OPTIMAL DECISIONS IN MARKETS AND PLANNED ECONOMIES

edited by

Richard E. Quandt and Dušan Tříska

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Equilibrium with Nonstandard Prices in Exchange Economies

Valeri Marakulin

1. Introduction

In this paper we study some models of economic equilibrium. We propose an approach based on the application of the techniques and methods of nonstandard analysis to economic equilibrium models. The main idea is the introduction and study of the properties of equilibria with "nonstandard prices," but standard consumption plans for economic agents. For this purpose, a method for estimating the cost of an agent's consumption plan is introduced into the model, which is finer than the traditional one and allows a more exact evaluation of economic states. This makes it possible to prove the existence of equilibria without Slater's condition or any of its analogues that are necessary for proving the existence of the usual equilibria. Detailed considerations of these results and other problems that appear in economic models with nonstandard prices are discussed in Marakulin (1988).

Nonstandard methods were applied for the first time in mathematical economics by Brown and Robinson (1975). In that paper, the set of economic agents is identified with the initial segment $\{1, 2, \ldots, n\}$ of nonstandard natural numbers, where n is an infinite integer. They proved Edgeworth's conjecture about the coincidence of the core of the economy and the set of Walrasian equilibria in the limit. These ideas have subsequently been developed by others (see Rashid, 1987). However, the technique of nonstandard analysis does not appear to have been applied to the investigation of economic equilibrium.

At present, there are three conceptions of nonstandard analysis: classical, nonclassical and radical. We follow the classical notation of Robinson found in Davis (1977). Methods of nonstandard analysis are based on including the "standard universum of mathematical reasoning" $\mathcal{U}$ in the "nonstandard universum" ${}^*\mathcal{U}$ which contains both infinitesimals and infinite objects.

The transfer principle is a meaningful concept of nonstandard analysis and establishes the connection between standard and nonstandard mathematics. This principle is as follows: any sentence (formula) will be true in standard mathematics if and only if it is true in nonstandard mathematics. The mapping, which adjoins standard objects to nonstandard ones, could be understood as a replenishment or expansion of standard sets. In this fashion, the mappings of standard sets would be interpreted as expansion mappings of nonstandard sets. In particular, $\Re$ is included in its own expansion ${}^*\Re$, which contain nonzero infinitesimals and infinite numbers, while retaining all the formal properties of $\Re$. For example, the number $\alpha \in {}^*\Re$ is said to be infinitesimal if $|\alpha| < |r|$ for all $r \in \Re$, $r \neq 0$. The relationship $p \approx q$ is equal to the infinitesimal $p - q$. The set of all numbers that are infinitesimally equal to p is known as a monad and denoted by $\mu(p)$. A nonstandard vector $p = (p_1, p_2, \dots, p_\ell)$ is a vector with nonstandard components.

2. The Models

The following formal model of an exchange economy is studied:

$$\mathcal{E}_0 = \langle N, \{X_i, \mathcal{P}_i\}_{i \in N}, w \rangle.$$

Here $N = \{1, 2, \dots, n\}$ is a set of economic agents, $X_i \subset \Re^\ell$ is the i^{th} agent's consumption set, $w \in \sum_N X_i$ is the initial endowment of the economy and ℓ is the number of products. Agent i's preferences are described by a point–to–set mapping

$$\mathcal{P}_i : \prod_N X_i \to 2^{X_i},$$

where the set $\mathcal{P}_i(x) \subset X_i$ is the collection of all product bundles strictly preferred to the bundle x_i by the i^{th} agent in the state $x = (x_1, x_2, \dots, x_n)$. The symbol $\mathcal{P}_i(x|x_i)$ denotes the set $\mathcal{P}_i(x)$ where the i^{th} component of vector x is equal to the i^{th} agent's consumption plan x_i.

The dual description of Pareto's states will be based on nonstandard prices. States of the economy that allow such a description are called equilibria by Pareto. From this point of view, the following theorem may be called the theorem on the coincidence of Pareto-optimal states and nonstandard Pareto-equilibria. Let $X = \prod_N X_i$ and

$$E(M,w) = \left\{x \in (\Re^\ell)^M \mid \sum_M x_i = w\right\}, \tag{1}$$

where $M \subset N$, $M \neq \emptyset$ and $w \in \Re^\ell$.

Definition 1. *A state $x \in X$ is said to be Pareto-optimal if $x \in E(N,w)$ and*

$$\prod_M \mathcal{P}_i(x) \cap E(M, x_M) = \emptyset \tag{2}$$

for all $M \subseteq N$, $M \neq \emptyset$, where $x_M = \sum_M x_i$.

The set of states that satisfy Definition 1 is called the Pareto-boundary. Note that this unusual definition of the Pareto-optimality of states is employed in order to allow for the possibility of satiation in agents' preferences, that is, to allow for the possibility that $\mathcal{P}_i(x) = \emptyset$ for some agents.

Theorem 1. *Let the following conditions hold for all $i \in N$: $\mathcal{P}_i$ are convex, $x_i \notin \mathcal{P}_i(x|x_i)$, and for all $\lambda \in (0,1)$ if $y \in \mathcal{P}_i(x|x_i)$, then*

$$\lambda y + (1-\lambda)x_i \in \mathcal{P}_i(x|x_i). \tag{3}$$

Then, $x \in E(N,w)$ is Pareto-optimal if and only if there exists a price vector $p \in {}^*\Re^\ell$, such that

$$\langle p, \mathcal{P}_i(x)\rangle > \langle p, x_i\rangle, \quad \forall\, i \in N. \tag{4}$$

Proof. We shall only sketch the main idea of the proof. A detailed proof follows standard arguments. We now prove (4) from the conditions of the theorem. It may be asserted that $x_i = 0$ for all $i \in N$, that is to say, $x = 0$. Define

$$M = \{i \in N \mid \mathcal{P}_i(0) \neq \emptyset\}, \quad \mathcal{P}^M = \prod_M \mathcal{P}_i(0).$$

The result is established by induction on the dimension of space T which includes X_i for all $i \in N$. When $\dim T = 1$, the result follows by applying the separation theorem to sets $E(M,0)$ and $\mathcal{P}^M$. Let $\dim T > 1$. By Definition 1,

$$\mathcal{P}^M \cap E(M,0) = \emptyset.$$

Using the separation theorem, we can find a functional f which separates these sets:

$$f(\mathcal{P}^M) \geqq f(E(M,0)).$$

Since $E(M,0)$ is a subspace of $(\Re^\ell)^M$, f must be zero on $E(M,0)$. Hence, the vector $\tilde{f} \in (\Re^\ell)^M$, which represents the functional, has the form

$$\tilde{f} = (\tilde{p}, \tilde{p}, \ldots, \tilde{p}), \quad \tilde{p} \in \Re^\ell,$$

since we have $\langle \tilde{p}, \mathcal{P}_i(0)\rangle \geqq 0$ for all $i \in N$, by the construction of f. Let $T' = \{y \in \Re^\ell \mid \langle \tilde{p}, y\rangle = 0\}$. According to the induction hypothesis, there exists nonstandard p', such that

$$\langle p', \mathcal{P}_i(0) \cap T'\rangle > 0, \quad i \in N.$$

Let

$$p = \tilde{p} + \epsilon \cdot p',$$

where $\epsilon > 0$ is a nonstandard number satisfying $\epsilon \cdot |p'| \approx 0$. Now we show that the vector p is the desired one. If $y \in \mathcal{P}_i(0)$ and $y \in T'$, then

$$\langle p, y\rangle = \langle \tilde{p}, y\rangle + \epsilon \cdot \langle p', y\rangle = \epsilon \cdot \langle p', y\rangle > 0.$$

For $y \notin T'$ we have

$$\langle p, y\rangle = \langle \tilde{p}, y\rangle + \epsilon \cdot \langle p', y\rangle \geqq \delta - \epsilon |p'| d > 0,$$

since $\epsilon |p'| d \approx 0$ when $d = ||y||$ and $\delta = \langle \tilde{p}, y\rangle > 0$ is a standard number. ∎

It can be seen from the proof of Theorem 1 that in the finite-dimensional case standard convex sets with an empty intersection can be strictly separated by a nonstandard hyperplane. In fact, this result is not connected with either the dimensional or the topological properties of the space (see Marakulin, 1988). We now extend model $\mathcal{E}_0$ by adding a formal cost mechanism, which is necessary for the definition of economic equilibrium, i.e., for defining ownership relations. This can be accomplished in the traditional way. Let Q be the set of all permissible prices and let $\alpha_i\colon Q \to \Re$ be the profit functions of the agents, where $\alpha_i(p)$ is the income of the i^{th} participant when the price is $p \in Q$. Thus, we have the following model:

$$\mathcal{E}_1 = \left\langle N, \left\{X_i, \mathcal{P}_i, \alpha_i\right\}_{i\in N}, Q, w\right\rangle.$$

The definition of a standard budget set with nonstandard prices is of vital importance for the introduction of an equilibrium with nonstandard prices but standard consumption plans for the agents. Let ${}^*Q$ be the set of all permissible nonstandard prices and let the agents' profit functions be replaced by their images

$${}^*\alpha_i\colon{}^*Q \to{}^*\Re, \quad i \in N.$$

It stands to reason that $Q \subset {}^*Q$ and that the functions α_i are "extended" from Q to the functions ${}^*\alpha_i$ defined on ${}^*Q$. This permits us later to use the symbol α_i (with the domain ${}^*Q$) instead of ${}^*\alpha_i$.

We now introduce nonstandard budget sets

$$\mathrm{Bud}_i(p) = \left\{x \in {}^*X_i \mid \langle x, p\rangle \leqq \alpha_i(p)\right\}, \quad p \in {}^*Q$$

by analogy with the standard case. Here $\mathrm{Bud}_i(p) \subset {}^*X_i$, that is, the budget set consists of nonstandard consumption plans. To return to the standard case, instead of $\mathrm{Bud}_i(p)$, we take its standard part, $\mathrm{st}\,\mathrm{Bud}_i(p)$, defined as

$$\mathrm{st}\,\mathrm{Bud}_i(p) = \left\{x \in X_i \mid \exists\, y \in \mathrm{Bud}_i(p)\colon y \approx x\right\}. \tag{5}$$

By definition, this is a standard set, but note that it can be empty for nonempty $\mathrm{Bud}_i(p)$.

There is another way of defining a standard budget set with nonstandard prices. Let us consider a usual budget mapping $B_i\colon Q \to 2^{X_i}$ that is treated as a point-to-point mapping and take its $*$-image

$${}^*B_i\colon {}^*Q \to {}^*(2^{X_i}).$$

Here ${}^*(2^{X_i})$ is the set of all internal subsets of ${}^*X_i$ (see Davis, 1977). Now the set ${}^*B_i(p)$ can be taken as a nonstandard budget set. However, the sets ${}^*B_i(p)$ have the same "structure" as those defined above by the transfer principle:

$${}^*B_i(p) = \left\{y \in {}^*X_i \mid \langle p, y\rangle \leqq {}^*\alpha_i(p)\right\} = \mathrm{Bud}_i(p).$$

Thus, the two definitions are equivalent, but instead of $\mathrm{Bud}_i(p)$ we now have the convenient notation ${}^*B_i(p)$, (which is not the $*$-image of $B_i(p)$ with $p \in Q$). Thus, a budget set with nonstandard prices means that

$$\mathrm{st}\,{}^*B_i(p) = \mathrm{st}\,\mathrm{Bud}_i(p).$$

Now we are ready to introduce the notion of a nonstandard equilibrium. This notion will be analogous to the usual idea of equilibrium in the standard case, with the condition that the budget sets of agents are replaced by sets of the form of (5).

Definition 2. *A pair $(\overline{x},\overline{p}) \in X \times {}^*Q$ is called an equilibrium with nonstandard prices if it satisfies the following conditions:*

1. Attainability:

$$\overline{x}_i \in \mathrm{st}\,{}^*B_i(\overline{p}), \quad i \in N, \tag{6}$$

2. Individual rationality:

$$\mathcal{P}_i(\overline{x}) \cap \mathrm{st}\,{}^*B_i(\overline{p}) = \emptyset, \quad i \in N, \tag{7}$$

3. Balance:

$$\sum_N \overline{x}_i = w. \tag{8}$$

If condition (8) is replaced by the requirement

$$\sum_N \overline{x}_i \leqq w, \tag{9}$$

we refer to the case as one of semiequilibrium.

The theorems about the existence of nonstandard equilibria will be stated with the following assumptions:

Assumption 1: The sets X_i are convex and closed in $\Re^\ell$, $i \in N$.

Assumption 2: (Continuity of preferences). Mappings $\mathcal{P}_i$ have open graphs in $X \times X_i$, $i \in N$.

Assumption 3: (Convexity and irreflexivity). For all $x = (x_1, \ldots, x_n) \in X$ and $p \in Q$, $x_i \notin \mathrm{conv}\,\mathcal{P}_i(x \mid x_i)$ for all $i \in N$.

Assumption 4: (Continuity of profits). The functions $\alpha_i : Q \to \Re$ are continuous for all $i \in N$.

Assumption 5: For every $p \in Q$ there exist $x_1 \in X_1, x_2 \in X_2, \ldots, x_n \in X_n$, such that $\langle p, x_i \rangle \leqq \alpha_i(p)$, $i \in N$, (that is, $B_i(p) \neq \emptyset$ for all $p \in Q$, $i \in N$).

Assumption 6: (Walras' law). For every $p \in Q$, $\sum_N \alpha_i(p) = \langle p, w \rangle$.

We first state an auxiliary proposition. Let $\mathcal{X}, \mathcal{Y}$ be metric spaces and $A \subset {}^*X$ and $\mathcal{L} \subset {}^*\mathcal{Y} \times {}^*\mathcal{X}$ be internal subsets. Let

$$B = \operatorname{si} \mathcal{L} = \{z \in \mathcal{Y} \times \mathcal{X} \mid \mu(z) \subset \mathcal{L}\},$$

where $\mu(z)$ is the monad of the point z. The set B can be treated as the graph of some correspondence $\tilde{B}: \mathcal{Y} \to \mathcal{X}$ (here $\tilde{B}(y)$ is a section of B along $\mathcal{X}$). The same is also true for $\mathcal{L}$, where $\tilde{\mathcal{L}}: {}^*\mathcal{Y} \to {}^*\mathcal{X}$ and $\operatorname{Gr} \tilde{\mathcal{L}} = \mathcal{L}$.

Proposition 1.

$$D = \{y \in \mathcal{Y} \mid \tilde{B}(y) \cap \operatorname{st} A = \emptyset\} \supset \operatorname{st} \{y' \in {}^*\mathcal{Y} \mid \tilde{\mathcal{L}}(y') \cap A = \emptyset\} = C$$

Proof. Let $y \in \mathcal{Y} \backslash D$. We can find $x \in \mathcal{X}$ such that $\mu(y, x) \subset \mathcal{L}$ and $x \in \operatorname{st} A$. Since $\mu(y, x) = \mu(y) \times \mu(x)$ and $\mu(x) \cap A \neq \emptyset$, there is a $z \in A$, $z \approx x$. From this it follows that $\mu(y) \times \{z\} \subset \mathcal{L}$, i.e., $z \in \tilde{\mathcal{L}}(y') \cap A$, $y' \approx y$. Thus, the condition

$$\tilde{B}(y) \cap \operatorname{st} A \neq \emptyset$$

implies

$$\cap_{y' \approx y} \tilde{\mathcal{L}}(y') \cap A \neq \emptyset$$

By construction, if $y \in C$, there is a $y' \in {}^*\mathcal{Y}$ such that $y' \approx y$ and $\tilde{\mathcal{L}}(y') \cap A = \emptyset$. Hence,

$$\cap_{y' \approx y} \tilde{\mathcal{L}}(y') \cap A = \emptyset$$

and

$$\tilde{B}(y) \cap \operatorname{st} A = \emptyset,$$

that is $y \in D$. ∎

Theorem 2. *If $\mathcal{E}_1$ satisfies Assumptions 2–6, $X_i = \Re^\ell_+$, $i \in N$, and $Q = \Re^\ell_+ \backslash \{0\}$, then a semiequilibrium with nonstandard prices exists.*

Proof. Consider the economy $\mathcal{E}_1$ in the case of the price set

$$Q(\epsilon) = \{p \in \Re^\ell \mid \sum_{j=1}^{\ell} p_j = 1,\ p_j \geqq \epsilon,\ j = 1, \dots, \ell\}, \quad \epsilon > 0$$

and compact consumption sets

$$X_i = \Re^\ell_+ \cap \{(x_1, \dots, x_\ell) \mid x_j \leqq c,\ j = 1, \dots, \ell\},$$

where c is a sufficiently large number. Furthermore, as is usual, we shall reduce the model to a noncoalition game with $n+1$ players. In this game, the $(n+1)^{\text{th}}$ player plays the role of a price-setting body. Other players correspond to the economic agents of $\mathcal{E}_1$, their objective being to "maximize their own utilities on their budget sets."[1] Now we search for the Nash equilibria of this game. Since the budget sets depend continuously on $p \in Q(\epsilon)$ (since $p \gg 0$ and Assumption 4 is true), and the other conditions for the existence of Nash equilibria are satisfied by Assumptions 1–3, we conclude (see Makarov, 1981 and Shafer and Sonnenschein, 1975) that there exists a Nash equilibrium (x^ϵ, p^ϵ), $x^\epsilon = (x_1^\epsilon, \dots, x_n^\epsilon) \in \prod_N X_i$, $p \in Q(\epsilon)$, such that

$$x_i^\epsilon \in B_i(p^\epsilon), \qquad i \in N; \tag{10}$$

$$\mathcal{P}_i(x^\epsilon) \cap B_i(p^\epsilon) = \emptyset, \qquad i \in N; \tag{11}$$

$$\langle p^\epsilon, \sum_N x_i^\epsilon - w \rangle = \max_{p \in Q(\epsilon)} \langle p, \sum_N x_i^\epsilon - w \rangle. \tag{12}$$

This conclusion holds for all $\epsilon \in \Re$, $\epsilon > 0$. Since the transfer principle is applicable to these conditions, we obtain the following true statement: for every $\epsilon \in {}^*\Re$, $\epsilon > 0$, there exists a pair $(x^\epsilon, p^\epsilon) \in {}^*X \times {}^*Q(\epsilon)$ such that conditions (10)–(12) are satisfied if all "standard constants" are replaced by their $*$-images (i.e., $*$-s are added to the B_i, $\mathcal{P}_i$, $Q(\epsilon)$). We consider $\epsilon > 0$, $\epsilon \approx 0$, and the corresponding pair (x^ϵ, p^ϵ) satisfying (10)–(12) (in a nonstandard sense). The set X is compact; hence (see Davis, 1977, Theorem I.6) every point of ${}^*X$ is situated near a standard point, i.e., there exists $\overline{x} \in X$ such that $\overline{x} \approx x^\epsilon$. Now we shall show that a pair $(\overline{x}, p^\epsilon)$ is a nonstandard semiequilibrium.

From the construction of the budget sets and (10) it follows that

$$\overline{x}_i \in \operatorname{st} {}^*B_i(p^\epsilon),$$

[1] This approach is considered in detail in Makarov (1981).

which proves (6). We apply Proposition 1 to establish (7). Let ${}^*\mathcal{P}_i = \tilde{\mathcal{L}}$, $\mathcal{Y} = \prod_N X_j$, $\mathcal{X} = X_i$, $A = {}^*B_i(p^\epsilon)$. From (11) and the above proposition we obtain

$$[\mathrm{si}\,{}^*\mathcal{P}_i](\overline{x}) \cap \mathrm{st}\,{}^*B_i(p^\epsilon) = \emptyset, \quad i \in N.$$

Now we use Assumption 2 which, together with the construction of si ${}^*\mathcal{P}_i$, yields $\mathcal{P}_i \subset \mathrm{si}\,{}^*\mathcal{P}_i$ (in fact, $\mathcal{P}_i = \mathrm{si}\,{}^*\mathcal{P}_i$). This proves (7).

Finally, we have to prove that (9) holds. Suppose that $\sum_N \overline{x}_i - w \nleqq 0$. Then, according to (12) and $\overline{x} \approx x^\epsilon$,

$$\langle p^\epsilon, \sum_N \overline{x}_i - w\rangle > \delta > 0, \tag{13}$$

where δ is a standard number. On the other hand, from (10) we have

$$\langle p^\epsilon, x_i^\epsilon\rangle \leqq \alpha_i(p^\epsilon), \qquad i \in N.$$

Adding these inequalities and using Assumption 6, we obtain

$$\langle p^\epsilon, \sum_N x_i^\epsilon\rangle \leqq \langle p^\epsilon, w\rangle.$$

Finally, after transition from x^ϵ to $\overline{x}$, we have

$$\langle p^\epsilon, \sum_N \overline{x}_i - w\rangle \leqq \beta$$

for some $\beta \approx 0$, which contradicts (13). ∎

In the analysis of Theorem 2, attention should be paid to Assumption 5, which guarantees nonempty budget sets. This assumption is weaker than traditional assumptions similar to Slater's condition and others. Assumption 5 is the main reason for the introduction and investigation of nonstandard equilibria, a new type of economic equilibrium. However, Assumptions 1–6 are not sufficient for the existence of equilibria with a balance requirement such as (8). The reason for this situation is a particular type of satiation arising in the nonstandard case. Makarov (1981) suggests a notion of equilibrium with transfer costs for the investigation of situations with satiated preferences. This notion may be treated as an extension of the usual Walrasian equilibrium. Still, the notion of equilibrium with transfer costs is a convenient instrument for examining the problems of existence of Walrasian equilibria. Since satiation is typical of the situations considered by us (but not strictly in the same sense as in the standard case), it is natural to use the idea of transfer costs in the nonstandard case. We turn to this task now.

Definition 3. *A triple* (x, p, δ), *where* $x \in X$, $p \in {}^*Q$, $\delta = (\delta_1, \ldots, \delta_n) \in {}^*\Re^N$, $\delta \geqq 0$, *is called a nonstandard equilibrium with transfer costs if it satisfies the following conditions:*

1. Attainability:

$$x_i \in \operatorname{st} {}^*B_i(p, \alpha_i(p) + \delta_i), \qquad i \in N; \tag{14}$$

2. Individual rationality:

$$\mathcal{P}_i(x) \cap \operatorname{st} {}^*B_i(p, \alpha_i(p) + \delta_i) = \emptyset, \qquad i \in N; \tag{15}$$

3. Balance:

$$\sum_N x_i = w. \tag{8'}$$

The transfer costs $\delta_i \geqq 0$, $\delta_i \in {}^*\Re$ consist essentially of a modification of the cost mechanism in the standard case. Those agents who consume their optimal bundle of goods, (i.e., agents who are satiated in their budget sets), can pass the remainig costs to other agents in the economy. These costs, which are equal to the difference between the agent's profits and the costs of his consumption plans, can be a number of the standard type or an infinitesimal. It is possible that the agent does not get satiated in the common sense that $\mathcal{P}_i(x) \neq \emptyset$, but for any increase of his utility, the incremental cost (possibly infinitesimal) is infinitely large with respect to the rest of the cost. This is the main innovation brought about by nonstandard methods with respect to the notions of prices and transfer costs. As a possible scheme for redistributing the remaining costs among agents, one can imagine that there exists a bank and that the agents give their surplus costs to this bank. The bank must return these costs to the owners, if they desire it, and credits the participants in the economy. This is the conventional view.

Theorem 3. *If the model $\mathcal{E}_1$ satisfies Assumptions 1–6, the sets X_i are bounded and $0 \in \operatorname{int} Q$, then there exist nonstandard equilibria with transfer costs.*

Proof. Let (1) the economy $\mathcal{E}_1$ be augmented by an agent with index 0; (2) $X_0 = X_0^\epsilon = \{x \in \Re^\ell \mid \|x\| \leqq \epsilon\}$, $\epsilon > 0$ be his consumption set; (3) $\alpha_0^\epsilon(p) = -(\epsilon/2)\|p\|$, $p \in Q$ be his profit function; (4) $\mathcal{P}_0(x) = \emptyset$, $x \in X^\epsilon$ be the preference of this agent, where $X^\epsilon = X_0^\epsilon \times \prod_N X_i$; and (5) $N_0 = N \cup \{0\}$.

We now define new profit functions for the agents in $\mathcal{E}_1$. Let

$$\alpha_i^\epsilon(p) = \alpha_i(p) + \frac{\epsilon}{2n}\|p\|, \qquad i \in N.$$

We can assume without loss of generality that Q is the ball with unit radius centered on 0. As a result, we obtain the following economy:

$$\mathcal{E}(\epsilon) = \langle N_0, \{X_i, \mathcal{P}_i, \alpha_i^\epsilon\}_{i \in N_0}, Q, w \rangle.$$

It follows from the construction that the functions α_i^ϵ are continuous,

$$\sum_{N_0} \alpha_i^\epsilon(p) = \sum_N \alpha_i(p) + n\frac{\epsilon}{2n}\|p\| - \frac{\epsilon}{2}\|p\| = \sum_N \alpha_i(p) = \langle p, w \rangle, \tag{16}$$

and if $p \neq 0$, then

$$\begin{gathered} \alpha_i^\epsilon(p) - \inf_{x \in X_i} \langle p, x \rangle = \alpha_i(p) - \inf_{x \in X_i} \langle p, x \rangle + \frac{\epsilon}{2n}\|p\| \geqq \frac{\epsilon}{2n}\|p\| > 0, \quad i \in N, \\ \alpha_0^\epsilon(p) - \inf_{x \in X_0} \langle p, x \rangle = -\frac{\epsilon}{2}\|p\| - (-\|p\|\epsilon) = \frac{\epsilon}{2}\|p\| > 0. \end{gathered} \tag{17}$$

From (16), (17) and the assumptions of Theorem 3, it follows that model $\mathcal{E}(\epsilon)$ constructed here satisfies all the conditions of Makarov's (1981) theorem on the existence of equilibria with transfer costs in the standard case. Note that Makarov's theorem can be proved by the method noted in the proof of Theorem 2. This technique reduces the original problem to the problem of existence of Nash equilibria in a game with $(n+1)+1$ players. Here, the continuity of the budget mappings at zero is ensured by replacing the profit functions of agents by the functions $\alpha_i^\epsilon(\cdot) + d(\cdot)$, where $d(\cdot)$ is a nonnegative, continuous function, which equals zero on the boundary of Q and $d(0) > 0$. Finally, we have the following statement: for all $\epsilon > 0$, $\epsilon \in \Re$, there exists $\delta_i(\epsilon) \in \Re$, $\delta_i(\epsilon) \geqq 0$, $i \in N$, a pair $(x_o^\epsilon, x^\epsilon) \in X_0^\epsilon \times X$ and a price vector p such that

$$\|x_o^\epsilon\| \leqq \epsilon, \quad x_i^\epsilon \in B_i(p^\epsilon, \alpha_i^\epsilon + \delta_i), \quad i \in N; \tag{18}$$

$$\mathcal{P}_i(x^\epsilon) \cap B_i(p^\epsilon, \alpha_i^\epsilon + \delta_i) = \emptyset, \quad i \in N; \tag{19}$$

$$\sum_N x_i^\epsilon + x_0^\epsilon = w, \tag{20}$$

where

$$B_i(p^\epsilon, \alpha_i^\epsilon + \delta_i) = \{x \in X_i \mid \langle p, x \rangle \leqq \alpha_i^\epsilon(p) + \delta_i(\epsilon)\}.$$

In this case, we can apply the transfer principle and add asterisks to all standard constants, which is a substitution of standard constants by their $*$-images. Now consider $\epsilon \approx 0$, and examine the pair (x^ϵ, p^ϵ) that satisfies (18)–(20). Since $x^\epsilon \in {}^*X$ and X is compact, x^ϵ is situated near a standard point. Let $\overline{x} = \operatorname{st} x^\epsilon$. Now we shall show that $(\overline{x}, p^\epsilon)$ is a nonstandard equilibrium of the model $\mathcal{E}_1$ with transfer costs $\delta_i' = \delta_i(\epsilon) + (\epsilon/2n)||p|| \geqq 0$, $i \in N$. After passing st into (18) we have

$$\overline{x}_i \in \operatorname{st} {}^*B_i(p^\epsilon, \alpha_i(\cdot) + \delta_i'), \quad i \in N.$$

A standardization of (20), with $x_0^\epsilon \approx 0$, gives

$$\sum_N \overline{x}_i = w.$$

Finally, using Proposition 1 in the case ${}^*\mathcal{P}_i = \tilde{\mathcal{L}}$, $\mathcal{Y} = \prod_N X_j$, $\mathcal{X} = X_i$, $A = {}^*B_i(p^\epsilon, \alpha_i + \delta_i')$ and relation (19), we have

$$[\operatorname{si} {}^*\mathcal{P}_i](\overline{x}) \cap \operatorname{st} {}^*B_i(p^\epsilon, \alpha_i + \delta_i') = \emptyset, \quad i \in N.$$

Finally, recalling that $\mathcal{P}_i \subset \operatorname{si} {}^*\mathcal{P}_i$ from Assumption 2, we get the relation (15) after replacing $\operatorname{si} {}^*\mathcal{P}_i$ by $\mathcal{P}_i$ in the last formula. ∎

3. Conclusion

In conclusion, we summarize the results that characterize the structure of the budget sets and the nonstandard properties of the budget mapping. More detailed results and proofs can be found in Marakulin (1988). Here we use the following notation. Let $\alpha: Q \to \Re$ be some function and let $X \subset \Re^\ell$. The budget set of the traditional type is

$$B(\alpha, p) = \{x \in X \mid \langle p, x \rangle \leqq \alpha(p)\}, \quad p \in Q$$

and its nonstandard analogue is

$$^*B(\alpha,p) = \{x \in {}^*X \mid \langle p,x\rangle \leqq {}^*\alpha(p)\}, \quad p \in {}^*Q,$$

where

$$\mathrm{st}\,^*B(\alpha,p) = \{x \in X \mid x \approx y,\ y \in {}^*B(\alpha,p)\}$$

is the budget set with nonstandard prices. Usual budget sets correspond to the budget mapping $B(\alpha)\colon Q \to 2^X$, where $[B(\alpha)](p) = B(\alpha,p)$.

Proposition 2. *The set* $\mathrm{st}\,^*B(\alpha,p)$ *is closed in* X *and, if* X *is convex, then* $\mathrm{st}\,^*B(\alpha,p)$ *is convex.*

Proposition 3. *Let* 2^X *have the topology of closed limits. Then the point-to-set mapping* $B(\alpha)$ *will be continuous at a point* $p \in Q$ *if and only if for all* $p' \in {}^*Q$, $p' \approx p$ *implies*

$$B(\alpha,p) = \mathrm{st}\,^*B(\alpha,p').$$

If the function $\alpha(\cdot)$ *is continuous at* $p \in Q$, *then*

$$\mathrm{st}\,^*B(\alpha,p') = \{x \in X \mid \langle \mathrm{st}\,p',x\rangle \leqq \mathrm{st}\,\alpha(p') = \alpha(p)\}, \quad p' \approx p.$$

Thus, the approach suggested in this paper for constructing the budget sets of the agents differs from the traditional way only at the points of discontinuity of the budget mapping $B(\alpha)$, i.e., at points where Slater's condition does not hold.

Proposition 4. *Every* $p \in {}^*\Re^\ell$ *unambiguously determines the collection* $\pi(p) = \{e_1,\ldots,e_k\}$ *of orthonormal standard vectors such that*

$$p = \lambda_1 e_1 + \cdots + \lambda_K e_K, \quad \lambda_j \in {}^*\Re, \quad j = 1,\ldots,K$$

and the coefficients $\lambda_j > 0$ *satisfy the conditions*

$$\frac{\lambda_{j+1}}{\lambda_j} \approx 0, \qquad j = 1,\ldots,K-1.$$

We examine the case where the profit function of the agent takes the form $\alpha(p) = \langle p, w \rangle$, where $w \in \Re^\ell$ is the vector of initial endowments. Here we write $B(w,p)$ instead of $B(\alpha, p)$. Let

$$B(m,w,p) = \{x \in X \mid x \cdot e_j = w \cdot e_j,\ j = 1,\dots,m-1,\ x \cdot e_m \leqq w \cdot e_m\}$$

for $m \leqq K$, and let

$$B(m,w,p) = \{x \in X \mid x \cdot e_j = w \cdot e_j,\ j = 1,\dots,m\}$$

for $m = K+1$. Here a natural K is defined by the vector p from Proposition 4.

Proposition 5. *If X is a convex polyhedron and $w \in \Re^\ell$ is a standard vector, then*

$$\mathrm{st}\,{}^*B(w,p) = B(m,w,p)$$

for some $m \leqq K+1$ and, if $m \leqq K$, there exists $x \in B(m,w,p)$ such that

$$\langle x, e_m \rangle < \langle e_m, w \rangle.$$

This result actually demonstrates a lexicographic organization of the set $\mathrm{st}\,{}^*B(w,p)$. Note the possibility of characterizing the sets taking the form $\mathrm{st}\,{}^*B(\delta,p)$, where $\delta \in {}^*\Re$ and

$${}^*B(\delta,p) = \{x \in {}^*X \mid \langle p, x \rangle \leqq \delta\}.$$

We do it with the help of the following simple method. Let us replace the set X by $X_1 = X \times \{1\}$ and the vector p by $p_1 = (p, -\delta)$ (where the dimension is $\ell + 1$). Now the condition $\langle p_1, x_1 \rangle \leqq 0$, $x_1 \in {}^*X_1$ will be equivalent to $\langle p, x \rangle \leqq \delta$, $x \in {}^*X$ and the set

$$\mathrm{st}\,{}^*B(0,p_1) = \{x \in X_1 \mid \exists y \in {}^*X_1 \colon p_1 y \leqq 0, x \approx y\}$$

is characterized by Proposition 5. Thus it is sufficient for us to take a projection of this set on the first ℓ components.

We present one more result.

Proposition 6. *Let X be a convex polyhedron and $\delta \in {}^*\Re$. Then, for every $y \in \mathrm{st}\,{}^*B(\delta,p)$,*

$$\mathrm{st}\,{}^*B(y,p) \subset \mathrm{st}\,{}^*B(\delta,p).$$

Then two consequences follow. (1) If X is a convex polyhedron and $\delta \in {}^*\Re$, then

$$\mathrm{st}\,{}^*B(\delta, p) = \bigcup_{y \in \mathrm{st}\,{}^*B(\delta,p)} \mathrm{st}\,{}^*B(y, p).$$

(2) If X is a convex polyhedron and $x \in {}^*X$ is placed near a standard point, then

$$\mathrm{st}\,{}^*B(\mathrm{st}\,x, p) \subset \mathrm{st}\,{}^*B(x, p)$$

and the reverse inclusion is false.

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