On (2,3)-generated groups

Maxim Vsemirnov

Steklov Institute of Mathematics at St. Petersburg

Group Theory Conference in honor of V. D. Mazurov
Novosibirsk, July 20, 2013

Outline

Outline

1. Definitions and motivations
2. (2,3)-generated finite (simple) groups
3. (2,3)-generated classical groups over $\mathbb{Z}$

(2,3)-generated groups

Definition

A (2,3)-generated group is a group generated by an involution and an element of order 3.

Definition

An (m, n)-generated group is a group generated by two elements of order m and n, respectively.

Why is the (2,3)-generation problem interesting?

Why do we look at $(2,3)$?

The modular group $\mathrm{PSL}_2(\mathbb{Z})$ is isomorphic to the free product of two cyclic groups, C_2 and C_3.

$$\mathrm{PSL}_2(\mathbb{Z}) = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\rangle \simeq C_2 * C_3.$$

Thus, apart from $\{1\}$, C_2, and C_3, all quotients of $\mathrm{PSL}_2(\mathbb{Z})$ are exactly the $(2,3)$-generated groups.

Comparison with $\mathrm{PSL}_n(\mathbb{Z})$, $n \geq 3$.

Remark

The normal subgroup structure of $\mathrm{PSL}_2(\mathbb{Z})$ differs dramatically from the normal subgroup structure of $\mathrm{PSL}_n(\mathbb{Z})$, $n \geq 3$.

Namely, for $n \geq 3$, any subgroup of finite index in $\mathrm{PSL}_n(\mathbb{Z})$ is a so-called congruence subgroup, i.e., contains the kernel of $\mathrm{PSL}_n(\mathbb{Z}) \mapsto \mathrm{PSL}_n(\mathbb{Z}/m\mathbb{Z})$ for some m.

In contrast, $\mathrm{PSL}_2(\mathbb{Z})$ contains many noncongruence normal subgroups (even of finite index).

Outline

Two different approaches

There are two different groups of methods in this area:

- constructive, i.e., when the corresponding generators are given explicitly;
- non-constructive, e.g., probabilistic, when only existence theorems are known (usually require a good knowledge of the characters and maximal subgroups).

Constructive methods can be also applied to infinite groups.

Known results

1. For any m, $\mathrm{PSL}_2(\mathbb{Z}/m\mathbb{Z})$ is (2,3)-generated (trivial)
2. A_n, $n \geq 4$, are $(2,3)$-generated except A_6, A_7 , and A_8 (Miller, 1901).
3. Sporadic groups are (2,3)-generated except M_{11}, M_{22}, M_{23}, and McL (Woldar, 1989)
4. ${}^2B_2(2^{2k+1})$ are not (2,3)-generated (trivial, since they do not contain elements of order 3)
5. Other exceptional Lie groups are $(2,3)$-generated (Malle, 1990, 1995, Malle and Lübeck, 1999)

Classical groups and the (2,3)-generation

Negative results for certain small groups:
$\mathrm{PSL}_2(9) \simeq \mathrm{Sp}_4(2)' \simeq A_6$, $\mathrm{PSL}_4(2) \simeq A_8$, $\mathrm{PSL}_3(4)$, $\mathrm{PSU}_3(9)$.

For n large enough, $\mathrm{SL}_n(q)$ are (2,3)-generated;
Tamburini, J. Wilson (1994–1995), $n \geq 14$;
Di Martino, Vavilov (1994–1996) for $n \geq 5$, $q \neq 3$, $q \neq 2^k$.

$\mathrm{PSp}_4(p^k)$ are (2,3)-generated if $p \neq 2, 3$ (Di Martino and Cazzola, 1993).

$\mathrm{PSp}_4(p^k)$ are not (2,3)-generated if $p = 2, 3$ (Liebeck and Shalev, 1996).

Almost all classical groups are $(2,3)$-generated (Liebeck and Shalev, 1996).

Probabilistic methods

Theorem (Liebeck, Shalev, 1996)

Let G run through some infinite set of finite classical groups, $G \neq \mathrm{PSp}_4(p^k)$. Then

$$\lim_{|G|\to\infty} \mathrm{Prob}(x^2 = y^3 = 1 \text{ and } G = \langle x, y\rangle) = 1.$$

Moreover, the result remains true

- if we fix the field and let the rank tend to infinity;
- if we fix the type and let the size of the field tend to infinity.

New examples of non (2,3)-generated groups

Theorem (V., 2011)

$\mathrm{PSU}_5(4)$ is not $(2,3)$-generated.

$|\mathrm{PSU}_5(4)| = 13,685,760 = 2^{10} \cdot 3^5 \cdot 5 \cdot 11$

A sketch of the proof.

1. $\dim \ker(x-1) = 3$ and $y \sim \mathrm{diag}(1, \omega, \omega, \omega^{-1}, \omega^{-1})$, $\omega^2 + \omega + 1 = 0$.

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 & c \\ 1 & 0 & 0 & 0 & -c \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 1 & 0 & -d \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 & 0 & a & 0 & b \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix},$$

for some a, b, c, d.

A sketch of the proof (cont.)

2. If $\det \begin{pmatrix} 3+a & ac+bd-b & a+b \\ -1 & b+cd+c & c-a-1 \\ -1 & 1-c+d^2+d & 1+d \end{pmatrix} = 0,$ then $\langle x, y \rangle$ has a 1-dimensional invariant space.

3. If $\langle x, y \rangle$ preserves a hermitian form then
$a = -d - d^\sigma - 1,$
$b = -c + c^\sigma + d + d^\sigma + d^2 + dd^\sigma - 1.$

4. There are 16 pairs of parameters (c, d).
For four of them, the group is defined over $\mathbb{F}_2$.
For ten of them, $\det(\cdots) = 0$.
For the remaining two, setting $z = yx$ we have

$$z^{11} = x^2 = (zx)^3 = (z^4xz^6x)^2 = 1,$$

a well-known presentation of $\mathrm{PSL}_2(11)$.

New examples of non (2,3)-generated groups

Theorem (Pellegrini, Tamburini Bellani, V., 2012)

$\mathrm{PSU}_4(9)$ is not $(2,3)$-generated.

Theorem (V., 2012)

$\Omega_8^+(2)$, $\mathrm{P}\Omega_8^+(3)$ are not $(2,3)$-generated.

$$|\mathrm{PSU}_4(9)| = 3,265,920 = 2^7 \cdot 3^6 \cdot 5 \cdot 7$$
$$|\Omega_8^+(2)| = 174,182,400 = 2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$$
$$|\mathrm{P}\Omega_8^+(3)| = 4,952,179,814,400 = 2^{12} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$$

Non (2,3)-generated finite simple groups

1. $\mathrm{PSp}_4(2^k)$
2. $\mathrm{PSp}_4(3^k)$, in particular $\mathrm{PSp}_4(3) \simeq \mathrm{PSU}_4(4)$
3. ${}^2B_2(2^{2k+1})$
4. $A_6 \simeq \mathrm{PSL}_2(9) \simeq \mathrm{Sp}_4(2)'$, A_7, $A_8 \simeq \mathrm{PSL}_4(2)$
5. $\mathrm{PSL}_3(4)$, $\mathrm{PSU}_3(9) \simeq G_2(2)'$
6. M_{11}, M_{22}, M_{23}, McL
7. $\mathrm{PSU}_5(4)$
8. $\mathrm{PSU}_4(9)$
9. $\Omega_8^+(2)$, $\mathrm{P}\Omega_8^+(3)$
10. ?

I strongly believe that the list is complete.

Outline

The main Theorem

Theorem

The groups $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{GL}_n(\mathbb{Z})$ are $(2, 3)$-generated precisely when $n \geq 5$

An overview of results

- $\mathrm{SL}_2(\mathbb{Z})$ is not $(2,3)$-generated as it contains no non-central involution.
- $\mathrm{SL}_4(\mathbb{Z})$ and $\mathrm{GL}_4(\mathbb{Z})$ are not $(2,3)$-generated as $\mathrm{SL}_4(2) = \mathrm{GL}_4(2) \simeq A_8$ is not (Miller, 1901)
- $\mathrm{SL}_3(\mathbb{Z})$ and $\mathrm{GL}_3(\mathbb{Z})$ are not $(2,3)$-generated (Nuzhin, 2001, Tamburini, Zucca, 2001).
- $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{GL}_n(\mathbb{Z})$ are $(2,3)$-generated for $n \geq 14$ (Tamburini, et al. 1994–1995, 2009)
- For $\mathrm{SL}_5(\mathbb{Z})$, $\mathrm{GL}_5(\mathbb{Z})$ and $\mathrm{SL}_6(\mathbb{Z})$ there are at most finitely many conjugacy classes of $(2,3)$-generators (Luzgarev, Pevzner, 2003, Vsemirnov, 2006).
- The groups $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{GL}_n(\mathbb{Z})$, $n = 5, \ldots, 13$ are $(2,3)$-generated (Vsemirnov, 2007–2009).

An idea of the proof

Two difficult problems:

- to guess the shape of $(2,3)$-generators;
- to show that they actually generate $\mathrm{SL}_n(\mathbb{Z})$.

The main idea: show that $\langle x, y \rangle$ contains some generating set of $\mathrm{SL}_n(\mathbb{Z})$.

For instance, one can show that $\langle x, y \rangle$ contains elementary transvections $t_{ij}(\alpha) = I + \alpha e_{ij}$, $i \neq j$.

One rather complicated example

$$x = \begin{pmatrix} -3 & 0 & 4 & 0 & 4 \\ -1 & -1 & 4 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 \\ 2 & 0 & -2 & -1 & -6 \\ -1 & 0 & 1 & 0 & 2 \end{pmatrix}, \quad y = \begin{pmatrix} 2 & -2 & 0 & 1 & 1 \\ 3 & -2 & -1 & 1 & -2 \\ 2 & -1 & -1 & 1 & 0 \\ -1 & 2 & -2 & -1 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{aligned} h_1 &= (yx)^3(y^2x)^3yxy^2x, \\ h_2 &= h_1^{-4} = t_{52}(2), \\ h_3 &= yxy^2xyxy^2, \\ h_4 &= yxyxyxy^2xyxyxy^2, \\ &\dots \end{aligned}$$

$$h_{51} = h_{47}h_{48}h_{49}h_{50}h_2^6h_{20}^{-1}h_{21}^{-13}h_{29}^{-15}h_{12}^{15}h_{36}^{-8} = t_{53}(1)$$

Two special cases

Let $M = Mat_n(\mathbb{Q})$. If $\langle x, y\rangle$ is absolutely irreducible then

$$\dim C_M(x) + \dim C_M(y) + \dim C_M(xy) \leq n^2 + 2.$$

Further analysis depends on whether

$$\dim C_M(x) + \dim C_M(y) + \dim C_M(xy) < n^2 + 2$$

or

$$\dim C_M(x) + \dim C_M(y) + \dim C_M(xy) = n^2 + 2.$$

In the latter case it is possible to classify all $(2,3)$-generating pairs of $\mathrm{SL}_n(\mathbb{Z})$ up to conjugation. This happens precisely when $n = 5$ and $n = 6$.

$\mathrm{SL}_5(\mathbb{Z})$

Theorem (V., 2007)

Any $(2,3)$-generating pair of $\mathrm{SL}_5(\mathbb{Z})$ is conjugate in $\mathrm{GL}_5(\mathbb{Z})$ to one of the pairs $-X$, Y, and any $(2,3)$-generating pair of $\mathrm{GL}_5(\mathbb{Z})$ is conjugate to one of the pairs X, Y, where

$$X = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 & 0 & 0 & a_1 \\ -1 & -1 & 0 & 0 & a_2 \\ 0 & 0 & 0 & 1 & a_3 \\ 0 & 0 & -1 & -1 & a_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and (a_1, a_2, a_3, a_4) is one of the sets

$$\begin{array}{rl} (1,-1,-2,-2), & (0,-1,-2,-2), \\ (-1,1,-2,-2), & (0,1,-2,-2), \\ (1,-1,1,-3), & (0,-1,0,-1). \end{array}$$

$\mathrm{SL}_6(\mathbb{Z})$

Theorem (V., 2012)

Any $(2,3)$-generating pair of $\mathrm{SL}_6(\mathbb{Z})$ is conjugate in $\mathrm{GL}_6(\mathbb{Z})$ to one of the pairs $\pm X$, Y, where

$$X = \begin{pmatrix} 0 & I_2 & B \\ I_2 & 0 & -B \\ 0 & 0 & I_2 \end{pmatrix}, \qquad Y = \begin{pmatrix} I_2 & 0 & A \\ 0 & 0 & -I_2 \\ 0 & I_2 & -I_2 \end{pmatrix},$$

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix},$$

and $(b_1, b_2, b_3, b_4, a_1, a_2, a_3, a_4)$ is either $(0, 2, -2, -3, 3, 1, -1, 1)$ or $(1, -3, 3, -4, 1, 1, -1, 3)$.

A version of the ping-pong lemma

Lemma (V., 2007)

Let $x, y \in \mathrm{GL}_n(\mathbb{Z})$, $n > 3$, $x^2 = y^3 = I$. Assume that for some $\mathcal{W} \subseteq \mathbb{R}^n$ and $w \in \mathbb{R}^n \setminus \mathcal{W}$, we have
(i) $xy\mathcal{W} \subseteq \mathcal{W}$, $xy^2\mathcal{W} \subseteq \mathcal{W}$;
(ii) $xy \cdot w \in \mathcal{W}$, $xy^2 \cdot w \in \mathcal{W}$.
Then $\langle x, y\rangle \simeq \mathrm{PSL}_2(\mathbb{Z})$. In particular, $\langle x, y\rangle \neq \mathrm{GL}_n(\mathbb{Z})$, $\langle x, y\rangle \neq \mathrm{SL}_n(\mathbb{Z})$.

Symplectic case

Theorem (Vasiliev, Vsemirnov, 2008–2011)

- $\mathrm{Sp}_2(\mathbb{Z})$, $\mathrm{Sp}_4(\mathbb{Z})$, and $\mathrm{Sp}_6(\mathbb{Z})$ are not $(2,3)$-generated.
- $\mathrm{Sp}_8(\mathbb{Z})$, $\mathrm{Sp}_{10}(\mathbb{Z})$ are $(2,3)$-generated.
- $\mathrm{Sp}_{2n}(\mathbb{Z})$ are $(2,3)$-generated for $n \geq 25$.

Cases $2n = 12, 14, \ldots, 48$ remain open. We expect the positive answer.