PRODUCTION EQUILIBRIA IN VECTOR LATTICES WITH UNORDERED PREFERENCES : AN APPROACH USING FINITE-DIMENSIONAL APPROXIMATIONS

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I. Introduction

The pioneering T.F.Bewley's paper [2] has initiated extensive investigation of the economic models with infinite number of commodities. T.F.Bewley has proved the existence of equilibria assuming that the consumption set of an agent coincides with the positive cone of the space $L_\infty(\mathbf{R}^l)$. His paper brightened up the importance of Alaoglu's theorem and the role of weak topologies (weak* and Mackey) for the existence of equilibria. It is beyond the scope of our paper to describe the stream of results which have appeared during twenty-year period and we shall mention only some of them. It was O.M.Kreps [6] who was the first to use ordered vector spaces explicitly in the equilibrium analysis. A.Mas-Colell in his remarkable paper [7] has introduced the important notion of uniform properness of preferences and extended the framework of the analysis up to topological vector lattices. One of the fundamental contributions has been done by C.A.Aliprantis, D.J.Brown & O.Burkinshow [1]. They paid special attention to the duality of vector lattices (Riesz spaces) and their locally solid topologies. These topologies are namely the linear ones that makes the lattice operations *uniformly continuous* (see also [10] for general overviews). Recently A.Mas-Colell and S.F.Richard [9], and S.F.Richard [11] have done the next step in proving the existence of equilibria in vector lattices. A term "linear vector lattice" assumes *only the continuity of lattice operations.* Not only have their papers weakened the topological assumptions on the commodity space but also covered some other results, earlier not included in the general theory (see C.Huang & D.Kreps [4] and L.Jones [5]). Mas-Colell-Richard's arguments have been based essentially on the ordered properties of the agent's preferences because of the explicit usage of the utility functions representing them.

The aim of the paper is to prove the existence of production quasi-equilibria for economy with finite many agents and with commodity space L which may be infinite dimensional being described as linear vector lattice. We generalize S.F.Richard's result [11] in three directions. First, we equip the economy with the modern weakest assumptions on preferences, which may be nontransitive and incomplete. Second, we admit that consumption sets $X_i \subset L$ may differ from the positive cone L_+ of commodity space, in contrast with [11]. Third, S.F.Richard has assumed that each production set $Y_j \subset L$ is *uniformly proper* with respect to some convex lattice $Z_j \supset Y_j$, called a *pretechnology set* (a concept originally introduced by Mas-Colell in [8]), a set satisfying the additional condition: there exists such $\epsilon > 0$ that $(1+\epsilon)Y_j \subset Z_j$. Assuming instead that each Y_j is *uniformly proper* with respect to some lattice $Z_j \supset Y_j$ (we call Y_j an *upper uniformly proper*), we avoid the second part of S.F.Richard's assumption, i.e. we *do not assume that* $(1+\epsilon)Y_j \subset Z_j$ for some $\epsilon > 0$. Our generalization extends the equilibrium existence theorem (and the equilibrium model, consequently) to many interesting cases. The simplest example of production set, which we admit but S.F.Richard's theorem does not, is the following one: $Y_j = \{y \in L | y \leq z \ \& \ f(y) \leq 0\}$ for some fixed $z \geq 0$ $z \neq 0$ and some linear continuous functional $f \geq 0$. Concerning the consumption sets we impose (similar to the production case) assumptions, that each X_i is *uniformly proper* (we call X_i *lower uniformly proper*, see definitions below) with respect to some lattice Z_i, which may be symmetrically called a *preconsumption set.* All other model's features and assumptions are quite standard.

Our arguments to state the existence theorem are based on the special "semi"-finite-dimensional approximating theorem, proved for the given "upper and lower boundaries $z_i \in Z_j$, $z_i \in Z_i$" with the help of Kakutani's fixed point theorem. Then we are doing double passing to the limits: by the net of finite-dimensional subspaces $\mathcal{L} \subset L$ and by proper boundaries $z_t \in Z_t$.

2. The model and main result

We consider a typical production economy in which the commodity space L is a partial ordered vector space equipped with a Hausdorff, locally convex topology τ. Let $N = \{1, ..., n\}$ denote the set of consumers, and let $M = \{1..., m\}$ be the set of firms. Each consumer $i \in N$ is characterized by a consumption set $X_i \subset L$ and a preference relation described by the point-to-set mapping $\mathcal{P}_i : X_i \Rightarrow X_i$, so that $\mathcal{P}_i(x_i)$ means the set of all consumption bundles strictly preferred by the i-th agent to the bundle x_i. We also will use the notation $y_i \underset{i}{\succ} x_i$ which is equivalent to $y_i \in \mathcal{P}_i(x_i)$. A consumer i is also endowed with a commodity bundle $w_i \in L$ and with a share $\theta_i^j \geq 0$ of firm $j \in M$, where $\sum_{i \in N} \theta_i^j = 1$ for all $j \in M$. Denote $\theta_i = (\theta_i^1, ..., \theta_i^m)$. A producer (a firm) j is characterized by production set $Y_j \subset L$. We require prices π to be chosen in the topological dual of L, denoted by L^*. Thus, the model under study is a 5-tuple

$$\mathcal{E} = (N, M, \langle L, L^* \rangle, \ \{\mathcal{P}_i(\cdot), X_i, w_i, \theta_i\}_{i \in N}, \{Y_j\}_{j \in M}).$$

The assumptions the economy is required to satisfy are divided into several groups. The first one consists of

STRUCTURAL ASSUMPTIONS (SA).

(i) L *is a linear vector lattice (or Riesz space);*

(ii) L_+ *is a closed cone in* τ*-topology of* L*;*

(iii) L^* *is a sublattice of the order dual to* L*;*

It is worth special noticing that *(i)* means that the order operations (such as $x \vee y, x \wedge y, x, \ y \ \in \ L$ and others) are *continuous* with respect to the topology τ and that they *may not be uniformly continuous*, as it is commonly required in most of current results, i.e. we do not assume the topology τ to be locally solid. Note, that if L were a locally solid Riesz space then requirements *(ii), (iii)* would be automatically valid. Since we avoid the solidness hypothesis we need to require it directly. For more specific explanations and references the reader is referred to [1], [10].

The second group of assumptions concerns agents' preferences.

ASSUMPTIONS ON PREFERENCES (PA)

For each $i \in N$:

(i) upper hemicontinuity:

$$\mathcal{P}_i(x) \ \ is \ \ \tau - open \ for \ each \ x \in X_i;$$

(ii) lower hemicontinuity: for each $x \in X_i$ the set

$$\mathcal{P}_i^{-1}(x) = \{y \in X_i \mid x \in \mathcal{P}_i(y)\} \ \ is \ \ \sigma(L, L^*)^1 - \ open;$$

(iii) weak convexity, irreflexivity, and nonsatiation: for each $x \in X_i$

$$x \notin conv\ \mathcal{P}_i(x)\ \&\ x \in cl(conv\mathcal{P}_i(x));^2$$

(iv) monotonicity: for each $x \in X_i$

$$\mathcal{P}_i(x) + L_+ \subset \mathcal{P}_i(x);$$

(v) $\mathcal{P}_i(\cdot)$ is uniformly proper on X_i.

All these assumptions are well known in the literature and do not require special explanations. The last one is essentially important for infinite commodity models. The notion of preference properness which we used in *PA(v)* is a little bit more specific in our context and we give below a precise definition. The preference $\mathcal{P}(\cdot)$ is said to be *v-proper* at the point x if there exists a vector $v \in L$, $v \neq 0$, and a τ-neighborhood of zero $V \subset L$, such that $[x - \alpha v + z \in conv\mathcal{P}(x)$ for $\alpha > 0]$ implies that $z \notin \alpha V$. The latter may be also written in the form

$$(x - \Gamma) \bigcap conv\ \mathcal{P}(x) = \emptyset, \tag{1}$$

where Γ is the conic hull of $(v + V)$, i.e. $\Gamma = con(v + V)$.

Note also that if $\mathcal{P}$ satisfies *PA* and if v & V are a vector and a neighborhood existing due to *PA(v)*, then for any $v' \geq v$ and for any neighborhood $V' \subset V$, for $\Gamma' = con(v' + V')$ we have

$$x - \Gamma' = x - con(v' - v + v + V') \subset x - L_+ - con(v + V') \subset x - L_+ - \Gamma.$$

However, the monotonicity condition *PA(iv)* implies that (1) is equivalent to

$$(x - L_+ - \Gamma) \bigcap conv\ \mathcal{P}(x) = \emptyset.$$

This, together with the previous relation, means that $\mathcal{P}(.)$ *may be considered proper with respect to v' and V'.*

If for the given domain of $\mathcal{P}(\cdot)$ the vector v and the neighborhood V can be chosen independently of x, then the preference is called *v-uniformly proper.* Originally the latter

[1] This denotes *weak* topology on L.

[2] *conv* A denotes the convex hull of the set A and clA is its closure.

notion was introduced by Mas-Colell [7], motivated by the empty interior of positive cone in many interesting cases (see also [1] for more on these definitions).

Before formulating the assumptions on consumption sets and on production sector, we need to extend the notion of properness to the subsets of L. We will call the vector $z \in L$ *v-proper upper boundary (p.u.b.)* of $X \subset L$ if there exists a vector $v \in L$, $v \neq 0$, and a τ-neighborhood of zero $V \subset L$, such that $[x \in X,\ x - \alpha v + y \leq z$ for $\alpha > 0$, and $y \in \alpha V]$ implies that $x - \alpha v + y \in X$.

This condition may be equivalently written in the form

$$(X - \Gamma)\bigcap(z - L_+) \subset X, \quad \Gamma = con(v + V). \tag{2}$$

The vector $z \in L$ is called *v-proper lower boundary (p.l.b.)* of $X \subset L$ if there exists a vector $v \in L$, $v \neq 0$, and a τ-neighborhood of zero $V \subset L$, such that $[x \in X$ and $x + \alpha v + y \geq z$ for $\alpha > 0$, and $y \in \alpha V]$ implies that $x + \alpha v + y \in X$.

The equivalent form of this is

$$(X + \Gamma)\bigcap(z + L_+) \subset X, \quad \Gamma = con(v + V). \tag{3}$$

At last, the set X is called *Z-uniformly upper proper* if $X \subset Z$, the set $Z \subset L$ satisfies free disposal condition $Z - L_+ \subset Z$, each $z \in Z$ is *p.u.b.* and both v & V are independent of z. Note also that the equivalent form of *Z-upper properness* is the following:

$$(X - \Gamma)\bigcap Z \subset X.$$

Symmetrically, the set X is called *Z-uniformly lower proper* if $X \subset Z$, the set $Z \subset L$ satisfies free disposal condition $Z + L_+ \subset Z$, each $z \in Z$ is *p.l.b.* and both v & V are independent of z. The equivalent form of this kind of *Z-properness* is the following:

$$(X + \Gamma)\bigcap Z \subset X.$$

ASSUMPTIONS ON CONSUMPTION SETS (ACS)

For each $i \in N$:

(i) $X_i \subset L$ *is convex, closed and* $w_i \in X_i$;

(ii) *free disposal:* $X_i + L_+ \subset X_i$;

(iii) *there exists such sublattice* $Z_i \subset L$, *that* X_i *is* Z_i*-uniformly lower proper.*

ASSUMPTIONS ON PRODUCTION SETS (APS)

For each $j \in M$:

(i) $Y_j \subset L$ *is convex, closed and* $0 \in Y_j$;

(ii) *free disposal:* $Y_j - L_+ \subset Y_j$;

(iii) *there exists such sublattice* $Z_j \subset L$, *that* Y_j *is* Z_j*-uniformly upper proper.*

Note, that if X_i satisfies *ACS* and v_i & V_i are a vector and a neighborhood existing by *ACS(iii)*, then in view of free disposal condition *ACS(ii)* for any $v' \geq v_i$ and a neighborhood $V' \subset V_i$ we have

$$X_i + \Gamma' = X_i + L_+ + con(v' - v_i + v_i + V') \subset X_i + L_+ + con(v_i + V') \subset X_i + con(v_i + V_i),$$

where $\Gamma' = con(v' + V')$. Therefore by (3) the set X_i will be *uniformly proper with respect to v' and V'*. The same thing can be said about production sets.

REMARK. It is worth noting here, repeating the *Introduction*, that our assumptions imposed on consumption and production sets are substantially relaxed in comparison with the assumptions given by S.F.Richard in [9]. First, S.F.Richard postulated $X_i = L_+$, that we avoid. Second, in addition to assumption *APS (iii)* imposed here on production sets, S.F.Richard requires a value $\epsilon > 0$, such that $(1+\epsilon)Y_j \subset Z_j$ to exist for every j. This extra assumption is too strong and essentially restricts possible applications. To explain this idea let us turn to the example suggested in the introduction. In this example, production sets have the form:

$$Y_j = \{y \in \ \mid y \leq z_j,\ f_j(y) \leq 0\}, \quad z_j \geq 0,\ z_j \neq 0,\ f_j \geq 0,$$

where the linear functionals f_j are continuous. One can see that the most natural choice of Z_j is to put $Z_j = \{y \in L \mid y \leq z_j\}$. However, sets of this kind, being lattices, do not satisfy Mas-Colell-Richard's condition. Moreover, if L_+ has empty interior in L, then we can not also choose $Z_j = Z_j^\epsilon = \{y \in L \mid y \leq (1+\epsilon)z_j\}$, $\epsilon > 0$, since for these sets there is no τ-open cone Γ satisfying

$$(Y_j - \Gamma) \bigcap Z_j^\epsilon \subset Y_j.$$

On the other hand, it would seem rather strange if equilibria were not existing for the considered production sets. In fact, if some production set has the form $\{y \in L \mid y \leq z\}$ or $\{y \in L \mid f(y) \leq 0\}$ for some linear continuous f, then it obviously does not obstruct to the existence of equilibria. But for their intersection the modern existence theory has no answer. The paper in particular is aimed to fill this gap.

Now we are going to formulate the result. Let us denote $w = \sum w_i$ and let

$$X(\mathcal{E}) = \{(x,y) \in \prod_N X_i \times \prod_M Y_j \mid \sum_N x_i = w + \sum_M y_j\}$$

be the set of all *feasible allocations.*

DEFINITION. A triplet (x, y, π) is said to be quasi-equilibrium iff $(x,y) \in X(\mathcal{E})$, $\pi \in L_+^*$, $\pi \neq 0$ and

(i) $\langle \mathcal{P}_i(x_i), \pi \rangle^3 \geq \pi x_i$, $i \in N$;

(ii) $\pi x_i = \pi(w_i + \sum_{j \in M} \theta_i^j y_j)$, $i \in N$;

[3] $\langle A, \pi \rangle$ denotes the set $\{< a, \pi > \mid a \in A\}$, and $A \geq b$ means $a \geq b$ for all $a \in A$.

(iii) $\langle Y_j, \pi \rangle \leq \pi y_j, \quad j \in M.$

Our main result is

THEOREM 1. Let $\mathcal{E}$ satisfy assumptions (SA), (PA), (ACS) and (APS) and let $X(\mathcal{E})$ be $\sigma(L^{n+m}, (L^*)^{n+m})$-compact. In addition there exist proper lower and upper boundaries $z_i \in Z_i$ & $z_j \in Z_j$ and proper vectors, such that $x_i \geq z_i$ & $y_j \leq z_j$ for some feasible allocation $(x, y) \in X(\mathcal{E})$, so that

$$v \leq \sum_M z_j - \sum_N z_i + \sum_N w_i \tag{4}$$

holds for each proper vector v. Then quasi-equilibrium does exist.

The requirement, imposed in *Theorem 1* on the choice of agent' proper vectors v_t is weaker[4] than the property that all of them are chosen from the linear hull of the set:

$$(\sum_N X_i) \bigcap (\sum_M Y_j + \sum_N w_i).$$

It follows from the fact that γv_t is proper for proper v_t and any $\gamma > 0$. It is easy to see that the above requirement is not stronger (even weaker!) than those considered in most of existing results, where the notion of *w-properness* is used. Indeed, if $X_i = L_+$, then setting $v_t = w$ for all t, and choosing $z_i = x_i = 0$, $y_j = -w/m$, any $z_j \in Z_j$, $z_j \geq 0$ we see that (4) is fulfilled, therefore the condition of *Theorem 1* holds with respect to *w-properness*.

3. Strategy of proof, auxiliary results and discussion

In [9] A.Mas-Colell and S.F.Richard suggested an attractive idea of representing an equilibrium price of trade economy as the *supremum of the "individual"* supporting prices. They used such an approach in their *Lemma 1* and *Proposition.* They also constructed the compacts, containing supporting "individual" prices for any given weak-optimal allocation, explicitly using w-uniform properness of preferences. In [11] S.F.Richard using similar approach generalized this result to production economies. Borrowing these ideas, we apply them in a different way. Our method is based on the direct usage of mappings, which put into correspondence to the given i-th consumption bundle the sets of continuous functionals, supporting all the i-th agent preferred points. For every producer we use "effective" points, situated at the boundary of production set and define the mappings of "individual production prices". The main theorem is proved due to multi-stage passing to limits. At the first stage we consider some finite-dimensional subspace $\mathcal{L} \subset L$ of commodity space and an auxiliary map $*(\cdot)$, which projects the elements of cones, defined by proper boundaries, onto consumption sets and onto effective boundaries of production sets. Using $*(\cdot)$ we construct the mentioned above mappings of "individualized consumption and production prices". Then, applying an appropriate approximation of consumers'

[4]Take into account that we impose the requirement on proper boundaries but not on allocations.

and firms' profits, we construct a point-to-set mapping, satisfying the conditions of Kakutani's fixed point theorem. It is shown that these fixed points approximate equilibrium points and that the corresponding collections of individual prices may be included in some weak compact. This result is formulated as *Theorem 2* and probably is the main technical novelty of this paper. *Theorem 2* allows us to pass to weak limits by the net of finite-dimensional subspaces $\mathcal{L} \subset L$ and derive the existence of bounded equilibria, considered with respect to any given bundle of proper feasible boundaries (they confine consumption and production sets). At last, to state the main theorem, we pass to limits by $z_i \in Z_i$ & $z_j \in Z_j$. Since each Z_t, $t \in N \cup M$ is a lattice, it is possible. Now we are going to describe the steps of proof in more details.

Let us take and fix any feasible proper lower and upper boundaries $z_i \in Z_i$, $i \in N$ and $z_j \in Z_j$, $j \in M$, existing due to *ACS(iii)* and *APS(iii)*. We may think that $w_i \geq z_i$ and $z_j \geq 0$ for all i, j and due to *Theorem 1* assumption think, without lost of generality, that

$$v \leq \sum_M z_j - \sum_N z_i + \sum_N w_i = h \tag{5}$$

for each agent's *proper vector v*. Further we also assume, without lost of generality, that all upper proper boundaries satisfy $z_j \notin Y_j$. In fact, if $Y_j = Z_j$ for some $j \in M$, then from the weak compactness of $X(\mathcal{E})$ and other assumptions of *Theorem 1* it can be easily proved that $Y_j = z_j - L_+$ for some $z_j \in L$[5]. But in such a situation we can consider z_j as a kind of initial endowment and reduce consumer i's endowments to $\theta^i_j z_j + w_i$, thus eliminating the j-th producer from the model. We recall again that everywhere in this section we think the boundaries z_i & z_j fixed.

Below we intend to use Kakutani's fixed point theorem, applied to a specially constructed point-to-set mapping, the fixed points of which satisfy the necessary "equilibrium conditions". This idea encounters many problems and one of them is the one-sided continuity of the inner product $\langle p, x\rangle = p \cdot x$, $p \in L^*$, $x \in L$ with respect to the weak* topology $\sigma(L^*, L)$. It is for this reason that we confine for the moment our considerations to the *finite-dimensional subspaces* of commodity space. One can state directly the following

PROPOSITION. Let $\mathcal{L}$ be any finite-dimensional subspace of L. Then the map $\langle \cdot, \cdot \rangle$: $L^* \times \mathcal{L} \to R$ is continuous with respect to both variables and to the weak topology $\sigma(L^*, L)$ for L^*.*

Let V_j, V_i and V_i^p be open convex and circled (i.e., $V_t = -V_t$) neighborhoods of zero in L and let v_j, v_i & v_i^p be appropriate chosen vectors, defined by the uniform properness of agents' sets and preferences (the upper index p means that it is chosen for preferences). Put

$$V = \bigcap_{N \cup M} V_k \bigcap_N V_i^p, \quad v = \bigvee_{N \cup M} v_k \bigvee_N v_i^p \vee 0. \tag{6}$$

[5] From the compactness of $X(\mathcal{E})$ we conclude that in the case $Y_j = Z_j$ there exists such $\bar{z} \in Z_j$ that $(\bar{z} + L_+) \cap Z_j$ is weak compact and therefore there exists such weak convergent directness $z_\xi \in Z_j$, $z_\xi \to z_j \in Z_j$ that for every given $z \in Z_j$ $\exists \xi :$ $z \leq z_\xi$, $\xi \in \Xi$. Now if $z \in Z_j$, then $pz \leq \lim_\Xi pz_\xi = pz_j$ for every $p \in L^*_+$. From this due to SA(ii) we conclude that $z \leq z_j$ (otherwise using the separation theorem we are coming to contradiction).

As we have seen above, the agents' sets and preferences may be considered to be uniformly proper with respect to v and V. It is easy to see also that $v \neq 0$ (otherwise (1) is false for some $x_i \in X_i$). Note also that by (4), (5) we have $h \geq 0$ and $v \leq h$ by v-specification. Now we fix v and V for all below considerations and put

$$\Gamma = con(v + V), \quad \Gamma_+ = \Gamma + L_+.$$

Now we take and fix any *finite-dimensional* subspace $\mathcal{L} \subset L$ which contains all vectors z_t, v, w_i and fixed agents' plans, which existence was postulated in *Theorem 1*. Denote

$$\mathcal{L}_+ = L_+ \cap \mathcal{L}$$

the positive cone of $\mathcal{L}$, induced by the order in L. Note, that the ordered space $\mathcal{L}$ *may not be a lattice.*

For every $z \geq 0, z \in \mathcal{L}$ and $g = (p, q) \in (L_+^*)^{n+m}$ let us denote

$$s^g(z) = s(g, z) = \sup_{\sum x_i' + \sum y_j' = z, \ x_i', y_j' \in \mathcal{L}_+} \left(\sum p_i x_i' + \sum q_j y_j'\right).$$

Also denote

$$D_j = [z_j - 2h, z_j] \bigcap \mathcal{L}, \quad j \in M.$$

The major auxiliary tool to prove *Theorem 1* is the following

THEOREM 2. Let $\mathcal{L} \subset L$ be some finite-dimensional subspace, satisfying the conditions described above, and let the proper upper and lower boundaries z_i & z_j be chosen satisfying the conditions of Theorem 1, while $z_j \notin Y_j$ for each $j \in M$. Then there exist consumption plans $x_i \in \mathcal{L} \cap X_i$, $x_i \geq z_i$, $i \in N$, and production plans $y_j \in \mathcal{L} \cap Y_j$, $y_j \leq z_j$, $j \in M$, and the bundle of individual prices $g = (p, q) = (p_1, ..., p_n, q_1, ..., q_m) \in (L_+^*)^{n+m}$, satisfying the following conditions:

(i) $\langle p_i, conv\mathcal{P}_i(x_i)\rangle \geq p_i x_i$;

(ii) $p_i(x_i - z_i) \geq s^g(w_i - z_i) + \sum_M \theta_j^i \mu_j(g)$, *where* $\mu_j(g) = s^g(z_j + h) - \inf_{y \in Y_j \cap D_j} s^g(z_j + h - y)$;

(iii) $\langle q_j, (z_j - Y_j)\rangle \geq \inf_{y \in Y_j \cap D_j} s^g(z_j + h - y) - s^g(h)$;

(iv) $\sum p_i(x_i - z_i) + \sum q_j(z_j - y_j) \geq \sum p_i x_i' + \sum q_j y_j'$ *for all* $x_i', y_j' \in \mathcal{L}_+$ *such that* $\sum x_i' + \sum y_j' = \sum_M z_j - \sum_N z_i + \sum_N w_i = h$;

(v) $\sum_N x_i = \sum_M y_j + \sum_N w_i$;

(vi) $| \langle (p, q), V^{n+m}\rangle | \leq \sum p_i v + \sum q_j v = 1$, *where the neighborhood of zero $V \subset L$ and vector v are specified by (6).*

This theorem gives us an opportunity to prove main *Theorem 1* passing to limits with respect to the triplets $(x^{\mathcal{L}}, y^{\mathcal{L}}, g^{\mathcal{L}})_{\mathcal{L} \subset L}$, (specified in *Theorem 2* and indexed by the finite-dimensional subspaces), satisfying the conditions of *Theorem 2* and with respect to the proper lower $z_i \in Z_i$ and upper boundaries $z_j \in Z_j$ for all i, j. Below we describe the proof of *Theorem 2* using the series of *Lemmas* which proofs are given in the next section.

Now let us consider the following sets (one of them has been already defined above):

$$C_i = [z_i, z_i + 2h] \bigcap \mathcal{L}, \qquad D_j = [z_j - 2h, z_j] \bigcap \mathcal{L},$$

where the vector h is chosen by (5) and $[.,.]$ means the order interval in L. Put

$$\mathbf{I} = \prod_N C_i \times \prod_M D_j, \quad w = \sum_N w_i. \tag{7}$$

Further, if $(x, y) \in \mathcal{L}^{n+m}$ is some bundle of agents' plans, such that

$$z_i \leq x_i, \quad y_j \leq z_j \quad \& \quad \sum_N x_i \leq \sum_M y_j + w + \gamma h, \quad 0 \leq \gamma \leq 1 \tag{8}$$

then

$$z_i \leq x_i \leq \sum_M y_j - \sum_{k \neq i} x_k + w + \gamma h \leq \sum_M z_j - \sum_N z_k + w + z_i + \gamma h = z_i + (1+\gamma)h,$$

$$z_j \geq y_j \geq \sum_N x_i - \sum_{k \neq j} y_k - w - \gamma h \geq \sum_N z_i - \sum_M z_k - w + z_j - \gamma h = z_j - (1+\gamma)h,$$

which means that the set $\mathbf{I}$ contains all bundles from $\mathcal{L}^{n+m}$, which satisfy (8). The compactness of $\mathbf{I}$ is stated in the following

LEMMA 1. The sets C_i and D_j are convex compacts in $\mathcal{L}$.

At the next stage we intend to introduce the map $*(\cdot)$, which projects the plans $x_i \in C_i$ and $y_j \in D_j$ onto $C_i \bigcap X_i$ and onto the effective boundary of $D_j \bigcap Y_j$, respectively.

For fixed i let us define $* : C_i \to X_i \bigcap C_i$ by the formula:

$$*(x_i) = x_i^* = \begin{cases} x_i, & x_i \in X_i, \\ x_i + t^*(z_i + 2h - x_i), & x_i \notin X_i, \end{cases}$$

where

$$t^* = \inf\{t \mid x_i + t(z_i + 2h - x_i) \in X_i\},$$

i.e. $x_i \notin X_i$ is projected onto the boundary of X_i as it is shown in figure 1(a).

For fixed j the projection $*(\cdot) : D_j \to D_j \bigcap Y_j$ operates onto the effective boundary of Y_j and is specified as follows:

$$*(y_j) = y_j^* = \begin{cases} y_j + t'(z_j - y_j), & y_j \in Y_j, \\ y_j + t''(z_j - 2h - y_j), & y_j \notin Y_j, \end{cases}$$

where

$$\begin{aligned} t' &= \sup\{t \mid y_j + t(z_j - y_j) \in Y_j\}, \\ t'' &= \inf\{t \mid y_j + t(z_j - 2h - y_j) \in Y_j\}. \end{aligned}$$

Geometrically the map $*(\cdot)$ is represented in figure 1(b).

Figure 1:

To correctly introduce the mappings of individual agents' prices in further considerations, we will apply the following property of "proper sets":

LEMMA 2. Let the set $X \subset L$ be the Z-upper uniformly proper, where $Z \subset L$ is the lattice and let $\Gamma = con(v + V)$ be the conic hull of a vector v and a convex neighborhood of zero V in L, defined by the properness of X. Then

$$X \bigcap (z + \Gamma) = \emptyset$$

for each $z \in Z \setminus X$. Symmetrically, if X is the Z-lower uniformly proper, then

$$X \bigcap (z - \Gamma) = \emptyset$$

for each $z \in Z \setminus X$.

Lemma 2 formulated above directly implies the following

COROLLARY 1. If X_i satisfies ACS then

$$(x_i^* - \Gamma_+) \bigcap X_i = \emptyset,$$

for each $x_i \in C_i \setminus X_i$. Symmetrically, if Y_j satisfies APS and $z_j \notin Y_j$ then

$$(y_j^* + \Gamma_+) \bigcap Y_j = \emptyset,$$

for each $y_j \in D_j$. Here the neighborhood V of zero in L and the vector v are defined by the uniform properness ACS(iii), APS(iii) of X_i, Y_j, and (6).

Now we can introduce the mappings of individual prices. For each $x_i \in C_i \bigcap X_i$ let us define

$$P_i(x_i) = \{p \in L_+^* \mid \langle p, x_i - \Gamma_+ \rangle \leq px_i \leq \langle p, conv\ \mathcal{P}_i(x_i) \rangle\} \tag{9}$$

and put

$$P_i(x_i) = \{p \in L_+^* \mid \langle p, x_i^* - \Gamma_+ \rangle \leq px_i^* \leq \langle p, X_i \rangle\} \tag{10}$$

for $x_i \in C_i \setminus X_i$.

Let us also specify the mapping of "supporting production prices" as follows. For $y_j \in D_j$ we put

$$Q_j(y_j) = \{q \in L_+^* \mid \langle q, y_j^* + \Gamma_+ \rangle \geq qy_i^* \geq \langle q, Y_j \rangle\}. \tag{11}$$

Directly below the following sets will play the crucial role:

$$P_i^\epsilon(x_i, \lambda_i) = \{p_i \in P_i(x_i) \mid \langle p, x_i - z_i + \epsilon h \rangle = \lambda_i\}, \tag{12}$$

$$Q_j^\epsilon(y_j, \alpha_j) = \{q \in Q_j(y_j) \mid \langle q, z_j - y_j + \epsilon h\rangle = \alpha_j\}. \tag{13}$$

They are defined for $\lambda_i \geq 0$, $\alpha_j \geq 0$, $\epsilon > 0$ and $i \in N$, $j \in M$.

LEMMA 3. The mappings $P_i(\cdot)$ & $Q_j(\cdot)$ have closed graphs.

Note that the latter lemma will be false if we try to expand the domain of the considered mappings up to the whole L.

To correctly define the domain of $P_i^\epsilon(\cdot,\cdot)$, $Q_j^\epsilon(\cdot,\cdot)$ let us introduce the set

$$\mathbf{\Delta} = \{(\lambda, \alpha) \in \mathbf{R}_+^{n+m} \mid 1/(n+m) \leq \sum_N \lambda_i + \sum_M \alpha_j \leq 1\}. \tag{14}$$

One more problem is that it is necessary that the range of "supporting" maps is a compact set. With this in mind for the given fixed $\epsilon > 0$ we consider the set

$$\begin{aligned} \mathbf{P}_\epsilon = \{(p,q) \in (L_+^*)^{n+m} \mid & \langle p_i, \Gamma\rangle \geq 0, \quad \langle q_j, \Gamma\rangle \geq 0, \quad i \in N, j \in M, \\ & 1/(1+\epsilon)(n+m) \leq \pi_\Sigma(h) \leq 1/\epsilon\}. \end{aligned}$$

Hereafter we use the notation

$$\pi_\Sigma = \sum_N p_i + \sum_M q_j.$$

The important property of "supporting maps" is formulated as

LEMMA 4. For any $(x, y, \lambda, \alpha) \in I \times \Delta$ the sets $P_i^\epsilon(x_i, \lambda_i)$ and $Q_j^\epsilon(y_j, \alpha_j)$ are convex, closed and non-empty. Moreover, for all $p_i \in P_i^\epsilon(x_i, \lambda_i)$ and $q_j \in Q_j^\epsilon(y_j, \alpha_j)$, $i \in N$, $j \in M$ the following estimation

$$\frac{\sum \lambda_i + \sum \alpha_j}{1+\epsilon} \leq \pi_\Sigma(h) \leq \frac{\sum \lambda_i + \sum \alpha_j}{\epsilon}$$

holds.

Now let us specify the total "supporting" mapping

$$\psi : \mathbf{I} \times \mathbf{\Delta} \Longrightarrow (L_+^*)^{n+m}$$

putting

$$\psi(x, y, \lambda, \alpha) = \prod_N P_i^\epsilon\ (x_i, \lambda_i) \times \prod_M Q_j^\epsilon\ (y_j, \alpha_j).$$

In terms of this map *Lemmas 3,4* yield

COROLLARY 2. $\psi(\cdot)$ *has closed graph. In addition* $\psi(x,y,\lambda,\alpha) \neq \emptyset$, *is convex, and*

$$\psi(x,y,\lambda,\alpha) \subset \mathbf{P}_\epsilon$$

for every $(x,y,\lambda,\alpha) \in \mathbf{I} \times \mathbf{\Delta}$.

The fact that $\mathbf{I}$ and $\mathbf{\Delta}$ are the compact sets has to be clear. The analogous property of $\mathbf{P}_\epsilon$ is stated in the following

LEMMA 5. $\mathbf{P}_\epsilon \subset L^{*(n+m)}$ *is convex non-empty compact in* $\sigma(L^{*(n+m)}, L^{(n+m)})$.

Now, assembling the specified sets we have the following convex compact

$$\mathcal{Z} = \mathbf{I} \times \mathbf{\Delta} \times \mathbf{P}_\epsilon.$$

We need to construct a point-to-set mapping from $\mathcal{Z}$ into itself. This mapping is represented as a product of three maps. One of them $\psi(\cdot)$ was specified above. To specify the second mapping of "agents' profits" we consider the following function. Remind that for every $z \geq 0, z \in \mathcal{L}$ and $g = (p,q) \in (L_+^*)^{n+m}$ we have put

$$s^g(z) = s(g,z) = \sup_{\sum x_i' + \sum y_j' = z,\ x_i', y_j' \in \mathcal{L}_+} \left(\sum p_i x_i' + \sum q_j y_j'\right). \tag{15}$$

It has to be clear the value $s^g(z)$ approximates the value of functional $\pi = \vee_N p_i \vee_M q_j$ on the element z.

Let $\mathcal{L}(h)$ denote the linear hull of $[0,h] \cap \mathcal{L}$.

LEMMA 6. The map $s(\cdot,\cdot)$ *is continuous in every* $(g,z) \in \mathbf{P}_\epsilon \times \mathcal{L}(h)$ *such that* $\gamma h \geq z \geq \sigma h$ *for some* $\gamma > \sigma > 0$ *and satisfies*
(i) $\quad s(g,z_1) + s(g,z_2) \leq s(g, z_1 + z_2), \quad z_1, z_2 \geq 0;$
(ii) $\quad s(tg,z) = t\ s(g,z), \quad s(g,tz) = t\ s(g,z) \quad \forall\ t > 0;$
(iii) $\quad \frac{1}{n+m}\ \pi_\Sigma(z) \leq s(g,z) \leq \pi_\Sigma(z), \quad z \in \mathcal{L}_+.$

The j's firm profit under prices $\vee_N p_i \vee_M q_j$ for given fixed $\beta > 0$ one can approximate by the value

$$\mu_j(g) = \mu_j^\beta(g) = s^g(z_j + h + \beta h) - \inf_{y \in Y_j \cap D_j} s^g(z_j + h + \beta h - y). \tag{16}$$

Note that since $0 \in Y_j \cap D_j$ we have $\mu_j(g) \geq 0$. We approximate the consumers' endowments adding the value βh to each w_i. Then the consumer i's "income" from the initial endowments, (considered with respect to the new origin z_i), may be calculated as $s^g(w_i + \beta h - z_i)$. Being normalized, the total consumer's income may be represented in the form:

$$\lambda_i(g) = \lambda_i^\beta(g) = \frac{s^g(w_i + \beta h - z_i) + \sum_M \theta_j^i \mu_j^\beta(g)}{s^g(1 + (n+m)\beta)h)}. \tag{17}$$

Producer j's values α_j we specify as

$$\alpha_j(g) = \alpha_j^{\beta}(g) = \frac{\underset{y \in Y_j \cap D_j}{inf} \; s^g(z_j + (1+\beta)h - y) - s^g(h)}{s^g(1 + (n+m)\beta h)}. \tag{18}$$

The properties of $\lambda_i(\cdot)$ and $\alpha_j(\cdot)$ are stated in the following

LEMMA 7. The maps $\lambda_i(\cdot), \alpha_j(\cdot) : \mathbf{P}_\epsilon \to \mathbf{R}$ defined by (16)-(18) are continuous and
(i) $\lambda_i(g) > 0, \quad \alpha_j(g) > 0$,
(ii) $1 \geq \sum_M \lambda_i(g) + \sum_M \alpha_j(g) \geq 1/(n+m)$
holds for every $g \in \mathbf{P}_\epsilon$ and $i \in N, \; j \in M$.

We see that

$$\delta(g) = (\lambda_1(g), ..., \lambda_n(g), \alpha_1(g), ..., \alpha_m(g)) \in \mathbf{\Delta},$$

for $g \in \mathbf{P}_\epsilon$ due to *Lemma 7* and to the set $\mathbf{\Delta}$ definition (see (14)).

The last mapping that we need to specify is the following

$$r^{\beta}(g) = \text{argmax} \;\; \{\textstyle\sum p_i x_i - \sum q_j y_j | \;\; x_i, y_j \in \mathcal{L}, x_i \geq z_i, \; y_j \leq z_j, i \in N, j \in M : \sum x_i = \sum y_j + w + \beta(n+m)h \;\}. \tag{19}$$

This mapping $r^{\beta}(\cdot) : \mathbf{P}_\epsilon \Rightarrow \mathbf{I}$ has a closed graph. Due to the choice of the upper and lower boundaries, satisfying (4), by *Proposition* and *Lemma 1*, this mapping has nonempty and convex images for every point of its domain (it can be shown in a routine way).

Now, assembling the specified above maps, we construct a point-to-set mapping from $\mathcal{Z}$ into itself represented as a product of three maps:

$$\varphi : (x, y, \lambda, \alpha, p, q) \Longrightarrow r \;(p, q) \times \delta \;(p, q) \times \psi \;(x, y, \lambda, \alpha).$$

Now we also choose and fix ϵ and β so that

$$\epsilon + (n+m)\beta \leq 1.$$

The fact that Kakutani's fixed point theorem can be applied to the mapping φ follows from *Lemmas*, by construction and by *Theorem 1* conditions.

So, we conclude that there exists the point $z \in \mathcal{Z}$ such that

$$z \in \varphi(z), \quad z = (x, y, \lambda, \alpha, p, q)$$

for all $\epsilon > 0$ and $\beta > 0$, satisfying the noted above condition. Next we prove that each fixed point of this kind satisfies the following properties:

(i) $p_i \in P_i(x_i), \;\; q_j \in Q_j(y_j)$;

(ii) $p_i(x_i - z_i) + \epsilon p_i(h) \geq s^g(w_i - z_i + \beta h) + \sum_M \theta_j^i \mu_j^\beta(g)$, where

$$\mu_j^\beta(g) = s^g(z_j + h + \beta h) - \inf_{y \in Y_j \cap D_j} s^g(z_j + h + \beta h - y) \text{ is j's firm "profit";}$$

(iii) $q_j(z_j - y_j) + \epsilon q_j(h) \geq \inf_{y \in Y_j \cap D_j} s^g(z_j + (1+\beta)h - y) - s^g(h)$;

(iv) $\sum p_i(x_i - z_i) + \sum q_j(z_j - y_j) = s^g((1 + (n+m)\beta)h)$;

(v) $\sum x_i = \sum y_j + w + \beta(n+m)h$.

We start the checking of (i)-(v) from item (iv). By the fixed point property we have $(x, y) \in r^\beta(g)$, that gives

$$\sum p_i x_i - \sum q_j y_j \geq \sum p_i x_i' - \sum q_j y_j'$$

for all $x_i', y_j' \in \mathcal{L}$, such that $x_i' \geq z_i$, $y_j' \leq z_j$ and $\sum x_i' \leq \sum y_j' + w + \beta(n+m)h$. Now one can subtract $\sum p_i z_i$ and add $\sum q_j z_j$ from the left and to the right-hand sides of the latter inequality. As a result, one can see that the value on the right-hand side obtained is equal to the value $s^g((1+(n+m)\beta)h)$ by its definition (15). Further, let us check items (ii),(iii). Again, by the fixed point property we have $p_i \in P_i^\epsilon(x_i)$ and $q_j \in Q_j^\epsilon(y_j)$ that by (12),(13) yields

$$\langle p_i, x_i - z_i + \epsilon h \rangle = \lambda_i, \quad \langle q_j, z_j - y_j + \epsilon h \rangle = \alpha_j, \quad i \in N, j \in M.$$

Summing these inequalities and taking into account $(\lambda, \alpha) \in \mathbf{\Delta}$ and $\mathbf{\Delta}$-definition (see (14)), we conclude

$$\sum p_i(x_i - z_i) + \sum q_j(z_j - y_j) + \epsilon \pi_\Sigma h = \sum_N \lambda_i + \sum_M \alpha_j \leq 1 \Rightarrow \sum p_i(x_i - z_i) + \sum q_j(z_j - y_j) \leq 1$$

that in view of (iv) implies $s^g((1 + (n+m)\beta)h) \leq 1$. Applying this to $\lambda_i(g)$ and $\alpha_j(g)$ specification by (17), (18) we see that the denominator on the right-hand side of the identity is positive and no more than 1. This proves (ii) and (iii). Since (i), (v) are true automatically by the fixed point property and by the $P_i(\cdot)$, $Q_j(\cdot)$ and $r^\beta(\cdot)$ specification, we have stated (i)-(v).

Now we are going to show that for each $\epsilon > 0$ all fixed points of the map φ may be included into a common compact which *does not depend on the choice of* $\epsilon > 0$. It will allow us to let $\epsilon \to 0$ and pass to limits by fixed points.

Indeed, the summation of normalizing equalities in (12) and (13), together with proved above (iv) gives:

$$1/(n+m) \leq s^\beta((1 + (n+m)\beta)h) + \epsilon \pi_\Sigma h = \sum_N \lambda_i + \sum_M \alpha_j \leq 1$$

that due to the choice of ϵ, β and by *Lemma 6(iii)* yields

$$1/(n+m) \leq \pi_\Sigma((1 + (n+m)\beta)h) + \epsilon \pi_\Sigma h \Rightarrow \pi_\Sigma(h) \geq \frac{1}{2(n+m)}$$

and

$$\frac{1}{(n+m)}\pi_\Sigma((1+(n+m)\beta)h)+\epsilon\pi_\Sigma h\le 1\Rightarrow \pi_\Sigma(h)\le n+m.$$

As a result we obtain the estimation:

$$\frac{1}{2(n+m)}\le \pi_\Sigma(h)\le n+m. \tag{20}$$

Further, applying (i) and (9), (10), and since V is chosen circled, we see

$$\langle p_i;\Gamma_+\rangle\ge 0\Rightarrow\langle p_i,v+V\rangle\ge 0\Rightarrow\mid\langle p_i,V\rangle\mid\le p_iv.$$

The same thing can be analogously stated for firms' prices:

$$\mid\langle q_j,V\rangle\mid\le q_jv.$$

Now since $h\ge v$ we obtain $\pi_\Sigma(h)\ge\pi_\Sigma(v)$, that due to (20), to the former and latter relations allow us to apply Alaoglu's theorem, concluding that *all individual prices, corresponding to fixed points for all ϵ small enough, may be included into some common weak compact.* Further, since other fixed point parameters by construction belong obviously also to some compact, we may pass to limits letting $\epsilon\longrightarrow 0$. Furthermore, in view of *Proposition, Lemmas 6,7* and (20), the properties described in items (i)-(v) will be fulfilled for limit fixed points also. The only difference is that in (ii), (iii) the second addend in the left-hand side of inequalities vanishes (because ϵ is zero now). For the convenience of the references below we reproduce these inequalities here:

(ii)' $p_i(x_i-z_i)\ge s^g(w_i-z_i+\beta h)+\sum_M\theta^i_j\mu^\beta_j(g)$, where

$$\mu^\beta_j(g)=s^g(z_j+h+\beta h)-\inf_{y\in Y_j\cap D_j}s^g(z_j+h+\beta h-y)\text{ is j's firm profit;}$$

(iii)' $q_j(z_j-y_j)\ge\inf_{y\in Y_j\cap D_j}s^g(z_j+(1+\beta)h-y)-s^g(h).$

Below we call the obtained limit points β-equilibria. Next we are going to prove that the β-equilibria satisfy the additional condition:

(vi) $x_i\in X_i,\ \ y_j\in Y_j,\ \ i\in N,\ j\in M$[6].

To prove the consumer's part of (vi) let us suppose that $x_i\notin X_i$ for some $i\in N$. Now in view of $x^*_i-x_i=t^*(z_i+2h-x_i)\ge 0$ we have $p_i(x^*_i-x_i)\ge 0$. Since $w_i\in X_i$ by (10) and (i) we have $p_iw_i\ge p_ix^*_i$. Now taking into account *Lemma 6(i)*, the fact that $w_i\ge z_i$

[6]One can see that we applied the ϵ-approximation to obtain the range of supporting mappings to be a compact set. Then we apply Kakutani's fixed point theorem and find the compact which does not depend on ϵ. The extra β-approximation is used only to obtain (vi).

by the choice of z_i, and (ii)', we can conclude that the following chain of inequalities is true

$$s^g(w_i - z_i) \geq p_i(w_i - z_i) \geq p_i(x_i^* - z_i) \geq p_i(x_i - z_i) \geq s^g(w_i - z_i + \beta h) \geq s^g(w_i - z_i) + \beta s^g(h).$$

But this implies $\beta s^g(h) = 0$, that in view of $\beta > 0$ is possible only if $s^g(h) = 0$, which contradicts (20) and *Lemma 6(iii)*. Therefore $x_i \in X_i$ for all $i \in N$.

Now let us suppose that $y_j \notin Y_j$ for some $j \in M$. Here we have $y_j^* = y_j + t''(z_j - 2h - y_j)$ (see the $*(\cdot)$ specification), where $t''(z_j - 2h - y_j) \leq 0$. Therefore by (11) and (i) we have $q_j y_j^* \leq q_j y_j$, that by (iii)' and *Lemma 6(i)* for every fixed $y' \in Y_j \cap D_j$ gives us the following true chain of inequalities:

$$s^g(z_j - y') \geq q_j(z_j - y') \geq q_j(z_j - y_j^*) \geq q_j(z_j - y_j) \geq$$

$$\inf_{y \in Y_j \cap D_j} s^g(z_j + (1+\beta)h - y) - s^g(h) \geq \inf_{y \in Y_j \cap D_j} s^g(z_j + \beta h - y).$$

Now minimizing by $y' \in Y_j \cap D_j$ the first member of the latter relation we obtain

$$\inf_{y \in Y_j \cap D_j} s^g(z_j - y) \geq \inf_{y \in Y_j \cap D_j} s^g(z_j + \beta h - y) \geq \inf_{y \in Y_j \cap D_j} s^g(z_j - y) + \beta s^g(h).$$

The latter is possible only if $s^g(h) = 0$, that contradicts (20) and *Lemma 6(iii)*.

So, we have stated the existence of β-equilibria satisfying (i), (ii)', (iii)', (iv)-(vi). Now, using arguments similar to those above, due to Alaouglu's theorem, we can conclude that *all these equilibria, for all $\beta > 0$ small enough may be included into some common weak compact.* Therefore we may pass to limits for $\beta \to 0$ (without lost of generality!). Furthermore, due to the facts obtained and to *Lemmas* proved, we may pass to limits also in (i), (ii)', (iii)', (iv)-(vi) & (20). As a result, we obtain the limit point $(x, y, p, q) \in \mathbf{I} \times \mathcal{L}_+^{*(n+m)}$ which satisfies the following properties:

*(i)** $p_i \in P_i(x_i), \; q_j \in Q_j(y_j)$;

*(ii)** $p_i(x_i - z_i) \geq s^g(w_i - z_i) + \sum_M \theta_j^i \mu_j(g)$, where

$$\mu_j(g) = s^g(z_j + h) - \inf_{y \in Y_j \cap D_j} s^g(z_j + h - y) \text{ is j's firm profit;}$$

*(iii)** $q_j(z_j - y_j) \geq \inf_{y \in Y_j \cap D_j} s^g(z_j + h - y) - s^g(h)$;

*(iv)** $\sum p_i(x_i - z_i) + \sum q_j(z_j - y_j) = s^g(h)$;

*(v)** $\sum x_i = \sum y_j + w$;

*(vi)** $x_i \in X_i, \; y_j \in Y_j, \; i \in N, \; j \in M$;

$(vii)^* \ \frac{1}{2(n+m)} \leq \sum p_i h + \sum q_j h.$

One can see, that the obtained point $z = (x, y, p, q) \in \mathbf{I} \times \mathcal{L}_+^{*(n+m)}$ will satisfy all requirements of *Theorem 2* if we can show that

$$\langle q_j, z_j - Y_j \rangle \geq \inf_{y \in Y_j \cap D_j} s^g(z_j + h - y) - s^g(h).$$

Let us do it. By construction, $(iv)^*$, and by $s^g(\cdot)$ specification (15), for each $0 \leq t \leq 1$ we have

$$\begin{aligned} \kappa_1 = s^g(h) - (1-t)q_j(z_j - y_j) = \textstyle\sum_N p_i(x_i - z_i) + \sum_{k \neq j} q_k(z_k - y_k) + tq_j(z_j - y_j) \leq \\ \leq s^g[\textstyle\sum_N (x_i - z_i) + \sum_{k \neq j}(z_k - y_k) + t(z_j - y_j)] = s^g(h - (1-t)(z_j - y_j)), \end{aligned} \tag{21}$$

and also

$$\begin{aligned} \kappa_2 = s^g(h) + (1-t)q_j(z_j - y_j) = \textstyle\sum_N p_i(x_i - z_i) + \sum_M q_k(z_k - y_k) + \\ +(1-t)q_j(z_j - y_j) \leq s^g(h + (1-t)(z_j - y_j)). \end{aligned} \tag{22}$$

Now using the property $(iv)^*$ and *Lemma 6 (ii)* we have $\kappa_1 + \kappa_2 = s^g(2h)$. Summing (21) and (22) we conclude

$$s^g(2h) \leq s^g(h - (1-t)(z_j - y_j)) + s^g(h + (1-t)(z_j - y_j)).$$

However, in view of *Lemma 6(i)* the inverse inequality is true everywhere. Therefore in (21) and (22) the equalities are realized and (22) yields

$$(1-t)q_j(z_j - y_j) = s^g(h + (1-t)(z_j - y_j)) - s^g(h).$$

On the other hand, by the y_j^*-specification we have $z_j - y_j^* = (1 - t_j')(z_j - y_j)$, that in view of the $(i)^*$, $Q_j(y_j)$-specification, the previous one and $0 \leq 1 - t_j' \leq 1$, yields

$$\langle q_j, z_j - Y_j \rangle \geq q_j(z_j - y_j^*) = (1 - t_j')q_j(z_j - y_j) = s^g(h + z_j - y_j^*) - s^g(h).$$

Since $y_j^* \in Y_j$ the right-hand side of the latter relation can not be more than

$$\inf_{y \in Y_j \cap D_j} s^g(z_j + h - y) - s^g(h)$$

and we have that wanted to prove. To finish the proof of *Theorem 2* and to show property *(vi)* it is enough to apply $(i)^*$, that standardly yields

$$|\langle p_i, V \rangle| \leq p_i v, \quad |\langle q_j, V \rangle| \leq q_j v.$$

Renormalize (p, q) putting $\sum p_i v + \sum q_j v = 1$, which is possible since $(vii)^*$ implies

$$\pi_\Sigma(h) > 0 \Rightarrow (p, q) \neq 0 \Rightarrow \pi_\Sigma v > 0.$$

Q.E.D.

4. Proofs

Proof of Theorem 1. It is based on passing to limits by the net of finite-dimensional subspaces and proper boundaries $z_i \in Z_i$ & $z_j \in Z_j$, $i \in N$, $j \in M$ (with respect to them the consumption and production sets are uniformly proper, see ACS(iii) & APS(iii)). Since all Z_t, $t \in N \cup M$ are assumed to be lattices, the boundaries form the net and passing to limits is admissible procedure.

At the first stage we apply *Theorem 2* and realize the passing to limits by the net of "equilibrium points" $(x, y, p, q)^{\mathcal{L}}$ stated in this theorem and indexed by the finite-dimensional subspaces $\mathcal{L} \subset L$.

Really, by *Theorem 2* there exist $(x, y, p, q)^{\mathcal{L}}$ satisfying the conditions *(i)-(vi)*, where $\mathcal{L}$ is formed as described in *Theorem 2* while the order in $\{\mathcal{L} | \mathcal{L} \subset L\}$ is specified by the inclusion $\mathcal{L}' \leq \mathcal{L}'' \iff \mathcal{L}' \subset \mathcal{L}''$. In view of *(vi)* all $(p, q)^{\mathcal{L}}_{\mathcal{L} \subset L}$ are included in

$$K = \{(p, q) \in (L^*_+)^{n+m} | \mid \langle (p, q), V^{n+m} \rangle \mid \leq \sum_N p_i v + \sum_M q_j v = 1\}.$$

Now applying Alaoglu's theorem we conclude that K is weak compact. Also, in *Theorem 1* we suppose that $X(\mathcal{E})$ is weak compact. Therefore, by *(v)* of *Theorem 2* we can think, without lost of generality, that $(x, y, p, q)^{\mathcal{L}}$ is weak a converging net and

$$((x, y)^{\mathcal{L}})_{\mathcal{L} \subset L} \underset{weak}{\longrightarrow} (x^z, y^z) \quad \& \quad ((p, q)^{\mathcal{L}})_{\mathcal{L} \subset L} \underset{weak}{\longrightarrow} (p^z, q^z).$$

where $z = (z_1, ..., z_{n+m})$ is the bundle of fixed proper boundaries. Now we are going to study these limits and show that they have some "equilibrium properties".

First of all we see that

$$\sum_M x^z_i = \sum_M y^z_j + \sum_N w_i. \tag{23}$$

In view of *(i)-(iii) Theorem 2* and *Lemma 6(i)* , since $z_j \geq 0$ & $h \geq 0$, for every fixed $\mathcal{L} \subset L$ we have

$$\langle p_i, x'_i - z_i \rangle \geq s^g_{\mathcal{L}}(w_i - z_i) + \sum_M \theta^i_j (s^g_{\mathcal{L}}(h) + s^g_{\mathcal{L}}(z_j) - \inf_{y \in Y_j \cap D_j} s^g_{\mathcal{L}}(z_j + h - y)) \quad \forall x'_i \in \mathcal{P}_i(x_i), \tag{24}$$

for each $i \in N$, and

$$\langle q_j, z_j - y'_j \rangle \geq \inf_{y \in Y_j \cap D_j} s^g_{\mathcal{L}}(z_j + h - y) - s^g_{\mathcal{L}}(h) \quad \forall y'_j \in Y_j, \tag{25}$$

for each $j \in M$.

Further, for all fixed $0 \leq u_t \leq \nu$, $t = 1, ..., n + m$ such that $\sum u_t = \nu$, one can find the finite-dimensional subspace $\mathcal{L} \subset L$, which contains these vectors, that by the $s^g_{\mathcal{L}}(\nu)$-specification gives

$$s^g_{\mathcal{L}}(\nu) \geq \sum_{t=1}^{t=n} p_t u_t + \sum_{t=n+1}^{t=n+m} q_{t-n} u_t.$$

Therefore, if we put

$$\pi^z = \vee_N p_i^z \vee_M q_j^z,$$

then we can conclude that

$$\underset{\mathcal{L}\subset L}{limsup}\, s_{\mathcal{L}}^g(\nu) \geq \pi^z(\nu) \quad \forall \nu \in L_+.$$

Further, in view of PA(ii) if $x_i' \in \mathcal{P}_i(x_i^z)$, then $x_i' \in P_i(x_i^{\mathcal{L}})$ for all $\mathcal{L}$ big enough. This allows us to pass to limits in (24) and using arguments similar to the previous ones, for all fixed $x_i' \in \mathcal{P}_i(x_i^z)$ and for each $i \in N$ we obtain

$$\langle p_i^z, x_i' - z_i\rangle \geq \pi^z(w_i - z_i) + \sum_M \theta_j^i(\pi^z(z_j) + \underset{\mathcal{L}\subset L}{limsup}[s_{\mathcal{L}}^g(h) - \underset{y\in Y_j\cap D_j}{inf} s_{\mathcal{L}}^g(z_j + h - y)]). \quad (26)$$

Also passing to limits in (25) we conclude that

$$\begin{aligned}\langle q_j, z_j - y_j'\rangle \geq \underset{\mathcal{L}\subset L}{liminf}[\underset{y\in Y_j\cap D_j}{inf} s_{\mathcal{L}}^g(z_j + h - y) - s_{\mathcal{L}}^g(h)] = \\ -\underset{\mathcal{L}\subset L}{limsup}[s_{\mathcal{L}}^g(h) - \underset{y\in Y_j\cap D_j}{inf} s_{\mathcal{L}}^g(z_j + h - y)] \quad \forall y_j' \in Y_j,\end{aligned} \quad (27)$$

for each $j \in M$.

Now summing inequalities (26) – (27) we obtain

$$\sum_N p_i^z(x_i' - z_i) + \sum_M q_j^z(z_j - y_j') \geq \sum_N \pi^z(w_i - z_i) + \sum_M \pi^z(z_j) \quad (28)$$

and the inequality holds for all $x_i' \in cl(conv\mathcal{P}_i(x_i^z))$ and $y_j' \in Y_j$. Furthermore, if in addition $x_i' \geq z_i$ and $y_j' \leq z_j$, then since $\pi^z \geq p_i^z$ and $\pi^z \geq q_j^z$ for all i, j we have

$$\pi^z(x_i' - z_i) \geq p_i^z(x_i' - z_i) \;\;\&\;\; \pi^z(z_j' - y_j) \geq q_j^z(z_j' - y_j).$$

Now, since x_i^z and y_j^z satisfy these requirements, due to PA(iii) we can substitute them instead of x_i' and y_j' and conclude

$$\pi^z(h) \geq \sum_N p_i^z(x_i^z - z_i) + \sum_M q_j^z(z_j - y_j^z) \geq$$

$$\geq \sum_N \pi(w_i - z_i) + \sum_N\sum_M \theta_j^i(\pi(z_j) + \underset{\mathcal{L}\subset L}{limsup}(s_{\mathcal{L}}^g(h) - \underset{y\in Y_j\cap D_j}{inf} s_{\mathcal{L}}^g(z_j + h - y))) -$$

$$-\sum_M \underset{\mathcal{L}\subset L}{limsup}(s_{\mathcal{L}}^g(h) - \underset{y\in Y_j\cap D_j}{inf} s_{\mathcal{L}}^g(z_j + h - y)) \geq \sum_N \pi^z(w_i - z_i) + \sum_M \pi^z(z_j) = \pi^z(h).$$

We see that the left-hand side of the latter relation is equal to the right-hand side. It implies that if we substitute x_i^z in (26) and y_j^z in (27), then the equalities are realized.In particular we conclude that

$$\langle q_j^z, z_j - y_j^z\rangle = \langle \pi^z, z_j - y_j^z\rangle = -\underset{\mathcal{L}\subset L}{limsup}[s_{\mathcal{L}}^g(h) - \underset{y\in Y_j\cap D_j}{inf} s_{\mathcal{L}}^g(z_j + h - y)] \quad \forall j \in M. \quad (29)$$

The latter one together with (27) proves

$$\langle q_j^z, Y_j\rangle \le q_j^z y_j^z. \tag{30}$$

Applying the identity (29) to the right-hand side of (26), and after simple transformation we obtain that consumers' consumption bundles satisfy the following property:

$$\langle p_i^z, conv\mathcal{P}_i(x_i^z) - z_i\rangle \ge \pi^z(w_i - z_i) + \sum_M \theta_j^i \pi^z y_j^z \quad i \in N. \tag{31}$$

We need to show also one more property. For each fixed $x_i' \ge z_i$ & $y_j' \le z_j$ such that $\sum x_i' = \sum y_j' + w \Rightarrow \sum(x_i' - z_i) + \sum(z_j - y_j') = h$ holds, in view of (28), PA(iii) and $\pi^z = \vee_N p_i^z \vee_M q_j^z$ – specification, we conclude

$$\sum p_i^z(x_i^z - z_i) + \sum q_j^z(z_j - y_j^z) = \sum \pi^z(x_i^z - z_i) + \sum \pi^z(z_j - y_j^z) = \pi^z h \ge$$
$$\sum p_i^z(x_i' - z_i) + \sum q_j^z(z_j - y_j').$$

Omitting the identical terms and due to (30), (31) we obtain

$$\sum p_i^z x_i - \sum q_j^z y_j \ge \sum p_i^z x_i' - \sum q_j^z y_j' \quad \forall x_i \in \mathcal{P}_i(x_i^z),\ \&\ y_j \in Y_j \tag{32}$$

for all $x_i', y_j' \in L$ such that $x_i' \ge z_i,\ y_j' \le z_j\ \&\ \sum x_i' = \sum y_j' + \sum w_i$.

Below we intend to realize passing to limits by $z_i \in Z_i$ & $z_j \in Z_j$,. To do it let us specify

$$Z = \prod_{i\in N} Z_i \prod_{j\in M} Z_j.$$

The set Z may be naturally ordered by $z' \ge z'' \iff z_i' \le z_i''\ i \in N\ \&\ z_j' \ge z_j'',\ j \in M$. Since each Z_i & Z_j is lattice, we can conclude that $(Z, \ge)$ is *directed.* Hence it is possible to pass to limits by $z \in Z$. We have proved above the existence of "equilibrium quadruples", which, being denoted now by $(x^z, y^z, p^z, q^z)_{z\in Z}$, are included in some common compact (it may be shown by similar to given above arguments) and satisfy properties (29) – (32). Now we intend to estimate firms' profits written in the right-hand side of (31) in the following way.

If $j \in M$ then by (29), (30) for every fixed $y_j \in Y_j$ and $0 \le z_j' \le z_j$, such that $y_j \le z_j'$, since $\pi^z \ge q_j^z$ we can write

$$\pi^z y_j^z = \pi^z z_j - \pi^z(z_j - y_j^z) = \pi^z z_j - q_j^z(z_j - y_j^z) \ge \pi^z z_j - q_j^z(z_j - y_j) =$$
$$\pi^z z_j' + \pi^z(z_j - z_j') - q_j^z(z_j - z_j') - q_j^z(z_j' - y_j) \ge \pi^z z_j' - q_j^z(z_j' - y_j).$$

For $i \in N$ and fixed $x_i \in \mathcal{P}_i(x_i^z)$, if $z_i' \le w_i$ is chosen so that $z_i \le z_i' \le x_i$ then, since $\pi^z \ge p_i^z$, we have

$$\pi^z(w_i - z_i) = \pi^z(w_i - z_i') + \pi^z(z_i' - z_i) \ge \pi^z(w_i - z_i') + p_i^z(z_i' - z_i).$$

As a result, omitting identical terms, one can obtain from (31) the following:

$$\langle p_i^z, x_i - z_i'\rangle \ge \pi^z(w_i - z_i') + \sum_M \theta_j^i(\pi^z z_j' - q_j^z(z_j' - y_j)) \quad \forall x_i \in \mathcal{P}_i(x_i^z),\ y_j \in Y_j, \quad i \in N,$$

where $z_i' \in Z_i$ and $z_j' \in Z_j$ satisfy the additional conditions

$$z_i \le z_i' \le x_i, \quad z_i' \le w_i \quad \& \quad 0 \le z_j' \le z_j, \quad y_j \le z_j' \tag{33}$$

for all $i \in N, \ j \in M$. Let

$$(x^z, y^z, p^z, q^z)_{z \in Z} \underset{weak}{\longrightarrow} (\hat{x}, \hat{y}, \hat{p}, \hat{q}),$$

then, taking into account $\underset{z \in Z}{limsup}\, \pi^z u \ge \hat{\pi} u$ for every $u \ge 0$ and the weak openness of $\mathcal{P}_i^{-1}(\hat{x}_i)$ by PA(ii), after passing to limits the former relation yields

$$\langle \hat{p}_i, x_i - z_i' \rangle \ge \hat{\pi}(w_i - z_i') + \sum_M \theta_j^i (\hat{\pi} z_j' - \hat{q}_j(z_j' - y_j)), \quad i \in N,$$

that in view of $\hat{\pi}(z_j' - y_j) \ge \hat{q}_j(z_j' - y_j)$ implies

$$\langle \hat{p}_i, x_i - z_i' \rangle \ge \hat{\pi}(w_i - z_i') + \sum_M \theta_j^i \hat{\pi} y_j, \quad i \in N$$

for all $x_i \in \mathcal{P}_i(\hat{x}_i), \ y_j \in Y_j$, satisfying (33). Furthermore, since $\hat{\pi}(x_i - z_i') \ge \hat{p}_i(x_i - z_i')$ by the $\hat{\pi}$-definition and (33), then after omitting the identical terms, the latter one implies

$$\hat{\pi} x_i \ge \hat{\pi} w_i + \sum_M \theta_j^i \hat{\pi} y_j, \quad i \in N. \tag{34}$$

This relation does not depend on the choice of z_i' & z_j' and, since $X_i \subset Z_i$ & $Y_j \subset Z_j$, where Z_i, Z_j are lattices by ACS (iii) & APS (iii) we have proved (34) *for all* $x_i \in \mathcal{P}_i(\hat{x}_i)$ & $y_j \in Y_j$.

Further, clear that we have $\sum \hat{x}_i = \sum \hat{y}_j + w$. Let us remember (32). Using PA(ii) we may think that all consumption and production bundles in (32) are fixed and pass to limit by $z \in Z$ in (32). Due to PA(iii) we conclude that

$$\sum \hat{p}_i x_i - \sum \hat{q}_j y_j \ge \sum \hat{p}_i x_i' - \sum \hat{q}_j y_j', \ \ \forall\, x_i \in cl(conv\mathcal{P}_i(\hat{x}_i)) \ \& \ \ \forall\, y_j \in Y_j \tag{35}$$

holds for all $x_i' \in Z_i - L_+, \ i \in N$ & $y_j' \in Z_j - L_+, \ j \in M$ satisfying the condition $\sum x_i' = \sum y_j' + \sum w_i$. Substituting now in (35) $\hat{x}_i$ instead of $x_i', \ x_i$ and $\hat{y}_k$ instead of y_k for all $k \ne j, \ k \in M$, and $\hat{y}_t$ instead of y_t' for $t \in M$, we conclude that

$$\langle \hat{q}_j, Y_j \rangle \le \langle \hat{q}_j, \hat{y}_j \rangle, \quad j \in M.$$

Analogously we yield

$$\langle \hat{p}_i, \mathcal{P}_i(\hat{x}_i) \rangle \ge \langle \hat{p}_i, \hat{x}_i \rangle, \quad i \in N.$$

Now let us show that

$$\langle \hat{\pi}, Y_j \rangle \le \langle \hat{\pi}, \hat{y}_j \rangle, \quad j \in M.$$

Choosing any $z_j' \in Z_j$ and $z_i' \in Z_i$, satisfying $\hat{y}_j \le z_j'$ & $\hat{x}_i \ge z_i'$ for $j \in M, \ i \in N$ and using similar arguments we may conclude from (35) that

$$\sum \hat{p}_i(\hat{x}_i - z_i') + \sum \hat{q}_j(z_j' - \hat{y}_j) \ge \sum \hat{p}_i(x_i' - z_i') + \sum \hat{q}_j(z_j' - y_j')$$

for all $x'_i, y'_j \in L$ such that $x'_i \geq z'_i$, $y'_j \leq z'_j$ & $\sum x'_i = \sum y'_j + \sum w_i$. The last standardly implies that $\hat{q}_j(z'_j - \hat{y}_j) = \hat{\pi}(z'_j - \hat{y}_j)$. But we have already proved

$$\langle \hat{q}_j, Y_j \rangle \leq \langle \hat{q}_j, \hat{y}_j \rangle \Rightarrow \langle \hat{q}_j, z'_j - Y_j \rangle \geq \langle \hat{q}_j, z'_j - \hat{y}_j \rangle, \quad j \in M,$$

that, in view of $\hat{\pi}(z'_j - y_j) \geq \hat{q}_j(z'_j - y_j)$ for every $y_j \in Y_j$ such that $z'_j \geq y_j$ (because of $\hat{\pi} \geq \hat{q}_j$), yields

$$\langle \hat{\pi}, z'_j - y_j \rangle \geq \langle \hat{q}_j, z'_j - \hat{y}_j \rangle = \langle \hat{\pi}, z'_j - \hat{y}_j \rangle \Rightarrow \langle \hat{\pi}, y_j \rangle \leq \langle \hat{\pi}, \hat{y}_j \rangle \quad j \in M.$$

But z'_j was chosen rather arbitrary and since Z_j is a lattice and $Y_j \subset Z_j$ in view of APS(iii) we conclude

$$\langle \hat{\pi}, Y_j \rangle \leq \langle \hat{\pi}, \hat{y}_j \rangle, \quad j \in M.$$

Having proved the latter relations we may maximize the right-hand side in (34) by $y_j \in Y_j$, $j \in M$ and conclude that

$$\langle \hat{\pi}, \mathcal{P}_i(\hat{x}_i) \rangle \geq \hat{\pi} w_i + \sum_M \theta^i_j \hat{\pi} \hat{y}_j, \quad i \in N.$$

The fact that $\langle \hat{\pi}, \hat{x}_i \rangle \geq \hat{\pi} w_i + \sum_M \theta^i_j \hat{\pi} \hat{y}_j$, $\quad i \in N$ can be standardly proved using the latter inequality and PA(iii). This, in view of $\sum_N \hat{x}_i = \sum_M \hat{y}_j + \sum_N w_i$, yields $\langle \hat{\pi}, \hat{x}_i \rangle = \hat{\pi} w_i + \sum_M \theta^i_j \hat{\pi} \hat{y}_j$, $i \in N$ (otherwise we are coming to contradiction). As a result, summing up three latter conclusions, we have proved that $(\hat{x}, \hat{y}, \hat{\pi})$ is a quasi-equilibrium.
Q.E.D.

Proof of Proposition. We need to show that the map $\langle \cdot, \cdot \rangle : L^* \times \mathcal{L} \to R$ is continuous with respect to both variables and to the weak* topology $\sigma(L^*, L)$ for L^*. To do it let us take any directness $p_\xi \in L^*$ & $x_\xi \in \mathcal{L}$, $\xi \in \Xi$ such that

$$p_\xi \underset{\Xi, weak}{\longrightarrow} p^* \ \ \& \ \ x_\xi \underset{\Xi}{\longrightarrow} x^*.$$

We need to show that

$$\lim_{\Xi} \langle p_\xi, x_\xi \rangle = \langle p^*, x^* \rangle.$$

We assumed $\mathcal{L}$ to be finite-dimensional, therefore choosing and fixing some linear basis $\nu_t \in \mathcal{L}$, $t = \overline{1, \dim \mathcal{L}}$ we can write $x_\xi = \sum_t \beta^\xi_t \nu_t$. Without loss of generality we can assume that $\beta^\xi_t \underset{\Xi}{\longrightarrow} \beta_t$, that entails

$$p^\xi x^\xi = \sum_t \beta^\xi_t (p^\xi \nu_t) \underset{\Xi}{\longrightarrow} \sum \beta_t (p^* \nu_t) = p^* x^*$$

because $p^\xi \nu_t \underset{\Xi}{\longrightarrow} p^* \nu_t$ for each t. **Q.E.D.**

Proof of Lemma 1. Let $p \in L^*$. Then, since L^* is the lattice, there exist p^-, p^+ & $|p| \in L^*$ and

$$z \in C_i \Longrightarrow z_i \leq z \leq z_i + 2h, \quad \& \quad h \geq 0$$

that implies

$$pz = p^+ z + p^-(-z) \leq p^+(z_i + 2h) + p^-(-z_i) \leq |p||z_i| + p^+(2h) \leq |p|(|z_i| + 2h).$$

Analogously it may be shown that $pz \geq -|p|(|z_i| + 2h)$ and we yield

$$-|p|(|z_i| + 2h) \leq \langle p, C_i \rangle \leq |p|(|z_i| + 2h), \qquad \forall\, p \in L^*,$$

that means *weak*-boundedness of C_i in L. In view of the finite-dimensionality and the closeness of C_i (as the intersection of closed sets), we conclude the compactness of C_i (since all separable topologies are equivalent for finite-dimensional spaces). The compactness of D_j is proved analogously. **Q.E.D.**

Proof of Lemma 2. Suppose

$$x \in X \bigcap (z + \Gamma), \quad \Gamma = con(v + V),$$

for some $z \in Z \setminus X$. Then $x = z + \alpha v + y$, $y \in \alpha V$ for some $\alpha > 0$ and we have $z = x - \alpha v - y$. Since $-y \in \alpha V$ and z is *p.u.b.*, by the definition of properness we conclude $z \in X$, that contradicts the choice of z. The second part of *Lemma* is proved symmetrically. **Q.E.D.**

Proof of Corollary 1. It is sufficient to consider the case of production sets. Indeed, by y_j^*-definition we have

$$*(y_j) = y_j^* = \begin{cases} y_j + t'(z_j - y_j), & y_j \in Y_j, \\ y_j + t''(z_j - 2h - y_j), & y_j \notin Y_j, \end{cases}$$

where

$$\begin{aligned} t' &= \sup\{t \mid y_j + t(z_j - y_j) \in Y_j\}, \\ t'' &= \inf\{t \mid y_j + t(z_j - 2h - y_j) \in Y_j\}. \end{aligned}$$

For $y_j \in Y_j$, since $z_j \notin Y_j$ and in view of *Lemma 2* we have

$$(y_j + t(z_j - y_j) + \Gamma_+) \bigcap Y_j = \emptyset, \quad t' < t \leq 1.$$

In view of $(z_j - y_j) \geq 0 \Rightarrow (z_j - y_j) \in \Gamma_+$ we have

$$(y_j^* + \Gamma_+) = \bigcup_{t' < t \leq 1} (y_j + t(z_j - y_j) + \Gamma_+)$$

that together with the previous relation implies

$$(y_j^* + \Gamma_+) \bigcap Y_j = \emptyset.$$

Analogously for $y_j \notin Y_j$ we have

$$(y_j + t(z_j - 2h - y_j) + \Gamma_+) \bigcap Y_j = \emptyset, \quad 0 \leq t < t''$$

and since $z_j - 2h - y_j \leq 0$, one can conclude that the latter relation is true again. **Q.E.D.**

Proof of Lemma 3. We start the proof of *Lemma 3* from the case of producer. Now we fix the index j but omit it sometimes to simplify the notations below. Note, that in view of *Proposition* we can pass to limits if it is necessary. Let

$$q_\xi \in Q_j(y_\xi) \;\; \& \;\; y_\xi \in D_j, \quad \xi \in \Xi$$

be some directed sets, such that

$$y_\xi \longrightarrow y \;\; \& \;\; q_\xi \longrightarrow q$$

holds. We need to prove that $q \in Q_j(y)$. Without lost of generality we may thing that one of two alternatives is true:

(i) $y_\xi \in Y_j, \quad \xi \in \Xi,$

(ii) $y_\xi \notin Y_j, \quad \xi \in \Xi.$

In both cases by (11) we have

$$\langle q_\xi, y_\xi^* + \Gamma_+ \rangle \geq q_\xi y_\xi^* \geq \langle q_\xi, Y_j \rangle.$$

The result will be proved via passing to limits in the latter formula and using the standard arguments. We should prove merely that

$$\lim_{\xi \in \Xi} \langle q_\xi, y_\xi^* \rangle = q y^*. \tag{36}$$

Let us do it. In the case (i) by the $*(\cdot)$-definition we have

$$y_\xi^* = y_\xi + t'_\xi (z_j - y_\xi), \quad t'_\xi = \sup\{t \mid y_\xi + t(z_j - y_\xi) \in Y_j\}.$$

Since $y^* \in Y_j$ we obtain

$$q_\xi y^* \leq q_\xi (y_\xi + t'_\xi (z_j - y_\xi)) \;\; \forall\, \xi \in \Xi. \tag{37}$$

But we have also

$$y^* = y + t'(z_j - y), \quad t' = \sup\{t \mid y + t(z_j - y) \in Y_j\}.$$

Now if $\bar{t}$ is any limit point of t'_ξ, then since $\bar{y} \in Y_j$, where $\bar{y} = y + \bar{t}(z_j - y) = \lim_{\Xi'} y_\xi^*$, we conclude that $\bar{t} \leq t' \Rightarrow \bar{y} \leq y^*$. Since $q \geq 0$ we obtain $qy^* \geq q\bar{y}$. But passing to limits in (37) by $\xi \in \Xi' \subset \Xi$ implies $qy^* \leq q\bar{y}$. This, together with the previous relation yields $qy^* = q\bar{y}$ and proves (36).

In the case (ii) we have

$$y_\xi^* = y_\xi + t''_\xi (z_j - 2h - y_\xi), \quad t''_\xi = \inf\{t \mid y_\xi + t(z_j - 2h - y_\xi) \in Y_j\}.$$

Again, if $\bar{t}$ is some limit point of t''_ξ, then

$$\bar{y} = \lim_{\xi \in \Xi'} y_\xi^* = y + \bar{t}(z_j - 2h - y) \in Y_j \Rightarrow \bar{t} \geq t'' \Rightarrow y^* \geq \bar{y} \Rightarrow qy^* \geq q\bar{y}.$$

But from $q_\xi \in Q_j(y_\xi)$ we have

$$q_\xi y_\xi^* \geq q_\xi y^* \quad \Rightarrow \quad q\bar{y} \geq qy^*.$$

This and the latter relation yield (36).

At last, let $x_\xi \in X_i$ & $x_\xi \to x$ and $p_\xi \in P_i(x_\xi)$ & $p_\xi \to p$. By specification we have

$$\langle p_\xi, x_\xi - \Gamma_+ \rangle \leq p_\xi x_\xi \leq \langle p_\xi, \mathcal{P}_i(x_\xi) \rangle.$$

We may pass to limits in this relation and *Lemma 3* will be proved if we show that $px \leq px'$ for every $x' \in \mathcal{P}_i(x)$. To do it let us remember PA(ii). For fixed x' it implies $x' \in \mathcal{P}_i(x_\xi)$ for all ξ big enough. We conclude $p_\xi x' \geq p_\xi x_\xi$. Passing to limits gives us the result. **Q.E.D.**

Proof of Lemma 4. First we show that there exist $p_i \in P_i(x_i)$ and $q_j \in Q_j(y_j)$ such that $p_i v > 0$, & $q_j v > 0$ for every $x_i \in C_i$ & $y_j \in D_j$.

In view of assumptions PA(iii)-(v) and (9)-(10), by *Corollary 1* we can conclude that

$$(x_i - \Gamma_+) \bigcap conv\ \mathcal{P}_i(x_i) = \emptyset, \quad x_i \in X_i$$

and

$$(x_i^* - \Gamma_+) \bigcap X_i = \emptyset, \quad x_i \notin X_i.$$

Therefore we may apply the separation theorem, which yields the existence of such $p_i \in L_+^*$ that $p_i \neq 0$ and

$$\langle p_i, (x_i - \Gamma_+) \rangle \leq \langle p_i, conv\mathcal{P}_i(x_i) \rangle, \ x_i \in X_i \ \ \& \ \ \langle p_i, (x_i^* - \Gamma_+) \rangle \leq \langle p_i, X_i \rangle, \ x_i \notin X_i.$$

Since Γ_+ is an open cone the latter is possible only if

$$\langle p_i, \Gamma_+) \rangle > 0 \Rightarrow \langle p_i, v \rangle > 0.$$

Now since $x_i - z_i \geq 0$ & $h \geq v$ and because of $p_i \geq 0$ we obtain

$$\langle p_i, x_i - z_i + \epsilon h \rangle \geq \epsilon p_i v > 0.$$

Therefore for each given $\lambda_i \geq 0$ the functional p_i may be normalized so that

$$\langle p_i, x_i - z_i + \epsilon h \rangle = \lambda_i$$

and we see that $p_i \in P_i^\epsilon(x_i, \lambda_i)$. The case of $Q_j^\epsilon(\cdot)$ is considered symmetrically.

Now let us consider the second part of *Lemma 4* and let $p_i \in P_i^\epsilon(x_i, \lambda_i)$ and $q_j \in Q_j^\epsilon(y_j, \alpha_j)$ be fixed. By specification we have

$$\langle p_i, x_i - z_i + \epsilon h \rangle = \lambda_i \ \ \& \ \ \langle q_j, z_j - y_j + \epsilon h \rangle = \alpha_j$$

for all i, j. Since $h \geq x_i - z_i \geq 0$ & $p_i \geq 0$ we obtain $p_i(\epsilon h) \leq \lambda_i$ and $p_i((1+\epsilon)h) \geq \lambda_i$ that yields

$$\frac{\lambda_i}{1+\epsilon} \leq p_i(h) \leq \frac{\lambda_i}{\epsilon} \quad i \in N.$$

Analogously we conclude

$$\frac{\alpha_j}{1+\epsilon} \le q_j(h) \le \frac{\alpha_j}{\epsilon} \quad j \in M.$$

Summing the former and the latter inequalities we have proved the result. **Q.E.D.**

Proof of Lemma 5. By definition for $(p,q) \in \mathbf{P}_\epsilon$ it can be standardly shown that

$$|\langle p_i, V\rangle| \le \langle p_i, v\rangle \quad \& \quad |\langle q_j, V\rangle| \le \langle q_j, v\rangle$$

for all i, j. Now since $h \ge v$ and $p_i \ge 0$ & $q_j \ge 0$ we obtain $\pi_\Sigma(h) \ge \pi_\Sigma(v)$, that yields

$$|\langle (p,q), V^{n+m}\rangle| \le \pi_\Sigma(h) \le \frac{1}{\epsilon}.$$

Applying Alaoglu's theorem we state the result. **Q.E.D.**

Proof of Lemma 6. First we prove the continuity of $s(\cdot,\cdot)$ in every point $(g,z) \in \mathbf{P}_\epsilon \times \mathcal{L}(h)$ if $\gamma h \ge z \ge \sigma h$ for some $\gamma > \sigma > 0$. In view of [3] (see Th.7 Chapter 2) and by (15) it is sufficient to prove the continuity of the mapping

$$z :\Rightarrow F(z) = \{y \in \mathcal{L}_+^{n+m} | \sum_{t=1}^{t=n+m} y_t = z\}$$

for every z satisfying the imposed properties. Let us do it.

The upper hemicontinuity follows from the fact that the induced order onto $\mathcal{L}$ is defined due to the *closed cone* $\mathcal{L}_+ = \mathcal{L} \cap L_+$, since it implies that in the inequalities

$$0 \le \sum_{t=1}^{t=n+m} y_t^\xi = z_\xi, \quad z_\xi \ge y_t^\xi \ge 0$$

we may pass to limits by ξ.

The proof of lower hemicontinuity is more involved. Due to the imposed assumptions z is the *interior point* of the set

$$\{y \in \mathcal{L} | (1-\nu)z \le y \le (1+\nu)z\}$$

for $\nu > 0$ in the space $\mathcal{L}(h)$ formed as the linear hull of the order interval $[0,h]$. Therefore, if $z_\xi \to z$ & $z_\xi \in \mathcal{L}(h)$ then one can find such $\nu_\xi > 0$ & $\nu_\xi \to 0$ that $(1-\nu_\xi)z \le z_\xi \le (1+\nu_\xi)z$. Now if $y \in F(z)$ then $(1-\nu_\xi)y \in F((1-\nu_\xi)z)$ and in view of $z_\xi - (1-\nu_\xi)z \ge 0$ we conclude that

$$y_\xi = ((1-\nu_\xi)y_1, ..., (1-\nu_\xi)y_{n+m-1}, z_\xi - (1-\nu_\xi)z + (1-\nu_\xi)y_{n+m}) \in F(z_\xi).$$

By specification $y_\xi \to y$ as we wanted to prove.

Now let us turn to show items (i)-(iii). The first two of them immediately follow from specification (15). Consider item (iii). By (15) for every $z \in \mathcal{L}_+$ we have

$$s^g(z) \ge p_i z, \quad i \in N \quad \& \quad s^g(z) \ge q_j z \quad j \in M.$$

The summation of these inequalities proves the left-hand side of (iii). On the other hand, in view of $p_i \geq 0,\ q_j \geq 0$, since $x_i' \leq z$ & $y_j' \leq z$ we have $p_i x_i' \leq p_i z$ and $q_j y_j' \leq q_j z$. The summation of these inequalities by (15) proves the right-hand side of (iii). **Q.E.D.**

Proof of Lemma 7. In view of *Lemma 6* the function $s(g, z_j + (1+\beta)h - y)$ is continuous by $g \in \mathbf{P}_\epsilon$ and $y \in Y_j \cap D_j$ (due to $y \leq z_j \leq h$ the required in *Lemma 6* properties hold). Since $Y_j \cap D_j$ and $\mathbf{P}_\epsilon$ are compact sets by *Lemmas 1,5*, we can conclude that $s(g, z_j + (1+\beta)h - y)$ is *uniformly continuous* on $\mathbf{P}_\epsilon \times (Y_j \cap D_j)$ for each $j \in M$. It proves the continuity of $\inf_{y \in Y_j \cap D_j} s^g(z_j + h + \beta h - y)$ that by (16) - (18) implies the continuity of $\lambda_i(\cdot)$ & $\alpha_j(\cdot)$ on $\mathbf{P}_\epsilon$ for all i, j.

Now we turn to items (i),(ii). Indeed, by (16) and due to $0 \in Y_j$ we see that $\mu_j(g) \geq 0$. This by (17),(18), *Lemma 6(ii)*, and $w_i - z_i \geq 0$ yields

$$\lambda_i(g) = \frac{s^g(w_i + \beta h - z_i) + \sum_M \theta^i_j \mu^\beta_j(g)}{s^g(1 + (n+m)\beta)h)} \geq \frac{s^g(\beta h)}{s^g(1 + (n+m)\beta)h)} = \frac{\beta}{1 + \beta(n+m)} > 0.$$

Since $z_j - y \geq 0$ for $y \in D_j$, then

$$\alpha_j(g) = \frac{\inf_{y \in Y_j \cap D_j} s^g(z_j + (1+\beta)h - y) - s^g(h)}{s^g(1 + (n+m)\beta h)} \geq \frac{s^g(\beta h)}{s^g(1 + (n+m)\beta)h)} = \frac{\beta}{1 + \beta(n+m)} > 0.$$

To prove (ii) let us sum these values:

$$\sum \lambda_i(g) + \sum \alpha_j(g) = \frac{\sum_N s^g(w_i - z_i + \beta h) + \sum_M s^g(z_j + (1+\beta)h) - m s^g(h)}{s^g(1 + (n+m)\beta h)}.$$

Here in view of *Lemma 6(i)* the sum of the first and second summands in the numerator is no more than $s^g[(m+1+(n+m)\beta)h]$. This applying *Lemma 6 (ii)* yields the left-hand side of inequality in (ii). On the other hand, applying *Lemma 6(i)&(iii)* this numerator may be estimated also as

$$\sum_N s^g(w_i - z_i + \beta h) + \sum_M s^g(z_j + (1+\beta)h) - m s^g(h) \geq \sum_N s^g(w_i - z_i + \beta h) + \sum_M s^g(z_j + \beta h) \geq$$

$$\frac{1}{n+m}\Big(\sum_N \pi_\Sigma(w_i - z_i + \beta h) + \sum_M \pi_\Sigma(z_j + \beta h)\Big) = \frac{1}{n+m}(\pi_\Sigma(1 + (n+m)\beta h)).$$

Since

$$s^g(1 + (n+m)\beta h) \leq \pi_\Sigma(1 + (n+m)\beta h),$$

then comparing these inequalities we have stated the right-hand side of (ii). **Q.E.D.**

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