Hypergraphs of Prime Models of Small Theories

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Let $\mathcal{M}$ be an infinite model of a small (i.e., with countably many pure types) theory $T$. Denote by $H(\mathcal{M}^{eq})$ the set of universes of all prime submodels $\mathcal{M}_a \preceq \mathcal{M}^{eq}$ over elements $a \in \mathcal{M}^{eq}$. Consider a kernel function

$$\ker_{\mathcal{M}} : H(\mathcal{M}^{eq}) \to \mathcal{P}(\mathcal{M}^{eq}),$$

acting by the following rule:

$$\ker_{\mathcal{M}}(\mathcal{M}_a) := \{ b \in \mathcal{M}_a \mid \mathcal{M}_b \neq \mathcal{M}_a \}.$$

The sets $\ker_{\mathcal{M}}(\mathcal{M}_a)$ are called kernels of the models $\mathcal{M}_a$ (w.r.t. $\mathcal{M}$).

Consider for the model $\mathcal{M}^{eq}$ all possible formulas $\varphi(x, y)$, for which $\vdash \varphi(x, y) \rightarrow \neg(x \approx y)$ and there exists an element $a \in \mathcal{M}^{eq}$ such, that the formula $\varphi(a, y)$ is principal. Define for each such formula $\varphi(x, y)$ the binary relation $R_\varphi := \{(a, b) \mid \mathcal{M}^{eq} \models \varphi(a, b)\}$. If $(a, b) \in R_\varphi$ then $(a, b)$ is called a $\varphi$-arc. If $\varphi(a, y)$ is a principal formula, the $\varphi$-arc $(a, b)$ is principal. If in addition $\varphi(x, b)$ is principal, the set $[a, b] := \{(a, b), (b, a)\}$ is a principal $\varphi$-edge. The principal arcs $(a, b)$, for which pairs $(b, a)$ are not principal arcs, are called to be irreversible. The model $\mathcal{M}^{eq}$ with the universe $\mathcal{M}^{eq}$ and all possible binary relations $R_\varphi$ is called a principal graph of $\mathcal{M}^{eq}$.

For the given model $\mathcal{M}$ the 4-tuple

$${\mathcal{H}}(\mathcal{M}) := (\mathcal{M}^{eq}, H(\mathcal{M}^{eq}), \ker_{\mathcal{M}}, \Sigma(\neg \mathcal{M}^{eq}))$$

(where $\Sigma(\neg \mathcal{M}^{eq})$ is the set of all binary relations of principal graph $\neg \mathcal{M}^{eq}$) is called a hypergraph of prime models with a kernel function and a principal binary structure, or a HPKB-hypergraph.

Considering $\mathcal{H}(\mathcal{M})$ we observe, that kernels $\ker_{\mathcal{M}}(\mathcal{M}_a)$ of prime models $\mathcal{M}_a$ over realizations $a$ of a fixed 1-type $p(x)$ form maximal connected graphs on the set $\mathcal{M}^{eq}$. These graphs correspond to connected components $C$ of undirected graphs with colored edges formed by principal edges. Restrictions of components $C$ to the set of realizations of $p(x)$ form graphs with colored edges, and having a saturated model $\mathcal{M}$ we call these graphs to be kernel graphs over the type $p$.

We get the following theorems.

**Theorem 1** ($p$-decomposition theorem). 1. The structure of set of realizations of any complete 1-type $p(x)$ over $\emptyset$ in a HPKB-hypergraph $\mathcal{H}(\mathcal{M})$ is formed

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by pairwise disjoint kernel graphs over \( p(x) \) as well as by a partial order \( \leq \) on the set of these kernel graphs such that \( \leq \) is defined by irreversible principal arcs.

2. The kernel graph on the set of realizations of type \( p(x) \) in the model \( \mathcal{M}^{eq} \), corresponding the countable saturated model \( \mathcal{M} \), is unique, if the structure of kernel graph, restricted to \( p(\mathcal{M}^{eq}) \), has finitely many 2-types and \( p(x) \) is a principal type. Otherwise there are infinitely many such kernel graphs. The kernel graphs over \( p(x) \) are pairwise isomorphic.

3. An equality of the partial order \( \leq \) to the identical relation is equivalent to an absence of irreversible principal arcs, connecting realizations of type \( p(x) \), or, that the same, to the symmetry of the relation of semiisolation \( SI_p \) on the set of realizations of \( p(x) \). If \( \leq \) is not identical, every kernel graph belongs to an infinite \( \leq \)-sequence.

**Theorem 2** (decomposition theorem). 1. The structure of set of realizations of any complete 1-type \( p(x) \) over \( \emptyset \) in the HPKB-hypergraph \( \mathcal{H}(\mathcal{M}) \) is formed by pairwise disjoint principal connected components (defined by principal edges) as well as by a partial order \( \leq \) on the set of these connected components such that \( \leq \) is defined by irreversible principal arcs.

2. Principal connected components in the model \( \mathcal{M}^{eq} \), corresponding the countable saturated model \( \mathcal{M} \), are unique, if the structure \( \mathcal{M} \) \( \omega \)-categorical. Otherwise there are infinitely many principal components. The principal components, containing realizations of the same type, are pairwise isomorphic.

3. An equality of the partial order \( \leq \) to the identical relation is equivalent to an absence of irreversible principal arcs, or, that the same, to the \( \omega \)-categoricity of the given theory. If \( \leq \) is not identical, then every principal connected component belongs to an infinite \( \leq \)-sequence. The set of all principal connected component is directed downwards by the partial order \( \leq \).

Using decomposition theorems we explain the roles of key definable relations in models of Ehrenfeucht theories [1–2] like the clarification [3] of the key role of ordered trees and constants for theories having three countable models and infinite \( dcl(\emptyset) \).

**References**


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