Products of modal logics

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In the early days of modal logic (before 1980s) there was interest in studying multiple particular systems. Contemporary modal logic also investigates classes of logics and general constructions combining different systems.

Products were introduced in the 1970s; their intensive study started in the 1990s.

Motivations for studying products of modal propositional logics

• A natural type of combined modal logics
• Connection to first-order classical logic
• Connection to first-order modal logics
• Connection to relation algebras
• Connection to description logics

The main reference for products (BOOK03)

PRODUCTS OF FRAMES

Kripke n-frames: \((W,R_1,...,R_n)\) (relational system with n binary relations).

Def. The product of two Kripke frames

\((W,R_1,...,R_n) \times (V,S_1,...,S_m) := (W \times V,R_{11},...,R_{n1},S_{12},...,S_{m2})\)

where

\((x_1,y_1)R_{i1}(x_2,y_2) \iff x_1R_ix_2 \& y_1=y_2\)

\((x_1,y_1)S_{j2}(x_2,y_2) \iff x_1=x_2 \& y_1S_jy_2\)

Multiple products \(F_1 \times ... \times F_n\) are defined in an obvious way. The multiplication is associative up to isomorphism.
Normal $n$-modal logics are defined as sets of modal formulas in the propositional language with unary modal connectives $\Box_1,\ldots,\Box_n$ containing the minimal logic and closed under standard rules.

Every Kripke frame is associated with a modal logic – the set of all valid formulas:

$$L(F) := \{ A \mid F \models A \}.$$

Logics of this form are called (Kripke) complete.

If $F$ is finite, $L(F)$ is called tabular.

For a class of frames $C$

$$L(C) := \bigcap \{ L(F) \mid F \in C \}.$$

If all frames in $C$ are finite, $L(C)$ has the finite model property (fmp).
A modal logic $L$ defines the class of $L$-frames

$$V(L) := \{ F \mid F \vDash L \}.$$ 

$L$ is called elementary if $V(L)$ is an elementary (first-order definable) class in the classical sense.

Remark $L$ is complete iff $L = L(V(L))$. 
Some particular complete logics

\( K_n \) is the minimal \( n \)-modal logic, \( K = K_1 \).
\( K.t_n \) is the minimal \( n \)-temporal logic, \( K.t = K.t_1 \).
\( K.t_n \) -frames are \((W, R_1, (R_1)^{-1},..., R_n, (R_n)^{-1})\).
\( T = K + \Box p \to p = L \) (all reflexive frames)
\( K4 = K + \Box p \to \Box \Box p = L \) (all transitive frames)
\( S4 = K4 + \Box p \to p = L \) (all transitive reflexive frames).
\( K4.3 = L \) (all transitive non-branching frames)
    = \( L \) (all strict linear orders)
\( S4.3 = K4.3 + \Box p \to p = L \) (all linear orders)
\( S5 = S4 + \Diamond \Box p \to p = L \) (all equivalence frames)
    = \( L \) (all universal frames)
\( Grz = L \) (all finite posets)
\( GL = L \) (all strict finite posets)
\( Grz3 = L \) (all finite chains)
\( GL3 = L \) (all strict finite chains)
Def. The product of two modal logics

\[ L_1 \times L_2 := L(\{F_1 \times F_2 \mid F_1 \models L_1, F_2 \models L_2 \}). \]

Similarly we can define multiple products

\[ L_1 \times \ldots \times L_n := L(\{F_1 \times \ldots \times F_n \mid F_1 \models L_1, \ldots, F_n \models L_n \}). \]

However, multiplication of logics is probably non-associative (an open problem).

**AXIOMATIZATION: FIRST RESULTS**

AXIOMATIZATION PROBLEM: to find axioms of \( L_1 \times \ldots \times L_n \) given the axioms of \( L_1, \ldots, L_n \).

Theorem 1 (Sh 1987, Gabbay&Sh 1998)

Classes of frames \( C_1, \ldots, C_n \) are elementary \( \Rightarrow \) \( L(C_1 \times \ldots \times C_n) \) is RE.

(So, \( L_1, \ldots, L_n \) are Kripke complete and elementary \( \Rightarrow \) \( L_1 \times \ldots \times L_n \) is RE.

Corollary 1.1 (Sh 1987) \( L((Q, <)^2), L((Q, \leq)^2) \) are RE.
Def. The fusion of two modal logics with disjoint modalities

\[ L_1 \ast L_2 := \text{the smallest logic containing } L_1 \cup L_2 \]

**Remarks on fusions**

Fusion of logics preserves many properties:


Bad news: products do not preserve any of these properties.

Good news: sometimes they still do.

Def. The commutative join of two modal logics with disjoint modalities

\[ \square_i \ (1 \leq i \leq n), \ \Box_k \ (1 \leq j \leq m) \]

is

\[ [L_1,L_2] := L_1 \ast L_2 + \]

\[ \Diamond_i \Box_k p \rightarrow \Box_k \Diamond_i p + \Box_i \Box_k p \leftrightarrow \Box_k \Box_i p \quad \text{for any } i, k \]
Remark. If the modalities are not disjoint, we rename them.

The additional axioms are Sahlqvist formulas expressing the following properties of the relations in the product frame

\[(R_{i1})^{-1} \circ S_{k2} \subseteq S_{k2} \circ (R_{i1})^{-1}\] (Church - Rosser property)

\[R_{i1} \circ S_{k2} = S_{k2} \circ R_{i1}\] (commutativity)

**Def.** $L_1, L_2$ are product matching if $[L_1, L_2] = L_1 \times L_2$
Def. A Horn sentence is a universal first order sentence of the form

\[ \forall x... (\varphi(x,y,z) \rightarrow R(x,y)), \]

where \( \varphi \) is positive, \( R(x,y) \) is atomic.

A modal formula \( A \) is Horn if it corresponds to a Horn sentence (i.e., the class of its frames \( \mathbf{V}(A) \) is definable by a Horn sentence).
Example Modal formulas of the form \((\Diamond \ldots \Diamond) \lozenge p \rightarrow (\Box \ldots \Box)p\)
correspond to Horn sentences

Logics with such axioms are always complete.

Def. A modal logic is Horn axiomatizable if it is axiomatizable by formulas that are either variable-free or Horn.

Completeness theorem for products
([Gabbay, Sh 1998]<< [BOOK03])

Theorem 3 If \(L_1, L_2\) are Kripke complete and Horn axiomatizable, then they are product matching.
Counterexamples

**Theorem 4** [Sh 1987 << Gabbay,Sh 1998]

Let L be a nontrivial 1-modal logic containing Grz. Then L and S5 are not product-matching.

**Theorem 5** [Kurucz & Marcelino 2011]

K4.3 and S5, S4.3 and S5 are not product-matching.
Stornger counterexamples: finite axiomatizability is not preserved (see later)

**FMP AND PRODUCT FMP**

Def. A QTC-logic is axiomatizable by variable-free formulas and formulas or axioms of the form \( \Diamond_i \Box_j p \rightarrow p, \Box_i p \rightarrow (\Box_i)^k p \).

**Theorem 6** [Sh 2005] If \( L_2 \) is a QTC-logic, then \( K.t_n \times L_2 = [K.t_n, L_2] \) has the fmp.

**Theorem 7** [Sh 2011] \( (K.t_n)^2 = [K.t_n, K.t_n] \) has the product fmp.
Theorem 8 (a) If \( C_1, C_2 \) are classes of \( K4.3 \)-frames containing some frames with descending \( \omega \)-chains and every frame in is Dedekind-complete (i.e., every bounded set has supremum), then \( L(C_1 \times C_2) \) is \( \Pi^1_1 \)-hard.

(b) If \( C_1, C_2 \) are classes of \( K4.3 \)-frames containing some frames with ascending \( \omega \)-chains and every frame in \( C_1 \) is Dedekind-complete, then \( L(C_1 \times C_2) \) is \( \Pi^1_1 \)-hard.

Corollary 8.1 \( GL3^2, Grz3^2, GL3 \times Grz3 \) are \( \Pi^1_1 \)-hard.

Thus products do not preserve any interesting property of modal logics.
Corollary 8.2 \( L(F \times G) \) is \( \Pi^1_1 \)-hard whenever \( F \) is \( R \) or \( \omega \), \( F \) is \( Q \), \( R \) or \( \omega \) (with \( < \) or \( \leq \)).

Theorem 9 If \( C_1, C_2 \) are classes of finite [strict] linear orders of unbounded length, then \( L(C_1 \times C_2) \) is \( \Pi^0_1 \)-complete.

Corollary 9.1 \( L((\omega,>)^2), L((\omega,\geq)^2) \) are \( \Pi^0_1 \)-complete.

**PRODUCTS WITH TABULAR LOGICS**

Theorem 10 (Sh 2013) (1) If \( L_1 \) has the fmp and \( L_2 \) is tabular, then \( L_1 \times L_2 \) has the product fmp.

(2) If \( L_1 \) is decidable, \( L_2 \) is tabular, then \( L_1 \times L_2 \) is decidable.
TRANSLATION INTO MODAL PREDICATE LOGIC

Consider n-modal predicate formulas with arbitrary predicates, but without equality, constants and function symbols.

Kripke frame semantics with constant domains

Propositional Kripke frames: \( F = (W,R_1,...,R_n) \)

Predicate Kripke frames with constant domains:
\( \Phi = (F,D) \), where \( D \) is nonempty.

\( F \) is the frame of worlds of \( \Phi \), \( D \) is the set of individuals.

Kripke models over \( \Phi \):
\( M = (\Phi,V) \), where \( V \) is a valuation:

\[ V(P) \subseteq D^n \times W \] for every n-ary predicate letter \( P \),

For every formula \( A(x_1,...,x_n) \) and \( d_i \in D \) we construct a D-sentence \( A(d_1,...,d_n) \)

Forcing relation \( M,u \models B \) between \( u \in W \) and a D-sentence \( B \) is defined by induction, in particular:

- \( M,u \models P(d_1,...,d_n) \) iff \( (d_1,...,d_n,u) \in V(P) \)
\begin{itemize}
  \item $M,u \models \Box_i B$ \iff $\forall v \in R_i(u) \ M,v \models B$
  \item $M,u \models \forall x \ B$ \iff $\forall d \in D \ M,u \models [d/x]B$
\end{itemize}

**Def** \quad $M \models A(x_1, \ldots, x_n)$ \iff $\forall u \in W \ M,u \models \forall x_1 \ldots \forall x_n A(x_1, \ldots, x_n)$

(validity in a frame) \quad $\Phi \models A$ \iff for any $M$ over $\Phi$, $M \models A$

$L(\Phi) := \{A \mid \Phi \models A\}$ is the modal predicate logic of $\Phi$.

**Wajsberg-type translation**

Wajsberg's translation interprets $S5$ in classical first-order logic. Similarly,

every propositional $(n+1)$-formula $A$ (with modalities $\Box_1, \ldots, \Box_n, \forall$)
is translated into

a first-order $n$-modal formula $A^\#(y)$ with (maybe) a parameter $y$: 

Every \( q \in PV \) is associated with a unary predicate letter \( Q \). Then

\[
q^\#(y) := Q(y) \quad (\text{for } q \in PV)
\]

\[
\bot^\#(y) := \bot
\]

\[
(A \rightarrow B)^\#(y) := A^\#(y) \rightarrow B^\#(y)
\]

\[
(\Box_i A)^\#(y) := \Box_i A^\#(y)
\]

\[
([\forall]A)^\#(y) := \forall y A^\#(y)
\]

**Lemma** \( F \times (D, D \times D) \models A \iff (F, D) \models \forall y A^\#(y) \)
Theorem 11 Let $L_1$ be an $n$-modal propositional logic, $CK(L_1)$ the class of all predicate Kripke frames $(F,D)$, with $F \models L_1$. Consider the corresponding predicate modal logic $L(CK(L_1))$. Then $L_1 \times S5$ is (polynomially) reducible to $L(CK(L_1))$: for any $(n+1)$-modal propositional $A$

\[ L_1 \times S5 \vdash A \iff \forall y A^\#(y) \in L(CK(L_1)) \]

(In other words, $L_1 \times S5$ specifies a fragment within $L(CK(L_1))$)

Completeness theorems for modal predicate logics yield a standard axiomatization of $L(CK(L_1))$ in some cases. $QL_1$ is the pure quantified version of $L_1$, $BF$ is the conjunction of Barcan schema for all modalities:

$\forall x \Box_i A \rightarrow \Box_i \forall x A$.

Theorem 12 (1) (Tanaka&Ono, 1999) If $L_1$ is complete and $V(L_1)$ is universally axiomatizable (in the classical sense), then $L(CK(L_1)) = QL_1 + BF$

(2) (Ono, 1983<<Gabbay&Skvorstov&Shehtman, 2009)
The same holds if $L_1$ is tabular.

In all these cases $L_1 \times S_5$ is RE, but now from Theorems 11, 12, 3 we obtain

**Corollary 12.1** $[L_1, S_5] \models A$ iff $QL_1 + BF \models \forall y A^#(y)$

whenever $L_1$ is Horn axiomatizable.
TRANSLATION INTO CLASSICAL PREDICATE LOGIC

This “square translation” resembles the well-known standard translation of modal formulas in the language of \((K_n)^2\) into classical first-order formulas with relativized quantifiers.

Consider the first order language with binary predicate letters \(R_1, \ldots, R_n, P_1, P_2, \ldots\). We associate a binary predicate letter \(P_i\) with every proposition letter \(p_i\).

\[
(p_i)^2(x,y) := P_i(x,y)
\]

\[
(A \rightarrow B)^2(x,y) := A^2(x,y) \rightarrow B^2(x,y)
\]

\[
\bot^2(x,y) := \bot
\]

\[
(\Box_i A)^2(x,y) := \forall z (R_i(x,z) \rightarrow A^2(z,y))
\]

\[
(\square_i A)^2(x,y) := \forall z (R_i(y,z) \rightarrow A^2(x,z))
\]
**Theorem 13** If $L$ is an elementary modal logic, $\varphi$ (first-order) axiomatizes $\mathcal{V}(L)$, then

1. $QCL + \varphi \vdash \forall x \forall y A^2(x,y) \iff L^2 \vdash A$.

   (Here $QCL$ is the classical first-order theory axiomatized by $\varphi$).

2. If $L^2$ has the product fmp, then the corresponding “square fragment” of $QCL + \varphi$ with binary predicates has the fmp (in the classical sense).
Axiomatizing some products of non-product-matching logics

Def. A propositional modal logic $L$ is called locally tabular if, up to equivalence in $L$, for any $m$ there are finitely many formulas in $m$ propositional variables.

It is well-known that every locally tabular logic has the fmp.

Def. A propositional 1-modal logic $L$ above $K4$ is of finite depth $< m$ if all $L$-frames are of depth $< m$.

$L$ is of depth $m$ if it is of depth $< m+1$, but not $< m$.

**Theorem 14** (Segerberg, 1971) Every logic of finite depth is locally tabular.

**Theorem 15** (Maksimova, 1974) The converse holds for extension of $K4$.

**Theorem 16** (Sh 2010) If $L$ is of finite depth, then $[\sim, S5]$ is locally tabular.

This allows us to axiomatize products of finite depth logics above $Grz$ with $S5$ in two ultimate cases: the catkin formula $ACk$ is
exactly what is missing. ACk is the Fine – Jankov formula of the following 2-frame (catkin):

Theorem 17 (Sh 2010) If
\[ L = \text{Grz} + A_{\text{depth}_n} \text{ (} \equiv L(\text{all posets of depth } n) \text{)} \]
or
\[ L = \text{Grz3} + A_{\text{depth}_n} \text{ (} \equiv L(\text{all chains of depth } n) \text{)}, \]
then
\[ L \times S_5 = [L, S_5] + ACk. \]

Corollary 17.1 These logics are decidable.
References


Theorem [Kurucz & Marcelino 2011]

\textbf{K4.3} \times \textbf{K}, \textbf{S4.3} \times \textbf{K} are not even axiomatizable in finitely many variables

QUESTION. Are the logics \textbf{K4.3} \times \textbf{S5}, \textbf{S4.3} \times \textbf{S5} finitely axiomatizable?
Squares of modal logics with additional connectives

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Def. The product of two Kripke frames

\((W,R_1,\ldots,R_n) \times (V,S_1,\ldots,S_m) := (W \times V, R_{11},\ldots,R_{n1},S_{12},\ldots,S_{m2}),\)

where
\[(x_1,y_1)R_{i_1}(x_2,y_2) \iff x_1R_ix_2 \land y_1=y_2\]
\[(x_1,y_1)S_{j_2} (x_2,y_2) \iff x_1=x_2 \land y_1S_jy_2\]

Def. The product of two modal logics

\[L_1 \times L_2 := L(\{F_1 \times F_2 \mid F_1 \models L_1, F_2 \models L_2\})\]

AXIOMATIZATION PROBLEM: to find axioms of \(L_1 \times L_2\) given the axioms of \(L_1, L_2\)

Def. The fusion of two modal logics with disjoint modalities

\[L_1 \ast L_2 := \text{the smallest logic containing } L_1 \cup L_2\]

Def. The commutative join of two modal logics with disjoint modalities

\[\Box_i (1 \leq i \leq n), \ Diamond_j (1 \leq j \leq m)\]

\[\lfloor L_1, L_2 \rfloor := L_1 \ast L_2 + \Diamond_i \Box_k p \vdash \Box_k \Diamond_i p + \Box_i \Box_k p \iff \Box_k \Box_i p\]

for any \(i, k\)
Remark. If the modalities are not disjoint, we can change them. These are Salqvist formulas expressing the following properties of the relations in the product frame:

\[ \Diamond_i \Box_k p \rightarrow \Box_k \Diamond_i p : \]

\[ (R_{i_1})^{-1} \circ S_{k_2} \subseteq S_{k_2 \circ (R_{i_1})^{-1}} \text{ (Church - Rosser property)} \]
\( \Box_i \Box_k p \leftrightarrow \Box_k \Box_i p: \)

\[ R_{i1} \circ S_{k2} = S_{k2} \circ R_{i1} \text{ (commutativity)} \]

Def. \( L_1, L_2 \) are product-matching if \( L_1 \times L_2 = [L_1, L_2] \)
SQUARES

For a class of frames $C$ put
$$C^2 := \{ F \times F \mid F \in C \}.$$ 

For a modal logic $\Lambda$ put
$$\Lambda^2 := \Lambda \times \Lambda$$

Proposition 1 [Gabbay, Sh 2000]
$$\Lambda^2 = L(\{ F \times F \mid F \models \Lambda \}).$$
("Squares of logics are determined by squares of frames".)

Proposition 2 [Gabbay, Sh 2000]
$$L_1 \times L_2 \text{ is embeddable in } (L_1 \star L_2)^2.$$ 
("Products are reducible to squares".)
These are square frames with additional functions. Krister Segerberg (1973) studied a special type - squares of frames with the universal relation.

He considered the following functions on squares.

\[ \sigma_{\circ}: (x,y) \mapsto (y,x) \] (diagonal symmetry)

\[ \sigma_{\Theta}: (x,y) \mapsto (y,y) \] (the first diagonal projection)

\[ \sigma_{\Phi}: (x,y) \mapsto (x,x) \] (the second diagonal projection)

These functions can be associated with extra modal operators \( \ominus, \Theta, \Phi \). So in square frames they are interpreted as follows:

\( (x,y) \models \ominus A \iff (y,x) \models A \)

\( (x,y) \models \Theta A \iff (x,x) \models A \)

\( (x,y) \models \Phi A \iff (y,y) \models A \)

Remark. Segerberg used the notation \( \boxtimes \) instead of \( \ominus \).
Formally we define the Segerberg square of a frame $F=(W,R_1,\ldots,R_n)$ as the $(2n+3)$-frame $F^{2\otimes} := (F^2,\sigma_\circ, \sigma_\ominus, \sigma_\exists)$ (where $\sigma_\circ,\sigma_\ominus,\sigma_\exists$ are the functions on $W^2$ described above).

Respectively, the Segerberg square of an n-modal logic $\Lambda$ is defined the logic of the Segerberg squares of its frames $\Lambda^{2\otimes} := L(\{F^{2\otimes} | F \models \Lambda\})$.

**TOMORROW (OR SUCCESSOR) LOGIC**

$SL := K + \diamond p \leftrightarrow \Box p$

(an equivalent form: $K + \neg \Box p \leftrightarrow \Box \neg p$)

This well-known logic is also due to Segerberg (1967). It is complete w.r.t. the frame

```
  .---.---.---.---.---
  |   |   |   |   |   |
  V---V---V---V---V
```

(the successor relation on natural numbers).
Every logic of a frame with a functional accessibility relation is an extension of SL.

**AXIOMATIZING SEGEBERG SQUARES**

**Soundness**  Every Segerberg square validates the following formulas

The corresponding semantic conditions for an arbitrary \((2n+3)\)-frame

\[ (V, X_1, \ldots, X_n, Y_1, \ldots, Y_n, f_\bigcirc, f_\Theta, f_\Phi) \]

are in the right column; here \(fg\) denotes the composition of functions: \((fg)(x)=f(g(x))\)

**(I)** The SL-axioms for the circles \(\bigcirc, \Theta, \bigcirc\).

**(II)**

(Sg1) \(\bigcirc\bigcirc p \leftrightarrow p\) \hspace{1cm} \(f_\bigcirc f_\bigcirc = 1\) (the identity function on \(V\))

The "symmetry" \(f_\bigcirc\) is an involution.

(Sg2) \(\Theta\Theta p \leftrightarrow \Theta p\) \hspace{1cm} \(f_\Theta f_\Theta = f_\Theta\)
(Sg2') $\emptyset\emptyset p \iff \emptyset p \quad f_\emptyset f_\emptyset = f_\emptyset$

Both projections $f_\emptyset$, $f_\Phi$ are idempotent transformations of the square. In fact (Sg2') follows from (Sg1), (Sg2), (Sg3).

(Sg3) $O\Theta p \iff \Phi p \quad f_\Theta f_\Theta = f_\Phi$

(Sg4) $\Theta O p \iff \Theta p \quad f_\Theta f_\Theta = f_\Theta$

In Segerberg squares (Sg4) means that the image of $f_\emptyset$ consists of self-symmetric points (or: every diagonal point is self-symmetric). But in the general case not all self-symmetric points are in $f_\emptyset[V]$.

(Sg3), (Sg4) imply that $f_\emptyset f_\emptyset f_\emptyset = f_\emptyset$, i.e., the involution $f_\emptyset$ conjugates the projections $f_\emptyset$ and $f_\Phi$.

(Sg3) shows that $\emptyset$ is expressible in terms of $O$, $\Theta$. It also implies that $f_\emptyset[V] = f_\Phi[V]$.

(Sg4') $\emptyset\emptyset p \iff \emptyset p \quad f_\emptyset f_\emptyset = f_\emptyset$

This conjugate of (Sg4) is derivable from (Sg1), (Sg3), (Sg4).

(Sg5) $O\square_i O p \iff \square_i p \quad aR_{i1}b \iff f_\emptyset(a)R_{i2}f_\emptyset(b)$
(Sg5) $\Diamond\Box_i\Diamond p \iff \Box_i p \quad aR_{i1}b \iff f_{\Diamond}(a)R_{i2}f_{\Diamond}(b)$

Symmetry is an isomorphism between $R_{i1}$ and $R_{i2}$

(Sg6) $\Theta\Box_i(\Box_ip \rightarrow \Box p) \quad f_{\Theta}(a)R_{i1}b \Rightarrow bR_{i2}f_{\Theta}(b)$

If $(y,y)R_{i1}(x,y)$ (i.e. $yR_{i}x$), then $(x,y)R_{i2}(x,x)$.

(Sg7) $\Theta p \rightarrow \Box_i \Theta p \quad aR_{i1}b \Rightarrow f_{\Theta}(a) = f_{\Theta}(b)$

Horizontally accessible points are in the same horizontal row.

(Sg8) $\Box_i \bot \iff \Theta \Box_i \bot \quad (\exists b \ aR_{i2}b) \iff (\exists b \ f_{\Theta}(a)R_{i2}b)$

Vertical seriality is equivalent for $(y,y)$ and $(x,y)$.

The conjugates of (Sg6)-(Sg8) are derivable, so they are not written here.

Def. For a modal logic $\Lambda$, put

$[\Lambda, \Lambda]^\Theta :=
[\Lambda, \Lambda] + SL*SL*SL$ (for $\Diamond$, $\Theta$, $\Box$) + {$(Sg1),..., (Sg8)$}.

Def. A universal Horn sentence is a first order sentence of the form

$\forall x... (\varphi(x,y,z) \rightarrow R(x,y)),$
where $\varphi$ is positive, $R(x,y)$ is atomic.
Modal formulas corresponding to such sentences are conjunctions of formulas of the form

\[(\Diamond \ldots \Diamond) \Box p \rightarrow (\Box \ldots \Box)p\]

**Def. A modal logic is Horn axiomatizable** if it is axiomatizable by formulas that are either variable-free or correspond to universal Horn sentences.

**Completeness theorem for products** [Gabbay, Sh 1998]

If \(L_1, L_2\) are Horn axiomatizable, then they are product-matching.

**Theorem 1 (Completeness)** If a logic \(\Lambda\) is Horn axiomatizable, then \(\Lambda^{2\otimes} = [\Lambda, \Lambda]^{\otimes}\)
Remark Segerberg himself axiomatized the logic $\mathcal{B}$ of all frames of the form $(W, W \times W)^{2\otimes}$. In this case (Sg8) becomes trivial and (Sg6) should be replaced with a stronger axiom: $\Box p \rightarrow \Phi p$. So Segerberg's logic is not a Segerberg square in our sense; it is a proper extension of $\mathbb{S}5^{2\otimes}$.

Sketch of the proof of Theorem 1
Step 1. $(K_n)^{2\otimes}=[K_n, K_n]^{\otimes}$

Consider the case $n=1$. The logic $L:= [K, K]^{\otimes}$ is Sahhqvist, so it has the countable frame property, so it is determined by countable rooted $L$-frames. Let $F=(W,R_1,R_2, f_\circ,f_\otimes,f_\phi)$ be such a frame.

Now the goal is to construct a $p$-morphism from a Segerberg square onto $F$. We use a "rectification game" similar to the one described in [Sh 2005] and originally motivated by the games from [Many-dimensional modal logics, 2003] and [Relation algebras by games, 2002].
Let $T_\omega = (\omega^*,<)$ be the standard countable intransitive irreflexive tree, where

$\omega^*$ is the set of all finite sequences in $\omega$;

$\alpha < \beta$ iff $\exists n \in \omega \beta = \alpha n$.

Let $T_\omega + T_\omega$ be the disjoint union of its two copies:

$\{x\alpha \mid \alpha \in \omega^*\} \cup \{y\alpha \mid \alpha \in \omega^*\}$ with the relation $<$.

Consider the product frame

$$(T_\omega + T_\omega)^2 = ((\omega^* + \omega^*)^2, S_1, S_2).$$

A network over $F$ is a partial function from $(T_\omega + T_\omega)^2$ to $F$

$h: N \to V$

such that

- $\text{dom}(h) = N$ is symmetric:
  
  $\sigma_0[N] = N,$
  
  $\sigma_\emptyset[N] \subseteq N.$

- $N$ does not have gaps:
$(\alpha, \beta) \in \mathbb{N} \land (\alpha, \gamma) \in \mathbb{N} \land \beta <^+ \gamma \land \beta < \beta' \Rightarrow (\alpha, \beta') \in \mathbb{N}$

($<^+$ is the transitive closure of $<$)

- $h$ is monotonic:
  
  \[
  aSib \Rightarrow h(a)Rih(b), \\
  h(\sigma_\circ(a)) = f_\circ(h(a)), \\
  h(\sigma_\bullet(a)) = f_\bullet(h(a)).
  \]

The game between $A$ and $E$ constructs a countable increasing sequence of networks $h_0 \subseteq h_1 \subseteq \ldots$ according to the following rules.

1. $N_0 = \{(x, y), (y, x), (x, x), (y, y)\}$, where $h_0(x, y) = u_0$, the root of $F$; then $h_0(y, x), h_0(x, x), h_0(y, y)$ are uniquely determined.

Remark. If $u_0$ is self-symmetric, we don't need two copies, the game can start from $N_0 = \{(\lambda, \lambda)\}$, where $\lambda$ is empty.

2. The $(n+1)$th move of $A$ is of two types

Lift enquiry $(a, u, j, v)$, where $a \in N_n$, $u = h_n(a)$, $uR_jv$
The response of $E$ must be a network $h_{n+1}$ extending $h_n$ that
that $\exists b \in N_{n+1}(a S_j b \&$

**THE FINITE MODEL PROPERTY**

**Def. A QT-formula** is a modal formula of the form

$\Box_i p \rightarrow \Box_i^k p$ (generalized transitivity)

or

$\Diamond_i \Box_i p \rightarrow p$ (symmetry)
A QTC-logic is axiomatizable by formulas that are either variable-free or QT-formulas.

**Notation** $K_{±n}$ is the minimal $n$-temporal logic (axiomatized by $\Box_i^{-1}□_i p → p$, $\Box_i □_i^{-1} p → p$)

The fmp for products [Sh 2005]
If $L_2$ is a QTC-logic, then $K_{±n} \times L_2 = [K_{±n},L_2]$ has the fmp.

**Theorem 2** $(K_n)^{2\otimes}$ has the fmp.

**Conjecture** $(K_{±n})^{2\otimes}$ has the fmp.
THE PRODUCT FMP FOR SEGERBERG SQUARES

Def A logic $\Lambda^{2\otimes}$ has the product fmp if it is determined by finite Segerberg squares:

$$\Lambda^{2\otimes} = L(\{F^{2\otimes} \mid F \text{ is finite, } F \models \Lambda\}).$$

The product fmp for products [Gabbay, Sh 2002]

Every logic $K_{\pm n} \times K_m$ has the product fmp.

**Theorem 3** $(K_n)^{2\otimes}$ has the product fmp.

**Conjecture** $(K_{\pm n})^{2\otimes}$ has the product fmp.
REFERENCES

Ideas for the proofs of Theorems 1, 2, 3.
Relation algebras