Problems of Unification and Admissible Rules in Non-Classical Logics (with applications to AI, CS and CD)

V. Rybakov

School of Computing, Mathematics and DT, Manchester Metropolitan University,
John Dalton Building, Chester Street, Manchester M1 5GD, U.K.

V.Rybakov@mmu.ac.uk

Part Time: Siberian Federal University, Krasnoyarsk
History, Pre-History


(1975) Harvey Fridman problem: if $\text{IPC } H$ is decidable w.r.t admissible rules.

Solution: yes – Rybakov, 1984 - IPC, $S4, Grz +$ etc.

(1975) A.Kuztetsov problem: if $\text{IPC } H$ has a finite basis for inference rules: -

Solution: no – Rybakov 1985-86.
(1950???) P. Novikov Problem: decidability logical equations in IPC + etc.

Solution: yes - Rybakov - 1986


Short recall of definitions and notation

**Definition.** An inference rule $\varphi_1, \ldots, \varphi_n/\psi$ is said to be *admissible* in a logic $L$ if for any substitution $\varepsilon$ the following holds: if $\varepsilon(\varphi_1) \in L, \ldots, \varepsilon(\varphi_n) \in L$ then $\varepsilon(\psi) \in L$.

**Definition.** A formula $\varphi$ is unifiable in a logic $L$ if there is a substitution $\varepsilon$ (which is called a unifier for $\varphi$) such that $\varepsilon(\varphi) \in L$.

**Definition.** A unifier $\varepsilon$ (for a formula $\varphi$ in a logic $L$) is more general than a unifier $\varepsilon_1$ iff there is a substitution $\delta$ such that for any letter $x$, $[\varepsilon_1(x) \equiv \delta(\varepsilon(x))] \in L$. 
If a logic $L$ is decidable, to check the unifiability a formula in $L$ is (theoretically, not computationally) an easy task: it is sufficient to use only ground substitutions: mappings of variable-letters in the set $\{\bot, \top\}$. But the problem - how to find all unifiers - all solving substitutions - is not easy at all.

**Definition.** A set of unifiers $CU$ for a given formula $\phi$ in a logic $L$ is a complete set of unifiers, if the following holds. For any unifier $\sigma$ for $\phi$ in $L$, there is a unifier $\sigma_1$ from $CU$, where $\sigma_1$ is more general than $\sigma$.

**Definition.** A logic $L$ has unitary unification if any unifiable in $L$ formulas has a complete set of unifiers consisting of single formula (we call it mgu - most general unifier)

If a logic $L$ has finite computable set of unifiers for any unifiable formula, $L$ is decidable w.r.t. admissible rules: it is sufficient to verify only these unifiers for checking admissibility.
For $LTL$ (which possesses definable $\square$ and $\Diamond$, 
- $\Diamond x := (\top U x)$,  $\square \equiv \neg \Diamond \neg$

or modal logics over $S4$ we may formulate projectivity as follows:

**Definition.** A formula $\varphi$ is said to be **projective** in a logic $L$ if the following holds. There is a substitution $\sigma$ (which is called projective substitution, projective unifier) such that $\sigma$ is a unifier and $\square \varphi \rightarrow [x_i \equiv \sigma(x_i)] \in L$ for any letter $x_i$ from $\varphi$. 
**Proposition** If a substitution \( \sigma_p \) is projective for a formula \( \varphi \) in a logic \( L \), then \( \{ \sigma_p \} \) is a complete set of unifiers for \( \varphi \) (i.e. \( \sigma_p \) is most general unifier).

*Proof.* Indeed, let \( \sigma \) be a unifier for \( \varphi \) in \( L \). Since we assume \( \sigma_p \) is projective for \( \varphi \) in \( L \), we have \( \Box \varphi \rightarrow [x_i \equiv \sigma_p(x_i)] \in L \) for any letter \( x_i \) from \( \varphi \). Acting by \( \sigma \) on the formula above we get \( \sigma(\Box \varphi) \rightarrow [\sigma(x_i) \equiv \sigma(\sigma_p(x_i))] \in L \), that is \( \sigma(x_i) \equiv \sigma(\sigma_p(x_i)) \in L \). Q.E.D.

\( L \) has projective unification if every unifiable formula has a projective unifier.

Projectivity implies unitary unification, but not vise versa:

**Modal logic** \( S4.2 \) has unitary unification (Ghilardi, Sacchetti, 2006), but
Example. The unifiable formula

\[ \Box(\Box x \to \Box y) \to \Box x \lor \Box z \]

has an mgu in $S4.2$ but can not have a projective unifier in $S4.2$. 
Survey: Recent Solutions of open questions in 'standard' areas ...


admissibility in typical normal extensions of K4 (K4, GL, S4, S4Grz) and s.i. logics (IPC ++++) is coNEXP-complete (and in particular, strictly more complex than the drivability problem, under reasonable complexity-theoretic assumptions).


IPC, K4, GL, and S4, as well as all logics inheriting their admissible rules, have independent bases of admissible rules.
A move for new open problems, to new horizon

(i) Lukasiewicz logic:


Lukasiewicz multi-valued propositional logic: admissibility of multiple-conclusion rules in Lukasiewicz logic, as well as validity of universal sentences in free MV-algebras, is decidable (in PSPACE).


Explicit bases of single-conclusion and multiple-conclusion admissible rules of propositional Lukasiewicz logic, also – a proof that Lukasiewicz logic has no finite basis of admissible rules.
**Linear temporal logic LTL with UNTIL and NEXT**


LTL is decidable w.r.t admissible inference rules. As a consequence we obtain algorithms verifying the validity quasi-identities in varieties of corresponding algebras.


Provide an explicit (infinite) basis for rules admissible in LTL.
Any unifiable in LTL formula has a most general unifier (thus, LTL enjoys unitary unification). The algorithm of construction such MGU is provided. This solves unifiability problem for LTL and the admissibility problem.
Unification with Coefficients


Solution of the unification problem in these logics for formulas with coefficients (meta-variables).

\textit{LTL} with \textit{SINCE} and similar (e.g. simply temporal logics with nodes) - EASY via modeling universal modality:

Paraconsistent minimal Johanssons’ logic J and positive intuitionistic logic


This paper proves that the problem of admissibility for inference rules with coefficients (parameters)(as well as plain ones without parameters) is decidable for the paraconsistent minimal Johanssons’ logic J and the positive intuitionistic logic IPC+. Using obtained technique we show also that the unification problem for these logics is also decidable: we offer algorithms which compute finite complete sets of unifiers for any given unifiable formula.
Description logics


via Algebra

APPLICATIONS to AI, CS and CD


Logic $UIA_{LTL}$, which is a combination of the linear temporal logic LTL, a multi-agent logic with operation for passing knowledge via agents’ interaction, and a suggested logic based on operation of logical uncertainty. The logical operations of $UIA_{LTL}$ also include (together with operations from LTL)

- operations of strong and weak until,

- $UIA_{LTL}$ agents’ knowledge operations,
- $UIA_{LTL}$ operation of knowledge via interaction,

- $UIA_{LTL}$ operation of logical uncertainty,

- $UIA_{LTL}$ the operations for environmental and global knowledge.


CD - in terms of plausibility to discover in search.

Distance - from $k$ to $k + m$ steps - possible to discover in this distance.


Uncertainty via combination of evidences in future and past.
Projectivity in linear temporal logics LTL with UNTIL

(Origin) Linear Temporal Logic $LTL$ with Next and Until:

Amir Pnueli, 1977: LTL was first proposed for the formal verification of computer programs.


Moshe Y. Vardi. An Automata-Theoretic Approach to Linear Temporal Logic, since 1995
Short recall of definitions:

LTL is built up from a finite set of propositional variables AP, the logical operations $\neg$ and $\lor$, and the temporal modal operations $N$ (next time) and $U$ (until). Formally, the set of LTL formulas over AP is inductively defined as follows:

If $p \in AP$ then $p$ is a LTL-formula;

If $\psi$ and $\varphi$ are LTL-formulas then

$\neg \psi$, $\psi \lor \varphi$, $N\varphi$ and $\psi U \varphi$, are LTL formulas.
Semantics for \textbf{LTL} consists of runs of computation with given evaluations of propositional variables $AP$. Formally they may be viewed as Kripke models with base sets to be natural numbers, with standard understanding meaning of next (interpretation of $N$), and with a given valuation $V$ of $AP$. $M := \langle N, N, V \rangle$, $\forall p \in AP, V(p) \subseteq N$. Formally, the satisfaction (truth) relation between a word and an LTL formula is defined as follows: $\forall w \in N$,

\[ w \models_V p \iff p \in V(p); \]

\[ w \models_V \neg \varphi \iff \text{not}(w \models_V \varphi); \]

\[ w \models_V \varphi \lor \psi \iff w \models_V \varphi \ \text{or} \ w \models_V \psi \]

\[ w \models_V \text{N}\varphi \iff (w + 1) \models_V \varphi; \]

\[ w \models_V \varphi \text{U}\psi \iff \exists k \in N[(w + k) \models_V \psi \ \text{and} \ \forall n < k(w + n) \models_V \varphi]. \]

($\varphi$ must remain true until $\psi$ becomes true)
Linear modal logic $S4.3$ since (long ago (1983)) was known to be decidable about admissibility and structure of admissibility bases. But recently an algorithm for constructing projective unifiers in logics extending $S4.3$ was offered in


by a technique using Löwenheim substitutions and CNF-forms in such logics. So, any unifiable formula there is projective and hence has a computable mgu.
With linear temporal logic $LTL$ the case is more complicated:

**Proposition** (Rybakov, 2012) Formula $\varphi = \square(\square x \lor (\neg x \land \Diamond \square x))$ is unifiable in $LTL$ but not projective.

**Proof.** Substitution $x \mapsto \top$ is an obvious unifier for $\varphi$. Suppose now $\varphi$ is projective and $\pi$ is a corresponding projective unifier. Consider the run $N_V$ (starting from 0: $|N_V| := \{0, 1, 2, \ldots\}$):

$x \xrightarrow{\circ} \neg x \xrightarrow{\circ} \square x \xrightarrow{\circ} \square x \ldots$

Since $(N_V, 1) \models V \square \varphi$, then $(N_V, 1) \models V x \leftrightarrow \pi(x)$. Therefore, notwithstanding either $(N_V, 0) \models V \pi(x)$ or $(N_V, 0) \models V \neg \pi(x)$, we have that $(N_V, 0) \models V \neg \square \pi(x)$ and, at the same time, $(N_V, 0) \models V \neg N \square \pi(x)$. Thus $(N_V, 0) \models V \neg \pi(\varphi)$, hence $\pi$ cannot be an $\varphi$-unifier, a contradiction. Q.E.D.
Bearing in mind our task to push anyway projectivity to LTL, we will consider $\text{LTL}_U$ - the fragment of $LTL$ with the operation $U$ but without next - $N$ (that is formulas of this fragment do not contain $N$).

Since basic operation - until - $U$ is presented in $\text{LTL}_U$, we can again define basic modal operations - $\Box$ and $\Diamond$ and define projectivity as earlier.
**Theorem** Any unifiable in $\text{LTL}_U$ formula $\varphi$ is projective.

Take any $X \subseteq \text{Sub}(\Box \varphi)$, let

$$\psi(X) := \Box \varphi \land \bigwedge_{\psi U \xi \in X} \psi U \xi \land \bigwedge_{\psi, \xi \in \text{Sub}(\Box \varphi), \psi U \xi \notin X} \neg (\psi U \xi).$$

$$\sigma(x_i) := (\Box \varphi(x_1, \ldots, x_n) \land x_i) \lor (\neg \Box \varphi \land \Diamond \Box \varphi$$

$$\land \bigvee_{\psi(X) \in \text{Sat}} \Box[\neg \Box \varphi \land \Diamond \Box \varphi \rightarrow \neg \Box \varphi U \psi(X) \land T(\psi(X), x_i))] \lor$$

$$\neg \Diamond \Box \varphi \land T(x_i)).$$
Thus, $\text{LTL}_U$ enjoys projective unification and any unifiable formula has computable mgu. This solves open problem of recognizing rules admissible in $LTL$. 