Contracts and domination in competitive economies*

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Abstract

The goal of the paper is to propose and study a concept of contract-based domination by coalitions for competitive economies. The formal mathematical notion of contract (an elementary exchange of commodities), the contractual allocations of different kinds and their stability properties are introduced and investigated. It is shown that applying this approach one can describe different known classical concepts for a perfect economy — equilibria, core, fuzzy core etc. — in pure game-theoretical terms. For non-perfect economies, in which not every contract is permissible, it may serve as one of the model's primitives to refine and to solve various theoretical problems.

Keywords and Phrases: exchange economy, contract, contractual allocation, competitive equilibrium, core.

JEL Classification Numbers: C 62, D 51

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Introduction

There are a lot of real economy features which are not taken into account by the classical theory which, starting mainly from the early 80's, motivated constructing and investigating of *non-perfect* market models. In modern economic theory, one can see a number of *non-perfect* market models including incomplete (financial) markets (the trade with specific financial tools, so-called assets, is incorporated into the model), markets with informational asymmetry (about future events, etc.), sequential markets (time factor and trust), and so on.[1] In our opinion, the diversity of models and the difficulties in their analysis are caused, on one hand, by the complexity of the object (economy) and, on the other hand, by the absence of sufficiently universal tools for the model investigation. The latter resulted in a variety of solution concepts primarily related to the notion of domination (via coalition) and therefore to the concept of the core. The reason for this is that, following the classical tradition, the main attention is paid to the analysis of the final resource allocation. The commonly missed fact is that in real economy this allocation is a result of many exchange dealings among economic agents (coalitions). It is important that not every exchange is permissible in real economy — there are many reasons for this, institutional, physical, informational, ethical, behavioral, etc. We believe that the focus of the theory should be shifted in order to be concentrated directly on the exchange bargains of commodities, contracts, which should be included in the model as primitives and form (together with the other model elements) the basis for theoretical constructions, instead of allocations. Thus we propose a *contract-based approach* as a tool for economic modelling. We hope that this approach, as a supplement to the classical one, can help to clarify many problems of economic theory arising in the analysis of non-perfect markets, it may also help in understanding of perfect competition conditions delivering another view on this subject and we suggest this approach to be a corner stone of domination and core for the economic models of different types. Moreover this approach may also be fruitful for the analysis of market processes even in classical frameworks, for example, to understand better the tâtonnement process, to avoid the idea of auctioneer, who rules prices while they are not equilibrium ones (see, e.g., Arrow, Hahn (1991)).

The first attempts to introduce the formalized notion of contract in exchange economies were made by Polterovich (1970), Makarov (1980, 1982) and Kozyrev (1981, 1982). In Polterovich (1970) some kinds of contractual processes were studied, in Makarov (1980, 1982) general ideas of contractual approach as a kind of new language were suggested. Kozyrev (1981, 1982) first suggested partially broken contracts that implied the study of appropriate forms of stability and some preliminary positive results were obtained. My paper presents partially reconsidered and extended results of the first part of Marakulin (2003) where the problem of correct core for incomplete markets was studied.

In the framework of an ordinary pure exchange model, every contract is simply an elementary, possible and permissible exchange of commodities among consumers. Contracts may be added to one another and with every (finite) set of contracts, an allocation of resources can be associated — as a result of the summation of contracts and the initial endowments allocation. It is presumed that every feasible set of (permissible) contracts — let us call it '*a web of contracts*' — may be changed during economic life.

[1] See our list of references for examples of non-perfect market models.

Each consumer can *break contracts* in which he/she participates, and each coalition of consumers can also *sign a new contract(s)*.

Moreover consumer can be able to partially break contracts (as signed in the past) if it is beneficial for him. The partial breaking of a contract means its replacement by a smaller volume contract with the same exchange proportions. This leads to the concept of *'properly contractual allocations'* and approaches contractual processes to market processes under perfect competition conditions. An allocation is called properly contractual if it can be realized by a web of contracts which is stable relative to the procedure of both parties *partially breaking* existing contracts and signing new contracts. At the same time, the core allocations are described in terms of *'contractual allocations'*. These are the allocations that can be realized by a web of contracts and which are stable relative to the procedure of (fully) breaking contracts and signing new contracts. Thus the only difference between these two notions of contractual allocation is that in the first case, the partial breaking of contracts is allowed, while in the second case, only the complete breaking is possible. This way, an equilibrium can be described in pure game-theoretical terms and does not address any kind of value parameters. The mathematical nature of this phenomena is quite similar to the coincidence of equilibrium allocations and the fuzzy core elements (or Edgeworth's equilibria), which is one possible way to model the conditions of perfect competition.

The paper *develops the theory of contracts* first in the framework of an abstract economy and then applies the theory to classical markets. We consider and study the formal rules of operating with the sets of contracts. The difference in these rules corresponds to the difference in the types of a web's stability and therefore in the stability of allocations realized by webs. The types of these 'stabilities,' together with the property of contracts to be permissible, reflect different behavioral, physical and institutional principles formally given in a game-theoretical form, which one can find in real life and in neoclassical economic theory. So, different types of web stabilities correspond to different types of contractual allocations, as well as their modifications, which can relax or strengthen the property for an allocation to be stable. We introduce a new formalism and several original contractual concepts: coherent, perfect, fuzzy and complex contractual allocations; some relationships among them are revealed. This way, depending on the structure of permissible contracts, one can describe notions well known in economic theory such as core, competitive equilibria, Pareto boundary and so on in terms of a stable web of contracts. The relationship between the elements of fuzzy core and fuzzy contractual allocations is important in its own right: it provides a natural interpretation for fuzzy core and can be applied for infinity dimensional models, e.g. to state the existence of equilibria (see Marakulin (2006) generalizing results from Florenzano, Marakulin (2001)). Notice that our theory of contracts does not address to bargaining theory which results may be partially incorporated into it in further investigations.[2]

The *inadmissibility* of some exchange contracts is a specific feature of many modern non-classical models and it reflects the essence of our approach. However, in this paper we consider only the first possible application of the contract-based approach, we study contractual economies *without admissibility constraints* for contracts, this corresponds to the case of classical markets. I hope the reader will see himself the potential of our approach for analyzing true non-perfect economies (e.g. incomplete markets[3] (see Magill,

[2]It seems possible to apply the bargaining theory results to correctly define a specific contactual tâtonnement process that drives economy to equilibrium.

[3]The problem of defining a true core for incomplete markets was my primary goal of the research but

Shafer (1991), Magill, Quinzii (2002)) and sequential markets (see Gale (1978), Repullo (1988)), markets with informational asymmetry (see Schwalbe (1999), Economic Theory 18 (2001), DI-economies (2004), Marakulin (2009) etc.). In non-perfect economies the permissibility constraints for contracts are usually implemented as a requirement for contract to be situated in a subspace of all possible barter exchanges. It may be the subspace of measurable functions relative to informational structure (algebra) for economies with asymmetrically informed agents or a specific subspace formed via real assets in incomplete markets (because the direct commodity exchange between different states of the world is impossible) and so on.

The paper is organized as follows. In the next section we define the model, give the main definitions, and set assumptions in general framework of abstract *contract based* (briefly — *contractual*) exchange economy. The second section is devoted to the study of the standard pure exchange economy as the first possible application of our approach.

1 The model of a contractual exchange economy

We consider a typical exchange economy in which E denotes the (finite dimensional) *space of commodities.* Let $\mathcal{I} = \{1, \ldots, n\}$ be a set of agents (traders or consumers). A consumer $i \in \mathcal{I}$ is characterized by a consumption set $X_i \subset E$, an initial endowment $\omega_i \in E$, and a preference relation described by a point-to-set mapping $\mathcal{P}_i : X_i \Rightarrow X_i$ where $\mathcal{P}_i(x_i)$ denotes the set of all consumption bundles strictly preferred by the i-th agent to the bundle x_i. It is also applied the notation $y_i \succ_i x_i$ which is equivalent to $y_i \in \mathcal{P}_i(x_i)$. So, the pure exchange model may be represented as a triplet

$$\mathcal{E} = \langle \mathcal{I}, E, (X_i, \mathcal{P}_i, \omega_i)_{i\in\mathcal{I}} \rangle.$$

Let us denote by $\omega = (\omega_i)_{i\in\mathcal{I}}$ the vector of initial endowments of all traders of the economy. Denote $X = \prod_{i\in\mathcal{I}} X_i$ and let

$$\mathcal{A}(X) = \{\, x \in X \mid \sum_{i\in\mathcal{I}} x_i = \sum_{i\in\mathcal{I}} \omega_i \,\}$$

be the set of all *feasible allocations.*

Everywhere below we assume that the model $\mathcal{E}$ satisfies the following assumption.

(**A**) *For each $i \in \mathcal{I}$, X_i is a convex solid closed set, $\omega_i \in X_i$ and for every $x_i \in X_i$ there exists an open convex $G_i \subset E$ such that $\mathcal{P}_i(x_i) = G_i \cap X_i$ and if $\mathcal{P}_i(x_i) \neq \emptyset$ then $x_i \in \overline{\mathcal{P}}_i(x_i) \setminus \mathcal{P}_i(x_i)$*[4].

Notice that due to (**A**) preferences may be satiated, i.e., $\mathcal{P}_i(x_i) = \emptyset$ is possible for some agent i and $x_i \in X_i$. However if $\mathcal{P}_i(x_i) \neq \emptyset$, then preference is *local non-satiated* at the point x_i.

Let $L = E^{\mathcal{I}}$ denote the space of allocations of the economy $\mathcal{E}$. In the framework of model $\mathcal{E}$, we are going to introduce and study a formal mechanism of contracting and recontracting. This mechanism reflects the idea that any group of agents can find and realize some (permissible) within-the-group exchanges of commodities, referred to as contracts. The mechanism defines rules of contracting.

the obtained results are not published yet. I did not incorporate them in this paper in view of volume constraints.

[4]The symbol $\overline{A}$ denotes the closure of A and $\setminus$ is set for the set-theoretical difference.

By the formal definition, any reallocation of commodities $v = (v_i)_{i \in \mathcal{I}} \in L$, i.e. any vector $v \in L$ satisfying $\sum v_i = 0$, is called a *contract.*

Not every kind of possible reallocation may be realized in the economy; there are some institutional, physical, and behavioral restrictions in the economic models of different types. This is why we equip the abstract contractual economy model with a new element, the set of *permissible* contracts $\mathcal{W} \subset L$. Thus, the contractual (exchange) economy under study may be shortly represented by the 4-tuple

$$\mathcal{E}^c = \langle \mathcal{I}, E, \mathcal{W}, (X_i, \mathcal{P}_i, \omega_i)_{i \in \mathcal{I}} \rangle.$$

In addition to (**A**), we only assume everywhere below that for a contractual economy

(**C**) *The set $\mathcal{W}$ is star-shaped at zero in L, i.e,*

$$v \in \mathcal{W} \implies \lambda v \in \mathcal{W}, \quad \forall\, 0 \le \lambda \le 1.$$

The economy $\mathcal{E}^c$ as well as the economy $\mathcal{E}$ is called *smooth* if for every $i \in \mathcal{I}$,

$$\mathcal{P}_i(x_i) = \{y \in X_i \mid u_i(y) > u_i(x_i)\}, \quad \forall\, x_i \in X_i$$

for some *differentiable* quasi-concave function u_i defined on an open neighborhood of X_i.

For a contractual economy we study the sets of contracts which represent *feasible* allocations and introduce the operation of breaking a part of a given set of contracts. This motivates the following important definition.

A finite collection V of permissible contracts is called *a web of contracts relative to* $y \in \mathcal{A}(X)$ if

$$y + \sum_{v \in U} v \in X, \quad \forall\, U \subseteq V.$$

We denote by $x_y(U)$ the feasible allocation sustained by U relative to y, i.e., we put

$$x_y(U) := y + \sum_{v \in U} v.$$

Similarly, $U_y(x)$ denotes the web which realizes x relative to y.

A web of contracts V relative to ω is called *a web of contracts* or simply *a web.* Note that $V = \emptyset$ is a web relative to every $y \in \mathcal{A}(X)$. Denoting

$$\Delta(V) = \sum_{v \in V} v,$$

where V is an arbitrary collection of contracts (by convention, we write $\Delta(\emptyset) = 0$), we can write

$$x_y(V) = y + \Delta(V), \quad x(V) = x_\omega(V) = \omega + \Delta(V)$$

so that V being a web simply means that

$$x_\omega(U) \in X, \quad \forall\, U \subseteq V.$$

Now we are going to introduce the operations of breaking existing contracts and signing new ones. For any contract $v \in V$, let us set

$$S(v) = \operatorname{supp}(v) = \{i \in \mathcal{I} \mid v_i \neq 0\},$$

the support of the contract v. It is assumed that any contract $v \in V$ may be *broken* by any trader in $S(v)$, since he/she simply may not keep his/her contractual obligations. Also a non-empty group (coalition) of consumers can *sign* any number of new contracts. Being applied jointly, i.e., as a simultaneous procedure, these operations allow coalition $T \subseteq \mathcal{I}$ to yield new webs of contracts. The set of all such webs is denoted by $F(V, T)$. Formally, we require that each element $U \in F(V, T)$ has to satisfy the following properties:

(i) $v \in V \setminus U \;\Rightarrow\; S(v) \cap T \neq \emptyset$,

(ii) $v \in U \setminus V \;\Rightarrow\; S(v) \subseteq T$,

(iii) $\sum_{v \in U \setminus V} \lambda_v v \in \mathcal{W}$ for all $0 \leq \lambda_v \leq 1,\ v \in U \setminus V$.

Condition (i) means that only members of T can break contracts in V, condition (ii) means that only members of T may sign new contracts and (iii) is a kind of joint permissibility of new contracts, which is useful in applications of contractual economy. Notice also that due to the definition of a web of contracts, a coalition can break any subset of contracts of a given web that satisfies (i).[5]

Further, for the webs of contracts it is introduced the notion of domination via a coalition. This property, being written as $U \underset{T}{\succ} V$ (U dominates V via coalition T), means that

(i) $U \in F(V, T)$,

(ii) $x_i(U) \underset{i}{\succ} x_i(V)$ for all $i \in T$.

Definition 1 *A web of contracts V is called stable if there is no web U and no coalition $T \subseteq \mathcal{I}$, $T \neq \emptyset$ such that $U \underset{T}{\succ} V$.*

An allocation x is called contractual if $x = x(V)$ for a stable web V.

The property a web of contracts be stable may be relaxed as well as strengthened. The most important possibilities are described below.

Definition 2 *A web of contracts V is called lower stable if there is no web U and no coalition $T \subseteq \mathcal{I}$, $T \neq \emptyset$ such that $U \underset{T}{\succ} V$ and $U \subset V$.*

A web of contracts V is called upper stable if there is no web U and no coalition $T \subseteq \mathcal{I}$, $T \neq \emptyset$ such that $U \underset{T}{\succ} V$ and $V \subset U$.

An upper and lower stable web of contracts V is called weakly stable.

An allocation x is called lower, upper, or weakly contractual if $x = x(V)$ for some lower, upper, or weakly stable web V, respectively.

It has to be clear that all the above notions of stability and domination may be considered as "relative to some given feasible allocation", simply use this allocation instead of the initial ω. Also it has to be clear that the notion of a weakly stable web (weakly contractual allocation) is really weaker than the corresponding notion of a stable web (contractual allocation). The difference is that in the first case the operations of breaking existing contracts and signing new contracts are applied separately, whereas in

[5] Otherwise, it would occur that an allocation realized via breaking contracts is not feasible.

the second case they are applied simultaneously. In the framework of a market economy, we consider below the relationships among the sets of contractual, lower, upper, and weakly contractual allocations. They correspond to notions well known in economic theory.

How can the process of recontracting (breaking existing contracts and signing new ones) be expressed in economic terms? We can assume that this is something like a tâtonnement process (cooperative tâtonnement), which, for example, may be as follows. To simplify the argument let us imagine that there is an ordered list of all coalitions. At the first stage (iteration), the coalitions, in the given order of appearance, start to sign and/or break contacts (transition to a web in $F(V_\xi, T_\xi)$, where ξ is the order number of coalition T_ξ). Here the first coalition "starts" from the given initial endowment allocation ω and from the web $V_1 = \emptyset$ since there were no contracts signed earlier. The stage, iterative loop is finished when the last coalition has made its choice. Next, the second stage starts where the same process is going on assuming that the first coalition in the list deals with the web of contracts realized at the end of the first stage. The fixed points of this iterative process correspond to the contractual allocations and to the stable webs of contracts. Clearly, the order of coalitions' "appearance" during a stage is not essential. Moreover, a coalition can appear several times during one stage and the order of coalitions' appearance can vary from stage to stage. What is really important is that each coalition has a chance to appear in infinitely many iterations. In general, this scenario does not impose any time restrictions on the duration of an iteration and the number of iterations is potentially unlimited (therefore the duration of a stage is infinitely small). Informally, it is just presumed that the process finishes in "a reasonable time" and the economy transits to a stationary state. These potentially possible stationary states are the subject of our analysis. Notice also that if in the iterative process, starting from some stage, one forbids the signing of new contracts for coalitions, one can realize stationary states and webs of contracts which are lower stable (in one of the sense described above or below: it depends on which kind of contract breaking is permissible, i.e., whether one can break a contract only as a whole, partially, or even with a transition to equivalent contracts). Similarly, by forbidding breaking contracts or by breaking contracts in a mixed regime (at one stage forbidding signing new contracts, at another stage forbidding breaking them, and so on), one can realize upper and weakly stable webs and contractual allocations, respectively.

Remark 1 The similar views on commodity exchange processes, sometimes called barter processes, one can find in Madden (1975), Graham, et al. (1976) and other papers. However all these papers studied other problems and did not elaborate contract based approach properly. ■

Now we continue the list of stability concepts, strengthening the stability relative to the procedure of breaking contracts. It is clear that a web which is not lower stable cannot be long-living in a market. This is why we restrict our attention below only to the lower stable webs. First let us introduce an equivalence relation on the set of all such webs. This equivalence will allow us to partially divide some contracts. To this end, we can define a partial ordering on the set of all webs as follows

$$U \geq V \iff \exists \text{ a map } \mathbf{onto} \;\; f : U \to V \text{ such that}$$

(i) $\lambda f(u) = u$ for some $0 \leq \lambda \leq 1$ and for every $u \in U$,

(ii) $\sum_{u\in f^{-1}(v)} u = v$ for every $v \in V$.

One can easily see that the set of contracts $f^{-1}(v)$ is a *partition* of contract v and so the web U consists of (finite) *partitions* of contracts from V. The minimal elements of the set of all webs may be called *root webs*. Note that for $U \geq V$ due to definition we have $\Delta(U) = \Delta(V)$. Now the equivalence relation may be defined as follows:

$$U \simeq V \iff \exists \text{ a web } W \text{ such that } V \geq W \;\&\; U \geq W.$$

Clearly, $U \simeq V$ simply means that these webs have a common root web.

Definition 3 *An allocation x is called properly contractual (resp. lower properly contractual, weakly properly contractual) if there exists a web V such that $x = x(V)$ and for every $U \simeq V$ the allocation $x = x(U)$ is contractual (resp. lower contractual, weakly contractual).*

So, for properly contractual allocations we allow the agents to partially break contracts as well as to sign new contracts (simultaneously or separately). This may be interpreted in two ways. First, speaking *in behavioral terms*, the agents are going to sign many small volume contracts instead of signing one contract of large volume. This way they gain more economic freedom through the ability to break some of the small contracts if a necessity arises. The second way is to treat a proper contract as a kind of a *preliminary agreement*. In this agreement only the rates of exchange are rigidly defined in contrast to the volume of the contract which is flexible and will be defined rigidly at the end of the contracting procedure. It should be clear that due to the last definition, the property of an allocation to be stable (in any of the senses) is essentially strengthened when we add the word "proper" to the term "contractual" allocation. Below we refine and define the term "proper" for a single contract and for a web.

Definition 4 *Let V be a web. A contract $v \in V$ is coherent if every web U such that $U \simeq \{v\}$ is lower stable relative to $(x(V) - v)$, when it is taken as the initial endowments or, equivalently for ordered preferences, if*

$$x_i(V) \succeq_i x_i(V) - \lambda v_i, \quad \forall\, 0 \leq \lambda \leq 1, \quad \forall\, i \in \mathcal{I}.$$

A subweb $U \subseteq V$ consisting of coherent contracts is called coherent.

A subweb $U \subseteq V$ is called proper if for every web $W \simeq U$ the web $(V \setminus U) \cup W$ is lower stable.

An allocation x realized by a coherent web V, i.e., $x = x(V)$, is called (lower) coherent.

Notice that the only difference between coherent and proper webs is that in the first case the web is stable relative to the partial breaking of any *single* contract, whereas in the second case the agents may partially break *any number* of contracts in the web. In general these notions are not equivalent (see Example 1). Moreover, even the notions of coherent and proper contracts are not equivalent; in the latter case (for proper contracts) breaking more than one contract in the web is also allowed. The case of a proper web of contracts is geometrically presented in Figure 1 in the coordinate system of consumer $i \in \mathcal{I}$. Figure 2 presents the geometry of stable but non-coherent and non-proper webs. The difference is that while in the first case *the whole "parallelogram of contracts"* does not *intersect* $\mathcal{P}_i(\bar{x}_i)$, in the second case it *does intersect*, but *no vertex* belongs to $\mathcal{P}_i(\bar{x}_i)$.

The next proposition fully characterizes coherent contracts for convex contractual economies.

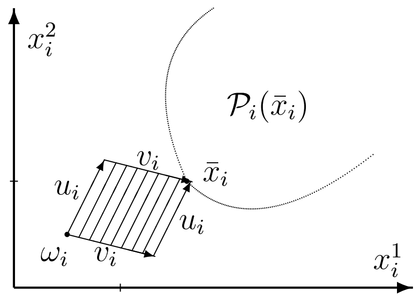

Figure 1: The web $\{u, v\}$ *is proper*

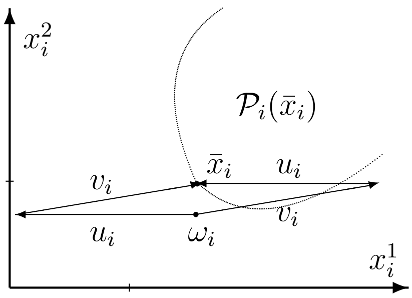

Figure 2: The web $\{u, v\}$ *is not proper*

Proposition 1 *Let V be a web. Contract $v \in V$ is coherent iff there exists linear functionals $p_i^v \neq 0$ such that*

$$\langle p_i^v, \mathcal{P}_i(x_i(V)) \rangle > \langle p_i^v, x_i(V) \rangle^6 \quad \& \quad \langle p_i^v, v_i \rangle \geq 0 \tag{1}$$

for every $i \in \mathcal{I}$. Moreover, if the utility functions are differentiable, (1) is fulfilled for $p_i^v = \operatorname{grad} u_i(x_i(V)) \neq 0$, $i \in \mathcal{I}$.

Proof of Proposition 1. Let us show that (1) is sufficient. Assume that for every $i \in \operatorname{supp}(v)$, inequalities (1) are true, but the contract v is not coherent. Then after partially breaking v, the broken part being $0 \leq \alpha_v \leq 1$, agents realize the new allocation

$$x_i^\alpha = x_i - \alpha_v v_i, \quad i \in \mathcal{I}$$

such that $x_i^\alpha \succ_i x_i$ for some $i \in \operatorname{supp}(v)$ so that the first part of (1) gives $p_i^v x_i^\alpha > p_i^v x_i$. But then using the second part of (1) for this i, we obtain $p_i^v x_i^\alpha \leq p_i^v x_i$, which contradicts the previous inequality.

To establish that (1) is necessary, assume that the contract $v \in V$ is coherent, i.e., the web V is stable relative to the partial breaking of contract v. For each consumer $i \in \mathcal{I}$, let us consider the set

$$\mathcal{U}_i(x) = \{\, x_i - \alpha_v v_i \mid 0 \leq \alpha_v \leq 1 \,\} \subset X_i.$$

By definition, the sets $\mathcal{U}_i$ are nonempty, convex and closed. Now, since the contract $v \in V$ is coherent, it follows that

$$\mathcal{P}_i(x_i) \cap \mathcal{U}_i(x) = \emptyset, \quad \forall\, i \in \mathcal{I}.$$

By assumption, there exists an open convex set $G_i \subset E$, such that in non-satiated case $G_i \cap X_i = \mathcal{P}_i(x_i) \neq \emptyset$, $x_i \in \overline{G}_i$ for the given $x \in \mathcal{A}(X)$ and i. Now for each non-satiated $i \in \mathcal{I}$ at x_i the last relations imply

$$G_i \cap \mathcal{U}_i(x) = \emptyset \;\; \& \; x_i \in \overline{G}_i.$$

[6] $\langle A, p \rangle$ denotes the set $\{\langle a, p \rangle \mid a \in A\}$ and $A > b$ ($A \geq b$) means $a > b$ ($a \geq b$) for all $a \in A$. By convention we assume $\langle p, \emptyset \rangle > b$, $\forall\, b \in \mathbb{R}$.

Therefore, by the separation theorem, for each non-satiated $i \in \mathcal{I}$ there exists $p_i^v \in E'$ such that $p_i^v \neq 0$ and

$$\langle p_i^v, G_i \rangle > \langle p_i^v, x_i \rangle \geq \langle p_i^v, \mathcal{U}_i(x) \rangle.$$

Clearly, for differentiable utility functions if $\operatorname{grad} u_i(x_i(V)) \neq 0$ then one can put $p_i^v = \operatorname{grad} u_i(x_i(V))$, $i \in \mathcal{I}$. Now the first inequality implies the first part of (1), and the second one gives

$$p_i^v x_i \geq p_i^v x_i - \alpha_v \langle v_i, p_i^v \rangle, \quad 0 \leq \alpha_v \leq 1.$$

Therefore, $\langle v_i, p_i^v \rangle \geq 0$. ■

The following corollaries characterize properly contractual allocations in terms of coherent webs.

Corollary 1 *Let $\mathcal{E}^c$ be a smooth contractual economy. Then allocation x is lower properly contractual iff there exists a coherent web V such that $x = x(V)$. Moreover, relation (1) is fulfilled for $p_i = p_i^v = \operatorname{grad} u_i(x_i(V)) \neq 0$ and for every $v \in V$ and $i \in \mathcal{I}$.*

Proof of Corollary 1. It is enough to check that every coherent web U, such that $x = x(U)$ is stable relative to the procedure of partial breaking *any number* of contracts, i.e., U is a proper web.

In fact, due to Proposition 1 for differentiable utilities, for *every* $v \in V_\omega(x)$ the functional (vector) $p_i^v = \operatorname{grad} u_i(x_i) = p_i \neq 0$ satisfies condition (1) for all $i \in \mathcal{I}$. This implies

$$G_i \cap \mathcal{M}_i(x) = \emptyset, \quad \forall\, i \in \mathcal{I},$$

where

$$\mathcal{M}_i(x) = \{\, x_i - \sum_{v \in V} \alpha_v v_i \;\mid\; 0 \leq \alpha_v \leq 1,\; v \in V \,\} \subset X_i.$$

The proof is completed. ■

In particular, the last corollary states that for smooth economies every coherent web is proper, i.e., stable relative to the partial breaking any number of contracts. Note that the assumption of differentiability of utilities cannot be dropped here; appropriate examples can be easily constructed (see Example 1, second part).

The next property of proper and coherent contracts is that under condition of saving aggregated exchange parameters they can be replaced by another proper web, keeping the lower stable property for the new web. This fact also follows from Proposition 1. Recall that $V_y(x) = V$ denotes a web realizing the allocation x relative to the initial endowments y, i.e., $x = y + \Delta(V)$.

Corollary 2 *Let $\mathcal{E}^c$ be a smooth economy and $x \in \mathcal{A}(X)$. Then any coherent web $V_\omega(x)$ has the following inheritance property: for every (coherent) contract $v \in V_\omega(x)$ and every coherent web $W_{x-v}(x)$, the new web $U = (V_\omega(x) \setminus \{v\}) \cup W_{x-v}(x)$ is lower stable relative to partially breaking (any number of) contracts and therefore is proper.*

Proof of Corollary 2. Due to Proposition 1 (necessity), for all $i \in \mathcal{I}$ there are unambiguously up to normalization defined functionals p_i satisfying (1) for any (coherent) contract $v \in V_\omega(x)$, as well as for contracts in the coherent web $W_{x-v}(x)$, since this web is stable relative to the partial breaking of contracts and $\sum_{w \in W_{x-v}(x)} w + x - v = x$. In other words, for every non-satiated i at x_i there is a $p_i \neq 0$ such that

$$\langle p_i, \mathcal{P}_i(x_i) \rangle > \langle p_i, x_i \rangle, \quad \langle p_i, v_i' \rangle \geq 0, \quad \forall\, v' \in V_\omega(x)$$

and

$$\langle p_i, w_i \rangle \geq 0, \quad \forall\, w \in W_{x-v}(x).$$

Hence, joining the second relation with the first one for contracts in U and applying Proposition 1 in the part of sufficiency yield the result via Corollary 1. ■

In applications of contractual economies one can also use contracts with a stronger stability property, so-called *perfect* contracts. To introduce this notion, let us first consider another kind of equivalence relation defined on the set of all proper webs. This (weak) equivalence relation may be define as follows: Let U and V be *proper* webs, then

$$U \sim V \iff \sum_{u \in U} u = \sum_{v \in V} v.$$

Clearly, $U \sim V$ simply means that these webs are proper and realize the same allocation. It also has to be clear that $U \simeq V$ implies $U \sim V$ for all proper webs U and V. Given a proper web V, a proper web U such that $U \sim V$ may be referred to as a *virtual* web (relative to V).

Definition 5 *An allocation x is called perfectly contractual if there exists a proper web V such that $x = x(V)$, and for every proper web U such that $U \sim V$ the allocation $x = x(U)$ is contractual.*

Perfectly contractual allocations corresponds with agents' perfectly contractual behavior which can be treated in the following way. When the agents are signing contracts, they should take care that not only these contracts would be short enough, as for the properly contractual behavior, but also would be *differently directed* (i.e., they have sufficiently many different exchange proportions) to provide the opportunity to break the "unluckily directed" ones. So agents are allowed not only to partially break contracts but also to change (in a sense) the "directions" of contracts without loss of low stable property. One can see an analogy with the hedge policy, the main difference being that here we are speaking about the exchange-of-goods based contracts (barter), by the signing of which and moving into agreements via "tacks or traverses," the agents reach a final resource reallocation.

This may also be treated in terms of an *optional agreement*. In fact, in a perfectly contractual allocation the society is protected from the possibility that some coalition initiates a new recontracting process. A coalition may hope that via recontracting, it will 'gather more profitable harvest,' i.e., find better consumption programs for its members. The following scenario may take place. A coalition, acting through its members may suggest to the non-members of the coalition, who are involved in the coalition contracting, that the contracts be rewritten so that

(i) the same allocation is realized,

(ii) nobody has incentives to *partially break* new contracts, i.e., the new web inherits the lower stable property of the initial web relative to the partial breaking of contracts.

In such a case, the non-members of the coalition may sign these new agreements as long as they have no revealed incentives to refuse from doing so (possibly the coalition members are good negotiators). Once these new agreements are signed, the coalition breaks a part of the contracts and signs a new contract that as a whole provides the coalition members with better consumption bundles. However, for a perfectly contractual allocation this hypothetical behavior of every coalition cannot be profitable.

Certainly the property of an allocation to be perfectly contractual is the strongest kind of stability. The following definition extends the notion of being *perfect* to a single contract.

Definition 6 *Let V be a web. A coherent contract $v \in V$ is perfect if every web U such that $U \sim \{v\}$ is stable relative to $(x(V) - v)$, which is taken as the initial endowments.*

A web (subweb) which contains only perfect contracts is called perfect.

Remark 2 We would also like to mention an alternative definition of a web's perfect subweb of contracts.

Let V be a web. A subweb $U \subset V$ is called perfect if it is proper and, for every web W proper relative to $(x(V) - \sum_{u \in U} u)$ and such that $W \sim U$, the web $(V \setminus U) \cup W$ is stable. A contract $v \in V$ is called perfect if the subweb $\{v\}$ is perfect.

Note that according to this definition every web containing at least one perfect contract is stable. Notice also the difference between the two following statements for $U \subset V$ which arises in this case: "U is a perfect subweb" and "U is a perfect web relative to $(x(V) - \sum_{u \in U} u)$". The first one implies the second, but in general the reverse is not true (since in the first case it is allowed to break contracts in $V \setminus U$, but in the second one it is not).

If one assumes this definition, the contracts and their webs (subwebs), perfect in the sense of Definition 6, may be renamed as perfectly coherent. Note also that due to Proposition 1 and its corollaries, for *smooth economies* both *variants of the definitions of a perfect contract and of a perfect web (subweb) are equivalent.* ■

Further we consider the concept of *fuzzy contractual* allocation. The notion of properly contractual allocation presumes that agents are able to partially break contracts in such a way that every contract may be divided in several contracts with equal exchange proportions and some of these contracts may be broken, i.e., instead of contract $v \in V$ the agents may deal with a finite family of contracts $\{u_\xi\}$, such that $\sum u_\xi = v$ and $u_\xi = \lambda_\xi v$ for some real $\lambda_\xi \geq 0$ for all ξ. Thus for partial breaking of contract v the members of coalition $S = \operatorname{supp}(v)$ have to coordinate their actions. Relaxing this coordination requirement for fuzzy contractual allocation we allow the agents to break contracts *asymmetrically* and together with $\sum u_\xi = v$ to require $(u_\xi)_i = \lambda_{\xi i} v_i$ for a real $\lambda_{\xi i} \geq 0$ for all ξ and i. Notice that now vectors u_ξ *may not be contracts* at all, since $\sum_{i \in \mathcal{I}} u_{\xi i} = 0$ may not hold.

Let V be a web of contracts. For every $v \in V$ consider and put into correspondence a n-dimension vector

$$t^v = (t_1^v, t_2^v, \ldots, t_n^v), \quad 0 \leq t_i^v \leq 1, \quad \forall\, i \in \mathcal{I},$$

and let

$$v^t = (t_1^v v_1, t_2^v v_2, \ldots, t_n^v v_n)$$

be the vector of commodity bundles formed from contract $v = (v_i)_{i \in \mathcal{I}}$ when all agents "break" individual bundles (fragments) of this contract in shares $(1 - t_i^v)_{i \in \mathcal{I}}$. Denote $T(V) = T = \{t^v \mid v \in V\}$ and introduce

$$V^T = \{v^t \mid v \in V,\ t^v \in T\}, \quad \Delta(V^T) = \sum_{v^t \in V^T} v^t. \tag{2}$$

Definition 7 *An allocation $x \in \mathcal{A}(X)$ is called fuzzy contractual if there exists a proper web V such that $x = x(V)$ and for every $T(V)$ the allocation $x^T = \omega + \Delta(V^T)$ is upper contractual.*

In economic terms this notion can be explained in the following way. During recontracting agents may make mistakes, coordination among coalition members may work imperfectly and so on. As a result an agent i can (erroneously) think that after partial breaking of current contracts he/she will have a commodity bundle x_i^T and that commodities from x_i^T may be exchanged in a new (mutually beneficial) contract. If allocation $x(V)$ is not fuzzy contractual then the last may (potentially) destroy agreements and allocation will be changed. Thus fuzzy contractual allocations are protected from this kind of agreements destructions.

Finishing the gallery of various kinds of allocation stability in a contractual economy, let us assume that the set of permissible contracts can be represented as a (finite) union of star-shaped sets, i.e,

$$\mathcal{W} = \cup \mathcal{V}_\xi.$$

Note that $\mathcal{W}$ is then a star-shaped set itself and $\mathcal{V}_\xi$ may, in particular, be convex sets or, moreover, subspaces of L (as it is the case for incomplete markets).

Now, for a given web V, we can associate with each $v \in V$, $v \in \mathcal{W}$, certain sets in $\{\mathcal{V}_\xi\}$ and can require that if $v \in \mathcal{V}_\xi$ for a given ξ, then the contract v has to be coherent and either proper or perfect or neither proper nor perfect. When such an association is established, the allocation $x(V)$ is called *complex* (composition) contractual. In other words, a complex contractual allocation is stable relative to both the procedure of appropriately breaking contracts (depending on the set which the contract belongs to) and the procedure of signing new (permissible) contracts. Moreover, some additional requirements may be imposed on a web realizing an allocation. These requirements always take the form of joint stability of contracts in the web. For example, in the case of an incomplete market economy we can identify the sets $\mathcal{V}_\xi$ with the subspaces of the commodity space which exactly correspond to the states of the world — trade exchanges (contracts) are allowed only in the present or at one and only one state in the future. For incomplete markets, one can establish (see Marakulin (2003)) that under some technical assumptions, the set of the GEI-equilibrium allocations coincides with the set of all complex contractual allocations such that their corresponding webs contain proper contracts in the present and perfect ones in future states. In these terms, a core allocation can be described as complex contractual for which perfect contracts are realized in future states and there are no restrictions in the present.

2 Contracts in a standard exchange economy

In the classical setting, it is indirectly assumed that for a pure exchange economy all kinds of commodity exchanges are allowed, the only restriction being that the realized consumption programs (bundles) have to belong to the agents' consumption sets, i.e., the allocations have to be feasible. This is why when one complements this model with a contract-based mechanism it is logical to think that all contracts are permissible, i.e., one may presume that $\mathcal{W} = L$, where L is the space of allocations. In fact sometimes it suffices to require a little bit less. So, speaking of a *standard* exchange economy, we always assume that the corresponding contractual economy is such that the set $\mathcal{W}$ of all permissible contracts is *radial* (absorbing)[7] at zero in L. In all other aspects the standard model coincides with model $\mathcal{E}$. Further let us recall some definitions.

A pair (x, p) is said to be a *quasi-equilibrium* of $\mathcal{E}$ if $x \in \mathcal{A}(X)$ and there exists a linear functional $p \neq 0$ onto E such that

$$\langle p, \mathcal{P}_i(x_i)\rangle \geq px_i = p\omega_i, \quad \forall\, i \in \mathcal{I}.$$

A quasi-equilibrium such that $x'_i \in \mathcal{P}_i(x_i)$ actually implies $px'_i > px_i$ is a *Walrasian or competitive equilibrium.*

On the other hand, $x \in \mathcal{A}(X)$ is said to be dominated (blocked) by a nonempty coalition $S \subseteq \mathcal{I}$ if there exists $y^S \in \prod_{i\in S} X_i$ such that $\sum_{i\in S} y_i^S = \sum_{i\in S} \omega_i$ and $y_i^S \in \mathcal{P}_i(x_i)\ \forall\, i \in S$.

The *core* of $\mathcal{E}$, denoted by $\mathcal{C}(\mathcal{E})$, is the set of all $x \in \mathcal{A}(X)$ that are blocked by no (nonempty) coalition.

Weak Pareto boundary for $\mathcal{E}$, denoted by $\mathcal{PB}^w(\mathcal{E})$, is the set of all $x \in \mathcal{A}(X)$ that cannot be dominated by the coalition $\mathcal{I}$ of all agents.

An allocation $x \in \mathcal{A}(X)$ is called *individual rational* if it cannot be dominated by singleton coalitions. $\mathcal{IR}(\mathcal{E})$ denotes the set of all these allocations.

The above definitions imply

$$\mathcal{C}(\mathcal{E}) \subset \mathcal{PB}^w(\mathcal{E}) \cap \mathcal{IR}(\mathcal{E}).$$

In general the reverse inclusion is true only for a two-consumer economy.

Next we would like to make several remarks on the concept of Pareto optimality (Pareto boundary). First recall a stronger concept of optimality, sometimes called the *strong Pareto optimality.* For preordered preferences[8] $\succeq_i$ an allocation $x = (x_i)_\mathcal{I} \in \mathcal{A}(X)$ is a *strong Pareto optimum* if there is no $z = (z_i)_\mathcal{I} \in \mathcal{A}(X)$ such that

$$z_i \succeq_i x_i \ \ \forall\, i \in \mathcal{I} \quad \& \quad \exists\, j \in \mathcal{I} : z_j \succ_j x_j.$$

Let us denote by $\mathcal{PB}^s(\mathcal{E})$ the *strong Pareto boundary*, the set of all strongly Pareto optimal allocations. The definitions imply $\mathcal{PB}^s(\mathcal{E}) \subset \mathcal{PB}^w(\mathcal{E})$.

There is one more possibility of defining the optimality concept in an economic model. It takes an intermediate position between the two notions considered above. We will see below that exactly this kind of optimality is realized by upper contractual allocations.

Let us call an allocation $x = (x_i)_\mathcal{I} \in \mathcal{A}(X)$ *strictly Pareto optimal* if there is no coalition $S \subseteq \mathcal{I}$ for which there exists an $y^S \in \prod_{i\in S} X_i$ such that $\sum_{i\in S} y_i^S = \sum_{i\in S} x_i$ and $y_i^S \succ_i x_i$ for each $i \in S$. In other words, x is an allocation in the core of the other economy, which differs from the original one in only one aspect, namely, allocation x is taken as the initial endowments. In our opinion, the last concept of optimality presents the most precise form of Pareto optimality.

[7]A set $A \subset L$ is radial at a point $a \in A$ if, for every $b \in L$, $\lambda b \in (A - a)$ for all real $0 \leq \lambda \leq \lambda_b$ and some $\lambda_b > 0$. Notice that a *convex* radial set is star-shaped at its every point.

[8]This is a reflexive, complete and transitive *non-strict* binary relation.

Denote by $\mathcal{PB}(\mathcal{E})$ the *strict Pareto boundary.* It is easily seen that

$$\mathcal{PB}^s(\mathcal{E}) \subset \mathcal{PB}(\mathcal{E}) \subset \mathcal{PB}^w(\mathcal{E}).$$

Therefore, if under some conditions one can show that an $x = (x_i)_{\mathcal{I}} \in \mathcal{PB}^w(\mathcal{E})$ is strongly Pareto optimal (this is the case if, for example, the preferences are locally non-satiated and $x \in \mathrm{int}X$),[9] then the allocation x is strictly Pareto optimal as well.

The next important notion, fruitfully working in the theory of economic equilibrium, is the concept of fuzzy core. Recall that any vector

$$t = (t_1, \ldots, t_n) \neq 0, \quad 0 \leq t_i \leq 1, \quad \forall\, i \in \mathcal{I}$$

may be identified with a fuzzy coalition, where the real number t_i being interpreted as the measure of agent i in the coalition. A coalition t is said to dominate (block) an allocation $x \in \mathcal{A}(X)$ if there exists $y^t \in \prod_{\mathcal{I}} X_i$ such that

$$\sum_{i\in\mathcal{I}} t_i y_i^t = \sum_{i\in\mathcal{I}} t_i \omega_i \iff \sum_{i\in\mathcal{I}} t_i (y_i^t - \omega_i) = 0 \tag{3}$$

and

$$y_i^t \succ_i x_i, \quad \forall\, i \in \mathrm{supp}(t) = \{i \in \mathcal{I} \mid t_i > 0\}. \tag{4}$$

The set of all feasible allocations which cannot be dominated by fuzzy coalitions is denoted by $\mathcal{C}^f(\mathcal{E})$ and is called the *fuzzy core* of the economy $\mathcal{E}$.

2.1 Preliminary results and an example

We start the below analysis with the theorem, which establishes relationships between the core and contractual allocations.

Theorem 1 *Let $\mathcal{E}^c$ be a contractual economy such that $\mathcal{W} = L$, and let x be an allocation. Then:*

(i) *x is contractual $\iff$ $x \in \mathcal{C}(\mathcal{E}) \cap \mathcal{PB}(\mathcal{E})$;*

(ii) *x is upper contractual $\iff$ $x \in \mathcal{PB}(\mathcal{E})$;*

(iii) *x is lower contractual $\iff$ $x \in \mathcal{IR}(\mathcal{E})$;*

(iv) *x is weakly contractual $\iff$ $x \in \mathcal{IR}(\mathcal{E}) \cap \mathcal{PB}(\mathcal{E})$.*

Proof of Theorem 1. The necessity of (i)–(iv) directly follows from the definitions. To check their sufficiency, let us consider the web $V_\omega(x)$ consisting of only one contract $v = x - \omega$. This is really a web since, $x \in \mathcal{A}(X)$ and by assumption, $x - \omega \in \mathcal{W}$. A routine checking of Definitions 1, 2 completes the proof. ■

The following theorem characterizes the equilibrium allocations in terms of the properly contractual ones. So, the result of this theorem allows to consider stability of an

[9] Moreover, it is well known that if the preferences are strictly monotonic and $X_i = \mathbb{R}^l_+$ (l is the number of commodities) for all i, then the concepts of strong and weak Pareto optimality are equivalent.

allocation relative to the partial breaking of contracts as a specific form of perfect competition conditions that delivers another contractual sight on this subject. This approach is developed via idea of fuzzy contractual allocations and there relationships with equilibria (they coincide under weaker assumptions) that is revealed at the end of section, see Propositions 2, 3 and Lemma 2.

Theorem 2 *Let $\mathcal{E}^c$ be a smooth contractual economy, the set $\mathcal{W}$ be radial at zero in L, and x be an allocation such that $x \in \mathrm{int}X$ and $\mathrm{grad}\, u_i(x_i) \neq 0$ for some i and all agents are non-satiated. Then the following statements are equivalent:*

(*i*) *x is an equilibrium allocation;*

(*ii*) *x is Pareto optimal and there exists a coherent web V realizing this allocation, i.e., $x = x(V)$ such that V is coherent and upper stable;*

(*iii*) *x is a properly contractual allocation;*

(*iv*) *x is a perfectly contractual allocation.*

Moreover, if (x, p) is an equilibrium and V is a web realizing $x = x(V)$, then V is a coherent web if and only if $pv_i = 0$, $\forall\, v \in V$, $\forall\, i \in \mathcal{I}$.

Remark 3 The analysis of the theorem's proof shows that the implication *(i)⇒(iii)* is true in the general case for the non-smooth, possibly satiated preferences and without the requirement $x \in \mathrm{int}X$, i.e., in the standard exchange economy every equilibrium is a properly contractual allocation.

Recall also that due to Theorem 1 (*ii*), statement (*ii*) of Theorem 2, in which the Pareto optimality of the allocation x is claimed, is equivalent to the existence of a coherent and upper stable web realizing this allocation. ■

Proof of Theorem 2. To establish the equivalence of *(i)–(iv)* recall that under the theorem's conditions and via Second Welfare Theorem, $x \in \mathcal{A}(X)$ is Pareto optimal iff there exists an $i \in \mathcal{I}$ such that for $p = \mathrm{grad}\, u_i(x_i)$

$$\langle \mathcal{P}_j(x_j), p\rangle > \langle p, x_j\rangle, \quad \forall\, j \in \mathcal{I}.^{10} \tag{5}$$

Let without loss of generality be $i = 1$.

Further let us notice that *(iv)⇒(iii)⇒(ii)*. To establish *(ii)⇒(i)* define $p = \mathrm{grad}\, u_1(x_1)$ and apply Proposition 1 to obtain $pv_i \geq 0$ for all $v \in V$ and $i \in \mathcal{I}$. Summing up over $v \in V$ for every $i \in \mathcal{I}$, we arrive at

$$\langle p, \Delta_i(V)\rangle \geq 0 \quad \Longrightarrow \quad px_i \geq p\omega_i.$$

This, due to the feasibility of x, implies $px_i = p\omega_i$ for all i which by (5) yields the equilibrium properties of (x, p).

Now let us prove *(i)⇒(iv)*. Let $v = v^r = (x - \omega)/r$, where the natural r is chosen based on the assumption that $\mathcal{W}$ is absorbing, and define the set V consisting of the r identical copies of the contract v. Clearly, V is a web. Now the equilibrium properties of the pair (x, p) imply $pv_i = 0$, $\forall\, v \in V$, $\forall\, i \in \mathcal{I}$ and therefore (1) is true (one can take

[10] In particular, it implies that non-zero $\mathrm{grad}\, u_i(x_i)$ and $\mathrm{grad}\, u_j(x_j)$ coincide up to a normalization for all $i \neq j$. It is easy to analyze the omitted case when $\mathrm{grad}\, u_i(x_i) = 0\ \forall i \in \mathcal{I}$.

$p = \text{grad}\, u_i(x_i)$ as the equilibrium price vector for x). Applying the sufficiency part of Proposition 1, one concludes that V is a coherent and, moreover, proper web. Now let $U \sim V$ for a proper web U. Once again due to Proposition 1 (necessity), the properness of U implies $pu_i \geq 0$, $\forall\, u \in U$, $\forall\, i \in \mathcal{I}$. But then the contract specification ($\sum_{i \in \mathcal{I}} u_i = 0$) implies $pu_i = 0$, $\forall\, u \in U$, $\forall\, i \in \mathcal{I}$. Finally, if for a $T \subseteq \mathcal{I}$, $T \neq \emptyset$ and a web $W \in F(U, T)$ we have $W \succ_T U$, then it follows from the equilibrium definition that

$$\langle p, y_i(W) \rangle > \langle p, x_i(U) \rangle, \quad \forall\, i \in T.$$

Consequently, summing up these inequalities over $i \in T$, we arrive at the contradiction with the contract specification of $w \in W \setminus U$ (since $S(w) \subseteq T$, because of $W \in F(U, T)$ and (ii)). The final part of theorem is also clear. ■

Further let us consider an example demonstrating the difference between the various notions of contractual allocation. Of course, for this difference to be realized when the partial breaking of contracts is allowed, the conditions of Theorem 2 have to be invalid, and either the utilities have to be non-smooth or the allocation has to belong to the boundary of X.

The following example borrowed from Kozyrev (1982), shows that for non-differentiable utility functions a properly contractual allocation may not be an equilibrium.

Example 1 Consider a two-commodities exchange economy with two consumers, where $X_i = \mathbb{R}^2_+$. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ denote the consumption bundles of 1st and 2nd agent respectively. Let preferences be defined onto $\mathbb{R}^2_+$ by the strictly monotonic utility functions

$$u_1(x_1, x_2) = 2\sqrt{x_1 x_2} + x_1 + x_2, \quad u_2(y_1, y_2) = 2\sqrt{y_1 y_2} + y_1 + y_2 + \min\{y_1, y_2\}.$$

For the initial endowments we take the vectors

$$\omega_1 = (1, 0), \quad \omega_2 = (0, 1), \quad \bar{\omega} = \omega_1 + \omega_2 = (1, 1), \quad \omega = (\omega_1, \omega_2) = ((1, 0), (0, 1)).$$

In further considerations, we make use of the "Edgeworth's box," the well known subset of $\mathbb{R}^2$:

$$EB(\bar{\omega}) = \{x \in \mathbb{R}^2 \mid 0 \leq x \leq (1, 1) = \bar{\omega}\}.$$

One can interpret $x \in EB(\bar{\omega})$ as the consumption of the first consumer and $(\bar{\omega} - x) = y$ as the consumption of the second one. The point x may also be associated with the allocation $(x, \bar{\omega} - x)$.

A simple analysis shows that in this example Pareto boundary is the set

$$\mathcal{PB} = \text{co}\{\, (0, 0), (1, 1) \,\} = \{\, x \in EB(\bar{\omega}) \mid x_1 = x_2 = \alpha,\ 0 \leq \alpha \leq 1 \,\},$$

i.e., it is the diagonal of $EB(\bar{\omega})$.

Since every equilibrium allocation is Pareto optimal and due to the fact that if it is an interior point of the box, the price vector has to coincide up to a normalization with $\text{grad}\, u_1(x)$, the vector $(1, 1)$ has to be an equilibrium price vector. Clearly, the points $(1, 1)$ and $(0, 0)$ are not equilibrium allocations. Consequently, $p = (1, 1)$ is *the only* (up to a normalization) *equilibrium price*. Using the budget constraints $px_i = p\omega_i$, one can easily find the unique equilibrium allocation which corresponds to the first agent's consumption bundle $(\frac{1}{2}, \frac{1}{2})$ in the Edgeworth's box.

The core in this economy with two consumers coincides with the set $\mathcal{PB}\cap\mathcal{IR}$ which, in turn, is the set of all contractual and weakly contractual allocations and can be easily calculated to be

$$\mathcal{PB}\cap\mathcal{IR} = \{\, x \in \mathcal{PB} \mid u_1(x) \geq u_1(\omega_1),\ u_2(\bar{\omega}-x) \geq u_2(\omega_2)\,\} = \mathrm{co}\{\,(\tfrac{1}{4},\tfrac{1}{4}),(\tfrac{4}{5},\tfrac{4}{5})\,\}.$$

Next let us find the set of all properly contractual allocations. Clearly, this is the set of all points $x=(\alpha,\alpha)$ in $\mathcal{PB}\subset EB$ for which the derivatives of the utility functions u_1 and u_2 are not positive in the directions of $h_1=\omega_1-(\alpha,\alpha)$ and $h_2=\omega_2-(1-\alpha,1-\alpha)$, i.e., we need to solve the system of equations

$$\partial_{h_1}u_1(\alpha,\alpha)\leq 0,\quad \partial_{h_2}u_2(1-\alpha,1-\alpha)\leq 0.$$

A direct calculation gives

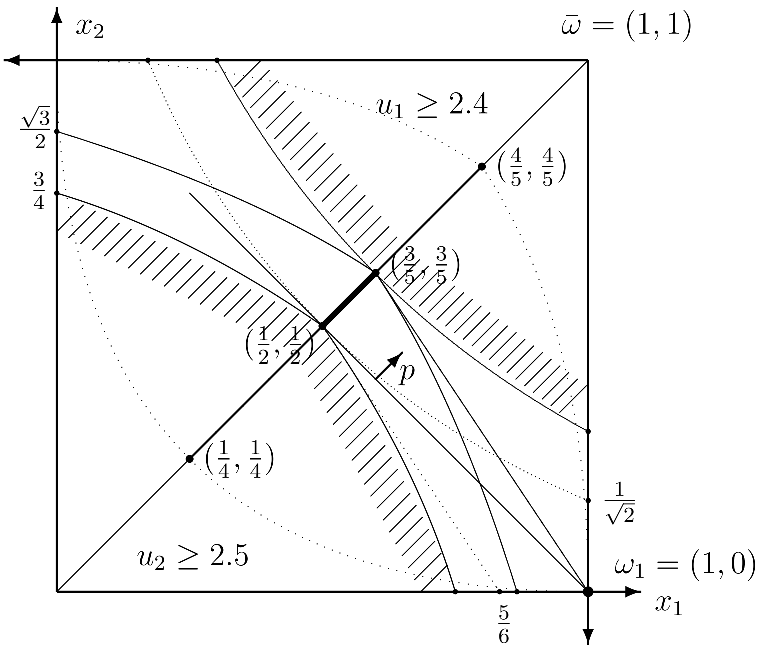

Figure 3: Non-smooth preferences

$$\mathrm{grad}\, u_1(\alpha,\beta) = (\sqrt{\beta/\alpha}+1,\sqrt{\alpha/\beta}+1)$$

for all $\alpha>0$, $\beta>0$ and

$$\mathrm{grad}\, u_2(1-\alpha,1-\beta) = (\sqrt{(1-\beta)/(1-\alpha)}+2\,,\ \sqrt{(1-\alpha)/(1-\beta)}+1)$$

for $\alpha>\beta>0$, $1-\alpha>0$. Calculating the inner products and substituting $\alpha=\beta$ (i.e., passing to the limit for $\beta\to\alpha$) yields

$$1-2\alpha\leq 0,\quad 5\alpha-3\leq 0.$$

As a result, the set of properly contractual allocations is described as $\mathrm{co}\{\,(\frac{1}{2},\frac{1}{2}),(\frac{3}{5},\frac{3}{5})\,\}$ and does *not coincide* with (but contains!) the set of *equilibrium* allocations. Figure 3 illustrates the analysis conducted.

The structure of the set of lower properly contractual allocations may also be clarified for this example. It is curious that this set is not convex, but due to the volume restrictions we omit these considerations here.

Let us continue the analysis and demonstrate the difference between the notions of coherent and proper webs as well as between the corresponding stability properties.

Let us consider an allocation, in which agent 1 consumes the bundle $(\frac{3}{5}, \frac{3}{5})$. Set $\hat{h}_1 = (0,1) - (\frac{3}{5}, \frac{3}{5}) = (-\frac{3}{5}, \frac{2}{5})$ and calculate the derivative of the function u_1 at the point $(\frac{3}{5}, \frac{3}{5})$ in the direction of $\hat{h}_1$:

$$\partial_{\hat{h}_1} u_1(\tfrac{3}{5}, \tfrac{3}{5}) = \langle (2,2), \hat{h}_1 \rangle = -\frac{2}{5} < 0.$$

For the allocation considered, the second agent's consumption bundle is a point of non-differentiability of his/her utility function, but its derivative in the direction of $\hat{h}_2 = (1,0) - (\frac{2}{5}, \frac{2}{5}) = (\frac{3}{5}, -\frac{2}{5})$ can be easily calculated as the limit for $(\alpha, \beta) \to (\frac{3}{5}, \frac{3}{5})$ of the inner product of the vector-gradient of u_2 calculated at the point $(1-\alpha, 1-\beta)$, $\beta > \alpha > 0$, and the vector $\hat{h}_2$. Since for $\beta > \alpha > 0,\ 1-\beta > 0$,

$$\operatorname{grad} u_2(1-\alpha, 1-\beta) = (\sqrt{(1-\beta)/(1-\alpha)} + 1, \sqrt{(1-\alpha)/(1-\beta)} + 2),$$

upon calculating the inner products and substituting $\alpha = \beta = \frac{3}{5}$, one obtains

$$\partial_{\hat{h}_2} u_2(\tfrac{2}{5}, \tfrac{2}{5}) = \langle (2,3), \hat{h}_2 \rangle = 0.$$

Next, for a sufficiently small $\varepsilon > 0$, take allocation

$$\omega^\varepsilon = \omega - \varepsilon w$$

to be the initial endowments and consider web $V^\varepsilon = \{v, \varepsilon w\}$ where

$$v = (v_1, v_2), \quad v_1 = -v_2 = (\tfrac{3}{5}, \tfrac{3}{5}) - (1,0) = (-\tfrac{2}{5}, \tfrac{3}{5}) = -h_1|_{\alpha=\frac{3}{5}} = h_2|_{\alpha=\frac{3}{5}}$$

and

$$w = (w_1, w_2), \quad w_1 = -w_2 = (3,-2) = -5\hat{h}_1 = 5\hat{h}_2$$

is the contract in which the first consumer exchanges 2 units of commodity 2 for 3 units of commodity 1. For example, one can take $\varepsilon = \frac{1}{6}$. Obviously

$$\omega^\varepsilon + v + \varepsilon w = ((\tfrac{3}{5}, \tfrac{3}{5}), (\tfrac{2}{5}, \tfrac{2}{5})),$$

i.e., the allocation considered is realized by the web V^ε relative to the endowments ω^ε. The above calculations of the derivatives in the directions of $\hat{h}_1$, $\hat{h}_2$ together with the previous calculations of the derivatives in the directions of h_1 and h_2 (for $\alpha = \frac{3}{5}$), show that each contract in the web V^ε is *coherent* relative to ω^ε. However, for every $\varepsilon \in (0, \frac{1}{6}]$ the web V^ε is not proper since after the breaking of a half of εw and after the partial breaking of the contract v, the broken part being $\delta = \frac{5}{2}\varepsilon < 1$, it realizes the allocation for which the first agent's consumption program has the form

$$\tfrac{1}{2}\varepsilon(3,-2) + (1 - \tfrac{5}{2}\varepsilon)(-\tfrac{2}{5}, \tfrac{3}{5}) + (1,0) - \varepsilon(3,-2) = (\tfrac{3}{5}, \tfrac{3}{5}) - \varepsilon(\tfrac{1}{2}, \tfrac{1}{2}) = x_1^\varepsilon.$$

Therefore, the proposed partial breaking of contracts is, in fact, profitable for the *second* agent since it results in an increase in her/his consumption by $\varepsilon(\frac{1}{2}, \frac{1}{2})$. This proves that the web V^ε *is not proper* relative to ω^ε. ■

The above example stimulates a more careful study of the mathematical properties of properly (and perfectly) contractual allocations in the situations where the conditions of Theorem 2 are invalid, that we are going to do below.

2.2 Properly and perfectly contractual allocations

Let us begin with the discussion of lower properly contractual allocations. Due to the definition, they are the allocations $x(V)$ which can be implemented by a web of contracts V, which is stable relative to the partially breaking contracts. As soon as for the standard market model every contract is permissible, web V can be replaced by web $\Delta V = \{u\}$ consisting of only one contract $u = \sum_{v \in V} v = x - \omega$. It can be easily seen that the low stability of the original web implies the same type of stability for web ΔV. Therefore, one can restrict the analysis of properly (not only lower) contractual allocations to the webs having the form $\{x - \omega\}$. Further it will be clear that all the conclusions can be easily applied to the case of webs consisting of multiple contracts. However, in the case of webs consisting of a single contact, one can directly conclude from the definition that an allocation $x \in \mathcal{A}(X)$ is *lower properly contractual* if and only if

$$[x_i, \omega_i] \cap \mathcal{P}_i(x_i) = \emptyset, \quad \forall\, i \in \mathcal{I}, \tag{6}$$

where

$$[x_i, \omega_i] = \{\lambda x_i + (1-\lambda)\omega_i \mid 0 \le \lambda \le 1\}.$$

Now via Theorem 1 (*ii*) the definitions of stability clearly imply that *weakly properly contractual* allocations are exactly the *Pareto optimal* allocations *satisfying* (6).

The properly contractual allocations can be characterized in similar terms as follows. Breaking the $(1-\lambda)$ part of the contract $x - \omega$ and signing a new contract v, the members of $S \subset \mathcal{I}$ realize the collection of consumption bundles $y^S = (y_i^S)_S$ such that $\sum_S y_i^S = \sum_S (\lambda x_i + (1-\lambda)\omega_i))$. Since the definition of properly contractual allocation forbids this type of domination, the allocation x must belong to the core of the economy with the initial endowments $\lambda x + (1-\lambda)\omega = \omega_x^\lambda = \omega_x \in [x, \omega]$, which we denote by $\mathcal{C}(\mathcal{E}_x^\lambda)$. Thus an allocation x is *properly contractual* if and only if it belongs to the core of each economy $\mathcal{C}(\mathcal{E}_x^\lambda)$, i.e., when

$$x \in \bigcap_{\omega_x \in [x,\omega]} \mathcal{C}(\mathcal{E}_x^\lambda).$$

We can continue by similarly describing the perfectly contractual allocations. In order to do this, we actually only need to understand to which kinds of the "initial endowments allocations" one can transit upon breaking a "virtual web" for $\{x - \omega\}$. First of all note that in this case we may restrict the analysis to the webs consisting of two contracts. To see this, let V be a proper virtual web realizing the allocation x, i.e., $V \sim \{x - \omega\}$, and let $U \subseteq V$ be the set of all contracts broken by some coalition. Form the web $W = \{\Delta(U), \Delta(V \setminus U)\}$ in which all broken contracts are aggregated into one contract and all preserved contracts into another one. Clearly, web W is proper, $W \sim V$, and by breaking contract $\Delta(U)$ one realizes the allocation that coincides with the allocation obtained from V as a result of breaking contracts $U \subseteq V$, which completes the argument. So, let $W = \{u, v\} \sim \{x - \omega\}$. Then by partially breaking contracts in this web, one can realize any point y in the convex hull of the set consisting of four points: x, ω, $\omega + v$, $\omega + u$. It is clear that the web $\{x - y, y - \omega\}$ constructed for this y is also proper. The opposite is also true: if this web is proper, $y \in \mathcal{A}(X)$ can be realized via breaking a part of the contracts in a virtual web realizing x. Thus if for $x \in \mathcal{A}(X)$ one defines

$$PC = \{y \in \mathcal{A}(X) \mid \text{web } \{x - y, y - \omega\} \text{ is proper}\},$$

then

$$x \text{ is perfectly contractual} \iff PC \neq \emptyset \;\;\&\;\; x \in \bigcap_{y \in PC} \mathcal{C}(\mathcal{E}_y), \tag{7}$$

where y is the vector of initial endowments in model $\mathcal{E}_y$. To better understand the meaning of this formula, the structure of the set PC needs to be clarified. Below we will do it in dual terms.

Further let us turn to the description of the objects under study in terms of dual cones. It is of mathematical interest in its own right and, as we will see below, can considerably contribute to the understanding of various concepts of contractual allocation in the situations we are interested in.

The cone

$$K^* = \{p \in E' \mid \langle p, K \rangle \geq 0\}$$

is said to be the dual cone of the set $K \subset E$. For every $i \in \mathcal{I}$, let us set

$$\Gamma(x_i) = \{p \in E' \mid \langle p, \mathcal{P}_i(x_i) - \{x_i\} \rangle \geq 0\}.$$

This is the dual cone of the $\mathcal{P}_i(x_i) - \{x_i\}$. It is well known (and easy to prove applying the separation theorem) that for every (strictly) Pareto optimal allocation, there corresponds a (non-zero) linear price functional $p \in E'$ such that

$$\langle p, \mathcal{P}_i(x_i) \rangle \geq \langle p, x_i \rangle, \quad \forall\, i \in \mathcal{I}.$$

The necessity part of this statement is always true (for convex, possibly satiated preferences), whereas its sufficiency part is true for the interior points of the consumption sets (if in addition the sets $\mathcal{P}_i(x_i)$ are open in X_i, that we have due to (**A**)). One can see that this description is very close to being a precise characterization of Pareto optimality. Accordingly, it is not inaccurate if we call the allocations which satisfy this property Pareto quasi-optimal. In the above terms, they can be described as the allocations satisfying

$$\Gamma(x) = \bigcap_{\mathcal{I}} \Gamma(x_i) \neq \{0\}. \tag{8}$$

For every $i \in \mathcal{I}$ set

$$G(x_i - \omega_i) = \{p \in E' \mid \langle p, x_i - \omega_i \rangle \geq 0\}, \quad G(x) = \bigcap_{\mathcal{I}} G(x_i - \omega_i).$$

Now one can easily see that x is a *quasiequilibrium* $\iff$

$$\bigcap_{\mathcal{I}} [G(x_i - \omega_i) \cap \Gamma(x_i)] \neq \{0\} \iff G(x) \cap \Gamma(x) \neq \{0\}. \tag{9}$$

Assume that $x \in \operatorname{int} X$.[11] Then similarly applying the separation theorem and (6), we see that an allocation x is *lower properly contractual* $\iff$

$$G(x_i - \omega_i) \cap \Gamma(x_i) \neq \{0\}, \quad \forall\, i \in \mathcal{I} \iff$$

$$\forall\, i \in \mathcal{I} \;\; \exists\, p_i \in E',\, p_i \neq 0 \;:\; \langle p_i, \mathcal{P}_i(x_i) \rangle \geq \langle p_i, x_i \rangle \;\;\&\;\; p_i x_i \geq p_i \omega_i. \tag{10}$$

Thus, in this case an allocation is *weakly properly contractual* if (8) holds in addition.

Further let us study the properties of the properly contractual allocations.

[11] This condition together with (**A**) is essential for establishing sufficiency; as for necessity, it may be dropped (in view of (**A**)).

Lemma 1 *If an allocation x is properly contractual then for each coalition $S \subseteq \mathcal{I}$, $S \neq \emptyset$, containing only non-satiated agents, there exists $p_S \in E'$, $p_S \neq 0$ such that*

$$\langle p_S, \mathcal{P}_i(x_i)\rangle \geq \langle p_S, x_i\rangle,\ \forall\, i \in S \ \ \&\ \ p_S \sum_S x_i \geq p_S \sum_S \omega_i. \tag{11}$$

If in addition $x \in \operatorname{int} X$, the opposite is true, i.e., if there exist linear $p_S \in E'$, $p_S \neq 0$ satisfying (11), $\forall\, S \subseteq \mathcal{I}$, then x is a properly contractual allocation.

In other words, the lemma states that for a properly contractual allocation, each coalition can find internal-coalition prices such that, first, they are "suitable" for every member of the coalition (the first inequality in (11), that may be treated as a form of the coalition efficiency) and, second, the contract $x - \omega$ is coalition-profitable relative to these prices (the second inequality in (11)). Thus, the properly contractual allocations are precisely the allocations which satisfy the condition of *coalition-profitability* (11). The statement of the lemma is similar to the description of the weakly properly contractual allocations given in (10), the only difference being that the lemma claims the existence of internal-coalition prices, satisfying (11) for *every coalition*, whereas in (10) just for *singleton coalitions*. This is why the weakly properly contractual allocations are just *individually profitable* and *Pareto optimal* (i.e., the coalition of all agents is profitable as well). Notice also that the statement of Lemma 1 may be rewritten in the equivalent form

$$\bigcap_S \Gamma(x_i) \bigcap G(\sum_S x_i - \sum_S \omega_i) \neq \{0\}, \quad \forall\, S \subseteq \mathcal{I}.$$

Of course, in the general case, this requirement is weaker than (9). This is why in order to establish that a properly contractual allocation is a (quasi)equilibrium, we need to make additional assumptions to guarantee that (9) is equivalent to the last relation (for example, that the utilities are differentiable and $x \in \operatorname{int} X$, which is actually a strong assumption, that we made in Theorem 2).

Proof of Lemma 1. It follows from the above analysis that an allocation x is properly contractual iff it cannot be improved upon by any coalition $S \subseteq \mathcal{I}$ relative to the endowments $x^\lambda = \lambda x + (1 - \lambda\omega)$ for all $\lambda \in [0, 1]$. Let $\mathcal{P}_S(y^S) = \prod_S \mathcal{P}_i(y_i)$ for $y^S = (y_i)_{i\in S} \in \prod_S X_i = X^S$ and let $x^\lambda_S = (x^\lambda_i)_{i\in S}$. Then for a fixed λ, the last property can be written in the form

$$\mathcal{P}_S(x^S) \bigcap (E_S + x^\lambda_S) = \emptyset, \ \ E_S = \{y^S \in E^S \mid \sum_S y_i = 0\}. \tag{12}$$

Therefore,

$$\mathcal{P}_S(x^S) \bigcap (E_S + [x^S, \omega^S]) = \emptyset, \ \ x^S = (x_i)_{i\in S}, \ \ \omega^S = (\omega_i)_{i\in S},$$

where $[x^S, \omega^S]$ is the linear segment in E^S connecting the points x^S and ω^S (the convex hull of two points). Due to assumption (**A**), since the set $E_S + [x^S, \omega^S]$ is convex and each member of S is non-satiated (hence $\operatorname{int} \mathcal{P}_S(x^S) \neq \emptyset$), one can apply the classical separation theorem, which gives the existence of a linear functional (vector) $p^S = (p_i)_{i\in S} \in (E')^S$, $p^S \neq 0$, such that

$$\langle p^S, \mathcal{P}_S(x^S)\rangle \geq \langle p^S, E_S + [x^S, \omega^S]\rangle.$$

The right-hand side of this inequality is a *bounded from above* subset of $\mathbb{R}$. Hence the inequality may be true only if the set $\langle p^S, E_S\rangle$ is bounded. Since E_S is a subspace of E^S, it follows that $\langle p^S, E_S\rangle = \{0\}$. A standard argument then implies that $p_i = p_j = p$ $\forall\, i, j \in S$ (because $p^S z^S = 0$ for all $z^S \in E^S$ such that $z_i^S = -z_j^S \in E$ and $z_t^S = 0$ for $t \neq i, j, t \in S$). Moreover, $p \neq 0$ since $p^S = (p, \ldots, p) \neq 0$. Next, assumption (**A**) implies that $\mathcal{P}_S(x^S)$ is convex and $x^S \in \overline{\mathcal{P}_S(x^S)}$. Therefore, it follows from the last inequality that

$$\langle p^S, x^S\rangle \geq \langle p^S, [x^S, \omega^S]\rangle \iff p\sum_S x_i \geq p\sum_S \omega_i.$$

Moreover, arguing by contradiction, we obtain

$$\langle p^S, \mathcal{P}_S(x^S)\rangle \geq \langle p^S, x^S\rangle = \sup\langle p^S, [x^S, \omega^S]\rangle \iff \langle p, \mathcal{P}_i(y_i)\rangle \geq \langle p, x_i\rangle, \quad \forall\, i \in S.$$

Now to complete the proof of the lemma's necessity, just substitute $p_S = p$.

The lemma's sufficiency follows from (**A**) and the condition $x \in \operatorname{int} X$, since in this case the inequalities in the first part of (11) are actually strict, which together with the second part of (11) implies (12) for all $\lambda \in [0, 1]$. ■

Lemma 1 allows us to discover new interesting (and sometimes unexpected) properties specific to properly contractual allocations. For example, it follows from this lemma that every properly contractual allocation in a 2-replicated economy is an equilibrium if the economy contains just two agents or, alternatively, if there is one agent with differentiable preferences.[12]

First we would like to recall the concept of a replicated economy. Given a natural $r \in \mathbb{N}$, the r-fold replica of $\mathcal{E}$ is the model $\mathcal{E}^r$ in which every consumer of the original model defines a *type of economic agent* represented by her/his r precise copies. For convenience, the agents in $\mathcal{E}^r$ are numbered by double indexes (i, m), $i \in \mathcal{I}$, $m = 1, \ldots, r$. It is assumed that $X_{im} = X_i$, $\omega_{im} = \omega_i$, and the preferences, being defined on and taking values in X_{im}, are defined by $\mathcal{P}_{im} = \mathcal{P}_i$. Notice that for every allocation $x = (x_i)_{\mathcal{I}}$ in the initial model, there canonically corresponds an allocation in the replica according to the rule $x_{im} = x_i$, $\forall\, i, m$. The opposite is also true if the allocation is *symmetric*, i.e., when identical agents consume equal bundles (equal treatment).

Replicas play an important role in the analysis of perfect competition, especially for proving the well known Edgeworth's conjecture which states that under perfect competition conditions the core and equilibria coincide. It is the replica's symmetric allocations and the corresponding allocations in the original model that is the main subject of this analysis, with every coalition in the replica being allowed to dominate the original allocations by not necessarily symmetric inter-coalition allocations.

Theorem 3 *Assume that an economy has two agents or, alternatively, there exists an agent with a smooth preference whose consumption choice is an interior point of his/her consumption set. Then every allocation, which is properly contractual in the 2-fold replica economy, is a quasiequilibrium if all agents are non-satiated.*

Proof of Theorem 3. Consider first the case of an economy with two agents. Let $\mathcal{I} = \{1, 2\}$ and let $x = (x_1, x_2)$ be a properly contractual allocation in the 2-fold replica

[12]This result was first proved in Kozyrev (1981), who applied the technique of subdifferential calculus (to *concave* utility functions, etc.), that restricts the generality. Lemma 1 and the proof of the next theorem are new.

economy. It suffices to establish (9) for x. To this end, apply Lemma 1 and relation (11) to the coalitions $S' = \{(1,1),(1,2),(2,1)\}$ and $S'' = \{(1,1),(2,1),(2,2)\}$. This results in the existence of *nonzero* vectors $p', p'' \in E'$ such that

$$p', p'' \in \Gamma(x_1) \cap \Gamma(x_2)$$

and

$$p'(2x_1 + x_2) \geq p'(2\omega_1 + \omega_2), \quad p''(x_1 + 2x_2) \geq p''(\omega_1 + 2\omega_2).$$

Since $x_1 + x_2 = \omega_1 + \omega_2$, the last inequalities are equivalent to

$$p'x_1 \geq p'\omega_1 \;\;\&\;\; p''x_2 \geq p''\omega_2.$$

If one of these inequalities is actually an equality, then due to the feasibility of x, either p' or p'' belongs to the intersection in (9). Suppose both inequalities are strict. Then, the first component of the *2-dimension* vector $(p'(x_1 - \omega_1), p'(x_2 - \omega_2))$ is strictly more than zero, the second one is strictly less than zero, and their sum is equal to zero. The same is true for the vector $(p''(x_1 - \omega_1), p''(x_2 - \omega_2))$, in which the first component is strictly less than zero. Next find a real $0 < \alpha < 1$ such that $\alpha p'(x_1 - \omega_1) + (1 - \alpha)p''(x_1 - \omega_1) = 0$ (set $\alpha = -p''(x_1 - \omega_1)/\ [p'(x_1 - \omega_1) - p''(x_1 - \omega_1)]$). Then it is clear that $[\alpha p' + (1 - \alpha)p''](x_2 - \omega_2) = 0$. Now let us set $p = \alpha p' + (1-\alpha)p'' \neq 0$. Hence, by construction, we obtain $p(x_i - \omega_i) = 0 \Rightarrow p \in G(x_i - \omega_i)$, $i = 1, 2$. Also $p \in \Gamma(x_i)$, $i = 1, 2$ due to the convexity of $\Gamma(x_i)$. This proves (9).

Now let us show (9), assuming that there exists an agent with a smooth preference. First of all note that due to this assumption, if there exists a nonzero vector $p \in \Gamma(x_i)$, $\forall\, i \in \mathcal{I}$, then it is *unique up to a normalization*. Now apply Lemma 1 and relation (11) to the coalitions $S^i = \{(i,2)\} \cup \mathcal{I} \times \{1\}$, $i \in \mathcal{I}$, i.e., all the coalitions in which the i-th type consumer is presented by two agents and all the other consumers just by one agent. It follows that there exists a nonzero vector $p \in \Gamma(x_i)$, $\forall\, i \in \mathcal{I}$, which is *common* for all coalitions S^i, such that for every $i \in \mathcal{I}$

$$p[\sum_{j \in \mathcal{I}} x_j + x_i] \geq p[\sum_{j \in \mathcal{I}} \omega_j + \omega_i] \quad \Longrightarrow \quad px_i \geq p\omega_i.$$

Since x is feasible, $px_i = p\omega_i$ for all i and the proof is complete. ■

Arguing along the lines of the proof of Lemma 1, one can obtain a dual description of perfectly contractual allocations. Having this in mind, let us make use of formula (7) to describe the proper web $\{x - y, y - \omega\}$ in dual terms. As in (6), we conclude that $\{x - y, y - \omega\}$ is proper $\iff$

$$\text{co}\{x_i, y_i, \omega_i, x_i - y_i + \omega_i\} \cap \mathcal{P}_i(x_i) = \emptyset \quad \forall\, i \in \mathcal{I}.$$

Now applying the separation theorem, we see that this web is proper if and only if for each i there exists a nonzero $p_i \in E'$ such that

$$\langle p_i, \mathcal{P}_i(x_i)\rangle \geq \langle p_i, x_i\rangle \geq \langle p_i, \text{co}\{x_i, y_i, \omega_i, x_i - y_i + \omega_i\}\rangle.$$

In this chain of inequalities, the second one is equivalent to $p_i x_i \geq p_i y_i \geq p_i \omega_i$. Thus we obtained a description of PC, the set of all allocations which can be realized via breaking a part of the contracts in a virtual web realizing x. We can apply this description to give a characterization of the perfectly contractual allocations.

In fact, being properly contractual, a perfectly contractual allocation has to satisfy (11) and, in addition, for every $y \in \mathcal{A}(X)$, the condition

$$\forall\, i \in \mathcal{I}\ \exists\, p_i \in E', p_i \neq 0\ :\ \langle p_i, \mathcal{P}_i(x_i)\rangle \geq p_i x_i\ \ \&\ \ p_i x_i \geq p_i y_i \geq p_i \omega_i \tag{13}$$

has to imply $x \in \mathcal{C}(\mathcal{E}_y)$. Moreover, since it has to be true for all $y' \in [x, y]$ (substitute y' for y in the last relations), i.e., since x is properly contractual relative to y, one can apply Lemma 1. Thus condition (13) has to imply that for each coalition $S \subseteq \mathcal{I}$, there exists a nonzero $p_S \in E'$ such that

$$\langle p_S, \mathcal{P}_i(x_i)\rangle \geq p_S x_i,\ \forall\, i \in S\ \ \&\ \ p_S \sum_S x_i \geq p_S \sum_S y_i.$$

Further we turn to a comparative analysis of the contractual allocations and the fuzzy core allocations.

2.3 Fuzzy core and fuzzy contractual allocations

We begin with a study of the specific properties of the fuzzy core allocations. The elements of fuzzy core are defined via conditions (3), (4) which for non-satiated preferences, i.e., when $\mathcal{P}_i(x_i) \neq \emptyset$, $\forall i \in \mathcal{I}$, may be equivalently rewritten in the form[13]

$$0 \in \sum_{i \in \mathcal{I}} t_i(\mathcal{P}_i(x_i) - \omega_i).$$

Thus in this case condition $x \in \mathcal{C}^f(\mathcal{E})$ is equivalent to[14]

$$0 \notin \mathrm{co}[\bigcup_{\mathcal{I}}(\mathcal{P}_i(x_i) - \omega_i)], \tag{14}$$

that after applying separation theorem allows to conclude that the elements of the fuzzy core are quasiequilibria. Below we propose other useful in applications characterizations of fuzzy core points presented in "geometrical" terms. To this end, let us consider the sets

$$\Omega_i(x_i) = \mathrm{co}(\mathcal{P}_i(x_i) \cup \{\omega_i\}),\quad i \in \mathcal{I}.$$

Due to the convexity of $\mathcal{P}_i(x_i)$, for $\mathcal{P}_i(x_i) \neq \emptyset$, conclude

$$\mathrm{co}(\mathcal{P}_i(x_i) \cup \{\omega_i\}) = \bigcup_{0 \leq \lambda \leq 1}[\lambda \mathcal{P}_i(x_i) + (1-\lambda)\omega_i] = \bigcup_{0 \leq \lambda \leq 1} \lambda(\mathcal{P}_i(x_i) - \omega_i) + \omega_i,\quad i \in \mathcal{I}.$$

This implies that the condition $z + \omega \in \prod_{\mathcal{I}} \Omega_i(x_i)$, where $\omega = (\omega_1, \ldots, \omega_n)$, is equivalent to the existence of $0 \leq \lambda_i \leq 1$ and $[y_i \in \mathcal{P}_i(x_i) \neq \emptyset$ and $y_i = \omega_i$, if $\mathcal{P}_i(x_i) = \emptyset]$, $i \in \mathcal{I}$ such that

$$z = (\lambda_1(y_1 - \omega_1), \ldots, \lambda_n(y_n - \omega_n)).$$

Hence, due to (3), (4)

$$x \in \mathcal{C}^f(\mathcal{E}) \iff \nexists\, z \in E^{\mathcal{I}},\ z \neq 0:\ \ z + \omega \in \prod_{\mathcal{I}} \Omega_i(x_i)\ \ \&\ \ \sum_{i \in \mathcal{I}} z_i = 0 \iff$$

[13] Admitting some inaccuracy in formulas here and below, we identify a vector with one-element set containing it.

[14] Clearly, for a dominating fuzzy coalition t one may always think that $\sum_{i \in \mathcal{I}} t_i = 1$.

$$\prod_{\mathcal{I}} \Omega_i(x_i) \bigcap \{(z_1, \ldots, z_n) \in E^{\mathcal{I}} \mid \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \omega_i\} = \{\omega\}. \quad (15)$$

Notice that characterization (15) is also valid for satiated preferences. In doing so we have proven the following

Proposition 2 *An allocation $x \in \mathcal{A}(X)$ is the element of fuzzy core if and only if relation (15) is true.*

In the case of a *2-agent economy*, condition (15) may be rewritten in the form

$$\Omega_1(x_1) \cap (\bar{\omega} - \Omega_2(\bar{\omega} - x_1)) = \{\omega_1\}, \quad \bar{\omega} = \omega_1 + \omega_2.$$

Hence,

$$(x_1, x_2) \notin \mathcal{C}^f(\mathcal{E}) \iff \exists \textit{ ray } \text{starting at the point } \omega_1, \text{ which intersects both sets, } \mathcal{P}_1(x_1) \text{ and } \bar{\omega} - \mathcal{P}_2(\bar{\omega} - x_1) = \widetilde{\mathcal{P}}_2(x_2).$$

Figure 4 presents a graphic illustration of conducted analysis in the Edgeworth's box for a 2-goods economy. In this case, an allocation x lying in the fuzzy core is equivalent to the convex hulls of $\mathcal{P}_1(x_1) \cup \{\omega_1\}$ and of $[\bar{\omega} - \mathcal{P}_2(\bar{\omega} - x_1)] \cup \{\omega_1\}$ having only one point, ω_1, in common.

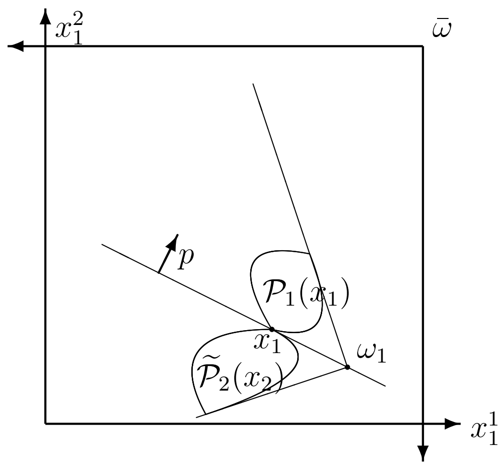

Figure 4: Fuzzy core

Further we are going to reveal the specific of fuzzy core allocations in pure contractual terms. We start from a preliminary result describing mathematical properties of fuzzy contractual allocations that is of interest in its own right.

Proposition 3 *A lower properly contractual allocation $x \in \mathcal{A}(X)$ is fuzzy contractual if and only if*

$$\prod_{\mathcal{I}} [(\mathcal{P}_i(x_i) + \text{co}\{0, \omega_i - x_i\}) \cup \{\omega_i\}] \bigcap \{(z_1, \ldots, z_n) \in E^{\mathcal{I}} \mid \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \omega_i\} = \{\omega\}. \quad (16)$$

Notice that in this proposition $\mathcal{P}_i(x_i) = \emptyset$ is possible for some $i \in \mathcal{I}$: by definition $\emptyset + A = \emptyset$ for any $A \subseteq E$. Also recall that being lower properly contractual an allocation $x \in \mathcal{A}(X)$ has to satisfy (6) that can be directly incorporated into the statement.

Figure 5 illustrates Proposition 3 result in the Edgeworth's box. Here $\widetilde{\mathcal{P}}_2(x_2) = \bar{\omega} - \mathcal{P}_2(\bar{\omega} - x_1)$ and one can see that preferred bundles are extended along linear segment with endpoints x_1, ω_1.

The statement of Proposition 3 can be reformulated in another form, that may be useful in applications.

Corollary 3 *An allocation $x \in \mathcal{A}(X)$ is fuzzy contractual* $\iff$

$$(\mathcal{P}_i(x_i) + \mathrm{co}\{0, \omega_i - x_i\}) \times \prod_{j \neq i, j \in \mathcal{I}} [(\mathcal{P}_j(x_j) + \mathrm{co}\{0, \omega_j - x_j\}) \cup \{\omega_j\}] \bigcap \mathcal{A}(E^{\mathcal{I}}) = \emptyset, \quad (17)$$

for each $i \in \mathcal{I}$, where $\mathcal{A}(E^{\mathcal{I}}) = \{(z_1, \ldots, z_n) \in E^{\mathcal{I}} \mid \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \omega_i\}$.

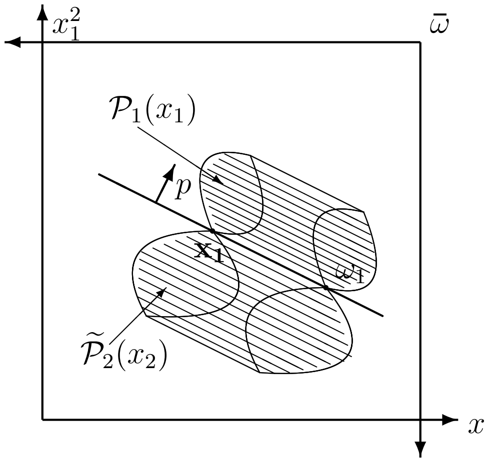

Figure 5: Fuzzy contractual allocations

Proof of Proposition 3. Let x be a fuzzy contractual allocation implemented by a proper web V, i.e., $x = x(V)$ for some web V, satisfying Definition 7. Suppose that (16) is false and therefore does exist $y = (y_i)_{\mathcal{I}} \neq \omega$ which belongs to the left part of equality (16). Consider coalition $S = \{i \in \mathcal{I} \mid y_i \neq \omega_i\}$. Notice $\mathcal{P}_i(x_i) \neq \emptyset$, $i \in S$ and find $z_i \in \mathcal{P}_i(x_i)$, $i \in S$ such that $y_i = z_i + \lambda_i(\omega_i - x_i)$, for some real $0 \leq \lambda_i \leq 1$, $i \in S$ and $y_i = \omega_i$, $i \notin S$. Determine $u_i = y_i - \omega_i$, $i \in \mathcal{I}$. Since $\sum_{i \in \mathcal{I}} y_i = \sum_{i \in \mathcal{I}} \omega_i$ then $\sum_{\mathcal{I}} u_i = 0$ and therefore $u = (u_i)_{i \in \mathcal{I}}$ is a contract with $\mathrm{supp}(u) = S \neq \emptyset$. One can write

$$z_i = y_i - \omega_i + \lambda_i(x_i - \omega_i) + \omega_i = u_i + \lambda_i \sum_{v \in V} v_i + \omega_i, \quad i \in S.$$

Now for all $v \in V$ put $t_i = t_i^v = \lambda_i$, $i \in S$, and $t_i = t_i^v \in [0, 1]$, $i \notin S$ and apply $T(V) = \{t^v\}_{v \in V}$ for allocation $x = x(V)$. We have $x^T = \omega + \Delta(V^T)$, where by construction $x_i^T = \omega_i + t_i(x_i - \omega_i)$, $\forall\, i \in \mathcal{I}$. Therefore by construction

$$u_i + x_i^T = z_i \in \mathcal{P}_i(x_i), \quad \forall\, i \in S.$$

Thus x^T is not upper contractual and this contradicts the fact that allocation x is fuzzy contractual.

Show that if a lower contractual allocation x satisfies (16) then it is fuzzy contractual relative to web $V = \{x - \omega\}$. Assume contrary and find $T = \{t\}$ and a contract $u = (u_i)_{\mathcal{I}}$, $\mathrm{supp}(u) = S \neq \emptyset$, such that

$$u_i + t_i(x_i - \omega_i) + \omega_i \in \mathcal{P}_i(x_i), \quad \forall\, i \in S \iff z_i = u_i + \omega_i \in \mathcal{P}_i(x_i) + t_i(\omega_i - x_i), \quad \forall\, i \in S.$$

Let us determine $z_i = \omega_i$ for $i \notin S$. Now due to contract's definition conclude $\sum_{i\in\mathcal{I}} z_i = \sum_{i\in\mathcal{I}} \omega_i$ that implies the allocation $z \neq \omega$ belongs to the left part of (16) and this is a contradiction. ■

Notice that as soon as for every feasible allocation $x = (x_i)_{\mathcal{I}}$ we have

$$\omega_i \in \Omega_i(x_i) \subset (\mathcal{P}_i(x_i) + \mathrm{co}\{0, \omega_i - x_i\}) \cup \{\omega_i\}, \quad \forall\, i \in \mathcal{I},$$

then due to Propositions 2, 3 every fuzzy contractual allocation belongs to fuzzy core of economy. However, in general, the property of an allocation to be fuzzy contractual is still a little bit stronger than being an element of fuzzy core. The following result clarifies the relationships between two fuzzy notions.

Lemma 2 *Let $x \in \mathcal{A}(X)$ and $\mathcal{P}_i(x_i) \neq \emptyset$ for all $i \in \mathcal{I}$. Then $x \in \mathcal{C}^f(\mathcal{E})$ implies:*

$$\prod_{\mathcal{I}}(\mathcal{P}_i(x) + \mathrm{co}\{0, \omega_i - x_i\}) \bigcap \{(z_1, \ldots, z_n) \in E^{\mathcal{I}} \mid \sum_{i\in\mathcal{I}} z_i = \sum_{i\in\mathcal{I}} \omega_i\} = \emptyset. \qquad (18)$$

The comparing of formulas (18) and (17) makes clearer the difference between fuzzy core allocation and fuzzy contractual one. One can see that this difference is not too big that allows us to interpret allocations from fuzzy core as fuzzy contractual ones.[15] Moreover, the fact that every element of fuzzy core is a quasiequilibrium (this is why fuzzy core is so popular in existence theory) can be also easily derived from formula (18). In fact, separating sets in (18) by a (non-zero) linear functional $\pi = (p_1, \ldots, p_n) \in E^{\mathcal{I}}$ one can conclude:

(i) $p_i = p_j = p \neq 0$ for each $i, j \in \mathcal{I}$; this is so because π is bounded on $\mathcal{A}(E^{\mathcal{I}}) = \{(z_1, \ldots, z_n) \in E^{\mathcal{I}} \mid \sum_{i\in\mathcal{I}} z_i = \sum_{i\in\mathcal{I}} \omega_i\}$. So, one can take p as a price vector.

(ii) Due to construction and in view of preferences are locally non-satiated at the point $x \in \mathcal{A}(X)$ the points x_i and ω_i belong to the closure of $\mathcal{P}_i(x) + \mathrm{co}\{0, \omega_i - x_i\}$. Therefore via separating property we have $\sum_{j\neq i} p\omega_j + px_i \geq \sum_{\mathcal{I}} p\omega_j \Rightarrow px_i \geq p\omega_i$ $\forall i \in \mathcal{I}$, that is possible only if $px_i = p\omega_i$ $\forall i \in \mathcal{I}$. So, we obtain budget constrains for consumption bundles.

(iii) By separation property for each i we also have $\langle p, \mathcal{P}_i(x) + \mathrm{co}\{0, \omega_i - x_i\}\rangle \geq p\omega_i$, that by *(ii)* implies $\langle p, \mathcal{P}_i(x)\rangle \geq px_i = p\omega_i$. So we proved that p is (quasi)equilibrium prices for allocation $x = (x_i)_{i\in\mathcal{I}}$.

As a result one can see that if an economic model is such that every quasiequilibrium is equilibrium then every fuzzy core allocation is fuzzy contactual one and therefore two fuzzy concepts are equivalent each other. Conditions delivering this fact are well known in literature, for example it is the case when an economy is irreducible.

Proof of Lemma 2. The argument in the proving of lemma's result is based on Propositions 2 and relation (15), which characterizes fuzzy core elements. We need to show that (15) implies (18).

[15] Earlier in literature allocations from fuzzy core were interpreted only as Edgeworth's equilibria and served more technical tool than an economic concept.

Assume that x satisfies (15) and suppose that (18) is false. This implies that there is a vector $t = (t_1, \ldots, t_n)$, $0 \leq t_i \leq 1$ and bundles $z_i \succ_i x_i$, $i \in \mathcal{I}$ such that

$$\sum_{\mathcal{I}} z_i + \sum_{\mathcal{I}} t_i(\omega_i - x_i) = \sum_{\mathcal{I}} \omega_i \tag{19}$$

holds. Now for a real $0 < \beta \leq \frac{1}{2}$ consider the vector $y = y(\beta) = (y_i)_{i \in \mathcal{I}}$, where

$$y_i(\beta) = \beta[z_i + t_i(\omega_i - x_i)] + (1 - \beta)x_i, \quad i \in \mathcal{I}.$$

In view of (19) and $x \in \mathcal{A}(X)$ we have $\sum_{\mathcal{I}} y_i(\beta) = \sum_{\mathcal{I}} \omega_i$ for every β. Now vectors $y_i(\beta)$ can be presented in the form

$$y_i(\beta) = (1 - \beta t_i)x_i + \beta t_i \omega_i + (1 - \beta t_i)\frac{\beta}{1 - \beta t_i}(z_i - x_i), \quad i \in \mathcal{I},$$

where by the choice of β we have $\mu_i = \frac{\beta}{1 - \beta t_i} \leq 1$. This due to (**A**) for $i \in \mathcal{I}$ implies

$$\mu_i(z_i - x_i) \in \mathcal{P}_i(x) - x_i \;\Rightarrow\; \exists\, \eta_i \in \mathcal{P}_i(x) : \mu_i(z_i - x_i) = \eta_i - x_i.$$

Therefore the previous formula gives

$$y_i = (1 - \beta t_i)\eta_i + \beta t_i \omega_i,$$

that implies $y_i \in \Omega_i(x_i)$, $i \in \mathcal{I}$. This allows us to apply relation (15), concluding $y = y(\beta) = \omega$ for *all* real $0 < \beta \leq \frac{1}{2}$. Write this equality componentwise and due to $y_i(\beta)$ specification find

$$\beta[z_i + t_i(\omega_i - x_i)] + (1 - \beta)x_i = \omega_i \;\Rightarrow\; z_i + t_i(\omega_i - x_i) = x_i + \frac{\omega_i - x_i}{\beta}$$

that has to be true for all $i \in \mathcal{I}$ and *all* $0 < \beta \leq \frac{1}{2}$. However these equalities (consider *different* β) can be true only if $x_i = \omega_i = z_i$, $i \in \mathcal{I}$, that due to the choice of z_i implies $x_i \succ_i x_i$ and contradicts to (**A**). Proof is completed. ■

Conclusion

The paper presented the basic elements of the barter contracts theory, developed for exchange economies. There was considered a series of stability concepts of contractual allocations and the webs of contracts and there were revealed their relationships with notions known in classical theory. Now equilibrium, Pareto boundary, core, fuzzy core obtain a clear description in contractual categories. It was shown that contract based approach is rather convenient to model perfect competition conditions and presents alternative forms of it. So, contract based approach seems to be a natural supplement to classical one and may work fruitfully in the investigations of modern non-classical models.

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