Atomic Rule Refinement in the Tableau Synthesis Framework

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Outline

1. Overview and motivation
2. Tableau synthesis framework
3. Constructive completeness
4. General rule refinement
5. Atomic rule refinement
6. Hypertableau
7. Conclusion
Prover Development Problem

- Different applications require different (logical) formalisms
- Logics need reasoning tools
- Implementation of provers is expensive
- Altering existing provers is hard
- Translational approach requires additional knowledge and skills for the user

Solution

Generation of a prover code from a specification of a logic.
Tableau-Based Reasoning

- Has a long tradition and is a well established method in automated reasoning
- Approach can be successfully used for a large number of logics:
  - Boolean logic, first-order logic, higher-order logics,
  - modal, description, hybrid, superintuitionistic logics, . . .,
  - temporal, dynamic, fix-point logics, . . .
- Multitude of different tableau approaches:
  - ground semantic tableau
  - free-variable tableau,
  - hypertableau, . . .
- Many implemented systems
Existing work for non-classical logics suggests that it should be possible to develop tableau calculi systematically for large classes of logics

- many variations
- many similarities
- important underlying principles tend to be the same

Questions:

- Can tableau calculi be developed automatically from the definition of logics?
- Can soundness and completeness be guaranteed?
- Can termination be guaranteed?

No, in each case.

Is it possible to develop tableau calculi automatically under certain restrictions?
Existing work for non-classical logics suggests that it should be possible to develop tableau calculi systematically for large classes of logics:
- many variations
- many similarities
- important underlying principles tend to be the same

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- many variations
- many similarities
- important underlying principles tend to be the same

Questions:

- Can tableau calculi be developed automatically from the definition of logics?
- Can soundness and completeness be guaranteed?
- Can termination be guaranteed?

No, in each case.

Is it possible to develop tableau calculi automatically under certain restrictions?
A method of synthesis of a sound and complete tableau calculus from a first-order semantic specification of a logic.

Calculus refinement methods.

A generic blocking mechanism which ensures termination of tableau algorithms for logics with the finite model property.
Refinement Methods

- Simplification of the tableau language
Simplification of the tableau language

Our focus: Reducing branching factor of the tableau rules
A multi-sorted first-order language (extending the language of given logic)

Connectives of the logic = functional symbols

Formulae of the logic = terms of the tableau language

Truth predicates $\nu_s$ for each sort $s$ of the logic

$$M \models \nu_f(\phi, a) \equiv M, a \models \phi$$

Equality predicates $\approx$ for each sort.
Basic Notions

- Tableau rule:
  \[
  \frac{X}{X_1 \mid \cdots \mid X_m}
  \]

- Tableau as a search space
- Tableau branch
- Soundness
- Completeness
- Termination
Tableau Synthesis

- Express semantics $S$ of a logic
- Transform semantics to a ‘well-defined’ form
- Eliminate quantifiers by Skolemisation
- Generate tableau calculus $T_S$ from the transformed semantics

**Theorem (Soundness)**

$T_S$ is sound for the logic specified by $S$, i.e.

if a formula $\phi$ is satisfiable in an $S$-model then

every tableau derivation for $\phi$ based on $T_S$ contains an open branch.
Generating Tableau Calculus for S4

\[
\begin{align*}
\forall x(\nu(\neg p, x) & \iff \neg \nu(p, x)) \\
\forall x(\nu(p \lor q, x) & \iff \nu(p, x) \lor \nu(q, x)) \\
\forall x(\nu(\Diamond p, x) & \iff \exists y(R(x, y) \land \nu(p, y))) \\
\forall x, y, z(R(x, y) & \land R(y, z) \rightarrow R(x, z)) \\
\forall x R(x, x)
\end{align*}
\]
Generating Tableau Calculus for S4

\[
\begin{array}{c}
\frac{\nu(\neg p, x)}{\nu(p, x)} & \frac{-\nu(\neg p, x)}{\nu(p, x)} \\
\end{array}
\]

\[
\forall x(\nu(p \lor q, x) \leftrightarrow \nu(p, x) \lor \nu(q, x))
\]

\[
\forall x(\nu(\Diamond p, x) \leftrightarrow \exists y(R(x, y) \land \nu(p, y)))
\]

\[
\forall x, y, z(R(x, y) \land R(y, z) \rightarrow R(x, z))
\]

\[
\forall x R(x, x)
\]
Generating Tableau Calculus for S4

\[
\begin{align*}
\frac{\nu(\neg p, x)}{-\nu(p, x)} & \quad \frac{-\nu(\neg p, x)}{\nu(p, x)} \\
\frac{\nu(p \lor q, x)}{\nu(p, x) \mid \nu(q, x)} & \quad \frac{-\nu(p \lor q, x)}{-\nu(p, x), \ -\nu(q, x)} \\
\forall x (\nu(\Diamond p, x) \iff \exists y (R(x, y) \land \nu(p, y))) & \\
\forall x, y, z (R(x, y) \land R(y, z) \rightarrow R(x, z)) & \\
\forall x R(x, x) & 
\end{align*}
\]
Generating Tableau Calculus for S4

\[ \nu(\neg p, x) \quad \neg \nu(p, x) \]
\[ \neg \nu(p, x) \quad \nu(p, x) \]
\[ \nu(p \lor q, x) \quad \neg \nu(p \lor q, x) \]
\[ \nu(p, x) \quad \nu(q, x) \quad \neg \nu(p, x), \quad \neg \nu(q, x) \]

\[ \forall x (\nu(\box p, x) \rightarrow (R(x, f(p, x)) \land \nu(p, f(p, x)))) \]
\[ \forall x, y (\neg \nu(\box p, x) \rightarrow (\neg R(x, y) \lor \neg \nu(p, y))) \]
\[ \forall x, y, z (R(x, y) \land R(y, z) \rightarrow R(x, z)) \]
\[ \forall x R(x, x) \]
Generating Tableau Calculus for S4

\[
\begin{align*}
\frac{\nu(\neg p, x)}{\neg \nu(p, x)} & \quad \frac{\neg \nu(\neg p, x)}{\nu(p, x)} \\
\frac{\nu(p \lor q, x)}{\nu(p, x) \mid \nu(q, x)} & \quad \frac{\neg \nu(p \lor q, x)}{\neg \nu(p, x), \ \neg \nu(q, x)} \\
\frac{\nu(\Diamond p, x)}{R(x, f(p, x)), \ \nu(p, f(p, x))} & \quad \frac{\neg \nu(\Diamond p, x)}{\neg R(x, y) \mid \neg \nu(p, y)} \\
\forall x, y, z (R(x, y) \land R(y, z) \rightarrow R(x, z)) & \quad \forall x R(x, x)
\end{align*}
\]
Generating Tableau Calculus for S4

\[ \begin{align*}
\frac{\nu(\neg p, x)}{\neg \nu(p, x)} & \quad \frac{\neg \nu(p, x)}{\nu(p, x)} \\
\nu(p \lor q, x) & \quad \frac{\neg \nu(p \lor q, x)}{\nu(p, x), \, \neg \nu(q, x)} \\
\frac{\nu(\diamond p, x)}{\neg \nu(\diamond p, x)} & \quad \frac{\neg \nu(p, f(p, x))}{R(x, f(p, x)), \, \nu(p, f(p, x))} \\
\frac{\neg R(x, y)}{\neg R(x, y) \, | \, \neg \nu(p, y)} & \quad \frac{\neg R(x, y) \, | \, \neg \nu(p, y)}{\neg R(x, y) \, | \, \neg \nu(p, y)} \\
\varforall x R(x, x)
\end{align*} \]
Generating Tableau Calculus for S4

\[
\frac{\nu(\neg p, x)}{\neg \nu(p, x)} \quad \frac{\neg \nu(p, x)}{\nu(p, x)}
\]

\[
\frac{\nu(p \lor q, x)}{\nu(p, x) \mid \nu(q, x)}
\]

\[
\frac{\nu(\Diamond p, x)}{R(x, f(p, x)), \; \nu(p, f(p, x))}
\]

\[
\frac{\neg \nu(\Diamond p, x)}{\neg R(x, y) \mid \neg \nu(p, y)}
\]

\[
\frac{\neg R(x, y) \mid \neg R(y, z) \mid R(x, z)}{R(x, x)}
\]
Generating Tableau Calculus for S4

\[
\frac{\nu(\neg p, x)}{\neg \nu(p, x)} \quad \frac{\neg \nu(\neg p, x)}{\nu(p, x)}
\]

\[
\frac{\nu(p \lor q, x)}{\nu(p, x) \mid \nu(q, x)} \quad \frac{\neg \nu(p \lor q, x)}{\neg \nu(p, x), \neg \nu(q, x)}
\]

\[
\frac{\nu(\Diamond p, x)}{R(x, f(p, x)), \nu(p, f(p, x))} \quad \frac{\neg \nu(\Diamond p, x)}{\neg R(x, y) \mid \neg \nu(p, y)}
\]

\[
\frac{\neg R(x, y) \mid \neg R(y, z) \mid R(x, z)}{R(x, x)}
\]

\[
\frac{\nu(p, x), \neg \nu(p, x)}{\nu(p, x), \neg \nu(p, x)} \quad \frac{R(x, y), \neg R(x, y)}{R(x, y), \neg R(x, y)}
\]

+equality rules
Constructive Completeness

Given open branch $\mathcal{B}$ in a derivation in $T_S$, build a model

$$\mathcal{I}(\mathcal{B}) \overset{\text{def}}{=} \langle \Delta^{\mathcal{I}(\mathcal{B})}, P^{\mathcal{I}(\mathcal{B})}, \ldots, \nu_S^{\mathcal{I}(\mathcal{B})}, \ldots \rangle$$

- $\|t\| \overset{\text{def}}{=} \{ t' \mid t \approx t' \text{ is in } \mathcal{B} \}$
- $\Delta^{\mathcal{I}(\mathcal{B})} \overset{\text{def}}{=} \{ \|t\| \mid t \text{ is in } \mathcal{B} \}$
- $(p, \|t\|) \in \nu_S^{\mathcal{I}(\mathcal{B})} \iff \nu(S, p, t) \in \mathcal{B}$
- $\|t\| \in P^{\mathcal{I}(\mathcal{B})} \iff P(t) \in \mathcal{B}$

$T_S$ is constructively complete iff (for every such $\mathcal{B}$)

$\mathcal{I}(\mathcal{B})$ is extendable to an $S$-model which reflects $\mathcal{B}$.\(^1\)

Constructive Completeness $\implies$ Completeness.

---

Theorem (Constructive Completeness)

(If $S$ is a well-defined semantic specification then)

$T_S$ is constructively complete.

---

\(^1\)i.e. validates all the literals in $\mathcal{B}$ under the valuation $t \mapsto \|t\|$.
Constructive Completeness

Given open branch $B$ in a derivation in $T_S$, build a model

$$\mathcal{I}(B) \overset{\text{def}}{=} \langle \Delta^{\mathcal{I}(B)}, P^{\mathcal{I}(B)}, \ldots, \nu_s^{\mathcal{I}(B)}, \ldots \rangle$$

- $\|t\| \overset{\text{def}}{=} \{ t' \mid t \approx t' \text{ is in } B \}$
- $\Delta^{\mathcal{I}(B)} \overset{\text{def}}{=} \{ \|t\| \mid t \text{ is in } B \}$
- $(p, \|t\|) \in \nu_s^{\mathcal{I}(B)} \iff \nu_s(p, t) \in B$
- $\|t\| \in P^{\mathcal{I}(B)} \iff P(t) \in B$

$T_S$ is **constructively complete** iff (for every such $B$)

$\mathcal{I}(B)$ is extendable to an $S$-model which reflects $B$.\(^1\)

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Constructive Completeness

Given open branch $B$ in a derivation in $T_S$, build a model

$$\mathcal{I}(B) \overset{\text{def}}{=} \langle \Delta^{\mathcal{I}(B)}, P^{\mathcal{I}(B)}, \ldots, \nu^{\mathcal{I}(B)}_S, \ldots \rangle$$

- $\|t\| \overset{\text{def}}{=} \{t' \mid t \approx t' \text{ is in } B\}$
- $\Delta^{\mathcal{I}(B)} \overset{\text{def}}{=} \{\|t\| \mid t \text{ is in } B\}$
- $(p, \|t\|) \in \nu^{\mathcal{I}(B)}_S \iff \nu_S(p, t) \in B$
- $\|t\| \in P^{\mathcal{I}(B)} \overset{\text{def}}{=} P(t) \in B$

$T_S$ is constructively complete iff (for every such $B$)

$\mathcal{I}(B)$ is extendable to an $S$-model which reflects $B$.\(^1\)

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Constructive Completeness

Given open branch $\mathcal{B}$ in a derivation in $T_S$, build a model

$$\mathcal{I}(\mathcal{B}) \triangleq \langle \Delta^{\mathcal{I}(\mathcal{B})}, P^{\mathcal{I}(\mathcal{B})}, \ldots, \nu_s^{\mathcal{I}(\mathcal{B})}, \ldots \rangle$$

- $\|t\| \triangleq \{t' \mid t \approx t' \text{ is in } \mathcal{B}\}$
- $\Delta^{\mathcal{I}(\mathcal{B})} \triangleq \{\|t\| \mid t \text{ is in } \mathcal{B}\}$
- $(p, \|t\|) \in \nu_s^{\mathcal{I}(\mathcal{B})} \iff \nu_s(p, t) \in \mathcal{B}$
- $\|t\| \in P^{\mathcal{I}(\mathcal{B})} \iff P(t) \in \mathcal{B}$

$T_S$ is constructively complete iff (for every such $\mathcal{B}$)

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**Theorem (Constructive Completeness)**

(If $S$ is a well-defined semantic specification then)

$T_S$ is constructively complete.

---

\(^1\) i.e. validates all the literals in $\mathcal{B}$ under the valuation $t \mapsto \|t\|$.
Non-Optimized Rules in S4

\[ \begin{align*}
\nu(\neg p, x) & \quad \neg \nu(\neg p, x) \\
\neg \nu(p, x) & \quad \nu(p, x) \\
\nu(p \lor q, x) & \quad \neg \nu(p \lor q, x) \\
\nu(p, x) \lor \nu(q, x) & \quad \neg \nu(p, x), \neg \nu(q, x) \\
\nu(\Box p, x) & \quad \neg \nu(\Box p, x) \\
R(x, f(p, x)), \nu(p, f(p, x)) & \quad \neg R(x, y) \lor \neg \nu(p, y) \\
\neg R(x, y) \lor \neg R(y, z) & \quad R(x, z) \\
\neg R(x, y) \lor \neg R(y, z) \lor R(x, z)
\end{align*} \]

+equality rules
Let $X_1 = \{ \psi_1, \ldots, \psi_k \}$ and

$$
 r \overset{\text{def}}{=} \frac{X}{X_1 \mid \cdots \mid X_m} \quad r_j \overset{\text{def}}{=} \frac{X \cup \{ \sim \psi_j \}}{X_2 \mid \cdots \mid X_m}
$$

$T'_S \overset{\text{def}}{=} (T_S \setminus \{ r \}) \cup \{ r_1, \ldots, r_k \}$.

For every open branch $B$ in a $T'_S$-tableau, if $E_1, \ldots, E_l$ are reflected in $I(B)$ then

$$(\dagger) \quad X(E, \bar{t}) \subseteq B \text{ implies } I(B) \models X_i (E, \parallel t \parallel) \text{ for some } i = 1, \ldots, m.$$

**Theorem (Refinement)**

$T'_S$ is sound and constructively complete whenever $T_S$ is.
General Rule Refinement

Let $X_1 = \{\psi_1, \ldots, \psi_k\}$ and

$$r \overset{\text{def}}{=} \frac{X}{X_1 | \cdots | X_m} \quad \text{and} \quad r_j \overset{\text{def}}{=} \frac{X \cup \{\sim \psi_j\}^1}{X_2 | \cdots | X_m}$$

$T'_S \overset{\text{def}}{=} (T_S \setminus \{r\}) \cup \{r_1, \ldots, r_k\}$.

For every open branch $B$ in a $T'_S$-tableau, if $E_1, \ldots, E_l$ are reflected in $I(B)$ then

\[(\dagger) \quad X(E, \bar{t}) \subseteq B \text{ implies } I(B) \models X_i(E, \|t\|) \text{ for some } i = 1, \ldots, m.\]

**Theorem (Refinement)**

$T'_S$ is sound and constructively complete whenever $T_S$ is.

\[\sim \phi \overset{\text{def}}{=} \psi \text{ if } \phi = \neg \psi \text{ and } \sim \phi \overset{\text{def}}{=} \neg \phi \text{ if } \phi \neq \neg \psi\]
General Rule Refinement

Let $X_1 = \{\psi_1, \ldots, \psi_k\}$ and

\[
\begin{align*}
  r & \overset{\text{def}}{=} \frac{X}{X_1 \mid \cdots \mid X_m} \\
r_j & \overset{\text{def}}{=} \frac{X \cup \{\sim \psi_j\}}{X_2 \mid \cdots \mid X_m}
\end{align*}
\]

$T'_S \overset{\text{def}}{=} (T_S \setminus \{r\}) \cup \{r_1, \ldots, r_k\}$.

For every open branch $\mathcal{B}$ in a $T'_S$-tableau, if $E_1, \ldots, E_l$ are reflected in $\mathcal{I}(\mathcal{B})$ then

\[(†) \quad X(\bar{E}, \bar{t}) \subseteq \mathcal{B} \text{ implies } \mathcal{I}(\mathcal{B}) \models X_i(\bar{E}, \|t\|) \text{ for some } i = 1, \ldots, m.\]

Theorem (Refinement)

$T'_S$ is sound and constructively complete whenever $T_S$ is.

\[1 \sim \phi \overset{\text{def}}{=} \psi \text{ if } \phi = \neg \psi \text{ and } \sim \phi \overset{\text{def}}{=} \neg \phi \text{ if } \phi \neq \neg \psi\]
**Atomic Rule Refinement**

- **$\mathcal{L}$-atomic** formula = atomic formula in $FO(\mathcal{L})$ which does not contain complex $\mathcal{L}$-terms:

  \[
  \nu_l(p, t) \quad R(t_1, t_2) \quad \nu_r(r, t_1, t_2, t_3)
  \]

- For every open branch $B$ in a $T'_S$-tableau, if $E_1, \ldots, E_l$ are reflected in $I(B)$ then

  \[
  X_0(\overline{E}, \overline{t}) \subseteq B \text{ implies } \\
  X_1(\overline{E}, \overline{t}) = \{\neg \xi_1, \ldots, \neg \xi_k\} \text{ and all } \xi_1, \ldots, \xi_k \text{ are } \mathcal{L}\text{-atomic.}
  \]

**Theorem (Atomic Refinement)**

$T'_S$ is sound and constructively complete whenever $T_S$ is.
Box Rule

\[
\frac{\neg \nu(\Diamond p, x)}{\neg R(x, y) \mid \neg \nu(p, y)}
\]
Box Rule

\[ \neg \nu(\Diamond p, x) \]

\[ \neg R(x, y) \mid \neg \nu(p, y) \]
For more expressive modal-like logics the rule
\[
\neg \nu(\Diamond p, x), \ R(x, y) \\
\hline 
\neg \nu(p, y)
\]
Box Rule

\[
\begin{align*}
\neg \nu(\Diamond p, x), \ R(x, y) \\
\Rightarrow \neg \nu(p, y)
\end{align*}
\]

For more expressive modal-like logics the rule

\[
\begin{align*}
\neg \nu_t(\Diamond p, x) \\
\Rightarrow \neg \nu_t(r, x, y) \mid \neg \nu_t(p, y)
\end{align*}
\]

may not be refinable, e.g. tableaux for logics with the relation complement.
Frame Conditions

\[ \neg R(x, y) \mid \neg R(y, z) \mid R(x, z) \]
Frame Conditions

\[ \neg R(x, y) \mid \neg R(y, z) \mid R(x, z) \]
Frame Conditions

\[
\frac{R(x, y)}{-R(y, z) \mid R(x, z)}
\]
Frame Conditions

\[
\frac{R(x, y)}{-R(y, z) \mid R(x, z)}
\]
Frame Conditions

\[
R(x, y), \quad R(y, z) \quad \overline{\quad} \quad R(x, z)
\]
Frame Conditions

\[
\frac{R(x, y), \ R(y, z)}{R(x, z)}
\]

For more expressive logics the rule

\[
\neg \nu_r(r, x, y) \ | \ \neg \nu_r(r, y, z) \ | \ \nu_r(r, x, z)
\]

may not be refinable.
Optimized Rules for S4

\[
\begin{align*}
\frac{\nu(\neg p, x)}{\neg \nu(p, x)} & \quad \frac{\neg \nu(\neg p, x)}{\nu(p, x)} \\
\frac{\nu(\neg p, x)}{\neg \nu(p, x)} & \quad \frac{\neg \nu(\neg p, x)}{\nu(p, x)} \\
\frac{\nu(p \lor q, x)}{\nu(p, x) \lor \nu(q, x)} & \quad \frac{\neg \nu(p \lor q, x)}{\neg \nu(p, x), \neg \nu(q, x)} \\
\frac{\nu(\boxdot p, x)}{R(x, f(p, x)), \nu(p, f(p, x))} & \quad \frac{\neg \nu(\boxdot p, x), R(x, y)}{\neg \nu(p, y)} \\
R(x, y), R(y, z) & \quad \frac{R(x, z)}{R(x, y), \neg R(x, y)} \\
R(x, x) & \quad \nu(p, x), \neg \nu(p, x) \\
\bot & \quad \bot
\end{align*}
\]

+equality rules
Optimized Rules for S4

\[
\begin{align*}
\frac{\nu(\neg p, x)}{\neg \nu(p, x)} & \quad \frac{\neg \nu(p, x)}{\nu(p, x)} \\
\frac{\nu(p \lor q, x)}{\nu(p, x) \mid \nu(q, x)} & \quad \frac{\neg \nu(p \lor q, x)}{\neg \nu(p, x), \neg \nu(q, x)} \\
\frac{\nu(\Diamond p, x)}{R(x, f(p, x)), \nu(p, f(p, x))} & \quad \frac{\neg \nu(\Diamond p, x), R(x, y)}{\neg \nu(p, y)} \\
\frac{R(x, y), R(y, z)}{R(x, z)} & \\
\frac{R(x, x), \nu(p, x), \neg \nu(p, x)}{\bot}\end{align*}
\]

+equality rules
Hypertableau Rule

Assumptions:

- The tableau for the logic contains rules:

\[
\frac{\nu_s(\neg p, \bar{x})}{\neg \nu_s(p, \bar{x})} \quad \text{and} \quad \frac{\nu_s(p \lor q, \bar{x})}{\nu_s(p, \bar{x}) \mid \nu_s(q, \bar{x})}
\]

- AC for disjunction (for simplicity)
Assumptions:

- The tableau for the logic contains rules:

\[
\frac{\nu_s(\neg p, \overline{x})}{\neg \nu_s(p, \overline{x})}
\quad \text{and} \quad
\frac{\nu_s(p \lor q, \overline{x})}{\nu_s(p, \overline{x}) \mid \nu_s(q, \overline{x})}
\]

- AC for disjunction (for simplicity)
Assumptions:

- The tableau for the logic contains rules:
  \[
  \frac{\nu_s(\neg p, \bar{x})}{\neg \nu_s(p, \bar{x})} \quad \text{and} \quad \frac{\nu_s(p \lor q, \bar{x})}{\nu_s(p, \bar{x}) \mid \nu_s(q, \bar{x})}
  \]

- AC for disjunction (for simplicity)

\[
\frac{\nu_s(p_1 \lor \cdots \lor p_k, \bar{x})}{\nu_s(p_1, \bar{x}) \mid \cdots \mid \nu_s(p_k, \bar{x})} (k > 1)
\]
Hypertableau Rule

Assumptions:

- The tableau for the logic contains rules:

\[
\frac{\nu_s(\neg p, \bar{x})}{\neg \nu_s(p, \bar{x})} \quad \text{and} \quad \frac{\nu_s(p \lor q, \bar{x})}{\nu_s(p, \bar{x}) \mid \nu_s(q, \bar{x})}
\]

- AC for disjunction (for simplicity)

\[
\frac{\nu_s(\neg p_1 \lor \cdots \lor \neg p_m \lor q_1 \lor \cdots \lor q_n, \bar{x})}{\neg \nu_s(p_1, \bar{x}) \mid \cdots \mid \neg \nu_s(p_m, \bar{x}) \mid \nu_s(q_1, \bar{x}) \mid \cdots \mid \nu_s(q_n, \bar{x})}
\]

\[(m + n > 1 \text{ and only atomic substitutions are allowed into } p_1, \ldots, p_m)\]
Assumptions:

- The tableau for the logic contains rules:

\[
\frac{\nu_s(\neg p, \bar{x})}{\neg \nu_s(p, \bar{x})} \quad \text{and} \quad \frac{\nu_s(p \lor q, \bar{x})}{\nu_s(p, \bar{x}) \mid \nu_s(q, \bar{x})}
\]

- AC for disjunction (for simplicity)

\[
\frac{\nu_s(\neg p_1 \lor \cdots \lor \neg p_m \lor q_1 \lor \cdots \lor q_n, \bar{x})}{\nu_s(p_1, \bar{x}) \mid \cdots \mid \nu_s(q_n, \bar{x})}
\]

\((m + n > 1 \text{ and only atomic substitutions are allowed into } p_1, \ldots, p_m)\)
Apply rules with higher branching factor less often.

For every instance of the hypertableau rule

\[
\begin{align*}
\nu_s(\neg p_1 \lor \cdots \lor \neg p_m \lor \phi_1 \lor \cdots \lor \phi_n, \overline{x}), & \quad \nu_s(p_1, \overline{x}), & \quad \cdots & \quad \nu_s(p_m, \overline{x}) \\
\nu_s(\phi_1, \overline{x}) & \quad | \quad \cdots & \quad | \quad \nu_s(\phi_n, \overline{x})
\end{align*}
\]

decompose \(\nu_s(\phi_i, \overline{x})\) by tableau rules.
Assume that for every $\phi$ from a large subclass of formulae of given logic:

$$\nu_s(\phi, \bar{x}) \leftrightarrow \bigwedge_{i=1}^{I} \left( \nu_{s_{ij}} \left( \bigvee_{j=1}^{J_i} \phi_{ij}, \bar{x} \right) \lor \bigvee_{k=1}^{K_i} L_{ik} \right).$$

Assume that there is an efficient reduction algorithm $\mathcal{A}$ for every such $\nu_s(\phi, \bar{x})$ to its equivalent clausal form.

Replace every new $\nu_s(\phi, \bar{x})$ in the derivation by its equivalent clausal form.
### Hypertableau for S4

<table>
<thead>
<tr>
<th>Rule</th>
<th>Hypertableau</th>
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</thead>
<tbody>
<tr>
<td>( \nu(\neg p, x) )</td>
<td>( \neg \nu(p, x) )</td>
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<tr>
<td>( \neg \nu(p, x) )</td>
<td>( \nu(p, x) )</td>
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<tr>
<td>( \nu(p \lor q, x) )</td>
<td>( \neg \nu(p, x), \neg \nu(q, x) )</td>
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<tr>
<td>( \nu(p, x) \lor \nu(q, x) )</td>
<td>( \neg \nu(p \lor q, x) )</td>
</tr>
<tr>
<td>( \nu(\Diamond p, x) )</td>
<td>( \nu(\Diamond p, x), R(x, y) )</td>
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<tr>
<td>( R(x, f(p, x)), \nu(p, f(p, x)) )</td>
<td>( \neg \nu(p, y) )</td>
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<td>( R(x, y), R(y, z) )</td>
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<td>( R(x, z) )</td>
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<td>( \neg R(x, x) )</td>
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<tr>
<td>( \nu(p, x), \neg \nu(p, x) )</td>
<td>( \bot )</td>
</tr>
<tr>
<td>+equality rules</td>
<td></td>
</tr>
</tbody>
</table>

\( \bot \)
Hypertableau for S4

\[ \frac{\nu(\neg p, x)}{\neg \nu(p, x)} \]
\[ \frac{\neg \nu(p, x)}{\nu(p, x)} \]
\[ \frac{\nu(\neg p_1 \lor \cdots \lor \neg p_m \lor q_1 \lor \cdots \lor q_n, x), \ \nu(p_1, x), \ \cdots, \ \nu(p_m, x)}{\nu(q_1, x) \ | \ \cdots \ | \ \nu(q_n, x)} \]

\( (m + n > 1 \text{ and only atomic substitutions are allowed into } p_1, \ldots, p_m) \)

\[ \frac{\neg \nu(p \lor q, x)}{\neg \nu(p, x), \ \neg \nu(q, x)} \]
\[ \frac{\nu(\Diamond p, x)}{R(x, f(p, x)), \ \nu(p, f(p, x))} \]
\[ \frac{\neg \nu(\Diamond p, x), \ R(x, y)}{\neg \nu(p, y)} \]
\[ \frac{R(x, y), \ R(y, z)}{R(x, z)} \]
\[ \frac{R(x, x)}{\nu(p, x), \ \neg \nu(p, x)} \]
\[ \bot \]

+equality rules
Conclusion

- Description of a tableau synthesis framework
- A generic method for refinement of tableau rules
- An atomic rule refinement condition to guarantee soundness and completeness
- Refinement of box rule and frame conditions in modal-like logics
- Application the atomic rule refinement method in order to obtain a hypertableau-like calculus
Thank You! Questions?
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