Writing out ’Best’ Unifiers in Modal and Temporal Logics. Consequences for problem of Admissibility

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UNIFIABLE FORMULAS:

**Definition.** A formula $\varphi$ is unifiable in a logic $L$ if there is a substitution $\varepsilon$ (which is called a unifier for $\varphi$) such that $\varepsilon(\varphi) \in L$ (sometimes, in the sequel, notation $\varphi^\varepsilon := \varepsilon(\varphi)$, if it is more readable in complex expressions, may be used).

Formulas $\varphi$ and $\psi$ are unifiable in a logic $L$ iff there is a substitution $\sigma$ such that $\sigma(\varphi) \equiv_L \sigma(\psi)$ (i.e. $\sigma(\varphi \equiv \psi) \in L$).

**Definition.** A unifier $\varepsilon$ (for a formula $\varphi$ in a logic $L$) is more general than a unifier $\varepsilon_1$ iff there is a substitution $\delta$ such that for any letter $x$, $[\varepsilon_1(x) \equiv \delta(\varepsilon(x))] \in L$. 
**Definition.** A set of unifiers $BU$ for a given formula $\varphi$ in a logic $L$ is a set of best unifiers, if the following holds. For any unifier $\sigma$ for $\varphi$ in $L$, there is a unifier $\sigma_b$ from $PU$, where $\sigma_b$ is more general than $\sigma$ (i.e. $\sigma$ is a substitutional example of $\sigma_b$).

Let $r := \varphi_1(x_1, \ldots, x_n), \ldots, \varphi_m(x_1, \ldots, x_n) / \psi(x_1, \ldots, x_n)$ be an inference rule.

**Definition.** Rule $r$ is admissible for (in) a logic $L$ if,

$$\forall \alpha_1 \in Form_L, \ldots, \forall \alpha_n \in Form_L,$$

$$[ \bigwedge_{1 \leq i \leq m} (\varphi_i(\alpha_1, \ldots, \alpha_n) \in L) ] \implies [\psi(\alpha_1, \ldots, \alpha_n) \in L].$$

Thus, $r$ is admissible if, for every substitution $\sigma$, $\sigma(\varphi_1) \in L, \ldots, \sigma(\varphi_n) \in L$ implies $\sigma(\psi) \in L$. 
CONNECTIONS BETWEEN: ....

Logical unification was situated at the focus of interest from logical community and computer science for a long time.

cf. F. Baader, F. Baader, and T. Nipkow,

F. Baader and W. Snyder (general theory + description logics)

S. Ghilardi (modal and intuitionistic logics)

S. Ghilardi studied extensively the unification in propositional modal logics over $K4$, with aim to describe all possible unifiers. Unification problem was solved for modal logics $K4$ and $S4$, intuitionistic logic $Int$. This approach and technique showed
to be very useful in dealing with admissibility and bases of admissible rules.


The admissibility problem (to determine for any given rule if this rule is admissible for a given logic) is closely related to unification in many ways.

This problem was in focus of interest for many logicians.

Origin – P.Lorentzen (1955)

Is Admissibility decidable in INT (H.Friedman Problem, 1975)?

Partial Results on sufficient conditions for admissible rules to be derivable in INT

Solution H.Friedman Problem for INT and S4 (V.Rybakov, 1984)
No-finite bases for admissible in INT (S4) rules (A.Kuznetsov Problem) (V.Rybakov, 1985).

Quasi-Charcteristic Inference Rules, basis for those admissible (A.Citkin, 1985)

Admissible Rules with Coefficients (for INT, S4, later K4 - Rybakov, 1986).

eetc... many of Rybakovs’ - temporal logics, LTL, Logics with Universal modality, logics of finite intervals etc...
What are explicit bases for admissible rules?

Explicit basis for rules admissible in INT (R. Iemhoff, 2001 – rules of Visser-de Johng)


Proof theory for admissible rules George Metcalfe + Rosalie Iemhoff (2003 - 2010)

Explicit independent basis for rules admissible in INT - E. Jerabek (2007)
New wave:


Basis for rules admissible in Multi-Valued Logics, E.Jerabek (2009)

Complexity of algorithms deciding admissibility, E. Jerabek (2005)


APLOGIZE if important results are not mentioned - time ...
In particular, in (Rybakov, APAL, 2008) the admissibility problem was solved for linear temporal logic $LTL$.

What concerning unification in $LTL$?

Recently we found a way to describe most general unifiers (mgu-s) in $LTL$ (Babenyshev and Rybakov), submitted. A bit later I found much more shorter, tricky solution, which I would like to describe today.

The language of $(LTL, - \text{ Boolean } + \text{ logical operations } \mathcal{N} (\text{next})$ and $\mathcal{U} (\text{until})$.

$\mathcal{N} \varphi$ means: $\varphi$ holds in the next time point (state)

$\varphi \mathcal{U} \psi$ means: $\varphi$ holds until $\psi$ will be true.
Linear Kripke structures – quadruples

\[ \mathcal{M} := \langle N, \leq, \text{Next}, V \rangle, \]

where \( N \) is the set of all natural numbers, \( \leq \) is the standard order on \( N \), Next is the binary relation, where \( a \) Next \( b \) means \( b \) is the number next to \( a \), and \( V \) is a valuation of letters.

For any Kripke structure \( \mathcal{M} \), the truth values can be extended from propositions of \( S \) to arbitrary formulas as follows:

\[ \forall p \in Prop \ (\mathcal{M}, a) \models_V p \iff a \in N \land a \in V(p); \]

\[ (\mathcal{M}, a) \models_V (\varphi \land \psi) \iff (\mathcal{M}, a) \models_V \varphi \land (\mathcal{M}, a) \models_V \psi; \]

\[ (\mathcal{M}, a) \models_V \neg \varphi \iff \text{not}[(\mathcal{M}, a) \models_V \varphi]; \]
The linear temporal logic LTL is the set of all formulas which are valid in all infinite temporal linear Kripke structures.
NECESSARY TECHNIQUE:

A formula $\varphi$ is said to be in an unfolded form if

$$
\varphi = \bigvee_{1 \leq j \leq m} \left( \bigwedge_{1 \leq i \leq n} [x^t_{i}(j,i,0) \land (\mathbb{N}x_i)^t(j,i,1) \land \
\bigwedge_{1 \leq k \leq n, k \neq i} (x_i \mathbb{U} x_k)^t(j,i,k,0)] \right),
$$

where $x_s$ are some variables (letters), and $t(j,i,z), t(j,i,k,0) \in \{0, 1\}$ and, for any formula $\alpha$ above, $\alpha^0 := \alpha, \alpha^1 := \neg \alpha$. 
Let $\phi_{uf}$ be a formula in the unfolded form.

$\phi_{uf}$ is an unfolded form for a formula $\phi$ iff

(i) $\text{Var}(\phi) \subseteq \text{Var}(\phi_{uf})$;

(ii) for any world $a$ from $N$ and any valuation $V$ in $N$, if $(N, a) \models_V \phi$ then there is an extension $V_1$ of $V$ to additional variables of $\phi_{uf}$, such that $(N, a) \models_{V_1} \phi_{uf}$;

(iii) for any valuation $V_1$ in $N$, if for all worlds $a$ from $N$, $(N, a) \models_{V_1} \phi_{uf}$, then, for all worlds $a$ from $N$, $(N, a) \models_V \phi$, where $V$ is restriction of $V_1$ to variables of $\phi$.

**Theorem A.** There exists an algorithm running in (single) exponential time, which, for any given formula $\phi$, constructs its unfolded form $\phi_{uf}$. 
Lemma 1. The following holds

(i) \( \varphi \) is unifiable in LTL iff \( \varphi_{uf} \) is unifiable in LTL.

(ii) if \( \varepsilon \) is a unifier for \( \varphi \) then an extension of \( \varepsilon \) to additional variables of \( \varphi_{uf} \) is a unifier for \( \varphi_{uf} \).

(iii) if \( \varepsilon \) is a unifier for \( \varphi_{uf} \) then the restriction of this unifier to only letters \( x \) of \( \varphi \) is an unifier for \( \varphi \).
For any formula $\varphi_{uf}(= \bigvee_{i \in I} \varphi_i)$, we denote by $M(\varphi)$ the set of all disjunctive members of $\varphi_{uf}$ (i.e. $M(\varphi) := \{\varphi_i \mid i \in I\}$).

For any $\varphi_1, \varphi_2 \in M(\varphi)$, we write $\varphi_1 \ Nt \ \varphi_2$ if

(i) $N x_j$ is a conjunct member of $\varphi_1$ iff $x_j$ is a conjunct member of $\varphi_2$;

(ii) if $x_i \cup x_j$ is a conjunct member of $\varphi_1$ and $\neg x_j$ is a conjunctive member of $\varphi_1$ then $x_i$ is a conjunctive member of $\varphi_1$ and $x_i \cup x_j$ is a conjunctive member of $\varphi_2$.

(iii) If $x_i \cup x_j$ is a conjunct member of $\varphi_2$ and $x_i$ is a conjunct member of $\varphi_1$ then $x_i \cup x_j$ is a conjunct member of $\varphi_1$. 
For a logic $L$, a structure $\mathcal{M}$ with a valuation defined for a set of letters $p_1, \ldots, p_k$ is called $k$-characterizing for $L$ if the following holds. For any formula $\varphi(p_1, \ldots, p_k)$ in letters $p_1, \ldots, p_k$, $\varphi(p_1, \ldots, p_k) \in L$ iff $\mathcal{M} \models \varphi(p_1, \ldots, p_k)$.

For $LTL$, if we take all Kripke structures $N_i, i \in I$ based on the frame $N$ (of all natural numbers) with all possible valuations $V$ of propositional letters $p_1, \ldots, p_k$, and the disjoint union $M_k := \bigsqcup_{i \in I} N_i$ of all such non-isomorphic Kripke structures, we immediately get $M_k$ is $k$-characterizing for $LTL$. 
Lemma 2. Let $\varphi$ be a formula and $\varphi_{uf}(= \bigvee_{i \in I} \varphi_i)$ be its unfolded form constructed in Theorem A. Let $\varphi_{uf}$ be unifiable. Let $\varepsilon$ be a unifier for $\varphi_{uf}$ in $n$ variables. Let $M_n$ be the $n$-characterizing model for LTL with valuation $V$. Let $\chi[M_\varepsilon]$ be the set of all $\varphi_i$ from $\varphi_{uf}$, such that each $\varepsilon(\varphi_i)$ is true at some $w \in M_n$.

Then, for any $\varphi_j \in \chi[M_\varepsilon]$, there is a repetition-free sequence $\varphi_1, \ldots, \varphi_k$ of formulas from $\chi[M_\varepsilon]$, where

(i) $\varphi_i \text{ Nt } \varphi_{i+1}$, (ii) $\varphi_1 \text{ Nt } \varphi_1$, and (iii) $\varphi_k = \varphi_j$ and

(iv) there is an $\varphi_0 \in \chi[M_\varepsilon]$ such that $\varphi_0 \text{ Nt } \varphi_0$ and any $x_l \mathbf{U} x_k$ is a positive conjunctive member of $\varphi_0$ iff $x_k$ is a positive conjunctive member of $\varphi_0$. 
Proof: follows easily from Rybakov (APAL, 2008).

Let $\varphi$, and consequently (Lemma 1) $\varphi_{uf}$, be unifiable. Recall that $\varphi_{uf} := \bigvee_{i \in I} \varphi_i$.

Let $Coherent(\varphi_{uf})$ be the set of all $X \subseteq I$, such that (1) $\forall j \in X$ there is a repetition-free sequence $\varphi_1, \ldots \varphi_k$ of formulas from $\{\varphi_s \mid s \in X\}$ satisfying conditions (i), (ii) and (iii) from Lemma 2, and (2) $\forall \varphi_i \in \{\varphi_s \mid s \in X\}$, the letter $x_\varphi$ is a conjunct member of $\varphi_i$. By Lemma 2 $Coherent(\varphi_{uf})$ is non-empty.

We chose and fix a numeration of sets from $Coherent(\varphi_{uf})$: $Coherent(\varphi_{uf}) = \{X_r \mid r \leq m_C\}$ (we essentially will use it to avoid disambiguating).
Let, for any $j \in X \in \text{Coherent}(\varphi_{uf})$, $\varphi_1(j, X), \ldots, \varphi_k(j, X)$ be a fixed sequence satisfying (i), (ii) and (iii) (for $\varphi_j$) mentioned above. Let $\varphi_0$ be any fixed disjunct member of $\varphi_{uf}$ satisfying (iv) from Lemma 2 (it exists by this lemma).

For any letter $x_i$ from $\varphi_{uf}$, $\text{sign}_i := \top$, if $x_i$ positively occurs in $\varphi_0$ and $\text{sign}_i := \bot$ otherwise. For any disjunct $\varphi_k$ from $\bigvee_{i \in I} \varphi_i$ and any letter $x_i$ from $\varphi_{uf}$, $\text{sign}_i(\varphi_k) := \top$ if $x_i$ positively occurs in $\varphi_k$ and $\text{sign}_i(\varphi_k) := \bot$ otherwise. Let (recall, for any formula $\alpha$, $\Box \alpha := \neg (\top \lor \neg \alpha)$)

$$\forall X \in \text{Coherent}(\varphi_{uf}), \Gamma(X) := [\Box \bigvee_{i \in X} \varphi_i];$$

$$\Gamma_1 := [\bigvee_{X \in \text{Coherent}(\varphi_{uf})} \Gamma(X)]; \quad \Gamma_2 := \neg \Diamond \Gamma_1;$$
\[ \forall X \in Coherent(\varphi_{uf}), \forall j \in X, \]

\[ \Gamma_3(X, j) := \Diamond [\neg \Gamma_1 \land N[\Gamma(X) \land \varphi_j]]. \]

For any letter \( x_i \) from \( \varphi_{uf} \), we put:

\[ \delta(x_i) := [\Gamma_1 \land x_i] \lor [\Gamma_2 \land sign_i] \lor \]

\[ \left[ \bigvee_{X_r, j \in X_r} \left( \Gamma_3(X_r, j) \land \bigwedge_{s < r} \neg \Gamma_3(X_s, j) \land \left[ \bigvee_{\varphi_t(j, X_r)} \Gamma(i, X_r, j, t) \right] \right) \right], \]

where
$$\Gamma(i, X_r, j, t) := [\{\Gamma_3(X_r, j) \land \{(sign_i(\varphi_t(j, X_r))) \land$$

$$\mathsf{N}^{kJ-t}[\varphi_j \land \Gamma(X_r)] \lor (\bigwedge_{q \leq kj} \{-\mathsf{N}^q[\varphi_j \land \Gamma(X_r)] \land sign_i(\varphi_1(j, X_r)))\})\}].$$

**Theorem 3.** $\delta$ is an unifier for $\varphi_{uf}$ in LTL.

Proof: half of page.
Theorem 4. \( \delta \) is more general than any other unifier \( \varepsilon \) for \( \varphi_{uf} \) in LTL (i.e. is, consequently, a most general unifier (mgu) for \( \varphi \).

Proof. Let \( \varepsilon_1 \) be an unifier for \( \varphi \), and \( \varepsilon \) be its extension to the unifier of \( \varphi_{uf} \) (cf. Lemma 1). We claim that \( \varepsilon(x_i) \equiv \varepsilon(\delta(x_i)) \). Indeed, take \( M_n \) (\( n \) is the number of letters in all formulas \( \varepsilon(x_i) \)) and a \( w \in M_n \). Since \( \varepsilon \) is a unifier for \( \varphi_{uf} \) we have \( w \models \varepsilon(\Gamma(X)) \) for some \( X \) by Lemma 2. Therefore \( w \models \varepsilon(x_i) \iff w \models \varepsilon(\delta(x_i)). \)

Notice that our definition above for \( \delta \) gives the explicit construction (i.e. even not just an algorithm of construction, but immediate way of writing out the unifier) of the mgu \( \delta \) by \( \varphi_{uf} \); and Theorem 1 gives algorithm (exponential) to construct \( \varphi_{uf} \) by \( \varphi \).
A formula \( \varphi \) is projective (in LTL) if there is an unifier \( \sigma : \text{Var}(\varphi) \rightarrow \text{For}_{LTL} \) for \( \varphi \) in LTL, such that \( \Box \varphi \rightarrow \bigwedge_{x \in \text{Var}(\varphi)}(x \leftrightarrow \sigma(x)) \in LTL \). (then we call \( \sigma \) a projective unifier).

LTL does not enjoy existence of projectivity for unifiable formulas:

**Example.** Formula \( \varphi = \Box(\Box x \lor (\neg x \land \Diamond \Box x)) \) is unifiable but not projective.
Theorem. Any unifiable in $S_{4.3}$ formula is projective.

Proof. Let a formula $\varphi(x_1, \ldots, x_n)$ built out of letters $x_i$ be unifiable in $S_{4.3}$ (by the way, easy to verify by ground substitutions).

Let for any $X \subseteq \text{Sub}(\varphi)$,

$$\Psi(X) := \left[ \bigwedge_{\psi \in X} \Diamond[\psi \land \Box \varphi] \right] \land \left[ \bigwedge_{\psi \in \text{Sub}(\varphi) \setminus X} \neg \Diamond[\psi \land \Box \varphi] \right].$$

If $\Psi(X)$ is satisfiable in $S_{4.3}$ (in the sequel we denote it by $\Psi(X) \in \text{Sat}$), we may compute a rooted finite model $M$ of bounded size for $\Psi(X)$ (using decidability of $S_{4.3}$ and, say, filtration), satisfying $\Psi(X)$ at a world of the minimal cluster. Take $w$ to be a world from next (up) cluster from $M$ to maximal one, where $\Box \varphi$ is false; if $\Box \varphi$ is true at all worlds, take $w$ from the minimal cluster of $M$.

Let $\text{Char}(\Psi(X)) := \{ x_i \mid (M, w) \models x_i \}$, and, for any $x_i$, $T(\Psi(X), x_i) := \top$ if $x_i \in \text{Char}(\Psi(X))$, otherwise $T(\Psi(X), x_i) := \bot$. 
Since $\varphi$ is unifiable in $S4.3$, $\varphi$ is true at the single world model with a valuation $V$. If $V(x_i)$ is empty, we set $T(x_i) := \bot$ and we set $T(x_i) := \top$ otherwise.

For any letter $x_i$ occurring in $\varphi$ we define the following substitution:

$$
\sigma(x_i) := (\Box \varphi(x_1, ..., x_n) \land x_i) \lor \\
\left(\neg \Box \varphi \land \Diamond \Box \varphi \land \bigvee_{\Psi(X) \in Sat} [\Psi(X) \land T(\Psi(X), x_i)] \right) \lor \\
(\neg \Diamond \Box \varphi \land T(x_i)).
$$

It is immediate to see that $\sigma$ is a projective substitution for $\varphi$. In fact, $\sigma$ unifiers $\varphi$, which is easy to verify. Indeed, take any finite $S4.3$-model $M_1$ with a valuation of letters of $\varphi$.

If (1) $\Box \varphi$ is true at all worlds of $M_1$, $\sigma$ does not change the truth values of $\varphi$, so $\sigma(\varphi)$ is true at all worlds of $M_1$.

If (2) $\Diamond \Box \varphi$ is not true at the minimal cluster of $M_1$, truth values of $\sigma(x_i)$ are constant and coincides with truth values of $T(x_i)$ and hence $\sigma(\varphi)$ is true all all worlds of $M_1$. 
(3) Assume $\diamondsuit \Box \varphi$ is true at the minimal cluster of $M_1$, bit $\Box \varphi$ – not, and $C$ is the minimal cluster of $M_1$ where $\Box \varphi$ is true. At all worlds accessible from $C$, $\sigma(\varphi)$ is true as in case (1) above. If $a$ is any world from $M_1$, which is not accessible from $C$, some unique $\Psi(X)$ is true at all such $a$. Then the truth values of all $\sigma(x_i)$ coincide at all such $a$ with truth values of $x_i$ at the world $w$ in $M$. And then, easy verifiable by induction on length of $\varphi$, $\sigma(\varphi)$ has the same truth value at all such $a$ as $\varphi$ at $w$ in $M$. This means $\sigma(\varphi)$ is true at all such $a$.

Thus, $\sigma(\varphi)$ is true at all finite $S4.3$ models. Hence $\sigma$ is projective. Q.E.D.
Thus, Theorem above gives as a construction of a most general unifier for any formula $\varphi$ unifiable in $S4.3$. The construction is given in the proof explicitly, – we just write out directly the formulas $\sigma(x_i)$ giving most general unifier.

**Unification of formulas with coefficients**

Consider formulas constructed out two sorts of letters: letters $x_i$ from potentially infinite set of letters $VL$ (we call them variable letters) and letters $p_j$ from a potentially infinite set of letters $MVL$ (which we call meta-variable letters or coefficients or parameters).

Any substitution $\varepsilon$, in what follows, always maps any meta-variable to itself, i.e. $\varepsilon(p_j) = p_j$.

Why it is interesting:

(i) Solution of equations in free algebras with constants for free variables (analog of A.Tarski Problem for Free Groups);

(ii) Possibility to express one specs in terms of another

$$\varphi(p_1, \ldots, p_n) \equiv \psi(x_1, \ldots, x_k, p_1, \ldots, p_n) \in L?$$
YET:

\[ \varphi(x_1, \ldots, x_k, p_1, \ldots, p_n) \equiv \psi(x_1, \ldots, x_k, p_1, \ldots, p_n) \in L? \]

COMPARE:

\[ x \ast x = x; \text{ and } a \ast x = b; \text{ or } a \ast x = b \]

or \( x^2 + x = 0 \) and \( a \ast x^2 + b \ast x + c = 0 \)

RESULTS:

**Theorem A.** The intuitionistic logic \( \text{Int} \) has finitary unification type for formulas with coefficients, and there is an algorithm witting a finite set of best unifiers for any given unifiable formula.

**Theorem B.** The super-intuitionistic logic \( \text{KC} \) has finitary unification type for formulas with coefficients. There is an algorithm witting a finite set of best unifiers for any given unifiable formula.


**Theorem C.** The modal logics \( K4, GL, S4, Grz \), have finitary unification type for formulas with coefficients. There is an algorithm witting a finite set of best unifiers for any given unifiable formula.

ESSENCE: change base (1-st) step on the inductive procedure for extension the definable valuation: put instead of filtrated finite model the open (generated) potentially infinite subset $V(\varphi)$ of $Ch_n(L)$. 
Conclusion, Problems

Problems: Admissibility/Unification/Bases for Admissible Rules

(i) LTL - with coefficients

(ii) Para-consistent Logics (Jonsson, etc., below and around INT)

(iii) Multi-modal logics, agents’ logics (as deep as possible, $S5^3$ itself is undecidable).

(iv) Temporal logics $T_{S4}$ etc...

(v) Intransitive modal logics, e.g. $K$.

(vi) Description logics in standard and extended language.

(vii) fragments of FOL?
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