathcal{I}} r_i, \tilde{y}^c + \beta y^c) \in \sum_{\mathcal{J}} Y_j,$$

that in view of (4.25) yields

$$(\sum_{\mathcal{I}} q_i)(\tilde{y}^c + \beta y^c) + p(\sum_S \tilde{r}_i + \beta \sum_{\mathcal{I}} r_i) \leq (\sum_{\mathcal{I}} q_i) y^c + p \sum_{\mathcal{I}} r_i \ \Rightarrow$$

$$(\sum_{\mathcal{I}} q_i)\tilde{y}^c + p \sum_S \tilde{r}_i \leq (1 - \beta)[(\sum_{\mathcal{I}} q_i) y^c + p \sum_{\mathcal{I}} r_i].$$

Further note that because of the assumption of the theorem all public goods are desirable for all agents, i.e., preferences are monotone in this commodity group.

Hence, by (4.22) it is easy to conclude that $q_i \geq 0$, $\forall\, i \in \mathcal{I}$. In addition, it was assumed in the theorem that public goods can be produced but not expended, i.e., $\tilde{y}^c \geq 0$, which together with the previous gives $\sum_{\mathcal{I}\setminus S} q_i\tilde{y}^c \geq 0$. Now applying (4.24) (summing the inequality over $i \in S$) one concludes that the right hand side of (4.26) is nonnegative, and finally one concludes

$$0 \leq (1-\beta)[(\sum_S q_i)y^c + p\sum_S r_i] < \sum_{\mathcal{I}\setminus S} q_i\tilde{y}^c + p\sum_S \tilde{r}_i + (\sum_S q_i)\tilde{y}^c \leq$$

$$\leq (1-\beta)[(\sum_{\mathcal{I}} q_i)y^c + p\sum_{\mathcal{I}} r_i].$$

Further, if $\beta = 1$ then one immediately has a contradiction, because a certain quantity is strictly greater than zero and simultaneously smaller one. For $\beta \neq 1$ one concludes

$$(\sum_{\mathcal{I}} q_i)y^c + p\sum_{\mathcal{I}} r_i > 0,$$

that contradicts to (4.25) and the theorem assumption on production sets which are the cones with the vertex at zero: in this case firms' profits (a value in the left hand side of inequality (4.25)) has to be zero. ■

## 4.4 Fuzzy contractual allocations

Contractual approach in general and especially the methodology and concepts related to the partial breaking of contracts in the previously studied models of economics is a specific way to model the conditions of perfect competition. Being much simpler well-known in the literature "classical" methods (nonatomic space of economic agents by Aumann, or a replica and Edgeworth equilibrium by Debreu–Scarf and Aliprantis, etc.), contractual method demonstrates high efficiency, leads to the same conclusions as in the previously studied situations, and to new ones in not well studied. Further I introduce the concept of fuzzy contractual allocation, which is really can be considered an alternative model of perfect competition. Consider first a meaningful scenario.

Imagine that at some intermediate moment of economic interaction, individuals intend to improve the structure of their contracts, partially breaking the old and entering into new contracts. Nobody controls their contractual activities and there is no coordinating body. Therefore, the break of contracts may take place asynchronously and in the planning stage in the representations of the individual may occur unrealizable asymmetrical agreements which are not contracts at all. However, at the stage of a new contract seeking agents can rely on the resources made through such bogus contractual options. This can motivate to sign new contracts and really to break old ones, but now contractual system as a whole breaks down, as the break always occurs at the highest possible option, because of all contracts are concluded on a voluntary basis, and they are voluntarily prolonged... The above situation may occur when the current allocation is not fuzzy contractual in the sense described below. Fuzzy contractual allocation is resistant to such perturbations of the contractual agreements. A formalization of this scenario is presented below.

Suppose, as above, $V$ is a web of barter and $W$ a web of production contracts of the model $\mathcal{E}^{pg}$, implementing allocation $z(V,W) = (x, x^c, y)$ by formulae (4.20) and (4.21). For simplicity and without loss of generality, I assume that both webs are singletons, i.e. $V = \{v\}$ and $W = \{w\}$. Suppose an individual $i \in \mathcal{I}$ intends to partially break the barter contract in the amount of $(1 - g_i^v)$, $0 \leq g_i^w \leq 1$ and production in a volume of $(1 - t_i^w)$, $0 \leq t_i^w \leq 1$. As a result, he/she will have the following resources bundle of private and public goods:

$$\tilde{x}_i(t_i^v, g_i^w) = \omega_i + g_i^v \cdot v_i + t_i^w \cdot r_i^w \quad \& \quad \tilde{x}^c(t_i^w) = \omega^c + t_i^w \cdot y_w^c.$$

Of course, the agent can not wish to break more contracts, i.e., as a result of the partial break procedure a lower properly contractual allocation (stability relative to a partial break) is realized, see Marakulin (2002, 2003a). Further the individual intends to sign a new barter $\upsilon = (s_1, \ldots, s_n)$ and a production $\vartheta = (\vartheta_1, \ldots, \vartheta_n, \eta^c)$ contract, which all together should lead to a preferred consumption: $(\xi_i, \xi^c) \succ_i (x_i, x^c)$, where

$$\xi_i = \xi_i(t_i^v, g_i^w) = \omega_i + g_i^v \cdot v_i + t_i^w \cdot r_i^w + s_i + \vartheta_i, \quad \xi^c = \xi^c(g_i^w) = \omega^c + t_i^w \cdot y_w^c + \eta^c.$$

The situation can be further simplified if one notes that due to the definition of barter contract $\sum s_k = 0$ and hence if $\vartheta$ was a *feasible* production contract, i.e. $(\sum \vartheta_k, \eta^c) \in Y$, then contract $(\vartheta_1 + s_1, \ldots, \vartheta_n + s_n, \eta^c)$ is also feasible. Therefore, *for domination* there is no need to use two new contracts, it is sufficient to apply *only one production contract* $\vartheta$. Moreover, it will also be valid for the original allocation: if instead of two contracts $v$ and $w$ one considers the only production contract $(r_1 + v_1, r_2 + v_2, \ldots, r_n + v_n, y_w^c)$ then the stability of allocation and the web of contracts can only be strengthened: due to the fact that now the partition of contracts is carried out only in the *equal amounts*. In other words, without loss of generality, I may always assume that $\upsilon = 0$. Further a formal definition is presented.

**Definition** 4.3 *An allocation $(x, x^c, y^p, y^c) \in \mathcal{A}(\mathcal{E}^{pg})$ is called* ***fuzzy contractual*** *if a production contract $w = (r_1, r_2, \ldots, r_n, y^c)$ implementing the allocation as*

$$x_i = \omega_i + r_i, \quad i \in \mathcal{I}, \quad x^c = \omega^c + y^c, \quad y^p = \sum_{\mathcal{I}} r_i, \quad (y^p, y^c) \in Y$$

*is so that for every $t = (t_i)_{i \in \mathcal{I}}$, $0 \leq t_i \leq 1$, $\forall\, i \in \mathcal{I}$ there is no another production contract $\vartheta = (\vartheta_1, \ldots, \vartheta_n, \eta^c)$, $(\sum \vartheta_k, \eta^c) \in Y$ such that for*

$$\xi_i^p = \xi_i^p(t, w, \vartheta) = \omega_i + t_i r_i + \vartheta_i, \quad \xi_i^c = \xi_i^c(t, w, \vartheta) = \omega^c + t_i y^c + \eta^c$$

$$(\xi_i^p, \xi_i^c) \succ_i (x_i, x^c) \quad \forall\, i \in \mathrm{supp}\,(\vartheta) \quad \textit{or if} \;\; t_i < 1. \tag{4.27}$$

*take place.*

Commenting the definition, I would like to note that applying (4.27) when $\vartheta = 0$ one can conclude that no individual may be interested only in a partial break of contracts, i.e. in accordance with the terminology of Marakulin (2003a) production contract could be called proper.[15]

The following lemma characterizes fuzzy contractual allocation in "geometrical" categories.

**Lemma** 4.3 *An allocation* $\bar{z} = (\bar{x}, \bar{x}^c, \bar{y}^p, \bar{y}^c) \in \mathcal{A}(\mathcal{E}^{pg})$ *is fuzzy contractual if and only if when it is lower stable relative to the partial breaking of contracts and*

$$\mathcal{L}^{pg} \bigcap \prod_{i \in \mathcal{I}} [(\mathcal{P}_i(x_i, x^c) + \mathrm{co}\{0, (\omega_i - x_i, \omega^c - x^c)\}) \cup \{(\omega_i, \omega^c)\}] \times Y = \{\tilde{\omega}\}. \tag{4.28}$$

*Here as above*

$$\tilde{\omega} = ((\omega_1, \omega^c), (\omega_2, \omega^c), \ldots, (\omega_n, \omega^c), 0) \in \mathbb{R}^{(n+1)(l+s)},$$

*and* $\mathcal{L}^{pg}$ *is the subspace corresponding to the balance constraints of economy with public goods, see (4.17):*

$$\mathcal{L}^{pg} = \{((z_1^p, z_1^c), (z_2^p, z_2^c), \ldots, (z_n^p, z_n^c), (y^p, y^c)) \mid$$
$$z_1^p + z_2^p + \cdots + z_n^p = y^p + \sum_{\mathcal{I}} \omega_i, \quad z_1^c = z_2^c = \cdots = z_n^c = y^c + \omega^c\}.$$

Now, since by Definition (4.15) one has

$$\Omega_i(x_i, x^c) \subset (\mathcal{P}_i(x_i, x^c) + \mathrm{co}\{0, (\omega_i - x_i, \omega^c - x^c)\}) \cup \{(\omega_i, \omega^c)\}, \quad i \in \mathcal{I}$$

and then due to Lemma 4.2 concludes

**Corollary** 4.2 *Every fuzzy contractual allocation belongs to the fuzzy core.*

**Remark** 4.4 It is a purely mathematical to note that in general (4.28) itself implies that an allocation is lower stable relative to the partial breaking of contracts. It is so (due to corollary) when all elements of fuzzy core are equilibria. So, in general (4.28) is alone sufficient for an allocation to be fuzzy contractual.

[15] In Marakulin (2003a) only pure exchange economies were studied.

*Proof of Lemma 4.3.* Let $\bar{z}$ be fuzzy contractual allocation according to Definition 4.3. Assume (4.28) is false and therefore there is $z = ((z_i^p, z_i^c)_{\mathcal{I}}, y^p, y^c) \neq \tilde{\omega}$, belonging to the left part of (4.28). Consider a coalition $S = \{i \in \mathcal{I} \mid z_i \neq (\omega_i, \omega^c)\}$. Notice that $\mathcal{P}_i(\bar{x}_i, \bar{x}^c) \neq \emptyset$, $i \in S$ and find $(\xi_i^p, \xi_i^c) \in \mathcal{P}_i(\bar{x}_i, \bar{x}^c)$, $i \in S$ such that $z_i = \xi_i + t_i[(\omega_i, \omega^c) - (\bar{x}_i, \bar{x}^c)]$ for some $0 \leq t_i \leq 1$, $i \in S$ and $z_i = (\omega_i, \omega^c)$, $i \notin S$. Define $\vartheta_i = z_i^p - \omega_i$, $i \in \mathcal{I}$. Since $\sum_{i\in\mathcal{I}} z_i^p = \sum_{i\in\mathcal{I}} \omega_i + y^p$, then $\sum_{\mathcal{I}} \vartheta_i = y^p$ and, therefore, if $\eta^c = y^c$ vector $\vartheta = (\vartheta_1, \ldots, \vartheta_n, \eta^c)$ presents a feasible production contract with $\operatorname{supp}(\vartheta) = S \neq \emptyset$. Now one can write

$$\xi_i^p = z_i^p - \omega_i + t_i(\bar{x}_i - \omega_i) + \omega_i = \vartheta_i + t_i r_i + \omega_i, \quad i \in S.$$

For the public sector now one has

$$z_i^c = \xi_i^c + t_i(\omega^c - \bar{x}^c) = y^c + \omega^c \quad \Rightarrow \quad \xi_i^c = y^c + \omega^c + t_i\bar{y}^c = \eta^c + \omega^c + t_i\bar{y}^c.$$

Thus, in accordance with the Definition 4.3 one finds a vector $t = (t_1, \ldots, t_n)$ and contract $\vartheta = (\vartheta_1, \ldots, \vartheta_n, \eta^c)$, $\operatorname{supp}(\vartheta) \neq \emptyset$, satisfying (4.27). One comes to contradiction.

Show that if a stable relative to the partial breaking contractual allocation $\bar{z} = (\bar{x}, \bar{x}^c, \bar{y}^p, \bar{y}^c) \in \mathcal{A}(\mathcal{E}^{pg})$ obeys (4.28), then it is fuzzy contractual with respect to the production web $V = \{(r_1, r_2, \ldots, r_n, \bar{y}^c)\}$, where $r_i = \bar{x}_i - \omega_i$, $i \in \mathcal{I}$. Assume contrary and find $t = (t_1, \ldots, t_n)$ and contract $\vartheta = (\vartheta_1, \ldots, \vartheta_n, \eta^c)$, $(y^p, y^c) = (\sum_{\mathcal{I}} \vartheta_i, \eta^c) \in Y$, $\operatorname{supp}(\vartheta) \neq \emptyset$ such that

$$\xi_i^p = \omega_i + t_i r_i + \vartheta_i, \quad \xi_i^c = \omega^c + t_i\bar{y}^c + \eta^c, \quad (\xi_i^p, \xi_i^c) \in \mathcal{P}_i(x_i, x^c), \quad \forall\, i \in \operatorname{supp}(\vartheta) \iff$$

$$z_i = (\omega_i, \omega^c) + (\vartheta_i, \eta^c) \in \mathcal{P}_i(\bar{x}_i, \bar{x}^c) + t_i((\omega_i, \omega^c) - (\bar{x}_i, \bar{x}^c)), \quad \forall\, i \in \operatorname{supp}(\vartheta).$$

Take $z_i = (\omega_i, \omega^c)$ for $i \notin \operatorname{supp}(\vartheta)$. Now via contract's definition one can conclude $\sum_{i\in\mathcal{I}} z_i^p = \sum_{i\in\mathcal{I}} \omega_i + y^p$, $z_i^c = \omega^c + \eta^c$. Thus, there is found allocation $z = ((z_i)_{\mathcal{I}}, (y^p, y^c)) \neq \tilde{\omega}$, belonging to the left side of (4.28), it is a contradiction. ■

The central result of the paper is the following theorem on the equivalence of Lindahl equilibrium and fuzzy contractual allocation.

**Theorem** 4.4 *Let $\mathcal{E}^{pg}$ satisfy* (**P**), (**M**), $(x_i, x^c) \in \operatorname{int}(X_i \times X^c)$, $i \in \mathcal{I}$ *and (4.5) be true (separate non-satiation). Then allocation $z = ((x_i)_{\mathcal{I}}, x^c, y^p, y^c) \in \mathcal{A}(\mathcal{E}^{pg})$ is fuzzy contractual if and only if when it is a Lindahl equilibrium allocation.*

*Proof of Theorem 4.4.* The proof of necessity follows immediately from the fact that the conditions of this theorem is identical to the conditions of Theorem 4.1 on the coincidence of equilibria with the elements of fuzzy core: therefore, being an element of the fuzzy core, the fuzzy contractual allocation is an equilibrium one. To prove sufficiency, consider as in Corollary 4.1, a production web $W$ consisting of the only contract $w = (r_1, \dots, r_n, y^c)$, that is defined by $r_i = x_i - \omega_i$ and let a barter contract be zero. Further proof repeats the arguments of Proposition 4.1, that I omit. ■

The following final statement of the section fully reveals the relationship between the elements of the fuzzy core and fuzzy contractual allocations.

**Lemma** 4.4 *Let $z = (x, x^c, y) \in \mathcal{A}(\mathcal{E}^{pg})$ and $\mathcal{P}_i(x_i, x^c) \neq \emptyset$ for all $i \in \mathcal{I}$. Then $(x, x^c, y) \in \mathcal{C}^f(\mathcal{E}^{pg})$ implies:*

$$\mathcal{L}^{pg} \bigcap \prod_{i \in \mathcal{I}} (\mathcal{P}_i(x_i, x^c) + \operatorname{co}\{0, (\omega_i - x_i, \omega^c - x^c)\}) \times Y = \emptyset, \tag{4.29}$$

*Here, as before $\mathcal{L}^{pg}$ is a subspace corresponding to the balance constraints of economy with public goods, see (4.17).*

Comparison of formulas (4.28) and (4.29) clarifies the difference between the fuzzy core allocations and fuzzy contractual ones. It is evident that this difference is not too large, that allows us to interpret the allocations of the fuzzy core as fuzzy contractual. Moreover, now the fact that every element of the fuzzy core is a quasi-equilibrium (it is the main reason why fuzzy core is so popular in existence theory) can be easily deduced from the formula (4.29).

*Proof of Lemma 4.4.* The proof is based on the Lemma 4.2 and relation (4.18), characterizing elements of the fuzzy core. One needs to show that (4.18) implies (4.29).

Assume that $z = (\bar{x}, \bar{x}^c, \bar{y}^p, \bar{y}^c)$ satisfies (4.18), but (4.29) is false. Then there is a vector $t = (t_1, \dots, t_n)$, $0 \leq t_i \leq 1$, production plan $(y^p, y^c) \in Y$ and bundles $\zeta_i = (\zeta_i^p, \zeta_i^c)$, $\zeta_i^p = \xi_i^p + t_i(\omega_i - \bar{x}_i)$, $\zeta_i^c = \xi_i^c + t_i(\omega^c - \bar{x}^c)$ satisfying $\xi_i = (\xi_i^p, \xi_i^c) \succ_i (\bar{x}_i, \bar{x}^c)$, $i \in \mathcal{I}$ and such that

$$\sum_{\mathcal{I}} \xi_i^p + \sum_{\mathcal{I}} t_i(\omega_i - \bar{x}_i) = \sum_{\mathcal{I}} \omega_i + y^p \quad \& \quad \xi_i^c + t_i(\omega^c - \bar{x}^c) = \omega^c + y^c. \tag{4.30}$$

Define $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n, y^p, y^c)$. Further for a real $0 < \beta \leq \frac{1}{2}$ consider a vector $\beta\zeta + (1-\beta)z = \rho(\beta) = \rho$, where for $i \in \mathcal{I}$ one has by construction

$$\rho_i^p(\beta) = \beta[\xi_i^p + t_i(\omega_i - \bar{x}_i)] + (1-\beta)\bar{x}_i, \quad \& \quad \rho_i^c(\beta) = \beta[\xi_i^c + t_i(\omega^c - \bar{x}^c)] + (1-\beta)\bar{x}^c.$$

Due to (4.30) and $z \in \mathcal{A}(\mathcal{E}^{pg})$ one has $\sum_{\mathcal{I}} \rho_i^p(\beta) = \sum_{\mathcal{I}} \omega_i + \beta y^p + (1-\beta)\bar{y}^p$ and $\rho_i^c(\beta) = \omega^c + \beta y^c + (1-\beta)\bar{y}^c$ for every $\beta$. In addition, $(\beta y^p + (1-\beta)\bar{y}^p, \beta y^c + (1-\beta)\bar{y}^c) \in Y$, i.e. a feasible production program corresponds to the vector $\rho(\beta)$. Further let us present vectors $\rho_i(\beta)$ in the form

$$\rho_i(\beta) = (1-\beta t_i)(\bar{x}_i, \bar{x}^c) + \beta t_i(\omega_i, \omega^c) + (1-\beta t_i)\frac{\beta}{1-\beta t_i}[(\xi_i^p, \xi_i^c) - (\bar{x}_i, \bar{x}^c)], \quad i \in \mathcal{I},$$

where by $\beta$ choice one has $\mu_i = \frac{\beta}{1-\beta t_i} \leq 1$. For $i \in \mathcal{I}$ the last due to (**A**) entails

$$\mu_i(\xi_i - (\bar{x}_i, \bar{x}^c)) \in \mathcal{P}_i(\bar{x}_i, \bar{x}^c) - (\bar{x}_i, \bar{x}^c) \;\Rightarrow$$

$$\exists\, \eta_i \in \mathcal{P}_i(\bar{x}_i, \bar{x}^c) :\; \mu_i(\xi_i - (\bar{x}_i, \bar{x}^c)) = \eta_i - (\bar{x}_i, \bar{x}^c).$$

Hence, from the previous formula one concludes

$$\rho_i = (1 - \beta t_i)\eta_i + \beta t_i(\omega_i, \omega^c),$$

that implies $\rho_i \in \Omega_i(\bar{x}_i, \bar{x}^c)$, $i \in \mathcal{I}$. Now one can apply (4.18) and conclude $\rho = \rho(\beta) = \tilde{\omega}$ for *all* real $0 < \beta \leq \frac{1}{2}$. Now write this equation by components and by definition of $\rho_i(\beta)$ find

$$\beta[\xi_i + t_i((\omega_i, \omega^c) - (\bar{x}_i, \bar{x}^c))] + (1 - \beta)(\bar{x}_i, \bar{x}^c) = (\omega_i, \omega^c) \quad \Rightarrow$$

$$\xi_i + t_i((\omega_i, \omega^c) - (\bar{x}_i, \bar{x}^c)) = (\bar{x}_i, \bar{x}^c) + \frac{(\omega_i, \omega^c) - (\bar{x}_i, \bar{x}^c)}{\beta},$$

that has to be true for all $i \in \mathcal{I}$ and *all* $0 < \beta \leq \frac{1}{2}$. However, these equalities hold for *different* $\beta$, that is possible only if $(\bar{x}_i, \bar{x}^c) = (\omega_i, \omega^c) = \xi_i$, $i \in \mathcal{I}$ that by the choice of $\xi_i$ implies $(\bar{x}_i, \bar{x}^c) \succ_i (\bar{x}_i, \bar{x}^c)$. This contradicts to (**A**) (preferences are irreflexive). Lemma 4.4 is proved. ■

# 5 Conclusion

In the paper there was proposed and analyzed contractual approach in the context of an economic model of Arrow–Debreu kind with the production sector. There were studied economies with convex and nonconvex production sets, as well as with public goods. The research has shown high potential of contractual approach, which presents the contractual description of a variate of known theoretical equilibrium concepts, they are: Walrasian equilibrium in a convex model, equilibrium with marginal cost pricing in a nonconvex setting (MCP–equilibrium) and, finally, Lindahl equilibrium in a model with public goods. At the same time there were proposed a number of new contractual concepts: specific variants of properly contractual, and also marginally contractual allocation. These concepts characterize equilibria in a cooperative terms and without value categories, and this is the main advantage of a contractual approach. The main results are the following:

1) Theorem 2.1 on the equivalence of properly contractual allocation and competitive equilibrium in a model of Arrow–Debreu with convex production;

2) Theorem 3.1 on the equivalence of MCP–equilibrium and marginally contractual allocation;

3) Theorems 4.1, 4.4 and Corollary 4.1 on the equivalence of Lindahl equilibria and fuzzy core and/or specific properly and fuzzy contractual allocations.

Thus, the theorem of 2) presents the theoretical foundations to justify the concept of MCP–equilibrium, which content still is not completely clear — increasing the credibility of the concept. The theorems of item 3) characterize Lindahl equilibrium do not involving the individual prices apparatus which is difficult to implement in practice; here, using an appropriate concept of contract there are embodied purely cooperative properties of the economic model and it does not appeal such notions as congested public goods and crowing in its provision. All this contributes to the further development of the contractual approach as a universal tool for constructing and analyzing the models of economic processes.

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