Algebraic approach to non-classical logics

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Abstract
Study of non-classical logics in Novosibirsk started in 60-th due to initiative and by supervision of A.I. Maltsev. His interest to this area was stimulated by the existence of an adequate algebraic semantics for the most known non-classical logics.
In the present paper inter-connections of syntactic properties of non-classical logics and categorial properties of appropriate classes of algebras are investigated. We consider such fundamental properties as the interpolation property, the Beth definability property, joint consistency property and their variants.
We consider propositional modal logics with many unary modalities, positive and superintuitionistic logics, and J-logics (i.e. logics extending Johansson’s minimal logic). Decidability results will be presented. Many of methods can be applied and many of results can be essentially generalized to other logical systems and classes of algebras.
Various versions of interpolation and Beth’s properties
Interpolation theorem proved by W. Craig in 1957 for the classical first order logic was a source of a lot of research results devoted to interpolation problem in classical and non-classical logical theories. Now interpolation is considered as a standard property of logics and calculi like consistency, completeness and so on. Interpolation is closely connected with Beth definability properties. These properties have as their source the theorem on implicit definability proved by E. Beth in 1953 for the classical first order logic: Any predicate implicitly definable in a first order theory is explicitly definable.
The original definition of interpolation admits various analogs which are equivalent in the classical logic but non-equivalent in other logics. The same is true for the Beth property. Here we consider several versions of interpolation and of Beth’s property.

Let $L$ be a logic, $\vdash_L$ deducibility relation in $L$. Suppose that $p$, $q$, $r$ are disjoint lists of non-logical symbols. We consider the languages which contain neither equality nor functional symbols but do contain at least one constant $\top$ (“truth”) or $\bot$ (“false”).
CIP. If $\vdash_L A(p, q) \rightarrow B(p, r)$, then there exists a formula $C(p)$ such that $\vdash_L A(p, q) \rightarrow C(p)$ and $\vdash_L C(p) \rightarrow B(p, r)$.

IPD. If $A(p, q) \vdash_L B(p, r)$, then there exists a formula $C(p)$ such that $A(p, q) \vdash_L C(p)$ and $C(p) \vdash_L B(p, r)$.

IPR. If $A(p, q), B(p, r) \vdash_L C(p)$, then there exists a formula $A'(p)$ such that $A(p, q) \vdash_L A'(p)$ and $A'(p), B(p, r) \vdash_L C(p)$.

WIP. If $A(p, q), B(p, r) \vdash_L \bot$, then there exists a formula $A'(p)$ such that $A(p, q) \vdash_L A'(p)$ and $A'(p), B(p, r) \vdash_L \bot$.

In our logics we have

$$CIP \Rightarrow IPD \Rightarrow IPR \Rightarrow WIP.$$
Beth’s definability properties have as their source the theorem on implicit definability proved by E.Beth in 1953 [1] for the classical first order logic: Any predicate implicitly definable in a first order theory is explicitly definable. We formulate some analogs of Beth’s property for propositional logics. Let $x, q, q'$ be disjoint lists of variables not containing $y$ and $z$, $A(x, q, y)$ a formula. We define two versions PB1 and PB2 of projective Beth’s property:

PB1. If $\vdash_L A(x, q, y) \& A(x, q', z) \rightarrow (y \leftrightarrow z)$, then $\vdash_L A(x, q, y) \rightarrow (y \leftrightarrow B(x))$ for some formula $B(x)$.

PB2. If $A(x, q, y), A(x, q', z) \vdash_L (y \leftrightarrow z)$, then $A(x, q, y) \vdash_L (y \leftrightarrow B(x))$ for some formula $B(x)$.

We get weaker versions B1 and B2 of Beth’s property by deleting $q$ in PB1 and PB2 respectively.
In NE(K):
PB1 ⇐⇒ B1 ⇐⇒ CIP ⇒ IPD ⇒ IPR ⇒ WIP,
PB1 ⇒ PB2 ⇒ IPR+B2,
the pairs B2 and IPD, PB2 and IPD, B2 and IPR are incomparable.

For logics over J and over Int⁺:
IPD ⇐⇒ CIP ⇒ PB1 ⇐⇒ PB2 ⇒ IPR;
PB1 is weaker than CIP, and all s.i.l. have WIP.
Intuitionistic and related calculi
The language of positive calculi contains $\&$, $\lor$, $\to$ and $\top$ as primitive, for the intuitionistic and minimal calculi the constant $\bot$ ("absurdity") is added, $\neg A \iff A \to \bot$.

A superintuitionistic logic is a set of formulas containing the set $\text{Int}$ of all intuitionistically valid formulas and closed under substitution and modus ponens. A consistent superintuitionistic logic is called an intermediate logic. A positive logic is a set of positive formulas containing the positive fragment $\text{Int}^+$ of $\text{Int}$ and closed under the same rules. A positive logic is determined by some set of axiom schemes added to $\text{Int}^+$. Johansson’s minimal logic $J$ is determined by the same axiom schemes and rules as $\text{Int}^+$ but its language has an extra constant $\bot$. Any logic containing $J$ is called a J-logic, and any J-logic containing $\bot$ is called a negative logic.
One can replace a finite set of axiom schemes with their conjunction. We denote by $L + A$ the extension of a logic $L$ by an extra axiom scheme $A$. For a given logic $L$, the set of all logics containing $L$ is denoted by $E(L)$.

Standard denotations:

$\text{Int}^+ = J^+$, $\text{Int} = J + (\bot \rightarrow \rho)$, $\text{Neg} = J + \bot$.

If $L$ is a calculus over $J$ or a positive calculus, $\Gamma \vdash_L A$ denotes that $A$ is derivable from $\Gamma$ and axioms of $L$ by modus ponens. Due to the deduction theorem, over $J$ and over $\text{Int}^+$ we have:

$\text{IPD} \iff \text{CIP}$, $\text{PB1} \iff \text{PB2}$, and $\text{B1} \iff \text{B2}$.

In addition, $\text{CIP} \implies \text{PB1} \implies \text{B1}$.
Proposition

Every logic in $E(J)$ or in $E(\text{Int}^+)$ has the Beth property B1. (Kreisel 1960 for s.i.l.) From Glivenko’s theorem:

Proposition

Every superintuitionistic logic has the weak interpolation property.

WIP is undefined for positive logics.
Normal modal calculi

The language: all the connectives of Int and the modal operators □ and ◊, where ◊A ⇔ ¬□¬A. The minimal normal modal calculus K is defined by adding one axiom schema □(A → B) → (□A → □B) and the necessitation rule A/□A to the postulates of Cl.

Γ ⊨ L A if there is a derivation of A from Γ and axioms of L by modus ponens and necessitation rules.
A normal modal logic is any set of modal formulas containing all the axioms of K and closed under modus ponens, necessitation and substitution rules.

$NE(L)$: the set of all n.m.l. containing $L$.

$K4 = K + (\square A \rightarrow \square \square A)$,

$S4 = K4 + (\square A \rightarrow A)$,

$Grz = S4 + (\square (\square (A \rightarrow \square A) \rightarrow A) \rightarrow A)$. 
Proposition

[Maks 92,98,2002,03,05] In the family of normal modal logics

1. CIP is equivalent to each of the properties PB1 and B1;
2. CIP implies the conjunction of B2 and IPD but the converse does not hold;
3. B2 and IPD are independent;
4. the conjunction of B2 and IPD implies PB2;
5. PB2 implies B2 but the converse is not true,
6. PB2 and IPD are independent,
7. PB2 implies IPR,
8. IPR implies WIP but the converse is not true.
Algebraic equivalents
It is well known that there is a duality between normal modal logics and varieties of modal algebras. The least normal modal logic $K$ is defined by the variety of all modal algebras. Also there is a duality between superintuitionistic logics and varieties of Heyting algebras, between J-logics and varieties of J-algebras, between positive logics and varieties of relatively pseudocomplemented lattices. For a given logic $L$, its associated variety $V(L)$ is determined by identities $A = \top$ for $A$ provable in $L$.

We find categorical properties of varieties equivalent to interpolation and so on.
Algebraization

Theorem

For any logic $L$ in $E(J)$, $E(\text{Int}^+)$ or in $\text{NE}(K)$:

1. $L$ has CIP iff $V(L)$ has the super-amalgamation property $\text{SAP}$ [Maks 1977, 91],
2. $L$ has IPD iff $V(L)$ has the amalgamation property $\text{AP}$ [Maks 1979, 92, Czelakowski 1982],
3. $L$ has IPR iff $V(L)$ has the restricted amalgamation property $\text{RAP}$ (Maks 2000, 01, 03),
4. for $L$ in $E(\text{Int})$ or in $\text{NE}(K)$, $L$ has WIP iff $V(L)$ has the property $\text{WAP}$ [Maks 2005],
5. $L$ has B2 iff $V(L)$ has epimorphisms surjectivity $\text{ES}^*$ [Maks 1992],
6. $L$ has PB2 iff $V(L)$ has strong epimorphisms surjectivity $\text{SES}$ [Maks 1998].
Recall that in NE(K): \( PB1 \iff B1 \iff CIP; \)
in \( E(J) \) or in \( E(\text{Int}^+) \): \( CIP \iff IPD, \ PB1 \iff PB2 \) and \( B1 \iff B2. \)

Other non-classical logics and algebraic logic:
A.Wronski, H.Ono, D.Pigozzi, J.Czelakowski, H.Andreka,
I.Nemeti, I.Sain, E.Hoogland, J.Madarasz, F.Montagna, etc.
AP: If \( A \) is a common subalgebra of algebras \( B \) and \( C \) in \( V \) then there exist \( D \) in \( V \) and monomorphisms \( \delta : B \to D \), \( \varepsilon : C \to D \) such that \( \delta(x) = \varepsilon(x) \) for all \( x \in A \).

SAP: AP with extra conditions:

\[
\delta(x) \leq \varepsilon(y) \iff (\exists z \in A)(x \leq z \text{ and } z \leq y),
\]
\[
\delta(x) \geq \varepsilon(y) \iff (\exists z \in A)(x \geq z \text{ and } z \geq y).
\]

StrAP: AP with an extra condition:

\[
\delta(B) \cap \varepsilon(C) = \delta(A).
\]
RAP: For each $A, B, C \in V$ such that $A$ is a common subalgebra of $B$ and $C$ there exist an algebra $D$ in $V$ and homomorphisms $\delta : B \to D$, $\varepsilon : C \to D$ such that $\delta(x) = \varepsilon(x)$ for all $x \in A$ and the restriction of $\delta$ onto $A$ is a monomorphism.

WAP: For each $A, B, C \in V$ such that $A$ is a common subalgebra of $B$ and $C$ there exist an algebra $D$ in $V$ and homomorphisms $\delta : B \to D$, $\varepsilon : C \to D$ such that $\delta(x) = \varepsilon(x)$ for all $x \in A$, where $D$ is non-degenerate whenever $A$ is non-degenerate.

Recall that a Heyting algebra or a modal algebra is *non-degenerate* if it contains at least two elements.
SES. For any $A,B$ in $V$, for any monomorphism $\alpha : A \to B$ and for any $b \in B - \alpha(A)$ there exist $C \in V$ and monomorphisms $\beta : B \to C$ and $\gamma : B \to C$ such that $\beta\alpha = \gamma\alpha$ and $\beta(b) \neq \gamma(b)$.

ES*: For any $A,B$ in $V$, for any monomorphism $\alpha : A \to B$ and for any $b \in B - \alpha(A)$, such that $\{b\} \cup \alpha(A)$ generates $B$, there exist $C \in V$ and monomorphisms $\beta : B \to C$ and $\gamma : B \to C$ such that $\beta\alpha = \gamma\alpha$ and $\beta(b) \neq \gamma(b)$.

ES* is equivalent to

ESM. For any $B$ in $V$, any its maximal subalgebra $A$ and for any $b \in B - A$, there exist $C \in V$ and monomorphisms $\beta : B \to C$ and $\gamma : B \to C$ such that $\beta(b) \neq \gamma(b)$ and $\beta(x) = \gamma(x)$ for all $x \in A$. 
In varieties of modal algebras we have:

\[ \text{SAP} \Rightarrow \text{StrAP} \Rightarrow \text{AP} \Rightarrow \text{RAP} \Rightarrow \text{WAP} \]

and the reverse arrows do not hold,

\[ \text{StrAP} \Leftrightarrow \text{AP} \& \text{ES}^* \Rightarrow \text{SES} \Rightarrow \text{RAP} \& \text{ES}^*, \]

\[ \text{ES}^* \nRightarrow \text{WAP} \text{ and } \text{AP} \nRightarrow \text{ES}^*. \]

In varieties of Heyting algebras, J-algebras and of relatively pseudocomplemented lattices:

\[ \text{SAP} \Leftrightarrow \text{StrAP} \Leftrightarrow \text{AP} \Rightarrow \text{SES} \text{ and } \text{SES} \nRightarrow \text{AP}. \]

All the considered varieties have CD and CEP.
Finitization

An algebra is *subdirectly irreducible* if it can not be represented as a subdirect product of its proper quotient algebras; it is *finitely indecomposable* if it can not be represented as a finite subdirect product of its proper quotient algebras. An algebra is *simple* if it is non-degenerate and has no non-trivial congruences. Finitely indecomposable and subdirectly irreducible algebras have the following useful characterization.

**Lemma**

Let $A$ be a non-degenerate $J$-algebra or a relatively pseudo-complemented lattice. Then

1. $A$ is finitely indecomposable iff it satisfies the condition:
   $$ x \lor y = T \Rightarrow (x = T \text{ or } y = T); $$

2. $A$ is subdirectly irreducible iff it has an opremum, i.e. an element which is the greatest in the set $\{x \in A \mid x < T\}$. 
An element $b$ of a modal algebra $A$ is said to be critical if $b \neq \top$ and for each $x \neq \top$ there is an $n$ such that $[n]x \leq b$. Here $[n]x = x \& \Box x \& \ldots \& \Diamond^n x$.

**Lemma**

Let $A$ be a non-degenerate modal algebra. Then

1. $A$ is finitely indecomposable iff for any $a, b \neq \top$ there exist $n$ such that $[n]a \lor [n]b \neq \top$ [7];

2. $A$ is subdirectly irreducible iff its set of critical elements is non-empty [Rautenberg].
Finitization Theorem. For any variety \( V \) of modal algebras, relatively pseudo-complemented lattices or \( J \)-algebras:

1. \( V \) has SAP iff for any algebras \( A, B, C \in FI(FG(V)) \) such that \( A \) is a common subalgebra of \( B \) and \( C \) and for any \( b \in B, c \in C \) satisfying the condition 
   \[ \neg \exists a \in A \,(b \leq z \land z \leq c) \] 
   there exist an algebra \( D \) in \( SI(FG(V)) \) and homomorphisms \( \delta : B \to D, \varepsilon : C \to D \) 
   such that \( \delta(x) = \varepsilon(x) \) for all \( x \in A \) and \( b \not\leq c \) [Maks 77,92,2003];
2. V has AP iff for any algebras $A, B, C \in FI(FG(V))$ such that $A$ is a common subalgebra of $B$ and $C$ and for any $c \in C$ different from $\top$ there exist an algebra $D$ in $SI(FG(V))$ and homomorphisms $\delta : B \to D$, $\varepsilon : C \to D$ such that $\delta(x) = \varepsilon(x)$ for all $x \in A$ and $c \neq \top$ [Maks 77,92,2003];

3. V has ES* iff $FI(FG(V))$ has ES* [Maks 92];

4. V has StrAP iff V has both AP and ES* [Maks 92];

5. V has RAP iff $FI(FG(V))$ has RAP iff $SI(FG(V))$ has RAP [Maks 2003];

6. V has SES iff $FI(FG(V))$ has SES and $SI(FG(V))$ has RAP [Maks 2003];

7. for varieties of HA or MA, V has WAP iff $FI(FG(V))$ has WAP iff the class of f.g. simple algebras in V has AP [M 2005].
Tabular logics and varieties
A logic (and a variety) is *tabular* if it is generated by finitely many finite algebras.

**Theorem**

1. *The following properties are decidable on the class of tabular modal, superintuitionistic or positive logics: CIP, IPD, IPR, WIP, PB1, PB2, B1, B2.*

2. *The following properties are decidable on the class of tabular varieties of modal or Heyting algebras, or implicative lattices: AP, SAP, StrAP, RAP, WAP, SES, ES*. 
Decidable problems

Algebraization and Finitization theorems allow to solve, for instance, interpolation problem for some important classes of logical calculi and, at the same time, amalgamation problem for varieties. In particular, the following problems are decidable:

- Craig’s interpolation property and deductive interpolation property for superintuitionistic and positive calculi and for modal calculi over the modal S4 logic,
- amalgamation and super-amalgamation properties for subvarieties of Heyting algebras, implicative lattices and closure algebras,
- projective Beth property and restricted interpolation property over the intuitionistic logic and over the Grzegorczyk logic,
- strong epimorphisms surjectivity for subvarieties of Heyting algebras, implicative lattices and of Grzegorczyk algebras.
- weak interpolation property over the modal K4 logic,
- weak amalgamation property for varieties of transitive modal algebras.
Conclusion
Many of methods can be applied and many of results can be essentially generalized to other logical systems and classes of algebras.


