Extended affine Weyl groups of BCD type, Frobenius manifolds and their Landau-Ginzburg superpotentials

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Main References:

[1]. B.Dubrovin, Geometry of 2D topological field theories, In:Springer Lecture Notes in Math. **1620** (1996), 120–348.

[2]. B.Dubrovin and Y-J.Zhang, Extended affine Weyl groups and Frobenius manifolds, Compositio Mathematica **111**(1998) 167–219.

[3]. B.Dubrovin, Ian Strachan, Y-J.Zhang and D.Zuo, Extended affine Weyl groups of BCD type, Frobenius manifolds and their Landau-Ginzburg superpotentials, **arXiv:1510.08690v1**.

Outline:

Part A. **Known Facts**

Part B. **Main Problem**

Part C. **Main Results**

Part D. **An Example** $[C_3, k = 1]$

Part A. Known Facts

Witten-Dijgraff-Verlinde-Verlinde equations

2-D TFT: find free energy $F = F(t^1, \cdots, t^n)$ satisfying WDVV equations of associativity (1991):

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^\delta \partial t^\gamma} = \frac{\partial^3 F}{\partial t^\delta \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^\alpha \partial t^\gamma},$$

with a quasihomogeneity condition

$$\mathcal{L}_E F = (3 - d)F + \text{quadratic polynomial in t},$$

and

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^1} = \eta^{\alpha\beta}$$

$(\eta^{\alpha\beta})$: constant nondegenerate.

Assume all the roots of $E(t)$ are simple.

Remark 1. If $\eta_{11} = 0$,then by a linear change of coordinates t^α the matrix $\eta_{\alpha\beta}$ can be reduced to the antidiagonal form

$$\eta_{\alpha\beta} = \delta_{\alpha+\beta,n+1}.$$

In these coordinates

$$F(t) = \frac{1}{2}(t^1)^2 t^n + \frac{1}{2} t^1 \sum_{\alpha=2}^{n-1} t^\alpha t^{n-\alpha+1} + f(t^2, \ldots, t^n)$$

for some function $f(t^2, \ldots, t^n)$;

Remark 2. If $\eta_{11} \neq 0$, the function F can be reduced by a linear change of t^α to the form

$$F = \frac{c}{6}(t^1)^3 + \frac{1}{2} t^1 \sum_{\alpha=1}^{n-1} t^\alpha t^{n-\alpha+1} + f(t^2, \ldots, t^n)$$

for a nonzero constant c.

Example 1.1. $n = 1$. $t = t^1$

$$F(t) = \frac{1}{6}t^3, \quad E = t\partial_t, \quad e = \partial_t, \quad \eta^{11} =< \partial_t, \partial_t >= 1.$$

Example 1.2. $n = 2$. Equations of associativity are empty. The other conditions specify the following general solutions of WDVV

$$\begin{aligned}
F(t_1, t_2) &= \frac{1}{2}t_1^2 t_2 + t_2^k, \quad k = \frac{3-d}{1-d}, \ d \neq -1, 1, 3 \\
F(t_1, t_2) &= \frac{1}{2}t_1^2 t_2 + t_2^2 \log t_2, \quad d = -1 \\
F(t_1, t_2) &= \frac{1}{2}t_1^2 t_2 + \log t_2, \quad d = 3 \\
F(t_1, t_2) &= \frac{1}{2}t_1^2 t_2 + e^{t_2}, \quad d = 1
\end{aligned}$$

(in concrete formulae I will label the coordinates t^α by subscripts for the sake of graphical simplicity). In the last case $d = 1$ the Euler vector field is $E = t_1\partial_1 + 2\partial_2$.

Frobenius manifold $(M, \bullet, \langle , \rangle, e, E)$ (**B.Dubrovin, 1990's**):

(i) $\eta := \langle\ ,\ \rangle$ is a flat pseudo-Riemannian metric;

(ii) $\bullet$ is $\mathbb{C}$-linear, associative, commutative product on $T_m M$ which depends smoothly on m;

(iii) e is the unity vector field for the product $\bullet$ and $\nabla e = 0$;

(iv) $(\nabla_w c)(x, y, z)$ is symmetric, where $c(x, y, z) := \langle x \bullet y, z\rangle$;

(v) A linear vector field $E \in Vect(M)$ must be fixed on M, i.e. $\nabla\nabla E = 0$ such that

$$\mathcal{L}_E \langle\ ,\ \rangle = (2 - d)\langle\ ,\ \rangle, \quad \mathcal{L}_E \bullet = \bullet, \quad \mathcal{L}_E\, e = -e.$$

THEOREM **(B.Dubrovin 1994)**.

FM $(M, \bullet, \langle\ ,\ \rangle, e, E) \ \leftrightarrow$ **Solution of WDVV** $(F(t), \eta, e, E)$

Main applications of Frobenius manifolds:

- ★ 2-dimensional topological field theory,
- ★ The theory of Gromov - Witten invariants,
- ★ Singularity theory,
- ★ Hamiltonian theory of integrable hierarchies,
- ★ **Differential geometry of the orbit spaces of groups**

Def. An **intersection form** of Frobenius manifold is a symmetric bilinear form on the cotangent bundle T^*M defined by

$$(\omega_1, \omega_2)^* = \mathbf{i_E}(\omega_1 \cdot \omega_2), \quad \omega_1, \omega_2 \in \mathbf{T^*M}.$$

Here the multiplication law on the cotangent planes is defined using the isomorphism

$$\langle\ ,\ \rangle : TM \to T^*M.$$

The **discriminant** Σ is defined by

$$\Sigma = \{t | \det(\ ,\ )|_{T_t^*M} = 0\} \subset M.$$

THEOREM (B.Dubrovin 1994). The metrics $\eta := \langle\ ,\ \rangle$ and $g := (\ ,\ )^*$ form a flat pencil on $M \setminus \Sigma$, i.e.,

1. The metric $h^{\alpha\beta} = \eta^{\alpha\beta} + \lambda g^{\alpha\beta}$ is flat for arbitrary λ and

2. The Levi-Civita connection for the metric $h^{\alpha\beta}$ has the form

$$\Gamma^{\alpha\beta}_{\delta_{(h)}} = \Gamma^{\alpha\beta}_{k_{(\eta)}} + \lambda \Gamma^{\alpha\beta}_{k_{(g)}},$$

where $\Gamma^{\alpha\beta}_{\delta_{(h)}} = -h^{\alpha\gamma}\Gamma^{\beta}_{\delta\gamma_{(h)}}$, $\Gamma^{\alpha\beta}_{\delta_{(g)}} = -g^{\alpha\gamma}\Gamma^{\beta}_{\delta\gamma_{(g)}}$, $\Gamma^{\alpha\beta}_{\delta_{(\eta)}} = -\eta^{\alpha\gamma}\Gamma^{\beta}_{\delta\gamma_{(\eta)}}$.

The holonomy of the local Euclidean structure defined on $M \setminus \Sigma$ by the intersection form $(\ ,\ )^*$ gives a representation

$$\mu : \pi_1(\mathbf{M} \diagdown \mathbf{\Sigma}) \to \mathbf{Isometries}(\mathbb{C}^{\mathbf{n}}).$$

Def. The group

$$\mathbf{W(M)} := \mu(\pi_1(\mathbf{M} \diagdown \mathbf{\Sigma})) \subset \mathbf{Isometries}(\mathbb{C}^{\mathbf{n}})$$

is called a monodromy group of Frobenius manifold.

$$\Longrightarrow \qquad \mathbf{M} \diagdown \mathbf{\Sigma} = \mathbf{\Omega}/\mathbf{W(M)}, \qquad \mathbf{\Omega} \subset \mathbb{C}^{\mathbf{n}}.$$

Question 1. Given a Frobenius manifold, how to find the monodromy group? (Some cases can be computed).

Eg.1. [W(M)=Coxeter group A_1] $n = 1$. $t = t^1$

$$F(t) = \frac{1}{6}t^3, \quad E = t\partial_t, \quad e = \partial_t, \quad \eta^{11} = < \partial_t, \partial_t > = 1.$$

Eg.2. $n = 2$. Quantum cohomology of $\mathbb{CP}^1$:

$$F(t) = \frac{1}{2}(t^1)^2 t^2 + e^{t^2}, E = t^1\partial_1 + 2\partial_2, e = \partial_1.$$

W(M)=Extended affine Weyl group $\widetilde{W}(A_1)$.

Summary. Give a solution of WDVV $(F(t), \eta, e = \partial_{t_1}, E)$, we have

$$\eta^{\alpha\beta} = \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^1}, \quad g^{\alpha\beta} = \mathcal{L}_E \left(\eta^{\alpha r} \eta^{\beta s} \frac{\partial^2 F(t)}{\partial t^r \partial t^s} \right), \quad W(M)$$

and

$$\mathbf{M} \diagdown \mathbf{\Sigma} = \mathbf{\Omega} / \mathbf{W}(\mathbf{M}), \qquad \mathbf{\Omega} \subset \mathbb{C}^{\mathbf{n}}.$$

Question 2. Which kind of groups can be served as the monodromy groups of some Frobenius manifolds?

♣ Coxeter groups [B.Dubrovin1996, M.Bertola(1998), Zuo 2007(B_l, D_l)]

♣ Extended affine Weyl groups
[B.Dubrovin and Y.-J. Zhang 1998, A-B-C-D-E-F-G]
[B.Dubrovin, Y.-J.Zhang and Zuo, arXiv:052365 (2005), B-C]

♣ Jacobi forms $J(A_n)$,$J(B_n)$, $J(G_2)$ [n=1, B.Dubrovin 1996, general n, M.Bertola 2000], $J(E_6)$, $J(D_4)$ [Satake.I 1993, 1998], Elliptic Weyl groups [Satake.I 2006]

FM and Coxeter groups

THEOREM (B.Dubrovin 1994). There exists a unique **Frobenius manifold structure** of charge $d = 1 - \frac{2}{h}$ on the orbit space of a **finite Coexter group polynomial** in $t^1, t^2, \cdots, t^n$ such that

1). The unity vector field e coincides with $\frac{\partial}{\partial t^n}$;

2). The Euler vector field has the form

$$E = \sum_{\alpha=1}^{n} d_\alpha t^\alpha \frac{\partial}{\partial t^\alpha},$$

where h is the Coxeter number and $\deg t^n = d_n = h > d_{n-1} > \cdots > d_1 = 2$ satisfying the *duality condition* $d_i + d_{n-i+1} = h + 2, \quad i = 1, \cdots, n.$

THEOREM **(B.Dubrovin's conjecture 1994; C.Hertling, 1999).** Any **irreducible semisimple polynomial Frobenius manifold** with positive invariant degrees is isomorphic to the orbit space of a **finite Coxeter group**.

THEOREM (D.Zuo[IMRN-2007]). For any fixed integer $1 \leq k \leq n$, there exists a unique Frobenius structure of charge $d = 1 - \dfrac{1}{k}$ on the orbit space $\mathcal{M} \setminus \{t^n = 0\}$ of a **Coexter group of type** B_n **(or** D_n**)** **polynomial** in $t^1, t^2, \cdots, t^n$, $\mathbf{\dfrac{1}{t^n}}$ such that

1). The unity vector field e coincides with $\dfrac{\partial}{\partial y^k} = \dfrac{\partial}{\partial t^k}$;

2). The Euler vector field has the form $E = \sum_{\alpha=1}^{n} \tilde{d}_\alpha t^\alpha \dfrac{\partial}{\partial t^\alpha}$, where $\tilde{d}_j = \frac{j}{k}, \quad j \leq k, \quad \tilde{d}_m = \frac{2k(n-m)+l}{2k(n-k)}, \quad m > k.$

RMK. **M.Bertola[1998]** used certain superpotential to give a construction.

Part B. Main Problem

How to construct Frobenius manifold structure on the orbit spaces of extended affine Weyl groups?

[DZ1998]. B.Dubrovin and Youjin Zhang, Compositio Mathematica 111(1998)167–219.

Let R be an irreducible reduced root system defined on $(V, (\ ,\ ))$;
$\{\alpha_j\}$: a basis of simple roots, $\{\alpha_j^\vee\}$: the corresponding coroots;
ω_j: the fundamental weights, $(\omega_i, \alpha_j^\vee) = \delta_{ij}$;

$W_a(R)$: affine Weyl group (the semi-direct product of Weyl group W by the lattice of coroots); $W_a(R) \curvearrowright V$: affine transformations

$$\mathbf{x} \mapsto w(\mathbf{x}) + \sum_{j=1}^{l} m_j \alpha_j^\vee, \quad w \in W,\ m_j \in \mathbb{Z}.$$

The **extended affine Weyl group** $\widetilde{W} = \widetilde{W}^{(k)}(R)$ acts on the extended space $\widetilde{V} = V \oplus \mathbb{R}$ and is generated by the transformations

$$\mathbf{x} = (\mathbf{x}, \mathbf{x_{l+1}}) \mapsto (\mathbf{w}(\mathbf{x}) + \sum_{\mathbf{j=1}}^{\mathbf{l}} \mathbf{m_j} \alpha_{\mathbf{j}}^{\vee}, \ \mathbf{x_{l+1}}), \quad \mathbf{w} \in \mathbf{W}, \ \mathbf{m_j} \in \mathbb{Z},$$

and

$$\mathbf{x} = (\mathbf{x}, \mathbf{x_{l+1}}) \mapsto (\mathbf{x} + \gamma\, \omega_{\mathbf{k}}, \ \mathbf{x_{l+1}} - \gamma).$$

Here $\gamma = 1$ except for the cases when $R = B_l, k = l$ and $R = F_4, k = 3$ or $k = 4$, in these three cases $\gamma = 2$.

$\mathcal{A} = \mathcal{A}^{(\mathbf{k})}(\mathbf{R})$ is **the ring of all $\widetilde{\mathrm{W}}$-invariant Fourier polynomials** of the form

$$\sum_{m_1,\dots,m_{l+1}\in\mathbb{Z}} a_{m_1,\dots,m_{l+1}} e^{2\pi i(m_1x_1+\cdots+m_lx_l+\frac{1}{f}m_{l+1}x_{l+1})}$$

that are bounded in the limit

$$\mathbf{x} = \mathbf{x}^0 - i\ \omega_k\tau, \quad x_{l+1} = x^0_{l+1} + i\ \tau, \quad \tau \to +\infty$$

for any $x^0 = (\mathbf{x}^0, x^0_{l+1})$, where f is the determinant of the Cartan matrix of the root system R.

Chevalley-Type Theorem (DZ1998). For a **particular choice of** α_k,

$$\mathcal{A}^{(k)}(R) \simeq \mathbb{C}[\tilde{y}_1, \cdots, \tilde{y}_{l+1}].$$

Here $\tilde{y}_{l+1}(x) = e^{\frac{2\pi i}{\gamma} x_{l+1}}$ and

$$\tilde{y}_j(x) = e^{2\pi i d_j x_{l+1}} \frac{1}{n_j} \sum_{w \in W} e^{2\pi i (\omega_j, w(\mathbf{x}))}, \quad j = 1, \ldots, l$$

where $d_j = (\omega_j, \omega_k), \quad n_j = \#\{w \in W | e^{2\pi i (\omega_j, w(\mathbf{x}))} = e^{2\pi i (\omega_j, \mathbf{x})}\}$.

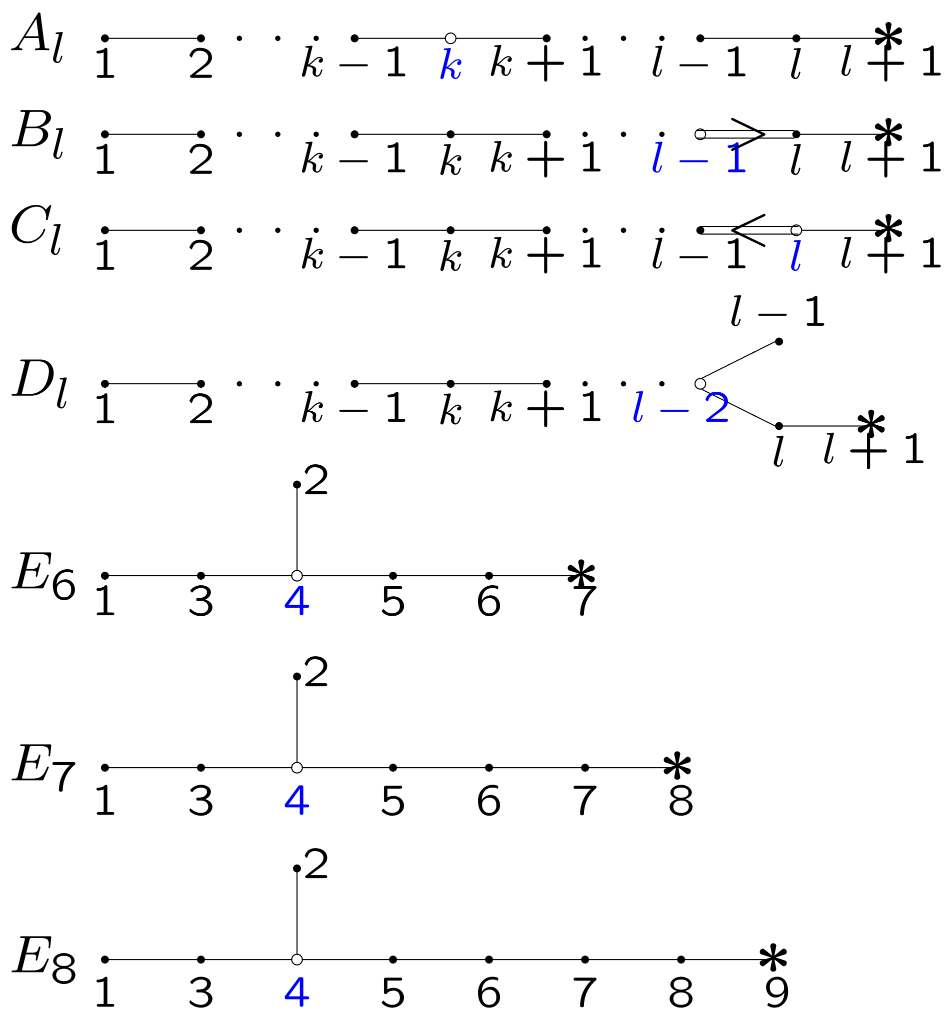
THEOREM(DZ1998). On the orbit space of $\widetilde{W}^{(k)}(R)$, there exist a Frobenius structure whose potential is a **quasi-homogeneous polynomial of** $t^1, \ldots, t^{l+1}, e^{t^{l+1}}$, where $t^1, \ldots, t^{l+1}$ are the flat coordinates of the Frobenius manifold.

The particular choice of α_k is based on the following observations:

(i). The Dynkin graph of $R_k := \{\alpha_1, \cdots, \widehat{\alpha_k}, \cdots, \alpha_l\}$ (α_k is omitted) consists of 1, 2 or 3 branches of A_r type for some r; and

(ii). $\mathbf{d_k > d_s, s \neq k}$.

Notice that for the root system of **type** A_l, there is in fact **no restrictions on the choice of** α_k. However, for the root systems of **type** B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2 there is **only one choice** for each.


A_l
1 2 . . . k − 1 k k + 1 . . . l − 1 l l + 1
B_l
1 2 . . . k − 1 k k + 1 . . . l − 1 l l + 1
C_l
1 2 . . . k − 1 k k + 1 . . . l − 1 l l + 1
D_l
1 2 . . . k − 1 k k + 1 . . . l − 2 l − 1 l l + 1
E_6
2
1 3 4 5 6 7
E_7
2
1 3 4 5 6 7 8
E_8
2
1 3 4 5 6 7 8 9

THEOREM **(P.Slodowy 1997, Unpublished preprint).** The above Chevalley-type theorem is a consequence of the results of Looijenga and Wirthmüller, and in fact **it holds true for any choice of the base element** α_k.

Problem: **(P.Slodowy 1997; B.Dubrovin, Y.Zhang 1998).**
“ Whether the geometric structures that were revealed by Dubrovin-Zhang’s construction also exist on the orbit spaces of the extended affine Weyl groups for an arbitrary choice of α_k**?”**

Part C. Main Results

THEOREM (DSZZ 2015). We give an elementary proof of the general Chevalley-type theorem of type B.C.D claimed by P.Slodowy.

THEOREM (DSZZ 2015). (Part of results in **arXiv:052365v1** for a special case of type B_l, C_l)

(i). For any fixed integer $0 \leq m \leq l-k$, there exists a unique Frobenius manifold structure, denoted by $\mathcal{M}_{k,m}(C_l)$, of charge $d=1$ living on the covering of the orbit space $\mathcal{M}\backslash\{t^{l-m}=0\}\cup\{t^l=0\}$ of $\widetilde{\mathbf{W}}^{(\mathbf{k})}(\mathbf{C_l})$ polynomial in $t^1,\cdots,t^{l+1},\frac{1}{t^{l-m}},\frac{1}{t^l},e^{t^{l+1}}$ for a suitable choice of flat coordinates $t^1,\cdots,t^{l+1}$ for the metric η.

(ii). Also for type $\mathbf{B_l}$ and $\mathbf{D_l}$.

Main steps of the Proof:

Take $G = \widetilde{W}^{(k)}(C_l)$, $G \curvearrowright V$, V is flat

$\rightsquigarrow$ $\mathcal{M} = V \otimes \mathbb{C}/G$ and an induced flat metric $(g_{ij}) = \langle , \rangle$ from V

$\rightsquigarrow$ another flat metric η_{ij}, e, E **Nontrivial**

$\rightsquigarrow$ Study the flat coordinates of η_{ij} **Nontrivial**

$\rightsquigarrow$ Describe analytic properties of $F(t)$.

Another Problem:

In **[DZ1998]**, the extended affine Weyl group $\widetilde{W}^{(k)}(A_l)$ describes the monodromy of roots of trigonometric polynomials - the superpotential - with a given bidegree being of the form

$$\lambda(\varphi) = \mathbf{e^{ik\varphi} + a_1 e^{i(k-1)\varphi} + \cdots + a_l e^{i(k-l)\varphi}}, \quad \mathbf{a_l \neq 0.}$$

A natural question is "does there exists a similar construction for the root systems of type B_l, C_l and D_l?"

Let us denote by $\mathfrak{M}_{k,m,n}$ the space of a particular class of cosine Laurent series of one variable being of the form

$$\lambda(\varphi) = \left(\cos^2(\varphi) - 1\right)^{-m} \sum_{j=0}^{k+m+n} a_j \cos^{2(k+m-j)}(\varphi), \quad a_0 a_{k+m+n} \neq 0,$$

where all $a_j \in \mathbb{C}$, $m, n \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{N}$.

THEOREM (DSZZ 2015). (i). The space $\mathfrak{M}_{k,m,n}$ carries a natural structure of Frobenius manifold.

(ii). There is an isomorphism of Frobenius manifolds between $\mathfrak{M}_{k,m,n}$ and $\mathcal{M}_{k,m}(C_{k+m+n})$.

RMKS. (1). When $m = 0$, $\mathfrak{M}_{k,0,n} \simeq \mathcal{M}_{k,0}(C_l)$, which is the Frobenius manifold structure constructed in arXiv:052365v1 (Dubrovin-Zhang-Zuo 2005).

(2). On the orbit space of the extended affined Weyl group $\widetilde{W}^{(k)}(D_{k+2})$, Dubrovin and Zhang constructed a quasi-homogenous polynomial Frobenius structure, denoted by $\mathcal{M}_{\mathrm{DZ}}^{(k)}(D_{k+2})$ which is isomorphic to $\mathfrak{M}_{k,1,1}$. Actually, in this case, there is a tri-polynomial description introduced in [P2010, T2010], also used in [DLZ2012].

[P2010] P.Rossi, Math. Ann., **348** (2010) no. 2, 265–287.

[T2010] A.Takahashi, Adv. Stud. Pure Math. **59** (2010) 371–388.

[DLZ2012] B.Dubrovin, S-Q.Liu and Y-J.Zhang, Russ. J. Math. Phys. **19** (2012), no. 3, 273–298.

Part D. An Example $[C_3, k = 1]$

Let R be the root system of type C_3. Take $k = 1$, then $d_1 = d_2 = d_3 = 1$ and

$$\mathcal{A}^{(\mathbf{1})}(\mathbf{C_3}) \simeq \mathbb{C}[\tilde{\mathbf{y}}_\mathbf{1}, \tilde{\mathbf{y}}_\mathbf{2}, \tilde{\mathbf{y}}_\mathbf{3}, \tilde{\mathbf{y}}_\mathbf{4}],$$

where

$$\begin{aligned}
\tilde{y}_1 &= e^{2\,i\pi\,x_4}\,(\xi_1 + \xi_2 + \xi_3)\,,\\
\tilde{y}_2 &= e^{2\,i\pi\,x_4}\,(\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3)\,,\\
\tilde{y}_3 &= e^{2\,i\pi\,x_4}\xi_1\xi_2\xi_3,\\
\tilde{y}_4 &= exp(2\,i\pi\,x_4),
\end{aligned}$$

where $\xi_j = e^{2\,i\pi\,(x_j - x_{j-1})} + e^{-2\,i\pi\,(x_j - x_{j-1})}$ and $x_0 = 0$, $j = 1, 2, 3$.

They form a system of global coordinates on $\mathcal{M}$. We now introduce a system of local coordinates on $\mathcal{M}$ as follows

$$y^1 = \tilde{y}_1, y^2 = \tilde{y}_2, y^3 = \tilde{y}_3,\ y^4 = \log \tilde{y}_4 = 2\pi i\, x_{l+1}. \tag{1}$$

They live on a covering $\widetilde{\mathcal{M}}$ of $\mathcal{M} \setminus \{\tilde{y}_4 = 0\}$. The projection

$$P : \tilde{V} \to \widetilde{\mathcal{M}} \tag{2}$$

induces a symmetric bilinear form on $T^*\widetilde{\mathcal{M}}$

$$(dy^i, dy^j)^{\tilde{}} \equiv g^{ij}(y) := \sum_{a,b=1}^{l+1} \frac{\partial y^i}{\partial x^a}\frac{\partial y^j}{\partial x^b}(dx_a, dx_b)^{\tilde{}}$$

with the metric of the form

$$((dx_i, dx_j)^{\tilde{}}) = \frac{1}{4\pi^2}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Q. To find a vector field e such that $\left(\eta^{ij}\right) = \left(\mathcal{L}_e g^{ij}(y)\right)$ is flat !

Case I. $m = 0$, i.e., $e = \frac{\partial}{\partial y^1} - 4\frac{\partial}{\partial y^2} + 4\frac{\partial}{\partial y^3}$.

We first introduce the variables

$$z^1 = y^1 + 6\, e^{y^4},\ z^2 = y^2 + 4\, y^1 + 12\, e^{y^4},$$
$$z^3 = y^3 + 2\, y^2 + 4\, y^1 + 8\, e^{y^4},\ z^4 = y^4.$$

Then the flat coordinates are given by

$$t_1 = z^1 - 2\, e^{z_4},\ \mathbf{t_2 = (z^2 - \frac{1}{6} z^3)(z^3)^{-\frac{1}{4}}},\ \mathbf{t_3 = (z^3)^{\frac{1}{4}}},\ t_4 = z^4$$

and **the intersection form** has the expression

$$g^{11} = 2\,t_2t_3\,e^{t_4} + \frac{1}{3}\,{t_3}^4 e^{t_4} + 4\,e^{2t_4},$$

$$g^{12} = \frac{7}{3}\,{t_3}^3 e^{t_4} + \frac{7}{2}\,t_2 e^{t_4}, \quad g^{13} = \frac{5}{2}\,t_3 e^{t_4}, \quad g^{14} = t_1,$$

$$g^{22} = 12\,{t_3}^2 e^{t_4} - \frac{1}{4}\,{t_2}^2 + \frac{1}{12}\,{t_3}^3 t_2 - \frac{1}{108}\,{t_3}^6 + \frac{\mathbf{1}}{\mathbf{4}}\,\frac{\mathbf{t_2}^{\mathbf{3}}}{\mathbf{t_3}^{\mathbf{3}}},$$

$$g^{23} = 2\,t_1 + 4\,e^{t_4} - \frac{1}{3}\,t_2t_3 + \frac{1}{72}\,{t_3}^4 - \frac{1}{4}\,\frac{{t_2}^2}{{t_3}^2},$$

$$g^{24} = \frac{3}{4}\,t_2, \quad g^{33} = \frac{\mathbf{1}}{\mathbf{4}}\,\frac{\mathbf{t_2}}{\mathbf{t_3}} - \frac{1}{12}\,{t_3}^2, \quad g^{34} = \frac{1}{4}\,t_3, \quad g^{44} = 1.$$

The metric (η^{ij}) is given by

$$\left(\eta^{\mathbf{ij}}\right) = \left(\mathcal{L}_{\mathbf{e}}\mathbf{g}^{\mathbf{ij}}(\mathbf{t})\right) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad e = \frac{\partial}{\partial t^1}.$$

The potential has the form

$$F = \frac{1}{2}{t_1}^2 t_4 + \frac{1}{2} t_1 t_2 t_3 - \frac{1}{48}{t_2}^2{t_3}^2 + \frac{1}{1440} t_2 {t_3}^5 - \frac{1}{36288}{t_3}^8 + t_2 t_3 e^{t_4} + \frac{1}{6}{t_3}^4 e^{t_4} + \frac{1}{2} e^{2t_4} + \mathbf{\frac{1}{48}\frac{{t_2}^3}{t_3}}$$

and **the Euler vector field** is given by

$$E = t_1\partial_1 + \frac{3}{4} t_2 \partial_2 + \frac{1}{4} t_3 \partial_3 + \partial_4.$$

Case II. $m = 1$, i.e., $e = \frac{\partial}{\partial y^1} - 4\frac{\partial}{\partial y^3}$.

Define

$$z^1 = y^1 + 2\,e^{y^4},\ z^2 = \frac{1}{2}y^2 + \frac{1}{4}y^3 + y^1 + 2\,e^{y^4},$$
$$z^3 = \frac{1}{4}y^3 - \frac{1}{2}y^2 + y^1 - 2\,e^{y^4},\ z^4 = y^4.$$

Then the flat coordinates are

$$t_1 = z^1 - 2e^{z^4},\ t_2 = \sqrt{z^2},\ t_3 = \sqrt{z^3},\ t_4 = z^4$$

and **the intersection form** is given by

$$g^{11} = 2\,t_2{}^2 e^{t_4} - 2\,t_3{}^2 e^{t_4} + 4\,e^{2t_4},$$
$$g^{12} = 3\,t_2 e^{t_4},\;\; g^{13} = -3\,t_3 e^{t_4},\;\; g^{14} = t_1,$$
$$g^{22} = 2\,e^{t_4} + t_1 - \frac{1}{4}\,t_3{}^2 - \frac{1}{4}\,t_2{}^2,\;\; g^{23} = -\frac{1}{2}\,t_2 t_3,$$
$$g^{33} = -2\,e^{t_4} + t_1 - \frac{1}{4}\,t_2{}^2 - \frac{1}{4}\,t_3{}^2,$$
$$g^{24} = \frac{1}{2}\,t_2,\;\; g^{34} = \frac{1}{2}\,t_3,\;\; g^{44} = 1.$$

The unit vector field e becomes $\mathbf{e} = \frac{\partial}{\partial \mathbf{t}^1}$ and **the metric** is

$$\left(\eta^{ij}\right) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The potential has the expression

$$F = \frac{1}{2} t_1 {t_2}^2 + \frac{1}{2} t_1 {t_3}^2 + \frac{1}{2} {t_1}^2 t_4 - \frac{1}{48} {t_2}^4 - \frac{1}{48} {t_3}^4 - \frac{1}{8} {t_2}^2 {t_3}^2 + {t_2}^2 e^{t_4} - {t_3}^2 e^{t_4} + \frac{1}{2} e^{2t_4}$$

and **the Euler vector field** is given by

$$E = t_1 \partial_1 + \frac{1}{2} t_2 \partial_2 + \frac{1}{2} t_3 \partial_3 + \partial_4.$$

This is isomorphic to the one given in Example 2.6 $[A_3, k = 2]$ of **[DZ1998]**.

Thanks for your attention !