Computability on structures and topological spaces

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Introduction

Study of computability properties of continuous objects such as reals, real-valued functions and functionals is one of the fundamental areas of Computer Science motivated by applications from Engineering; where the vast majority of objects are of a continuous nature. The classical theory of computation, which works with discrete structures, is not suitable for formalisation of computations that operate on real-valued data. This theory is well established, whereas the theory of computation on continuous data is still in its infancy. This has resulted in many different nonequivalent approaches to computability of continuous objects [4, 5, 9, 10, 12, 14, 17, 21, 22, 31, 35]. One of the most promising approaches is based on the notion of definability, where continuous objects and computational processes involving these objects can be defined using finite formulas in suitable abstract structures.
Main question

How to compute on structures?

Keep in mind that the theory should correspond to the real practice of computations and involve real analytical functions like exponential one.
Let us mention some of the beneficial features of semantic approach which differ from other approaches to computability.

- It does not depend on representations of elements of structures.
- It is flexible: sometimes we can change the language of the structure to obtain appropriate computability properties.
- We can employ model theory to study computability.
- Formulas can be seen as a suitable finite representation of the relations they define.

One of the most interesting and practically important types of definability is $\Sigma$-definability. The concept of $\Sigma$-definability is closely related to the generalised computability on abstract structures [1, 11, 29, 33], in particular on the real numbers [21, 22, 33]. Notions of $\Sigma$-definable sets or relations generalise those of computable enumerable sets of natural numbers, and play a leading role in the specification theory that is used in the higher order computation theory on abstract structures.
One of the most fundamental theorems in the area is Gandy’s theorem which states that the least fixed point of any positive $\Sigma$-operator is $\Sigma$-definable. This theorem allows us to treat inductive definitions using $\Sigma$-formulas. The role of inductive definability as a basic principle of general computability is discussed in [15, 28]. It is worth noting that for finite structures the least fixed points of definable operators give an important and well studied logical characterisation of complexity classes [8, 16, 34]. For infinite structures fixed point logics are also studied e.g. [7]. The case of modal propositional logics was detaily investigated by Mardaev [27]. Gandy’s theorem was first proven for abstract structures with the equality test (see [1, 11, 35]), the most popular case was related to the superstructure $HF(M)$ over structure $M$. In many cases it is natural to consider structures in the language without equality. This is not surprising, because infinite amount of information must be checked in order to decide that two given elements are equal.
After some period of joint work M. Korovina had shown that Gandy’s theorem holds for the set of hereditarily finite sets over abstract structures without the equality test in 2003. As a useful corollary we have obtained an existence of universal $\Sigma$–predicate for the set of hereditarily finite sets over any structure in finite relational language without the equality test. This gave us new tools for investigation of properties of computability on continuous data types and any structures.
Computability within HF, but without equality

We start by introducing basic notations and definitions. Let us consider an abstract structure $A$ in a finite language $\sigma_0$ without the equality test.

In order to do any kind of computation or to develop a computability theory one has to work within a structure rich enough for information to be coded and stored. For this purpose we extend the structure $A$ by the set of hereditarily finite sets $HF(A)$.

The idea that the hereditarily finite sets over $A$ form a natural domain for computation is quite classical and is developed in detail in [1, 11].

Note that such or very similar extensions of structures with equality are used in the theory of abstract state machines [3, 2] and in query languages for hierarchic databases [6].

We are working with an arbitrary structure $A = \langle A, \sigma_0 \rangle = \langle A, \sigma_P, \neq \rangle$, where $A$ contains more than one element, $\sigma_P$ is a finite set of basic predicates.
The idea that the hereditarily finite sets over $A$ form a natural domain for computation is quite classical and is developed in detail in [1, 11] for the case when $\sigma_0$ contains equality.

We construct the set of hereditarily finite sets, $HF(A)$, as follows:

1. $HF_0(A) \equiv A$,
2. $HF_{n+1}(A) \equiv \mathcal{P}_{\omega}(HF_n(A)) \cup HF_n(A)$, where $n \in \omega$ and for every set $B$, $\mathcal{P}_{\omega}(B)$ is the set of all finite subsets of $B$.
3. $HF(A) \equiv \bigcup_{n \in \omega} HF_n(A)$.

We define $HF(A)$ as the following model:

$$HF(A) \equiv \langle HF(A), U, \sigma_0, \in \rangle \equiv \langle HF(A), \sigma \rangle,$$

where the binary predicate symbol $\in$ has the set-theoretic interpretation. Also we add predicates symbols $U$ for urelements (elements from $A$).
The natural numbers 0, 1, ... are identified with the (finite) ordinals in \( \text{HF}(A) \) i.e. \( \emptyset, \{\emptyset\}, \ldots, \) so in particular, \( n + 1 = n \cup \{n\} \) and the set \( \omega \) is a subset of \( \text{HF}(A) \). The atomic formulas include \( U(x), \neg U(x), x \neq y, x \in s, x \notin s \) where \( s \) ranges over sets, and also, for every \( Q_i \in \sigma_P \) with the arity \( n_i \), \( Q_i(x_1, \ldots, x_{n_i}) \) which has the following interpretation:

\[
\text{HF}(A) \models Q_i(x_1, \ldots, x_{n_i})
\]

if and only if

\[
A \models Q_i(x_1, \ldots, x_{n_i})
\]

and, for every \( 1 \leq j \leq n_i \), \( x_j \in A \).

The set of \( \Delta_0 \)-formulas is the closure of the set of atomic formulas under \( \land, \lor \), bounded quantifiers \( (\exists x \in y) \) and \( (\forall x \in y) \), where \( (\exists x \in y) \) \( \Psi \) means the same as \( \exists x (x \in y \land \Psi) \) and \( (\forall x \in y) \) \( \Psi \) as \( \forall x (x \in y \rightarrow \Psi) \) where \( y \) ranges over sets. The set of \( \Sigma \)-formulas is the closure of the set of \( \Delta_0 \)-formulas under \( \land, \lor \), \( (\exists x \in y) \), \( (\forall x \in y) \) and \( \exists \), where \( y \) ranges over sets.
Remark

It is worth noting that all predicates $Q_i \in \sigma_P$ and $\neq$ occur only positively in $\Sigma$-formulas. We are going to consider existential formulas with the same restriction.

For an abstract structure $\mathcal{A}$, we introduce a topology, called $\tau_\Sigma^A$, with the base consisting of the subsets defined by existential formulas. We argue that an abstract structure could be considered as a topological space with the same carrier and the topology $\tau_\Sigma^A$, and, vice versa, a given topological space could be considered as an abstract structure in an appropriate language. We prove that, for subsets of $\mathcal{A}$, $\Sigma$-definability coincides with effective openness in the topology $\tau_\Sigma^A$. These principles show that $\Sigma$-definability has a natural meaning for abstract structures without equality and can be used for effective reasoning about computability over continuous data.
### Definition

1. A relation $B \subseteq \text{HF}(A)^n$ is $\Delta_0 (\Sigma)$-definable, if there exists a $\Delta_0 (\Sigma)$-formula $\Phi(\bar{a})$ such that

$$\bar{b} \in B \iff \text{HF}(A) \models \Phi(\bar{b}).$$

2. A function $f : \text{HF}(A)^n \to \text{HF}(A)^m$ is $\Delta_0 (\Sigma)$-definable, if there exists a $\Delta_0 (\Sigma)$-formula $\Phi(\bar{c}, \bar{d})$ such that

$$f(\bar{a}) = \bar{b} \iff \text{HF}(A) \models \Phi(\bar{a}, \bar{b}).$$

Let $S(\text{HF}(A))$ denote the set of all sets in $\text{HF}(A)$ and $S'(\text{HF}(A))$ denote the set of all nonempty sets in $\text{HF}(A)$. 
Lemma

1. The predicates $S(x) \iff x$ is a set,
   $\emptyset(x) \iff x$ is the empty set, $\neg\emptyset(x) \iff x$ is not the empty set
   and $n \in \omega$ are $\Delta_0$-definable.

2. The predicate $S'(x) \iff \text{“}x$ is a nonempty set\text{”}$ is $\Sigma$-definable.

3. The following predicates are $\Delta_0$-definable: $x = y$, $x = y \cap z$, $x = y \cup z$, $x = \langle y, z \rangle$, $x = y \setminus z$ where all variables $x, y, z$ range over sets.

4. A function $f : \omega^n \rightarrow \omega^m$ is computable if and only if it is $\Sigma$-definable.

5. Let $\text{Fun}(g)$ means that $g$ is a finite function. Then the predicate $\text{Fun}(g)$ is $\Delta_0$-definable.

6. If $\text{HF}(A) \models \text{Fun}(g)$ then the domain of $g$, denoted by $\text{dom}(g)$, is $\Delta_0$-definable.

7. The set $\{ \gamma : \omega \rightarrow S'(\text{HF}(M)) \mid \gamma$ is a finite function $\}$ is $\Sigma$-definable.
Let us recall Gandy’s Theorem for $\mathbf{HF}(A)$ which will be essentially used in all proofs of the main results. Let $\Phi(a_1, \ldots, a_n, P)$ be a $\Sigma$-formula, where $P$ occurs positively in $\Phi$ and the arity of $\Phi$ is equal to $n$. We think of $\Phi$ as defining an effective operator $\Gamma : \mathcal{P}(\mathbf{HF}(A)^n) \rightarrow \mathcal{P}(\mathbf{HF}(A)^n)$ given by

$$\Gamma(Q) = \{ \bar{a} | (\mathbf{HF}(A), Q) \models \Phi(\bar{a}, P) \}.$$ 

Since the predicate symbol $P$ occurs only positively we have that the corresponding operator $\Gamma$ is monotone and continuous with respect to Scott topology on $\mathcal{P}(\mathbf{HF}(A)^n)$. By monotonicity, the operator $\Gamma$ has a least (w.r.t. inclusion) fixed point which can be described as follows. We start from the empty set and apply operator $\Gamma$ until we reach the fixed point:

$$\Gamma^0 = \emptyset, \quad \Gamma^{n+1} = \Gamma(\Gamma^n), \quad \Gamma^\gamma = \bigcup_{n<\gamma} \Gamma^n,$$

where $\gamma$ is a limit ordinal.
One can easily check that the sets $\Gamma^n$ form an increasing chain of sets: $\Gamma^0 \subseteq \Gamma^1 \subseteq \ldots$. By set-theoretical reasons, there exists the least ordinal $\gamma$ such that $\Gamma(\Gamma^\gamma) = \Gamma^\gamma$. This $\Gamma^\gamma$ is the least fixed point of the given operator $\Gamma$.

**Theorem (Gandy’s Theorem for $HF(A)$)**

Let $\Gamma : \mathcal{P}(HF(A)^n) \rightarrow \mathcal{P}(HF(A)^n)$ be an effective operator. Then the least fixed-point of $\Gamma$ is $\Sigma$-definable and the least ordinal such that $\Gamma(\Gamma^\gamma) = \Gamma^\gamma$ is less or equal to $\omega$.

**Definition**

A relation $B \subseteq A^n$ is called $\Sigma$-inductive if it is the least-fixed point of an effective operator.

**Corollary**

Every $\Sigma$-inductive relation is $\Sigma$-definable.
Universal $\Sigma$-predicate

In order to obtain a result on the existence of a universal $\Sigma$-predicate we first prove $\Sigma$-definability of the predicate $TR^\forall$. We fix a standard effective Gödel numbering of formulas of the language $\sigma$ by finite ordinals which are elements of $HF(\emptyset)$. Let $\lceil \Phi \rceil$ denote the codes of a formula $\Phi$. It is worth noting that the type of an expression is effectively recognisable by its code. We also can obtain effectively from the codes of expressions the codes of their subexpressions and vice versa. Since equality is $\Delta_0$-definable in $HF(\emptyset)$, we can use the well-known characterisation which states that all effective procedures over ordinals are $\Sigma$-definable. Thus, for example, the following predicates

\[
\text{Code}_0(n, j) \iff n = \lceil U(x_j) \rceil,
\]
\[
\text{Code}_i(n, j_1, \ldots, j_{n_i}) \iff n = \lceil Q_i(x_{j_1}, \ldots, x_{j_{n_i}}) \rceil,
\]
\[
\text{Code}_\wedge(n, i, j) \iff n = \lceil \Phi \wedge \Psi \rceil \wedge i = \lceil \Phi \rceil \wedge j = \lceil \Psi \rceil
\]

are $\Sigma$-definable. Hence, in $\Sigma$-formulas we can use such predicates.
Theorem

For every $A$ of the cardinality $> 1$ there exists a $\Sigma$-definable set $TR^\forall \subseteq \omega \times [\omega \rightarrow S'(HF(A))]$ with the following properties.

1. If $n$ is the Gödel number of a $\Sigma$-formula $\Phi$ and $\gamma : \omega \rightarrow S'(HF(A))$ is a finite function defined by an assignment function $f : FV(\Phi) \rightarrow HF(A)$ as $\gamma(i) = \{f(x_i)\}$ for all $i : x_i \in \text{dom}(f)$ then $< n, \gamma > \in TR^\forall$.

2. If $< n, \gamma > \in TR^\forall$ then $n$ is the Gödel number of a $\Sigma$-formula $\Phi$ and $\gamma : \omega \rightarrow S'(HF(A))$ is a finite function such that, for every assignment function $f : FV(\Phi) \rightarrow HF(A)$ with the property $f(x_i) \in \gamma(i)$, it holds $HF(A) \models \Phi[f]$.

Theorem

For every $n \in \omega$ there exists a $\Sigma$-formula $Univ_{n+1}(m, x_0, \ldots, x_n)$ such that for any $\Sigma$-formula $\Phi(x_0, \ldots, x_n)$ it holds

$$HF(A) \models \Phi(r_0, \ldots, r_n) \iff Univ_{n+1}([\Phi], r_0, \ldots, r_n).$$
Theorem
A set $B \subseteq A^n$ is $\Sigma$-definable if and only if there exists an effective sequence of existential formulas in the language $\sigma_0$, $\{\varphi_s(\bar{x})\}_{s \in \omega}$, such that

$$(x_1, \ldots, x_n) \in B \iff \mathcal{A} \models \bigvee_{s \in \omega} \varphi_s(x_1, \ldots, x_n).$$

From now we are going to discuss how by a given structure to construct a topological space with the same carrier and vice versa. The case of reals is rather trivial, one should use the language of strictly ordered rings, changing functional symbols in atomic subformulas on corresponding epigraph and ordinate set.

In the case of $C[0, 1]$ we show how to pick up an appropriate finite language for the structure of $C[0, 1]$ in such a way that $\tau_{\Sigma}^C$ coincides with the topology induced by the norm. We consider the structure $\mathcal{C} = (C[0, 1], P_1, \ldots, P_{10}, \neq)$ where the predicates $P_1, \ldots, P_{10}$ have the following meanings for every $f, g \in C[0, 1]$. 
The case of $C[0, 1]$

The first group formalises relations between infimum and sumpemum of two functions.

$$\text{HF}(C) \models P_1(f, g) \iff \sup f < \sup g;$$  
$$\text{HF}(C) \models P_2(f, g) \iff \sup f < \inf g;$$  
$$\text{HF}(C) \models P_3(f, g) \iff \sup f > \inf g;$$  
$$\text{HF}(C) \models P_4(f, g) \iff \inf f > \inf g.$$

The second group formalises properties of operations on $C[0, 1]$.

$$\text{HF}(C) \models P_5(f, g, h) \iff f + g < h;$$  
$$\text{HF}(C) \models P_6(f, g, h) \iff f \cdot g < h;$$  
$$\text{HF}(C) \models P_7(f, g, h) \iff f + g > h;$$  
$$\text{HF}(C) \models P_8(f, g, h) \iff f \cdot g > h.$$

The third group formalises relations between functions $f$ and $\lambda x.x$.

$$\text{HF}(C) \models P_9(f) \iff f > \lambda x.x;$$  
$$\text{HF}(C) \models P_{10}(f) \iff f < \lambda x.x.$$
Theorem

For the structure $\mathcal{C} = (C[0, 1], P_1, \ldots, P_{10}, \neq)$, the following equivalence holds:

$$\tau^C_{\Sigma} = \tau_{|||}.$$

Proof $\subseteq$. It is easy to see that $\{\bar{x} | \text{HF}(\mathcal{C}) \models P_i(\bar{x})\} \in \tau_{|||}$ for every $1 \leq i \leq 10$. Since $(\mathcal{C}, d_{|||})$ is a metric space, a projection of an open set is again open. So, $\{\bar{x} | \text{HF}(\mathcal{C}) \models Q(\bar{x}), \ Q \text{ is a } \exists \text{-formula}\} \in \tau_{|||}$. By induction, $\tau^C_{\Sigma} \subseteq \tau_{|||}$.

$\supseteq$. First, recall that a base of the topology $\tau_{|||}$ consists of the following sets: $\{f | |f - p| < \epsilon\}$, where $p \in Q[t], \epsilon \in Q^+$. Since the set $Q[t]$ is dense in $C[0, 1]$, it is sufficient to show that $f > p$ and $f < p$ are $\exists$-definable. This claim follows from the following equivalences.
\[ f > 0 \iff f + f > f; \]
\[ f > 1 \iff \exists g \,(f \cdot g > f \land f > 0); \]
\[ f < 0 \iff f + f < f; \]
\[ f < 1 \iff \exists g \,(f < 0 \lor g > 0 \land f \cdot g < g); \]
\[ f > x^2 \iff \exists g \,(g > \lambda x.x \land f > g \cdot g); \]
\[ f < x^2 \iff \exists g \,(g < \lambda x.x \land f < g \cdot g); \]
\[ f > \frac{x}{n} \iff \exists g \,(g > \lambda x.x \land (f + \cdots + f) > g); \]
\[ f < \frac{x}{n} \iff \exists g \,(g < \lambda x.x \land (f + \cdots + f) < g). \]

So, the set \( \{ f \mid \| f - p_i \| < \epsilon, \ p_i \mbox{ is a polynom with rational coefficients, } \epsilon \in \mathbb{Q} \} \) is \( \exists \)-definable. Therefore \( \tau^C_{\Sigma} \supseteq \tau_{\| \|}. \)

**Remark**

Let us note that, for \( \mathcal{R}, \mathcal{R}^n \), the topologies \( \tau^A_{\Sigma} \) coincide with standard topologies, and, for the suitable structure \( \mathcal{A} \) with the carrier \( C(\mathcal{R}) \), the topology \( \tau^A_{\Sigma} \) coincides with the natural compact-open topology.
The following proposition shows that the topology $\tau^A_\Sigma$ is natural with respect to $\Sigma$-definability.

**Corollary**

*A subset of $C[0, 1]$ is $\Sigma$-definable if and only if it is effectively open.*
Suppose now \((X, \tau, \nu)\) is a topological space, where \(X\) is a non-empty set, \(\tau^* \subseteq 2^X\) is a base of the topology \(\tau\) and \(\nu : \omega \rightarrow \tau^*\) is a numbering. Below we use the notation \(\nu i\) for \(\nu(i)\).

We define the structure for this topological space as follows: \(\mathcal{X} = (X, \sigma_P, \neq)\) with \(\sigma_P = \{P_i\}_{i \in \omega}\), where every \(P_i(x)\) has the following interpretation: \(\mathcal{X} \models P_i(x)\) if and only if \(x \in \nu i\).

In order to study relation between \(\tau\) and \(\tau_X^{\Sigma}\), we consider a suitable fragment \(L_{\omega_1\omega}^{\Sigma}\) of the constructive infinitary language \(L_{\omega_1\omega}\) described below. We propose the inductive definition of the language \(L_{\omega_1\omega}^{\Sigma}\) as follows. Let \(Var\) be a fixed finite set of variables.
The set $L_{Var}$ of formulas over $Var$ includes the set of atomic formulas in the language $\sigma_0$ all of the variables of which belong to $Var$. In addition,

- if $\{\Phi_i|i \in A\}$ is an indexed family of atomic formulas in the language $\sigma_0$ all of the variables of which belong to $Var$ then $\bigvee_{i \in A} \Phi_i$ is a formula of $L_{Var}$.

- If $\Phi$ and $\Psi$ are formulas of $L_{Var}$ then $\Phi \land \Psi$ is a formula of $L_{Var}$.

- If $\Phi$ is a formula of $L_{Var}$ and $x \in Var$ then $\exists x \Phi$ is a formula of $L_{Var}$.

The language $L^{\Sigma}_{\omega_1 \omega}$ is the union of all $L_{Var}$ for all finite sets $Var$ of variables which range over $X$. 
Theorem (KK08)

For every $T_1$-space $\mathcal{X}$ it holds $\tau = \tau^\mathcal{X}_\Sigma$.

Proof $\subseteq$). By the definition of the topology $\tau^\mathcal{X}_\Sigma$, every $P_i$ is open. Hence, if $B \in \tau$ then $B \in \tau^\mathcal{X}_\Sigma$. So $\tau \subseteq \tau^\mathcal{X}_\Sigma$.

$\supseteq$). In order to show this inclusion we have to prove that every set definable by existential formulas is open in the topology $\tau$. The proof is based on a quantifier elimination for $L^{\Sigma}_{\omega_1\omega}$. Here we consider the formulas $\top$ (true), $\bot$ (false) as atomic sentences. It is sufficient to show how to construct a quantifier free formula $\Psi$ for the formula $\Phi(\bar{x}) \equiv \exists y \left( \land_{1 \leq j \leq n} y \neq x_j \land P_m(y) \right)$. The formula $\Phi(\bar{x})$ is equivalent to the disjunction $\Phi_{>n}(\bar{x}) \lor \lor_{1 \leq i \leq n} \Phi_i(\bar{x})$, where the subformulas are the following.
\[ \Phi_{>n}(\bar{x}) \iff \exists^n > y P_m(y); \]
\[ \Phi_i(\bar{x}) \iff \exists^=i y P_m(y) \land \left( \bigvee_{J \subseteq \{1,\ldots,n\}, |J| = n-i+1} \left( \bigwedge_{j \in J} \neg P_m(x_j) \right) \right) \]
for 1 \leq i \leq n.

If |\{x| \mathcal{X} \models P_m(x)\}| > n then \( \Psi \models \top \). Suppose \( \mathcal{X} \models \Phi_i(\bar{x}) \) for some 1 \leq i \leq n. Since we are working with \( T_1 \)-space, \( \{x| \mathcal{X} \models P_m(x)\} \) is closed and open at the same time. Hence, \( \mathcal{X} \models \neg P_m(x) \leftrightarrow \bigvee_{k \in K} P_k(x) \) for some set \( K \subseteq \omega \). So every subformula \( \bigwedge_{j \in J} \neg P_m(x_j) \) is representable as follows

\[ \Psi_J(x) \iff \bigvee_{\langle k_1,\ldots,k_{|J|}\rangle \in K^{|J|}} \left( \bigwedge_{r=1}^{|J|} P_{k_r}(x_{j_r}) \right) \text{ for } J = \{j_1,\ldots,j_{n-i-1}\}. \]
In this case the formula $\Psi$ is equivalent on $\mathcal{X}$ to a formula
$$\bigvee_{J \subseteq \{1, \ldots, n\}, |J|=n-i+1} \Psi_J,$$
i.e., to no more then countable disjunction of quantifier free formulas in the language $\sigma_0$. By induction, every formula of the language $L_{\omega_1 \omega}^{\Sigma}$ is equivalent to a quantifier free formula.

Assume $B \in \tau_{\mathcal{X}}^\mathcal{X}$. By quantifier elimination, $B$ is definable by no more then countable disjunction of quantifier free formulas of one variable in the language $\sigma_0$ with positive occurrences of basic predicates. So, $B \in \tau$. Theorem is proven.

The observations above give a view to a given structure as to a topological space and vice versa. Later we will show how it can be used for studying computability over structures.
The Uniformity Principle for $\Sigma$-definability over the Reals

In the previous section we have proposed the basic principles for $\Sigma$-definability which work for every abstract structure. In special cases we can use extra principles for effective reasoning about computability. In order to illustrate this we recall the Uniformity Principle for $\Sigma$-definability over the reals [26].

Now we are working with $\mathbb{R} = (\mathbb{R}, \sigma_P)$, where $\sigma_P = \{M^*_E, M^*_H, P^+_E, P^+_H, <\}$, where $M^*_E, M^*_H$ are interpreted as an open epigraph and an open hypograph of multiplication respectively, and $P^+_E, P^+_H$ are interpreted as an open epigraph and an open hypograph of addition respectively.

**Theorem (KK07, [26])**

For every $\Sigma$-formula $\varphi$ there exists a $\Sigma$-formula $\psi$ such that

$\mathsf{HF}(\mathbb{R}) \models \forall x \in [a, b] \varphi(x, y_1, \ldots, y_n)$ iff $\mathsf{HF}(\mathbb{R}) \models \psi(a, b, y_1, \ldots, y_n)$,

where free variables range over $\mathbb{R}$.

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Computability on structures and topological spaces
The Uniformity principle allows using of rational numbers, polynomials, computable real numbers, and computable real-valued functions in \( \Sigma \)-formulas without enlarging the class of \( \Sigma \)-definable sets. In other words we can extend the language of \( \Sigma \)-formulas by computable functions, e.g., \( \cos, \sin, \exp \), and Uniformity Principle allows eliminate them later. Moreover, in reasoning, we can use both universal and existential quantifiers bounded by computable compact sets and later eliminate them.
Effectively enumerable topological spaces

Let \((X, \tau, \nu)\) be a topological space, where \(X\) is a non-empty set, \(\tau^* \subseteq 2^X\) is a base of the topology \(\tau\) and \(\nu : \omega \rightarrow \tau^*\) is a numbering.

**Definition**

A topological space \((X, \tau, \nu)\) is **effectively enumerable** if the following conditions hold.

1. There exists a computable function \(g : \omega \times \omega \times \omega \rightarrow \omega\) such that

   \[
   \nu i \cap \nu j = \bigcup_{n \in \omega} \nu g(i, j, n).
   \]

2. The set \(\{i | \nu i \neq \emptyset\}\) is computably enumerable.
Definition

An effectively enumerable topological space \((X, \tau, \nu)\) is strongly effectively enumerable if there exists a computable function \(h: \omega \times \omega \to \omega\) such that

\[
X \setminus \text{cl}(\nu i) = \bigcup_{j \in \omega} \nu h(i, j).
\]

Theorem (KK08)

For every structure \(\mathcal{X}\) the following properties hold.

1. The topological space \((X, \tau^X_\Sigma)\) is effectively enumerable if and only if \(\text{Th}_\exists(X)\) is computable enumerable.

2. If \(\text{Th}_\exists(X)\) is decidable then \((X, \tau^X_\Sigma)\) is strongly effectively enumerable.
Theorem (KK08)

If $\mathcal{M} = (M, \nu, B, d)$ is a computable metric space then $(M, \tau_d, \nu^*)$ is a strongly effectively enumerable topological space.

Now we compare effectively enumerable topological spaces with $\omega$-continuous domains.

Lemma

For an $\omega$-continuous domain $\mathcal{D} = (D, \{b_i\}_{i \in \omega}, \sqsubseteq)$ the following properties hold.

1. If $a \ll x$ then there exists $n \in \omega$ such that $a \ll b_n \ll x$.

2. $(D, \tau, \nu)$ is a $T_0$-space, where $\tau$ is generated by the base $\tau^* = \{U_{b_n}\} \cup \{\emptyset\}$ and the numbering $\nu : \omega \to \tau^*$ is defined as follows: $\nu 0 = \emptyset$, $\nu k = U_{b_{k-1}} = \{x | b_{k-1} \ll x\}$, $k > 0$.

Definition

An $\omega$-continuous domain $\mathcal{D} = (D, \{b_i\}_{i \in \omega}, \sqsubseteq)$ is called weakly effective if $\{< n, m > | b_n \ll b_m\}$ is computably enumerable.
**Theorem (KK08)**

*Every weakly effective \(\omega\)-continuous domain is an effectively enumerable topological space.*

**Proof.** Let \(\mathcal{D} = (D, \{b_i\}_{i \in \omega}, \sqsubseteq)\) be a weakly effective \(\omega\)-continuous domain. The topology \(\tau\) is generated by the base \(\tau^* = \{U_{b_n}| n \in \omega\} \cup \{\emptyset\}\), where \(U_a = \{x| a \ll x\}\), and \(\nu: \omega \rightarrow \tau^*\) is the standard numbering. We show now that

\[
U_{b_n} \cap U_{b_m} = \bigcup_{b_s \gg b_n, b_m} U_{b_s}.
\]

If \(x \in U_{b_s}\) for \(b_s \gg b_n, b_m\) then, by definition, \(x \gg b_s\). So, \(x \in U_{b_n} \cap U_{b_m}\). Suppose \(x \in U_{b_n} \cap U_{b_m}\). By definition, \(x \gg b_n\) and \(x \gg b_m\). So there exist \(s_1\) and \(s_2\) such that \(x \gg b_{s_1} \gg b_n\) and \(x \gg b_{s_2} \gg b_m\).

Since \(\{b_i| b_i \ll x\}\) is directed, there exists \(b_s \gg b_n, b_m\) such that \(x \in U_{b_s}\). By weak effectiveness, the set \(\{n|U_{b_n} \neq \emptyset\}\) is computably enumerable.
Definition
A function $\Gamma_e : \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)$ is called **enumeration operator** if
\[
\Gamma_e(A) = B \iff B = \{j | \exists i \ c(i, j) \in W_e, D_i \subseteq A\},
\]
where $W_e$ is the $e$-th c.e. set, and $D_i$ is the $i$-th finite set.

Definition
Let $\mathcal{X} = (X, \tau, \alpha)$ be e.e. topological space and $\mathcal{Y} = (Y, \lambda, \beta)$ be e.e. $T_0$-space. A partial function $F : X \rightarrow Y$ is called **computable** if there exists an enumeration operator $\Gamma_e : \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)$ such that, $\forall x \in X$

1. $x \in \text{dom}(F) \implies \Gamma_e(\{i \in \omega | x \in \alpha i\}) = \{j \in \omega | F(x) \in \beta j\}$.
2. $x \not\in \text{dom}(F) \implies \forall y \in Y \ \cap_{j \in \omega} \{\beta j | j \in \Gamma_e(A_x)\} \neq \cap_{j \in \omega} \{\beta j | j \in B_y\}$, where
   \[A_x = \{i \in \omega | x \in \alpha i\}, \quad B_y = \{j \in \omega | y \in \beta j\}.\]
**Theorem**

Let $\mathcal{X} = (X, \tau, \alpha)$ be an effectively enumerable topological space and $\mathcal{Y} = (Y, \lambda, \beta)$ be an effectively enumerable $T_0$-space. For a total function $F : X \to Y$ FAE.

1. $F$ is computable;
2. There exists a computable function $h : \omega \times \omega \to \omega$ such that $F^{-1}(\beta j) = \bigcup_{i \in \omega} \alpha h(i, j)$.

**Corollary**

Let $\mathcal{X} = (X, \tau, \alpha)$ be an effectively enumerable topological space and $\mathcal{Y} = (Y, \lambda, \beta)$ be an effectively enumerable $T_0$-space. A total function $F : X \to Y$ is computable if and only if $F$ is effectively continuous.
In this section we compare our notion of computability with the notions of computability proposed in [32, ?, 10, ?]. We start with the classical computability, i.e., computability on the natural numbers \( \mathbb{N} \). We consider the structure \( \mathbb{N} = (\mathbb{N}, P_1, P_2, <) \), where \( P_1 \) and \( P_2 \) have the following meanings:

\[
P_1(x) \iff x = 0; \quad P_2(x, y) \iff x = y + 1.
\]

It is easy to see that \((\mathbb{N}, \tau^N_\Sigma)\) is an effectively enumerable topological space and the topology \( \tau^N_\Sigma \) coincides with the discrete topology.

**Theorem**

A total function \( f : \mathbb{N} \to \mathbb{N} \) is recursive if and only if \( f \) is computable.
Another example concerns functions $f : (\mathbb{N}, \tau^\Sigma) \rightarrow (2^\mathbb{N}, \tau_S)$, where $2^\mathbb{N}$ is identified with $P(\mathbb{N})$ and $\tau_S$ is standard (Scott) topology.

**Theorem**

A total function $f : \mathbb{N} \rightarrow 2^\mathbb{N}$ is computable if and only if the corresponding numbering $n \rightarrow f(n)$ is computable.

Now we compare our notion of computability with $(\rho^c_X, \rho^c_Y)$-computability for $F : X \rightarrow Y$, where $X$ and $Y$ are computable metric spaces and $\rho^c_X$, $\rho^c_Y$ are Cauchy-representations.

**Theorem (KK08)**

Let $\mathcal{X} = (X, \lambda, B_X, d_X)$ and $\mathcal{Y} = (Y, \beta, B_Y, d_Y)$ be computable metric spaces and $(X, \tau_X, \alpha^*)$, $(Y, \tau_Y, \beta^*)$ be corresponding effectively enumerable topological spaces. For every total function $F : X \rightarrow Y$ FAE.

1. $F$ is $(\rho^c_X, \rho^c_Y)$-computable;
2. $F$ is computable as a function from one effectively enumerable topological space to another.
Proof.

It is easy to see that there exists an effective procedure which given a Cauchy-representation \( \rho^c_X(z) \) produces \( A_z = \{ i | z \in \alpha^*i \} \) as well as there exists an effective procedure which given \( A_z \) produces a Cauchy-representation \( \rho^c_X(z) \) for every \( z \in X \). So, both computabilities coincide, details are routine.

**Theorem (KK08)**

Let \( D_1 = (D_1, \{ b_i \}_{i \in \omega}, \sqsubseteq) \) and \( D_2 = (D_2, \{ c_i \}_{i \in \omega}, \sqsubseteq) \) be weakly effective \( \omega \)-continuous domains and \( (D_1, \tau, \alpha) \), \( (D_2, \lambda, \beta) \) be corresponding them effectively enumerable topological spaces. For every total function \( F : D_1 \rightarrow D_2 \) FAE.

1. \( F \) is computable in the domain theoretical sense (dt-computable);
2. \( F \) is computable as a function from one effectively enumerable topological space to another.
Proof

1 → 2). Assume $F : D_1 \to D_2$ is $dt$-computable. First we show that $y \ll F(x)$ if and only if there exists $n \in \omega$ such that $y \ll F(b_n)$, where $b_n \in \{b_k| b_k \ll x\}$. Indeed, if $y \ll F(x)$, by Lemma 19, there exists $c_k$ such that $y \ll c_k \ll F(x)$. Since $F(x) = \sup\{F(b_n)| b_n \ll x\}$, there exists $n \in \omega$ such that $y \ll c_k \leq F(b_n)$, i.e., $y \ll F(b_n)$. The ”only if”-direction follows from monotonicity of $F$. So, for $A_x = \{i| x \in \alpha i\} = \{k| b_k-1 \ll x\}$, we have the following equivalence

$$i \in \{i| c_{i-1} \ll F(x)\} = B_{F(x)} \leftrightarrow \exists k (k \in \in A_x c_{i-1} \ll F(b_{k-1})).$$

Therefore, for an enumeration operator $\Gamma_e$, we have $\Gamma_e(A_x) = B_{F(x)}$. So, $F$ is computable.

2 → 1). Suppose $F : D_1 \to D_2$ is computable as a function from one effectively enumerable topological space to another. By definition, there exists an enumeration operator $\Gamma_e$ such that $\Gamma_e(\{k+1|x \in \alpha i\}) = \{i+1| F(x) \in \beta i\}$. 
By the constructions of $\alpha$ and $\beta$ (c.f. Theorem 21), 
$\Gamma_e(\{k|b_k \ll x\}) = \{i|c_i \ll F(x)\}$. It means that

$$c_i \ll F(x) \leftrightarrow \exists j\forall k \in D_j (c(j, i) \in W_e \land b_k \ll x).$$

So, the set $\{c(m, n)|c_m \ll F(b_n)\}$ is recursively enumerable, i.e., $F$ is $dt$-computable. Theorem is proven.

Now we consider the special case of real-valued functions. Let $\mathcal{I}_R$ be the interval domain

$\mathcal{I}_R = \{[a, b] \subseteq \mathbb{R} | a, b \in \mathbb{R}, a \leq b\} \cup \{\bot\}$, and $\omega$-continuous domain $\mathcal{I}_R^* = \mathcal{I}_R \cup \{\top\}$, where $x \ll \top$ for every $x \in \mathcal{I}_R$. 
Theorem

Let \((X, \tau, \alpha)\) be a effectively enumerable topological space and \(f : X \to \mathbb{R}\) be a partial function. Than FAE.

1. The function \(f\) is computable.
2. There exists a total computable function \(F : X \to \mathcal{I}_\mathbb{R}^*\) s.t.
   2.1 \(F(x) = f(x)\) for \(x \in \text{dom}(f)\)
   2.2 \(F(x) \neq \{y\}\) for \(x \notin \text{dom}(f), y \in \mathbb{R}\)
3. There exist effective open sets \(U, V \subseteq X \times \mathbb{R}\) such that
   3.1 for all \(x \in X\) and all \(b, c \in \mathbb{R}\) such that \(b \leq c\),
   \[
   \text{if } (x, c) \in U \text{ then } (x, b) \in U, \text{ if } (x, b) \in V \text{ then } (x, c) \in V;
   \]
   3.2 if \(x \in \text{dom} f\) then
   \[
   U(x) \Leftrightarrow \{y|(x, y) \in U\} = \{y|y < f(x)\} \text{ and } V(x) \Leftrightarrow \{y|(x, y) \in V\} = \{y|y > f(x)\};
   \]
   3.3 if \(x \notin \text{dom} f\) then the \(\mathbb{R} \setminus (U(x) \cup V(x))\) is not a singleton.
Why structures?

THESIS
Any 'natural' countably based $T_2$ space $(X, \tau)$ admits some open interpretation of symbols from some finite relational language in such a way that $\tau = \tau_X^\Sigma$ for corresponding structure $\mathcal{X}$.
Spaces as structures

Definition
We say that effectively enumerable space \((X, \tau, \nu)\) can be **structured** if it admits some effectively open interpretation of symbols from some finite relational language in such a way that \(\tau = \tau_X^\Sigma\) for corresponding structure \(X'\) and, additionally, sets \(\nu(n)\) are uniformly \(\Sigma\)—definable.

Definition
We say that effectively enumerable space \((X, \tau, \nu)\) can be **semi-structured** if the disjoint union \(M\) of \(X\) and \(\mathbb{N}\) admits some effectively open interpretation of symbols from some finite relational language (within \(X\)) and effectively open interpretation of new binary symbol \(P\) as subset of \(X \times \mathbb{N}\) together with standard interpretation of symbols \(P_1, P_2\) on \(\mathbb{N}\) and symbols \(X, N\) in such a way, that \(\tau\) coincides with the restriction of \(\tau_M^\Sigma\) for corresponding structure \(M\) on \(X\) and, additionally, sets \(\nu(n)\) are uniformly \(\Sigma\)—definable.
So, spaces $\mathbb{R}$, $C[0, 1]$ and many others (see previous and following slides) can be structured. Actually, any effectively enumerable space $(X, \tau, \nu)$ can be semi-structured, we just put

$$P(x, n) \leftrightarrow x \in \nu(n).$$

On the other hand, the short example of ”nonstructured” space is usual $\mathbb{N}$ with base topology, consisting of co-finite subsets. Indeed, any open predicate $P$ of arity $k$ on it should contain some power $(\omega \setminus m)^k$ for some fixed $m \in \omega$. By monotonicity and induction, any $\Sigma$-definable relation should contain corresponding power of $\omega \setminus m$, so, for finite language there are only finite number of $\Sigma$—definable subsets of $\mathbb{N}$, contrary to infinite topology base.

The partial confirmation of the thesis above is related to computable metric spaces, it is the subject of the following slide.
Let $\mathcal{M} = (M, \mathcal{B}, d)$ be a computable metric space. We define the following list of open predicates on $M$, using notation $D(y, z, v) \iff d(y, z) - d(y, v)$.

$$D_1(x, y, u, v) \iff d(x, y) < d(u, v),$$
$$D_2(y, z, v) \iff D(y, z, v) < 1,$$
$$D_3(y, z, t, w, s) \iff 2D(y, z, v) < D(t, w, s),$$
$$R_n(x, y, z, v) \iff 2d(x, b_n) < D(y, z, v) \land (\forall i < n) 2d(x, b_i) > D(y, z, v),$$
$$S(x, y, z, v, a, b, c, d) \iff \bigvee_{n} (R_n(x, y, z, v) \land R_{n+1}(a, b, c, d)).$$

**Theorem**

Relations $R_n$, $n \in \omega$ are uniformly $\Sigma$—definable in the structure $\mathcal{M} = (M, D_1, D_2, D_3, R_0, S)$. Similarly, balls $B(b_r, a)$, $a \in \mathbb{Q}^+$ are uniformly $\Sigma$—definable. So, $d$ is computable function on $\mathcal{M}$, the space $\mathcal{M}$ is structured.
Definition
Given two effectively enumerable presentations \((X, \tau, \alpha), (X, \tau, \beta)\) of the same space \((X, \tau)\), we use notation \(\alpha \leq \beta\) if there exists a computable function \(g : \omega \times \omega \to \omega\) such that
\[
(\forall n \in \omega) \alpha n = \bigcup_{k \in \omega} \beta g(k, n).
\]
We use notation \(\alpha \equiv \beta\) for \(\alpha \leq \beta \land \alpha \geq \beta\).

Definition
Given two effectively enumerable presentations \((X, \tau, \alpha), (X, \tau, \beta)\) of the same space \((X, \tau)\), we use notation \(\alpha \leq^h \beta\) if there exists a computable homeomorphism \(\varphi : (X, \tau, \alpha) \to (X, \tau, \beta)\) with condition \(\varphi \circ \alpha \leq \beta\)
Given a computable list $\bar{E}$ of effectively open relations $(E_n \mid n \in \omega)$ on effectively enumerable space $(X, \tau, \nu)$, this space is called computably categorical relatively to $\bar{E}$, if for any two effectively enumerable presentations $(X, \tau, \alpha), (X, \tau, \beta)$ of it, which preserve computability of the list $\bar{E}$, there exists a computable homeomorphism $\varphi : (X, \tau, \alpha) \rightarrow (X, \tau, \beta)$ with conditions:

1. for any $i \in \omega \varphi(E_i) = E_i$ (\bar{E}-invariance property);
2. $\varphi \circ \alpha \leq \beta$.

In the case of (computable) metric space $(M, d)$ there is standard way to form canonical list $\bar{E}$ consisting of binary relations of kind $E_r^- = \{(x, y) \mid d(x, y) < r\}$ and of kind $E_r^+ = \{(x, y) \mid d(x, y) > r\}, r \in \mathbb{Q}^+$. It can be noted, that mentioned relative categoricity is similar to standard notion of categoricity for computable metric spaces, especially in complete case.
Notions of c.e. space and of computable function between spaces can be define in $\text{qcb}$–spaces in natural way. It is similar to domain theory with effectively enumerable $\text{qb}$–spaces instead of domains. The advantage is cartesian closedness of the category of $\text{qcb}$–spaces, the disadvantage is the nonconstructive style of the construction of exponential in it. Are there any relations to structures in this way? It is an open question.
THANK YOU FOR ATTENTION!
J. Barwise.  
*Admissible sets and Structures.*  

A. Blass and Y. Gurevich.  
Background, reserve and Gandy machines.  

A. Blass, Y. Gurevich, and S. Shelah.  
Choiceless polynomial time.  

L. Blum, F. Cucker, M. Shub, and S. Smale.  
*Complexity and Real Computation.*  
Springer Verlag, Berlin, 1996.

V. Brattka and P. Hertling.  
Topological properties of real number representations.  
E. Dahlhaus and J. A. Makowsky.
Query languages for hierarchic databases.

A. Davar and Y. Gurevich.
Fixed-point logics.

H. Ebbinghaus and J. Flum.
*Finite Model Theory.*

A. Edalat and M. Escardo.
Integration in Real PCF.

A. Edalat and A. Lieutie.
Domain theory and differential calculus (function of one variable.)
Yu. L. Ershov.
*Definability and computability.*

H. Friedman and K. Ko.
Computational complexity of real functions.

R. Gandy.
Inductive definitions.

A. Grzegorczyk.
On the definitions of computable real continuous function.
P. G. Hinman.
Recursion on abstract structure.

N. Immerman.
*Descriptive Complexity.*

Ulrich Kohlenbach.
Proof theory and computational analysis.

M. Korovina.
Computational aspects of $\Sigma$-definability over the real numbers without the equality test.

M. Korovina.
Fixed points on the reals numbers without the equality test.
Margarita V. Korovina.
Gandy’s theorem for abstract structures without the equality test.

M. Korovina and O. Kudinov.
Characteristic properties of majorant-computability over the reals.

M. Korovina and O. Kudinov.
Some properties of majorant-computability.
In M. Arslanov and S. Lempp, editors, Recursion Theory and Complexity”, Proceedings of the Kazan-97 Workshop, July


Korovina M.V. and Kudinov O.V., Basic principles of $\Sigma-$definability and abstract computability. Schriften zur Theoretischen Informatik, Bericht Nr 08-01, Universität Siegen, 2008.


Y. N. Moschovakis.
Abstract first order computability I, II.


Y. N. Moschovakis.


Y. N. Moschovakis.


M. B. Pour-El and J. I. Richards.


H. Rogers, Jr.,

J. V. Tucker and J. I. Zucker.
Computable functions and semicomputable sets on many-sorted algebras.

M. Vardi.
The complexity of relational query languages.

Klaus Weihrauch.
Computable Analysis.