Computable numberings in the Ershov hierarchy
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Numberings as a tool of and as a subject to study

Gödel and Turing codings in logic

Definition. Numbering of a countable set $S$ is a surjective mapping $\nu : \omega \to S$.

$\eta_\nu$ stands for the numerical equivalence of $\nu$: $x \eta_\nu y \leftrightarrow \nu(x) = \nu(y)$.

General theory of numberings is the theory of equivalences with respect to effective transformations over them.
**Definition.** Let $S_0 \subseteq S$ and let $\nu : \omega \to S_0$ and $\nu : \omega \to S$ be two numberings. Numbering $\nu_0$ is reducible to numbering $\nu$ ($\nu_0 \leq \nu$) if $\nu_0(x) = \nu(f(x))$ for some computable function $f$ and all $x \in \omega$.

If $\nu_0 \leq \nu$ we say also that $\nu_0$ is computable relative to $\nu$.

Numberings $\nu_0, \nu_1$ are called equivalent ($\nu_0 \equiv \nu_1$) if $\nu_0 \leq \nu_1$ and $\nu_1 \leq \nu_0$. 
Approach of Goncharov-Sorbi (1997) to the notion of computable numbering.

Let $C$ be a family of constructive objects described by ’expressions’ (programs) of some language $\mathcal{L}$. Suppose that the language $\mathcal{L}$ is equipped with Gödel numbering $\gamma$ for ’expressions’ of $\mathcal{L}$. Let $i$ be an interpretation of the expressions from $\mathcal{L}$, i.e. let $i : \mathcal{L} \to C$ be any partial mapping. Then a numbering $\alpha : \omega \mapsto \mathcal{A} \subseteq C$ is called computable numbering (relative to an interpretation $i$) if there exists a computable function $f$ s.t. for every $n \in \omega$, $\alpha(n) = i(\gamma_f(n))$. 
Example: computable numberings in the arithmetical hierarchy.

Let $C$ be the class $\Sigma^0_{n+1}$, $\mathcal{L}$ be the set of all $\Sigma_{n+1}$-formulas of arithmetics of a free variable $x$, and let $i(n) = \{ a \mid \mathcal{N} \models \gamma_n(a) \}$. Then a numbering $\alpha$ of a family $\mathcal{A} \subseteq \Sigma^0_{n+1}$ is called $\Sigma^0_{n+1}$-computable if there exists a computable function $f$ s.t., for every $m \in \mathbb{N}$,

$$\alpha(m) = \{ x \mid \mathcal{N} \models \gamma_f(m)(x) \}$$
Theorem (Goncharov and Sorbi)

A numbering $\alpha$ of a family $A$ of $\Sigma^0_{n+1}$ sets is $\Sigma^0_{n+1}$-computable $\iff \{ (m, x) \mid x \in \alpha(m) \} \in \Sigma^0_{n+1}$. 

Theorem (Badaev and Goncharov, 2008)

A numbering $\alpha$ of a family $A$ of hyperarithmetical $\Sigma^0_a$ sets is $\Sigma^0_a$-computable $\iff \{ (m, x) \mid x \in \alpha(m) \} \in \Sigma^0_a$. Here $a$ is a computable ordinal.
Computable numberings in hierarchies

Theorem (Goncharov and Sorbi)

A numbering $\alpha$ of a family $\mathcal{A}$ of $\Sigma^0_{n+1}$ sets is $\Sigma^0_{n+1}$-computable
$\iff \{(m, x) \mid x \in \alpha(m)\} \in \Sigma^0_{n+1}$.

Straightforward modification yields a criterion:
A numbering $\alpha$ of a family $\mathcal{A}$ of $\Sigma^{-1}_{n+1}$ sets is $\Sigma^{-1}_{n+1}$-computable
$\iff \{(m, x) \mid x \in \alpha(m)\} \in \Sigma^{-1}_{n+1}$.
Computable numberings in hierarchies

**Theorem (Goncharov and Sorbi)**

A numbering $\alpha$ of a family $A$ of $\Sigma_{n+1}^0$ sets is $\Sigma_{n+1}^0$-computable $\iff \{ (m, x) \mid x \in \alpha(m) \} \in \Sigma_{n+1}^0$.

Straightforward modification yields a criterion:

A numbering $\alpha$ of a family $A$ of $\Sigma_{n+1}^{-1}$ sets is $\Sigma_{n+1}^{-1}$-computable $\iff \{ (m, x) \mid x \in \alpha(m) \} \in \Sigma_{n+1}^{-1}$.

**Theorem (Badaev and Goncharov, 2008)**

A numbering $\alpha$ of a family $A$ of hyperarithmetic $\Sigma_{\alpha}^0$ sets is $\Sigma_{\alpha}^0$-computable $\iff \{ (m, x) \mid x \in \alpha(m) \} \in \Sigma_{\alpha}^0$. Here $\alpha$ is a computable ordinal.
Criterion for computable numberings in the Ershov hierarchy

**Definition.** Let $a$ be a notation of a computable ordinal. Then a numbering $\alpha$ of a family $\mathcal{A}$ of $\Sigma_a^{-1}$ sets is $\Sigma_a^{-1}$-computable if
\[ \{(m, x) \mid x \in \alpha(m)\} \in \Sigma_a^{-1}. \]

**Theorem (Ospichev 2010).** A set of numbers $A$ is $\Sigma_a^{-1}$-set $\iff$ there are a computable function $f(z, t)$ and a partial computable function $\gamma(z, t)$ such that, for all $z$,

1. $A(z) = \lim_t f(z, t)$, with $f(z, 0) = 0$;
2. $\gamma(z, t) \downarrow \Rightarrow \gamma(z, t + 1) \downarrow$, and $\gamma(z, t + 1) \leq_0 \gamma(z, t) <_0 a$;
3. $f(z, t + 1) \neq f(z, t) \Rightarrow \gamma(z, t + 1) \neq \gamma(z, t)$.

The partial function $\gamma$ is called the *mind–change function* for $A$ relatively to $f$. 

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*Note:*

The theorem is based on the concept of computable ordinals and the Ershov hierarchy, which are fundamental in computability theory. The criterion provides a method to determine if a set of numbers is computable in a certain complexity class under a given numbering scheme.
Rogers-Ershov semilattices

For $\mathcal{A} \subseteq \Sigma^i_a$, let $\text{Com}^i_a(\mathcal{A})$ be the set of all $\Sigma^i_a$-computable numberings of $\mathcal{A}$. Here $a$ is an ordinal notation or $a \in \omega$ if $i = -1$ (hierarchy of Ershov), and $a \in \omega$ if $i = 0$ (arithmetical hierarchy).

$\langle \text{Com}^i_{n+1}(\mathcal{A}), \leq \rangle$ is the pre-ordered set.

**Definition.** Rogers-Ershov semilattice $\mathcal{R}^i_a(\mathcal{A})$ is the quotient structure $\langle \text{Com}^i_a(\mathcal{A})/\equiv, \leq \rangle$ w.r.t. equivalence of numberings.

Join in $\mathcal{R}^i_a(\mathcal{A})$ is easily induced by the direct sum of numberings:

$$(\alpha \oplus \beta)(2n) = \alpha(n), \quad (\alpha \oplus \beta)(2n + 1) = \beta(n)$$
A bit history on Rogers-Ershov semilattices

Rogers (1958): idea to study numbering up to equivalence, acceptable numberings
Mal’tsev (1961): discrete and effectively discrete families,
Ershov (1968): upper semilattices of computable numberings and open problems on them.

Problem of Ershov (1967–1968). Find a structural criterion for a family of c.e. sets to have one-element upper semilattice of computable numberings.
This problem is still open for infinite families but it seems me to be hopeless. Nevertheless it stimulated a lot of research in the field of computable numberings, especially in the fSU.
Cardinality of Rogers-Ershov semilattices in the classical case

**Theorem (Khutoretskii, 1971).** For every family $\mathcal{A}$ of c.e. sets, the cardinality of the Rogers-Ershov semilattice $\mathcal{R}_{1}^{0}(\mathcal{A})$ can be infinite or equal to 0 or 1 only.

**Theorem (Ershov)** If $\mathcal{A} \subset \Sigma_{1}^{0}$ is a finite family then the Rogers-Ershov semilattice $\mathcal{R}_{1}^{0}(\mathcal{A})$ is infinite $\iff$ $\mathcal{A}$ contains a pair of embedded sets.

**Theorem (Badaev,Goncharov,1998)** There exists an infinite family $\mathcal{A}$ of c.e. sets s.t. $\mathcal{A}$ contains the least set under inclusion and $|\mathcal{R}_{1}^{0}(\mathcal{A})| = 1$. 
Cardinality of Rogers-Ershov semilattices for the families in the arithmetical hierarchy

Generalized problem of Ershov. What is a possible cardinality of an upper semilattice of computable numberings?

Theorem (Goncharov and Sorbi, 1997) For every $n$ and every computable family $\mathcal{A} \subseteq \Sigma_{n+2}^0$, $\mathcal{R}_{n+2}^0(\mathcal{A})$ is infinite $\iff$ $\mathcal{A}$ contains at least 2 sets.
Cardinality of Rogers-Ershov semilattices for the families in the difference hierarchy

It is easy to find families in the Ershov hierarchy with infinite semilattice of computable numberings since $\mathcal{R}^{-1}_a(A)$ is isomorphic to an ideal of $\mathcal{R}^{-1}_b(A)$ if $a \prec b$.

**Theorem (Badaev, Talasbaeva, 2006)** There exists a family $\mathcal{A} = \{A \subset B\} \subseteq \Sigma_2^{-1}$ s.t. $|\mathcal{R}_2^{-1}(\mathcal{A})| = 1$.

**Main open problem.** Does there exist a family of sets in the Ershov hierarchy with non-trivial finite Rogers-Ershov semilattice?
On cardinality of Rogers-Ershov semilattices of the families of two embedded sets

Theorem (Badaev, Manat, Sorbi, in preparation) For every notation $a$ of a successor ordinal, there exists a family $\mathcal{A} = \{A \subset B\} \subseteq \Sigma_a^{-1}$ s.t. $|R_a^{-1}(\mathcal{A})| = 1$.

Theorem (Badaev, Manat, Sorbi, in preparation) If $A, B \in \Sigma_a^{-1}$, $A \subset B$, and $|a|_O$ is a limit ordinal s.t. there exists a partial computable function $\psi$ such that for every $b_0, b_1 < O a$,

$$\psi(b_0, b_1) \downarrow < O a \& |b_0|_O + O |b_1|_O = |\psi(b_0, b_1)|_O,$$

then $R_a^{-1}(\{A, B\})$ is infinite.
This theorem is true for the notations of $\omega$ and $\omega^2$. 
Witnesses of the extremal elements in the Rogers-Ershov semilattice

Let $\mathcal{A} \subseteq \Sigma^i_a$ and let $\alpha$ be any numbering from $\text{Com}_{\alpha}^i(\mathcal{A})$.

$\alpha$ is called a principal numbering of $\mathcal{A}$ if it induces the greatest element (if any) in the Rogers-Ershov semilattice $\mathcal{R}_a^i(\mathcal{A})$.

$\alpha$ is called a least numbering of $\mathcal{A}$ if it induces the least element (if any) in $\mathcal{R}_a^i(\mathcal{A})$.

$\alpha$ is called a minimal numbering of $\mathcal{A}$ if it induces a minimal element (if any) in $\mathcal{R}_a^i(\mathcal{A})$. 
Principal numberings: general facts and ideas

What does it mean that a numbering $\alpha$ is ”principal” (Maltsev), ”acceptable” (Rogers), ”covering” (Uspensky)?

**Theorem**

1. Each of the considered classes $\Sigma^i_a$ has a principal numbering $\pi^i_a$.
2. A numbering $\alpha$ of a family $A \subseteq \Sigma^i_a$ is computable $\iff$ $\alpha$ is computable relative to (i.e. reducible to) $\pi^i_a$.

Remind the approach of Goncharov-Sorbi:
Explanation in terms of the Goncharov-Sorbi approach

Let $\mathcal{C}$ be a family of constructive objects described by 'expressions' (programs) of some language $\mathcal{L}$. Suppose that the language $\mathcal{L}$ is equipped with Gödel numbering $\gamma$ for 'expressions' of $\mathcal{L}$.

Let $i$ be an interpretation of the expressions from $\mathcal{L}$, i.e. let $i : \mathcal{L} \rightarrow \mathcal{C}$ be any partial mapping.

Then a numbering $\alpha : \omega \mapsto A \subseteq \mathcal{C}$ is called computable numbering (relative to an interpretation $i$) if there exists a computable function $f$ s.t. for every $n \in \omega$, $\alpha(n) = i(\gamma f(n))$. 
Principal numberings of finite families in the arithmetical hierarchy

**Theorem (Lachlan, 1964)** Every finite family $\mathcal{A} \subset \Sigma_1^0$ has a principal numbering.

**Theorem (Badaev, Goncharov, Sorbi, 2003)** For every $n$, if $\mathcal{A} \subset \Sigma_{n+2}^0$ is a finite family then $\mathcal{A}$ has a principal numbering $\iff \mathcal{A}$ has the least set under inclusion.
Finite families with principal numberings in the Ershov hierarchy

Theorem (Abeshev, Badaev, 2009) For every $n$, every finite family which consists of finite extensions of any set $A \in \Sigma_{n+2}^{-1}$ has a principal numbering.

Theorem (Abeshev, Lempp, in preparation) If there are c.e sets $A_0, A_1, B_0, B_1$ and $A = A_0 \setminus A_1$ and $B = B_0 \setminus B_1$ such that

$$\forall x \ (x \in A_0 \Rightarrow x \notin A_1 \text{ or } x \notin B),$$
$$\forall x \ (x \in B_0 \Rightarrow x \notin B_1 \text{ or } x \notin A),$$

then the family $\{A, B\}$ has a principal numbering.
Finite families without principal numberings in the Ershov hierarchy

Theorem (Abeshev, Lempp, in preparation) There is a family which consists of disjoint two non-empty $\Sigma^1_{-12}$ sets and has no principal numberings.

Theorem (Badaev, in preparation) For every $n$, there exists a two-element family $\mathcal{A} \subset \Sigma^{-1}_{n+2}$ which has no principal numberings. (The sets in this family have non-empty intersection)
Minimal numberings in the classical case

**Theorem (Ershov)** Every finite family $\mathcal{A} \subset \Sigma^0_1$ has a least numbering.

There are a lot of computable families of c.e. sets with minimal numberings. Mostly, these families have up to equivalence either one or infinitely many minimal numberings.

**Theorem (Vjugin, 1971, Badaev, 1991)** There exist computable families of c.e. sets without minimal numberings.

**Open question (Ershov, the mid of 1960th)** What is a possible number of minimal numberings of a family of c.e. set?

**Open question (Goncharov, the mid of 1980th)** Is there any computable family of c.e. sets which has exactly one minimal but non-least numbering?
Theorem (Badaev, Goncharov, 2003) Every infinite family $\mathcal{A} \subset \Sigma^0_{n+2}$ has infinitely many computable minimal numberings.

Theorem (Abeshev, Badaev, Manat, in preparation) For every notation $a$ of a successor ordinal, there exists $\Sigma^1_a$-computable family without minimal numberings.
Minimal numberings of special types

Numbering $\alpha$ is called Friedberg (decidable, positive) numbering if the numerical equivalence $\eta_\alpha$ is identical (computable, computably enumerable).

For finite families, decidable $\iff$ positive $\iff$ minimal $\iff$ least. In what follows, we will consider the infinite families only. In this case, every decidable numbering is equivalent to some Friedberg numbering, and decidable $\Rightarrow$ positive $\Rightarrow$ minimal. In general, these arrows are not invertible.

**Theorem (Friedberg, Khutoretsky, Goncharov, Sorbi, Lempp, Solomon, Ospichev, and others)** Each of classes $\Sigma^i_a$ has infinitely many Friedberg numberings.

**Theorem (Khutoretsky, Ershov, Talasbaeva, Manat, Sorbi)** Each of classes $\Sigma^0_1$ and $\Sigma^{-1}_a$ has infinitely many positive non-decidable numberings.
Theorem (Goncharov, 1980) For every $n$, there exists a family of c.e. sets which has exactly $n$ Friedberg numberings.

Theorem (Badaev, 1994) If a family of c.e. sets has a Friedberg non-least numbering then it has infinitely many positive numberings.

Theorem (Badaev, Lempp, Kastermans, in progress) For every $n$, there exists a family of d.c.e. sets whose Rogers-Ershov semilattice has exactly $n$ minimal elements and these elements are induced by Friedberg numberings.
Families with unique Friedberg but non-least numberings

Theorem (Goncharov, 1988) There exists a family of c.e. sets with exactly one Friedberg numbering which is not the least.

Theorem (Badaev, Manat, Sorbi, in progress) For every notation $a$ of a successor ordinal, there exists $\Sigma_a^{-1}$-computable family with exactly one Friedberg numbering which is not the least.
Families without Friedberg numberings

**Theorem (Manat, Sorbi, 2011)** For every ordinal notation $a$, $a > \omega 1$, there exists $\Sigma_{a}^{-1}$-computable family which has no Friedberg numberings but has positive numberings.

**Theorem (Ospichev, 2011)** For every ordinal notations $a, b$ with $|b|_\omega = |a|_\omega + 1$, there exists a family of $\Sigma_{a}^{-1}$ sets which has no $\Sigma_{a}^{-1}$-computable Friedberg numberings but has a $\Sigma_{b}^{-1}$-computable Friedberg numbering.
Thank you for your attention!