Spatial Equilibrium on the Plane and an Arbitrary Population Distribution

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Abstract

The existence of immigration proof partition for communities (countries) in a multidimensional space is studied. This is a Tiebout type equilibrium which existence previously was studied under weaker assumptions (measurable density, fixed centers and so on). The migration stability suggests that the inhabitants of frontier have no incentives to change jurisdiction (an inhabitant at every frontier point has equal costs for all possible adjoining jurisdictions). It is required that inter-country border is represented by a continuous curve (surface). Assuming population is distributed in one or two dimension area (convex compact) and this distribution is described by Radons measure, we prove that for an arbitrary number of countries there exists stable partition into countries. The proof is based on Kakutani's fixed point theorem applied for specific approximation of initial problem with the subsequent passing to the limits.

Introduction

In the seminal paper [Alesina & Spolaore, 1997] a basic model of country formation was offered. In this model, the cost of the population is described as the sum of the two values—the ratio of total costs on the total weight of the population plus transportation costs to the center of the state. This model has been studied in a number of subsequent studies, but in each of them deals with the case of one-dimensional region and the interval-form countries (country formation on the interval $[0, 1]$).

The first progress in the resolution of the problem of existence was obtained in [Le Breton et al., 2010], where well known Gale–Nikaido–Debreu lemma was applied to state the existence of *nontrivial* immigration proof partition for interval countries, i.e. such that no one has incentive to change their country of residence. In [Le Breton et al., 2010] were made rather strong assumptions on the distribution of the population—continuous density, separated from zero. Next in [Marakulin, 2017] mathematical part of the approach was significantly strengthened and extended to the case of distribution of the population, described as a Radon measure (probability measure defined on the Borel σ-algebra). In [Savvateev et al., 2016] a new significant advancement was

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suggested, it disseminates the result (existence theorem) to the case of 2-dimensional (or more) region. The proof of [Savvateev et al., 2016] is very elegant and is based on the application of KKM-lemma (Knaster–Kuratowski–Mazurkiewicz), but the result is essentially limited by the presence of fixed in the space positions of capitals and absolute continuous measure of the population distribution. The goal of our paper is to show how this result can be extended to a more general case of Radon measure.

The problem of division of the rectangular area $\square ABCD$ in $\mathbb{R}^2$ into two countries at a given measurable random distribution of the population is studied. There is described a basic one-dimensional approximation, for which a fixed point (via Kakutani's theorem) can be found, and then the limit process gives the result. The existence theorem and its proof for the population distribution, which is absolutely continuous with respect to the Lebesgue measure, constitute the first section of the paper. The second section provides a further generalization and extends the existence result to an arbitrary distribution of the population, described as a Radon measure. The result generalises [Marakulin, 2017] to two-dimensional case. Indeed, the assumption for a measure to be absolutely continuous with respect to the Lebesgue measure seems unrealistic one: What happens with the urban population? There are no cities at all?.. and how the existence of unoccupied areas can hinder the division into countries? It is clear that the reason for these misunderstandings has purely mathematical nature and the solution can be found by applying an appropriate mathematical technique. This is the focus of this work. Now the distribution of inhabitants is described using the Radon measure μ—this is a countably additive probability measure defined on a Borel σ-algebra. By virtue of the theorem known in mathematical analysis, the so-called Lebesgue decomposition, the measure μ can be represented as a sum of a purely discrete measure ν and absolutely continuous with respect to the Lebesgue measure ϑ, i. e. $\mu = \nu + \vartheta$. The absolutely continuous component corresponds to the distribution of the rural population. At the same time, the purely discrete term ν corresponds to the urban population, for then the measure has a counting carrier and is represented as the sum of measures concentrated at a point (Dirac measure) of the form $\delta(\{a\}) = \alpha = \delta(A)$ for all $A \subseteq \square ABCD$ including the point a, and zero otherwise. Here α can be understood as a population (mass) of the city, concentrated (located) at the point a. Of course, from interesting considerations, we are only interested in the case of a finite number of cities. And the last: what kind of divisions into countries exists and can be realized?—in the formulation of [Le Breton et al., 2010], [Savvateev et al., 2016], this question does not matter because the measure of single-point sets is zero. In our case this is not so, since there are points with non-zero mass. In this case, the point-city can be divided by mass into two unequal parts, one of which belongs to one border state, the other to another one. It can also happen a curious thing such as the emergence of country-cities of zero size (area). Such cities as Jerusalem, Rome with the Vatican, previously—divided Berlin etc. can serve as a real illustration of the theoretical conclusion.

1 The partition into two countries on the plane

The division of the one-dimensional world on countries can not be considered as a satisfactory solution of the problem but two-dimensional formulation seems fundamentally more difficult. Now for a particular example of division of the rectangular area in two countries we consider an approximating design allowing to find a solution by passing to the limit.

First, we define the principle of stability applied for the country located on the plane. As in the case of one-dimensional world, it must be such division that border residents have no incentive to change their jurisdiction. Thus, the costs for any border resident should be the same with respect to any of the possible for her/him adjoining jurisdictions. It is assumed that the borders between the two countries allow continuous parametrization, i.e. they are the image of the interval from $\mathbb{R}$ for some *continuous one-to-one* mapping. As a result, as in the one-dimensional case, the function of individual costs of individuals should be continuous on *the whole field* of country division, that is, country partition must implement continuous “gluing” of country-depended individual costs.

For simplicity, we consider now a particular case of a rectangular area of possible settlement represented in the diagram 1 rectangle $\square ABCD$. We assume that $c_i(x, y, \cdot)$, $i = 1, \ldots, n$ be the functions of individual costs, depending on the place of individual location—defined via coordinates $(x, y) \in \square ABCD$, the weight of the resident jurisdiction $\mu_i(S_i)$, the location of its center $r_c(S_i)$, metrics $\rho(\cdot, \cdot)$ (to specify the distance to the center), cost of government g_i and so on. The basic model representation of these cost functions is

$$c_i(x, y, \mu_i(S_i), r_c(S_i)) = \frac{g_i}{\mu_i(S_i)} + \rho((x, y), r_c(S_i)), \quad g_i > 0, \quad i \in N. \tag{1}$$

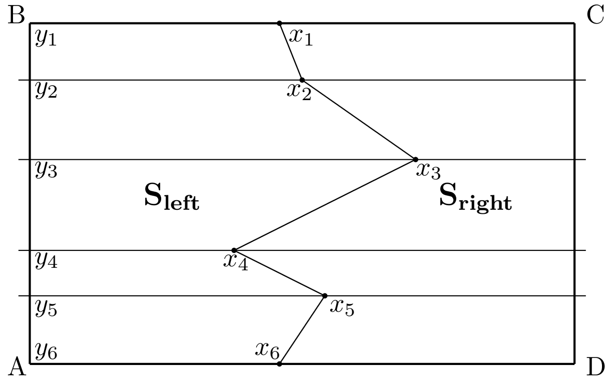

Figure 1: Possible division into two countries of the rectangular area $ABCD$, $m = 6$

In general these functions may have sufficiently general form but they always continuously depend on certain country parameters and obey some other specific assumptions, see [Savvateev et al., 2016, Marakulin, 2016].

For the cost functions $c_1(\cdot)$, $c_2(\cdot)$ specified by (1) possible inter-countries boundaries are defined by equation

$$||(x,y) - r_c(S_1)||_2 - ||(x,y) - r_c(S_2)||_2 = \frac{g_2}{\mu(S_2)} - \frac{g_1}{\mu(S_1)} = const. \tag{2}$$

Now a possible border has hyperbolic form and, for Euclidean distance, (2) defines the classic hyperbola. In the next subsection we shall assume

(**P**) *The distribution of population is described by an absolutely continuous probability measure μ such that* $\text{supp}(\mu) = \square ABCD$.[1]

1.1 Partition of an area on plane via one-dimensional approximation

The idea of approach is that given coordinate system (potentially curved), a stable partition relative to one-dimensional world appeared along every coordinate line. At the same time, the function of individual costs must be calculated relative to the position of "center" of the country and the general population distributed in *two-dimensional space*. Finding such a partition is not an easy task, to solve this we shall apply a special "one-dimensional approximation", relative which a country partition can be found by a fixed point theorem (Brouwer or Kakutani).

The construction as follows: specify $m-2$ straight lines parallel to the base of the rectangle, $m \geq 3$. Let the lower base has a number m, top one—the number 1 and all others are numbered from top to bottom. Each i-th segment is divided into two parts by the point x_i, which can be considered the point from interval $[0,1]$ (length of the base $\square ABCD$), $i = 1, \dots, m$. Straight line segments connecting consecutive points $x_1, \dots, x_m$, form a polygon line, which we accept as the border between the left and right countries. Now, if density $f(x,y)$ is presented then it is possible to integrate it over each of the country area, finding the weights (size) $\mu(S)$ of their populations.

Within each country its "center" (the capital) $r_c(S) \in S$ is specified, the position of which we will consider as *depending continuously* from a given country settings $\mathbf{x} = (x_1, \dots, x_m) \in [0,1]^m$. Thus we have:

$$\mu(\mathbf{S_{left}}) = \int_{\mathbf{S_{left}}} f(x,y)dxdy \geq 0, \quad r_c(\mathbf{S_{left}}) = r_{\mathbf{left}}(x_1, \dots, x_m) \in \mathbf{S_{left}}$$

$$\mu(\mathbf{S_{right}}) = \int_{\mathbf{S_{right}}} f(x,y)dxdy \geq 0, \quad r_c(\mathbf{S_{right}}) = r_{\mathbf{right}}(x_1, \dots, x_m) \in \mathbf{S_{right}}.$$

Moreover, without loss of generality

$$\mu(\mathbf{S_{left}}) + \mu(\mathbf{S_{right}}) = 1.$$

The fact that we are talking about the "mass of the population" (population) and the "distance to the center" (transport availability of capital) as the main parameter determining the costs of individuals in a country, it is only the interpretation of the cost function in the context of the main model variant. The same can be said about the property of the center of the country be located on its territory—it's just a natural variant of content,

[1]This combined means that $\mu(A) > 0 \iff \int_A dxdy > 0$ for every measurable $A \subseteq \square ABCD$.

from a mathematical point of view, the center could be anywhere. What is really important is (described below) certain properties of their individual costs.

Next, consider a point-to-set mapping, whose fixed point gives the desired country partition. The construction of mapping applies the ideas borrowed from the one-dimensional case, see [Marakulin, 2017]. Let

$$X = [0,1]^m,$$

and define a point-to-set mapping of X into itself.

Let $c_1(\cdot)$, $c_2(\cdot)$ be the functions of individual costs depending on the weight of the jurisdiction population $\mu_1(\mathbf{x})$, $\mu_2(\mathbf{x})$, location of its center $r_c(S_1)$, $r_c(S_2)$, metrics $\rho(\cdot,\cdot)$ (to determine the distance to the center) and a place of the individual location specified by coordinates $(x,y) \in \Box ABCD$. The basic model representation of these functions is (1). Now we shall think that they are functions of general form continuously depending on $\mathbf{x} = (x_1, \dots, x_m) \in [0,1]^m$ for $\mu(S_k(\mathbf{x})) > 0$, $k = 1, 2$. Additionally assume that

(*i*) $c_k(x, y, \mathbf{x}) > 0$ for $\mu(S_k) \neq 0$ and

(*ii*) $c_k(x, y, \mathbf{x}) \to +\infty$ if $\mu(S_k) \to 0$, $k = 1, 2$.

For the functions of (1) this condition is always satisfied. At the same time, if the density $f(x, y)$ of the population is so that $\int_A dxdy > 0$ implies $\int_A f(x,y)dxdy > 0$ for every measurable subset $A \subset \Box ABCD$ (i.e. each subset of nonzero area (Lebesgue measure) has a population of non-zero mass), the latter requirement is equivalent to

$$c_1(x, y, \mathbf{x}) \to +\infty \iff \mathbf{x} \to (0, \dots, 0) \;\; \& \;\; c_2(x, y, \mathbf{x}) \to +\infty \iff \mathbf{x} \to (1, \dots, 1). \tag{3}$$

For the boundary points $x_1, \dots, x_m$ of country areas let us find an excess cost of possible (two) jurisdictions (constants $y_1, \dots, y_m$ in the argument are excluded)

$$h_i(\mathbf{x}) = c_1(x_i, \mathbf{x}) - c_2(x_i, \mathbf{x}), \quad i = 1, \dots, m.$$

Notice that (3) implies that for all $i = 1, \dots, m$, $h_i(\mathbf{x}) \to +\infty$ for $\mathbf{x} \to 0$, and $\mathbf{x} \to \mathbf{1}$ when $h_i(\mathbf{x}) \to -\infty$.

Next we define the (single-valued) map $\varphi : X \to X = [0,1]^m$ putting

$$\varphi_i(\mathbf{x}) = \begin{cases} x_i - \frac{x_i}{2} \cdot \frac{h_i(\mathbf{x})}{1+h_i(\mathbf{x})}, & \text{for } \; h_i(\mathbf{x}) \geq 0, \\ x_i + \frac{1-x_i}{2} \cdot \frac{h_i(\mathbf{x})}{h_i(\mathbf{x})-1}, & \text{for } \; h_i(\mathbf{x}) \leq 0. \end{cases} \tag{4}$$

By construction, this mapping is well defined everywhere on X with the exception of two points $\mathbf{x} = \mathbf{0} = (0, \dots, 0)$ and $\mathbf{x} = \mathbf{1} = (1, \dots, 1)$, which values can be specified by continuity:

$$\varphi(\mathbf{0}) = (0, \dots, 0), \quad \varphi(\mathbf{1}) = (1, \dots, 1).$$

It is obvious that according to the construction these points are *trivial* fixed points of $\varphi(\cdot)$, that does not comply with the requirements of the division of rectangular area. Further construction and analysis will focus on the finding of the *nontrivial* fixed point corresponding to the division of the area into two countries with non-zero masses of the population.

Now we define a point-to-set mapping Φ from $\mathfrak{X} = X \times \Delta_2$ to X by formula: for $(\mu_1, \mu_2) = (\mu(\mathbf{S_{left}}(\mathbf{x})), \mu(\mathbf{S_{right}}(\mathbf{x})))$ specify

$$\Phi(\mathbf{x}, \nu) = \begin{cases} \{\frac{\nu_1}{\mu_1}\varphi(\mathbf{x})\}, & \text{if } \; \nu_1 \leq \mu_1, \;\; \mu_1 \neq 0, \\ \{\frac{\nu_2}{\mu_2}\varphi(\mathbf{x}) + \frac{\mu_2 - \nu_2}{\mu_2}(1, \dots, 1)\}, & \text{if } \; \nu_2 \leq \mu_2, \;\; \mu_2 \neq 0, \\ X, & \text{for } \; \nu_1 = \mu_1 = 0, \;\; \text{or} \;\; \nu_1 = \mu_1 = 1. \end{cases} \tag{5}$$

The second mapping $\Psi : X \rightrightarrows \Delta_2$ is specified as follows

$$\Psi(\mathbf{x}) = \underset{\nu \in \Delta_2}{\operatorname{argmax}} \langle H(\mathbf{x}), \nu \rangle. \tag{6}$$

where $H(\mathbf{x}) = (H_1(\mathbf{x}), H_2(\mathbf{x}))$ and

$$I_+ = \{i \mid h_i(\mathbf{x}) \geq 0, \; i = 1, \dots, n\}, \quad I_- = \{i \mid h_i(\mathbf{x}) \leq 0, \; i = 1, \dots, n\}$$

are defined by formulas[2]

$$H_1(\mathbf{x}) = [\inf_{i=1,\dots,m} h_i(\mathbf{x})]^+ + \sum_{i\in I_+} x_i \tfrac{h_i(\mathbf{x})}{h_i(\mathbf{x})+1}, \qquad I_+ \neq \emptyset$$
$$H_2(\mathbf{x}) = [\sup_{i=1,\dots,m} h_i(\mathbf{x})]^- + \sum_{i\in I_-} (1-x_i) \tfrac{h_i(\mathbf{x})}{h_i(\mathbf{x})-1}, \qquad I_- \neq \emptyset.$$

If $I_+ = \emptyset$ or $I_- = \emptyset$, then by definition $H_1(\mathbf{x}) = 0$ and $H_2(\mathbf{x}) = 0$ respectively. Constructed map is well defined everywhere except at zero and one for which we postulate

$$\Psi(\mathbf{0}) = (1,0), \qquad \Psi(\mathbf{1}) = (0,1).$$

Finally, we define the resulting mapping

$$\Upsilon : \mathfrak{X} \Rightarrow \mathfrak{X}, \qquad \Upsilon(\mathbf{x},\nu) = \Phi(\mathbf{x},\nu) \times \Psi(\mathbf{x},\nu),$$

which fixed points give us the desired result. The following lemmas describes the important properties of the mapping $\Upsilon(\cdot)$.

Lemma 1 *The mapping $\Upsilon : \mathfrak{X} \Rightarrow \mathfrak{X}$ is a Kakutani map, i.e. it has closed graph and for every $\kappa \in \mathfrak{X}$ takes non-empty convex values.*

Proof of the lemma is omitted.

Lemma 2 *Under the above assumptions, the map $\varphi(\cdot)$ has **nontrivial** fixed point in X such that the mass of the population of each country is **nonzero**.*

The proof of Lemma 2 can be found in [Marakulin, 2016]. Figure 2 illustrates the result of Lemma 2 and following Theorem 1.

Theorem 1 *Let the individual costs be given by (1) and centers be situated on a line parallel to the axis of abscissa. Then for each positive integer $m \in \mathbb{N}$ there exists the partition of $\square ABCD$ into two countries $\mathbf{S_{left}}(\mathbf{x})$ and $\mathbf{S_{right}}(\mathbf{x})$, with piecewise linear boundary formed by the points $x_k, \dots, x_l$, $1 < k+1 \leq l-1 < m$ where all $x_{k+1}, \dots, x_{l-1}$ are immigration proof.*

Corollary 1 *Assume that the costs in formula (1) are calculated relative to the Euclidean distance. Then in the conditions of Theorem 1 boundary points $x_{k+1}, \dots, x_{l-1}$ are suited on classical hyperbola and for a general form of the metric (e.g. p-norm), these points belong to a generalized hyperbola.*

Theorem 2 *Let for the rectangle $\square ABCD$ individual costs be defined by (1) and centers of the country be located on a line parallel to the axis of abscissa. Then there is immigration proof division into two countries $\mathbf{S_{left}}$ and $\mathbf{S_{right}}$ with a continuous border.*

It is immaterial fact that the considered area is a rectangular. This result holds for any convex closed bounded domain. It also follows that the centers can be located on any fixed line (turn the area so that the line is parallel to the axis of abscissa), and it can be *any pair of fixed points.*[3] Due to the volume of the paper constraints me omit (more or less standard reasonings via passing to the limits) the proof of this theorem.

1.2 General partition for a discontinuous population distribution

Following the construction of the first section, consider the rectangular region $\square ABCD$, which must be divided into two countries. Suppose that in this region there is a unit mass of inhabitants whose distribution *in contrast* to the previous section is described by the probability Radon measure μ.[4] In this case we *will not assume* the

[2]We use standard notations $z^+ = \sup\{z, 0\}$ and $z^- = \sup\{(-z), 0\}$ for any real z.
[3]Note that this is one of the fundamental differences between the two-dimensional setting and one-dimensional one.
[4]This is an internally regular measure defined on a Borel sigma-algebra. I will also recall that the Borel σ-algebra is specified by a topology. Set-theoretic operations are performed relative to open subsets, and in the countable number of times, obtaining different (and only such) elements of algebra. It is important that every continuous function is measurable (integrable) with respect to the Borel σ-algebra.

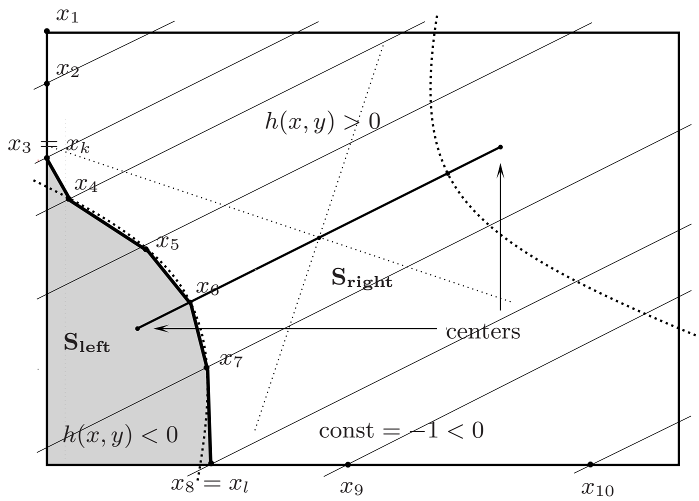

Figure 2: Partition according to (i)–(ii) for $const < 0 \iff g_2\mu(\mathbf{S_{left}}) < g_1\mu(\mathbf{S_{right}})$.

existence of density for the μ, but we will also postulate that the discrete component has a finite carrier. Indeed, it is known that for any (nonnegative) Borel measure there is an *Lebesgue decomposition* in a sum of continuous, discrete and singular components. According to an interpretation we are interesting in the case $\mu = \vartheta + \nu$, where

$$\vartheta(B) = \int_B h(x)dx, \quad B \in \mathcal{B},$$

for some Lebesgue measurable function $h : \square ABCD \to \mathbb{R}_+$ and, for an countable family of pairwise distinct points $y_k \in \square ABCD$, we have

$$\nu(\{y_k\}) > 0, \quad k = 1, 2, \ldots, \quad \nu(B) = \sum_{k: y_k \in B} \nu(\{y_k\}), \quad B \in \mathcal{B}.$$

Thus, the entire population of the region is divided into two parts—a village one, somehow approximately uniformly continuously distributed on $\square ABCD$ and the urban population distributed over not more than a countable number of points—the population of the k-th city is the value $\mu(\{y_k\}) > 0$, $k = 1, 2, \ldots$. Since the existence of a countable-infinite carrier for ν is untenable from a substantive point of view, we postulate that it is *finite*:

$$\mathrm{card}[\mathrm{supp}(\nu)] < +\infty.$$

In the first section it was proved that for a probabilistic absolutely continuous measure, an immigration-consistent partition into countries does exist, but what will happen in the general case? In the subsequent analysis we need some functional spaces and topologies that are relevant in the formulation of the general case. Indeed, for a compact set $K \subset \mathbb{R}^2$, the Radon measures form a simplex in the space $ca(K)$—it is the space of all ***c***ountably ***a***dditive measures defined on Borel σ-algebra on K. In turn, $ca(K)$ is isomorphic to the space of all linear continuous functionals over the space of continuous functions $C(K)$, considered with the norm topology (maximum or uniform convergence). The value of the functional φ_μ which is associated with the measure μ is presented by the formula

$$\varphi_\mu(f) = \int f(x)d\mu(x), \quad f \in C(K).$$

Thus we have the right to write $[C(K)]' = ca(K)$ and consider duality (or pairing) $\langle C(K), ca(K)\rangle$, in which the bilinear map $\langle \cdot, \cdot \rangle$ (inner product) is given by formula

$$\langle f, \mu \rangle = \int f(x)d\mu(x), \quad f \in C(K), \quad \mu \in ca(K).$$

For a duality, it is customary to consider and study the various topologies induced by it; the most important among them are weak topologies. Weak topology can be defined on the source space (here it is $C(K)$) or conjugate (now it is $ca(K)$), in the latter case it is usually called *weak star* topology. By definition, the weak topology is the weakest locally convex topology for which only functionals from the second duality specify continuous linear functionals. In our case, $\sigma(C(K), ca(K))$ is weak topology on $C(K)$ (by tradition is denoted by σ), is specified by the space $ca(K)$. A weak star topology $\sigma^*(ca(K), C(K))$ is defined for $ca(K)$ and is specified via $C(K)$. Several characterizations can be given for a weak topologies. It is convenient to characterize them in terms of convergence (a directed family or network). So for $\sigma^*(ca([0,1]), C([0,1]))$:

$$\mu_\xi \underset{\xi\in\Xi}{\longrightarrow} \mu \quad \Longleftrightarrow \quad \forall g \in C(K), \quad \int g(x) d\mu_\xi(x) \underset{\xi\in\Xi}{\longrightarrow} \int g(x) d\mu(x). \tag{7}$$

Weak topologies are often called topologies of pointwise convergence.

The general case of population distribution is analyzed through the passing to the limit for the already proven case with a continuous density, i. e. I want to find a family $f_\xi \in C(K)$ such that for the measures $\mu(f_\xi) = \mu_\xi$, defined by

$$\mu_\xi(B) = \int_B f_\xi(x) dx, \quad B \in \mathcal{B}, \tag{8}$$

we would have $\mu_\xi \underset{\xi\in\Xi}{\longrightarrow} \mu$. It is asserted that the space $C(K)$ is dense in $ca(K)$ relative to the weak* topology $\sigma^*(ca(K), C(K))$, that is, its closure in σ^* gives the whole $ca(K)$ and, therefore, any given measure can be realized as a limit of measures with continuous densities. Below one can find an (original) short proof, especially due to we need a little bit different statement: each *nonnegative* measure to be realized as a limit of measures with *positive* continuous densities.

Lemma 3 *The set $M \subset ca(K)$ of all measures with* ***positive*** *and continuous densities is dense in $ca_+(K)$*[5] *in weak* topology $\sigma^*(ca(K), C(K))$.*

Proof. Consider the closure of M in $\sigma^*(ca(K), C(K))$, which we denote by $\mathrm{cl}^*(M)$. It is clear that $\mathrm{cl}^*(M) \subseteq ca_+(K)$ and is convex and weakly* closed cone. Suppose the assertion of the lemma is false. Take $\nu \in ca_+(K) \setminus \mathrm{cl}^*(M)$. We are in the conditions of the classical second separability theorem (strict separability of closed convex sets, one of which is compact) and we can find a linear functional G, continuous in the weak* topology, strictly separating ν from $\mathrm{cl}^*(M)$, i. e.

$$\exists\, \gamma \in \mathbb{R}: \quad G(\mathrm{cl}^*(M)) \geq \gamma > G(\nu). \tag{9}$$

Now for the separating functional the following conclusions can be drawn:

(i) Since the functional is continuous in the weak* topology, it must be defined by some continuous function $g \in C(K)$ by formula

$$G(\mu) = \int g(x) d\mu(x), \quad \mu \in ca(K).$$

(ii) Since $G(\mathrm{cl}^*(M)) \geq \gamma$, then (by contradiction, M is a cone), we formally conclude

$$G(\mathrm{cl}^*(M)) \geq 0 \quad \Rightarrow \quad \int g(x) h(x) dx \geq 0 \ \forall h \in C_+(K) \quad \Rightarrow \quad g(x) \geq 0 \ \forall x \in K.$$

(iii) Due to (ii) and $\nu \geq 0$ one concludes $\int g(x) d\nu(x) \geq 0$, that via right hand side of (9) implies $\gamma \geq 0$.

Finally, since $0 \in \mathrm{cl}^*(M)$, the left inequality in (9) allows us to conclude $\gamma \leq 0$, which together with (iii) gives $\gamma = 0$. However for $\nu \geq 0$ and $\gamma \geq \int g(x) d\nu(x) \geq 0$ it implies $0 = \gamma = G(\nu)$ that contradicts to (9). ■

So it is proven the existence of a family of measures μ_ξ with positive continuous densities f_ξ, $\xi \in \Xi$ which weakly converges to μ, i. e. (8) and (7) are satisfied. To this family there corresponds a family of curves b_ξ defining the inter-country boundary and given by the following equation (with respect to $(x, y) \in \square ABCD$):

$$c_1((x,y), \mu_\xi(\mathbf{S}_{\mathbf{left}\,\xi}), r_c(\mathbf{S}_{\mathbf{left}\,\xi}), z_\xi) - c_2(\mu_\xi(\mathbf{S}_{\mathbf{right}\,\xi}), r_c(\mathbf{S}_{\mathbf{right}\,\xi}), z_\xi) = 0.$$

[5] It is the cone of all positive elements of the space $ca(K)$

It is assumed that the variable parameters depending on ξ and specifying this equation vary within bounded limits (in a compact set) and the functions c_1, c_2 are continuous. Hence, without loss of generality, we can assume that

$$\mu_\xi(\mathbf{S_{left}}_\xi) \to \mu_{\lim}(\mathbf{S_{left}}) = \delta_{\mathbf{left}},\ \mu_\xi(\mathbf{S_{right}}_\xi) \to \mu_{\lim}(\mathbf{S_{right}}) = \delta_{\mathbf{right}},$$

$$(r_c(\mathbf{S_{left}}_\xi), z_\xi) \to (r_c^{lim}(\mathbf{S_{left}}), z^{lim}).$$

Further, applying the regularity of the measure μ, one can prove that the family of boundary curves b_ξ converges to the limit curve b^{lim}, which can be specified by the equation

$$c_1((x,y), \delta_{\mathbf{left}}, r_c^{lim}(\mathbf{S_{left}}), z^{lim}) - c_2((x,y), \delta_{\mathbf{right}}, r_c^{lim}(\mathbf{S_{right}}), z^{lim}) = 0.$$

This is the boundary of the desired inter-country division. Now it is necessary to clarify only one problem. Everything is obvious in the case when the boundary curve does not pass through any city. However, what if the city (it is a point of $\Box ABCD$ with nonzero mass) is on the boundary? This city will be divided into two (unequal) parts, one of which belongs to one country, the other to another one. The proportions of this division can be found in the following way.

Since the carrier of $\mathrm{supp}(\nu)$ of the discrete component is finite, there exists $\varepsilon > 0$ such that for each point of the support $a \in \mathrm{supp}(\nu)$ in the ε-neighborhood $B_\varepsilon(a)$ there are no other points from the carrier of this point. Take $\varepsilon > 0$ and consider $\lim_{\xi\in\Xi} \mu_\xi(\mathbf{S_{left}}_\xi \setminus B_\varepsilon(m)) = \delta^\varepsilon_{\mathbf{left}}$. Next we find

$$\lim_{\varepsilon\to+0} \delta^\varepsilon_{\mathbf{left}} = \limsup \delta^\varepsilon_{\mathbf{left}} = \delta^a_{\mathbf{left}}.$$

This is the mass of the population living in the left country with the exception of "city" $a \in \mathrm{supp}(\nu)$. Thus, the mass of residents from this city living in the left country can be calculated as $\delta_{\mathbf{left}} - \delta^a_{\mathbf{left}} = \nu^a_{\mathbf{left}}$. Now in the right country will live $\mu(a) - \nu^a_{\mathbf{left}} = \nu^a_{\mathbf{right}}$. As a result we have proven the following

Theorem 3 *Let the cost functions satisfy the assumptions above and the population distribution is given by the Radon measure. Then there is an immigration-consistent partition of $\Box ABCD$ into any given number of countries.*

Corollary 2 *Under Theorem 2 conditions, the requirement* $\mathrm{supp}(\mu) = \Box ABCD$ *can be omitted.*

So, we summarize: in the country division of a compact convex area cities (points from a discrete component of carrier) those are located on the border may have a fractional affiliation (fall under jurisdiction) to the border countries. Moreover, zero-area countries may emerge. The proper modernization of the concept of immigration-wealthy division into countries taking into account these circumstances is carried out in a natural way.

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