Poisson Structures in Theories of Modular Forms, Elliptic Functions, and Invariant Theory

Dynamics in Siberia, 2/27-3/3/2017

Overview of the talk

It is very natural to consider geometric structures on moduli spaces or parameter spaces of deformations, or on the total space of the deformation family.

The simplest example is the family of elliptic curves in Weierstrass form:

$$y^2 = 4x^3 - g_2 x - g_3,$$

where the parameter space is the space of $(g_2, g_3) \in \mathbb{C}^2$, where $g_2^3 - 27g_3^2 = 0$ gives the discriminant locus, and the above equation defines a hyperplane in $(x, y, g_2, g_3) \in \mathbb{C}^4$ that is the universal family of elliptic curves.

Overview of the talk

We will report on some results obtained from considerations in the theory of modular forms and elliptic functions, prompted by a question for relationship between the theory of modular forms with the theory of Moyal products.

Via ideas from invariant theory, we arrive at some Poisson structures on the spaces mentioned on last slides.

Finally, we apply the construction of Poisson structures to the invariant theory of binary polyhedral groups.

Overview of the talk

The theories of modular forms, invariant theory and Moyal product seem to be three unrelated subjects, studied in different areas of mathematics. Nevertheless, as pointed out by Zagier, in each of these theories one can define a sequence of binary "bracket operations" that define algebraic structures of similar type.

These bracket operations are well studied in the latter two theories, while they are relatively less well understood in the theory of modular forms.

We will first to explain how to use the brackets in invariant theory and Moyal product to understand the brackets in modular form theory, then extend it to the theory of elliptic functions.

Modular Forms

A modular form of weight k is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ on the upper half plane $\mathbb{H}$, such that the following conditions are satisfied:

$$f(\tau) = \sum_{n \geq 0} a_n q^n, \quad q = e^{2\pi i \tau},$$

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau),$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

The Ring of Modular Forms

Denote by M_k the vector space of modular forms of weight k,

and set $M_* = \bigoplus_{k \geq 0} M_k$.

Then M_* is a commutative ring under multiplication, such that $M_k \cdot M_k \subset M_{k+l}$.

Construction of Modular Forms

One easy way to construct modular forms is to consider the Weierstrass $\mathcal{P}$-function:

$$\mathcal{P}(z;\tau) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \left(\frac{1}{(z+m+n\tau)^2} - \frac{1}{(m+n\tau)^2}\right),$$

where $\tau \in H$.

This is a meromorphic function on $\mathbb{C}$ with two periods 1 and τ, and double poles at $m+n\tau$, $m,n \in \mathbb{Z}$.

From Weierstrass $\mathcal{P}$-Function to Eisenstein series

Weierstrass $\mathcal{P}$-function has the following Laurent expansion at $z = 0$:

$$\mathcal{P}(z;\tau) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1) G_{2n+2}(\tau) z^{2n},$$

where the coefficients $G_{2k}(\tau)$ are the Eisenstein series:

$$G_{2k}(\tau) = \sum_{m,n}{}' \frac{1}{(m+n\tau)^{2k}}.$$

Differential Equation Satisfied by Weierstrass $\mathcal{P}$-Function

Let $x = \mathcal{P}(z;\tau)$, $y = \mathcal{P}'(z,\tau)$. Then

$$y^2 = 4x^3 - g_2 x - g_3,$$

where $g_2 = 60G_4$, $g_3 = 140G_6$.

Differentiate both sides:

$$\mathcal{P}''(z;\tau) \;=\; 6\mathcal{P}(z;\tau)^2 - \frac{g_2}{2},$$

From this one derives recursion relations:

$$G_{2n} = \frac{3}{(n-3)(2n-1)(2n+1)} \sum_{r=2}^{n-2} (2r-1)(2n-2r-1) G_{2r} G_{2n-2r}.$$

Relationship to KdV Equation

Differentiate again:

$$\mathcal{P}'''(z) \;=\; 12\mathcal{P}(z)\mathcal{P}'(z).$$

This is a special case of the KdV equation.

This is the starting point for the relationship between theta functions and solutions of KdV hierarchy.

It develops into Krichever construction.

Eisenstein Series as Modular Forms

The Eisensten series have the following properties ($k > 1$):

$$G_{2k}(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^{2k} G_{2k}(\tau),$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and

$$G_{2k}(\tau) = \frac{|B_{2k}|(2\pi)^{2k}}{(2k)!} + \frac{2(-1)^k(2\pi)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$

where $\sigma_k(n) = \sum_{d|n} d^k$.

So when $k > 1$, $G_{2k}(\tau)$ are modular forms of weight $2k$.

In fact, $M_* = \mathbb{C}[G_4, G_6]$.

Eisenstein Series E_2

For simplicity of notations, let

$$E_4(\tau) = \frac{3^2 \cdot 5}{\pi^4} G_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

$$E_6(\tau) = \frac{3^3 \cdot 5 \cdot 7}{2\pi^6} G_4(\tau) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n,$$

One also needs:

$$E_2(\tau) = 1 + 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n.$$

Derivatives of Modular Forms

Eisenstein series E_2 is not a modular form:

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) + \frac{12}{2\pi i}c(c\tau+d), \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

This leads to quasimodular forms and derivatives of modular forms. Differentiate both sides of

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau),$$

one gets

$$f'\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+2} f'(\tau) + kc(c\tau+d)^{k+1} f(\tau),$$

i.e., $f'(\tau)$ is no longer a modular form. It is close to being a modular form of weight $k+2$.

Ramanujan Identities

In a famous paper published in 1916, Ramanujan proved the following identities:

$$q\frac{d}{dq}E_2 = \frac{1}{12}(E_2^2 - E_4),$$
$$q\frac{d}{dq}E_4 = \frac{1}{3}(E_2E_4 - E_6),$$
$$q\frac{d}{dq}E_6 = \frac{1}{2}(E_2E_6 - E_4^2).$$

Notes. 1. These were actually proved first by Eisenstein.

2. $D = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$.

Serre Derivation

Based on Ramanujan identities, Serre introduced a differential operator acting on the space of modular forms:

$$\nabla : M_k \to M_{k+2}, \quad \nabla f = q\frac{d}{dq}f - \frac{k}{12}E_2 \cdot f.$$

If one rewrite a modular form $f(\tau)$ of weight k as a differential form $f(\tau)(d\tau)^k$, then it is holomorphic section of a holomorphic line bundle on the moduli space of elliptic curves. The Poincaré metric induces a Hermitian metric on this line bundle, then the Serre differential is just the Chern connection for this metric.

Rankin-Cohen Brackets and Rankin-Cohen ALgebras

In 1956, Rankin gave a general description of the differential operators which send modular forms to modular forms.

A very interesting special case is certain bilinear bracket operators on M_* introduced by Cohen in 1977.

In 1994, Zagier studied the algebraic relations among the Rankin-Cohen brackets and introduced a notion of Rankin-Cohen algebra.

In the last section of his paper, Zagier raised the question of finding Rankin-Cohen algebras in other areas in mathematics. He mentioned invariant theory, Moyal product and vertex operator algebra.

Rankin-Cohen Brackets

Let $f \in M_k$ and $g \in M_l$, the n-th Rankin-Cohen bracket of f and g is defined by:

$$[f,g]_n = \sum_{r+s=n} (-1)^r \binom{n+k-1}{s} \binom{n+l-1}{r} D^r f \cdot D^s g.$$

For example,

$$[f,g]_0 = fg,$$
$$[f,g]_1 = k \cdot f \cdot Dg - l \cdot Df \cdot g.$$

One can check that $[f,g]_n \in M_{k+l+2n}$.

The Rankin-Cohen Algebra

The Rankin-Cohen brackets satisfies many algebraic identities, e.g.,

$$
\begin{aligned}
&[f,g]_n = (-1)^n [g,f]_n,\\
&[[f,g]_1,h]_1 + [[g,h]_1,f]_1 + [[h,f]_1,g]_1 = 0,\\
&[[f,g]_0,h]_1 + [[g,h]_0,f]_1 + [[h,f]_0,g]_1 = 0,\\
&m[[f,g]_1,h]_0 + l[[g,h]_1,f]_0 + k[[h,f]_1,g]_0 = 0,\\
&[[f,g]_1,h]_1 = [[g,h]_0,f]_2 - [[h,f]_0,g]_2 + [[g,h]_2,f]_0 - [[h,f]_2,g]_0,
\end{aligned}
$$

etc., where $f \in M_k, g \in M_l, h \in M_m$.

From these one can extract a notion of a Rankin-Cohen algebra.

Kuznetsov-Cohen Series

For $f \in M_k$ define a formal power series:

$$\tilde{f}(\tau, X) = \sum_{n=0}^{\infty} \frac{D^n f(\tau)}{(n+k-1)!}(2\pi i X)^n.$$

This is called the Kuznetzov-Cohen series of f.

It satisfies the following transformation formula:

$$\tilde{f}\left(\frac{a\tau+b}{c\tau+d}, \frac{X}{(c\tau+d)^2}\right) = (c\tau+d)^k e^{cX/(c\tau+d)} \tilde{f}(\tau, X).$$

Generating Series of Rankin-Cohen Brackets

For $f \in M_k, \;\; g \in M_l$,

$$\tilde{f}(\tau, -X)\tilde{g}(\tau, X) = \sum_{n=0}^{\infty} \frac{[f,g]_n(\tau)}{(n+k-1)!(n+l-1)!}(2\pi i X)^n,$$

This resembles Hirota bilinear derivatives:

$$f(x+y)g(x-y) = \sum_{n=0}^{\infty} \frac{1}{n!}(D_x^n f \cdot g) y^n,$$

and so it suggests a connection with the theory of integrable hierarchies.

From Modular Forms to Modular Pseudodifferential Operators

For $f \in M_{2k}$, Cohen-Manin-Zagier associated a family of pseudodifferential operator parameterized by κ:

$$\mathcal{D}^\kappa[f] = \sum_{n=0}^{\infty} \frac{\binom{-k}{n}\binom{-k+\kappa-1}{n}}{\binom{-2k}{n}} D^n f(\tau) D^{-n-k}.$$

For $f \in M_{2k}$, $g \in M_{2l}$, one has

$$\mathcal{D}^\kappa[f] \cdot \mathcal{D}^\kappa[g] = \sum_{n=0}^{\infty} t_n^\kappa(k,l)\mathcal{D}[[f,g]_n],$$

where the coefficients $t_n^\kappa(k,l)$ are given by:

$$t_n^\kappa(k,l) = \frac{1}{\binom{-2l}{n}} \sum_{r+s=n} \frac{\binom{-k}{r}\binom{-k-1+\kappa}{r}}{\binom{-2k}{r}} \frac{\binom{n+k+l-\kappa}{s}\binom{n+k+l-1}{s}}{\binom{(2n+2k+2l-2}{s}}.$$

Multiplications on M_* Defined by Rankin-Cohen Brackets

Since the multiplication on the space of pseudodifferential forms is associative, it follows the following multiplication defined on M_* is associative:

$$f *_\kappa g = \sum_{n=0}^{\infty} t_n^\kappa(k,l)[f,g]_n, \quad (f \in M_{2k}, g \in M_{2l}).$$

By taking $\kappa = \frac{1}{2}$ or $\frac{3}{2}$, one can see that

$$f *_\hbar g = \sum_{n=0}^{\infty} \hbar^n [f,g]_n$$

is associative. This is called the Eholzer product.

Classical Invariant Theory

Classical invariant theory studies polynomials which are fixed by certain group action.

A typical example is binary invariants, which are polynomials in two variables invariant under the action of a subgroup $\Gamma \subset SL_2(\mathbb{C})$.

In other word $f(x, y) \in \mathbb{C}[x, y]^\Gamma$ iff

$$f(ax + by, cx + dy) = f(x, y), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

The Ring of Invariants

It is clear that $\mathbb{C}[x,y]^{\Gamma}$ is a commutative ring, and can be decomposed into subspaces consisting of homogeneous polynomials:

$$\mathbb{C}[x,y]^{\Gamma} = \bigoplus_{n\geq 0} \mathbb{C}[x,y]^{\Gamma}_n.$$

For example, for

$$\Gamma = \left\{ \begin{pmatrix} e^{2\pi ik/n} & 0 \\ 0 & e^{-2\pi ik/n} \end{pmatrix} \; 0 \leq k < n \right\}$$

$$\mathbb{C}[x,y]^{\Gamma} = \mathbb{C}[u = x^n, v = xy, w = y^n] = \mathbb{C}[u,v,w]/(uw - v^n).$$

Transvectants

The m-th transvectant of two functions $F(x,y)$ and $G(x,y)$ is the function:

$$(F,G)^{(m)} = \sum_{i+j=m} (-1)^i \binom{m}{i} \frac{\partial^m F}{\partial x^j \partial y^i} \frac{\partial^m G}{\partial x^i \partial y^j}.$$

This defines an operation: $\mathbb{C}[x,y]_k^\Gamma \times \mathbb{C}[x,y]_l^\Gamma \to \mathbb{C}[x,y]_{k+l-2m}^\Gamma$.

Cayley's Ω Process

Cayley's Ω operator in two variables is

$$\Omega = \begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial y_1} \end{vmatrix} = \frac{\partial}{\partial x_1}\frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_1}\frac{\partial}{\partial x_2}.$$

Its action on $F(x,y)$, $G(x,y)$ is given by:

$$\begin{aligned} & \Omega(F(x_1,y_1)G(x_2,y_2)) \\ = \; & \frac{\partial F(x_1,y_1)}{\partial x_1} \cdot \frac{\partial G(x_2,y_2)}{\partial y_2} - \frac{\partial F(x_1,y_1)}{\partial y_1} \cdot \frac{\partial G(x_2,y_2)}{\partial x_2}. \end{aligned}$$

The m-th transvectant is given by taking the Ω process m times:

$$(F,G)^{(m)} = \Omega^m(F(x_1,y_1)G(x_2,y_2))|_{x_1=x_2=x, y_1=y_2=y}.$$

Moyal $*$-Product

The Moyal product is defined as follows:

$$\begin{aligned} F *_\hbar G &= \sum_{m=0}^{\infty} \frac{\hbar^m}{m!} (F, G)^{(m)} \\ &= \exp(\hbar\Omega)(F(x_1, y_1)G(x_2, y_2))|_{x_1=x_2=x, y_1=y_2=y}. \end{aligned}$$

The Moyal $*$-product is associative. Indeed,

$$\begin{aligned} & (F *_\hbar G) *_\hbar H \\ = \;& \exp(\hbar\Omega_{12} + \Omega_{13} + \Omega_{23})(F(x_1, y_1)G(x_2, y_2)H(x_3, y_3))|_{x_i=x, y_i=y}, \\ & F *_\hbar (G *_\hbar H) \\ = \;& \exp(\hbar\Omega_{12} + \Omega_{13} + \Omega_{23})(F(x_1, y_1)G(x_2, y_2)H(x_3, y_3))|_{x_i=x, y_i=y}, \end{aligned}$$

where $\Omega_{ij} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial y_j} - \frac{\partial}{\partial x_j}\frac{\partial}{\partial y_i}$.

Modular Forms as Homogeneous Functions of Two Variables

Modular forms $f \in M_{2k}$ of weight $2k$ are in one-to-one correspondence with homogeneous functions of weight $-2k$:

$$F(v,u) = u^{-2k} f(\frac{v}{u}), \qquad\qquad f(\tau) = F(\tau, 1).$$

Using this correspondence, one can understand modular forms as functions on the following space

$$\mathcal{B} = \{(v,u) \in \mathbb{C} \mid \text{Im}(\frac{v}{u}) > 0\}.$$

On this space we consider the following action of $SL_2(\mathbb{Z})$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (v,u) = (av + bu, cv + du).$$

Modular Forms as Invariants of $SL_2(\mathbb{Z})$ in Two Variables

So modular forms of weight $2k$ correspond to $F(v, u)$ such that

$$F(av + bu, cv + du) = F(v, u), \qquad F(\lambda v, \lambda u) = \lambda^{-2k} F(v, u).$$

In other words, modular forms should be understood as invariants of degree $-2k$ on the space $\mathcal{B}$. This subspace of $\mathbb{C}^2$ is invariant under the action of $SL_2(\mathbb{Z})$ which is a subgroup of the linear symplectic group that preserves the symplectic structure on $\mathbb{C}^2$.

The Rankin-Cohen Brackets as Transvectants

Denote the map that sends $f(\tau)$ to $F(v,u)$ by $H(f)$, then it is not hard to see that

$$H([f,g]_m) = \frac{1}{m!}(H(f),H(g))^{(m)}.$$

As a corollary, the Eholzer product becomes the Moyal product:

$$H(f *_\hbar g) = H(f) *_\hbar H(g).$$

The associativity of the Eholzer product is then automatic.

From Modular Forms to Polynomials in Two Variables

Recall $M_* = \mathbb{C}[E_4, E_6]$, i.e., every modular form can be written as a polynomial in E_4 and E_6 in a unique way.

It follows that if $f \in M_{2k}$ and $g \in M_{2l}$, then there are weighted homogeneous polynomials P and Q of degree $2k$ and $2l$ respectively, such that

$$f = P(E_4, E_6), \qquad\qquad g = Q(E_4, E_6),$$

where $\deg E_4 = 4$, $\deg E_6 = 6$. Then for each $n \geq 0$, there is a weighted homogeneous polynomial $R_n(x, y)$ of degree $2k+2l+2n$ such that $[f, g]_n = R_n(E_4, E_6)$.

Moyal Product on the Plane Induced by Rankin-Cohen Brackets

It follows that the Rankin-Cohen brackets and the product $*_\hbar$ induce bracket operations and a star-product on $\mathbb{C}[x,y]$.

In other words, there is an isomorphism $G : M_* \to \mathbb{C}[x,y]$, $P(E_4, E_6) \mapsto P(x,y)$, such that

$$[P,Q]_n = G([G^{-1}(P), G^{-1}(Q)]),$$
$$P *_\hbar Q = G(G^{-1}(P) *_\hbar G^{-1}(Q)).$$

For example,

$$\begin{aligned} x *_\hbar y &= xy + 2(y^2 - x^3)\hbar + \frac{245}{18}(y^2 - x^3)x\hbar^3 + \frac{175}{24}(x^3 - y^2)y\hbar^4 \\ &+ \frac{154}{3}(y^2 - x^3)x^2\hbar^5 + \frac{119}{3}(x^3 - y^2)xy\hbar^6 \\ &+ (y^2 - x^3)(\frac{6125}{24}x^3 - \frac{2645}{24}y^2)\hbar^7 + \frac{1045}{8}(x^3 - y^2)x^2y\hbar^8 + \cdots \end{aligned}$$

More Examples of Moyal Product Induced by Rankin-Cohen Brackets

$$\begin{aligned} x *_\hbar x &= x^2 + \frac{25}{9}(x^3 - y^2)\hbar^2 + \frac{175}{18}(x^3 - y^2)x\hbar^4 + \frac{70}{3}(x^3 - y^2)x^2\hbar^6 \\ &+ (x^3 - y^2)(\frac{40975}{432}x^6 - \frac{21175}{432}y^2)\hbar^8 + \cdots, \end{aligned}$$

$$\begin{aligned} y *_\hbar y &= y^2 + \frac{49}{4}(y^2 - x^3)x\hbar^2 + \frac{147}{2}(y^2 - x^3)x^2\hbar^4 \\ &+ (y^2 - x^3)(\frac{290521}{576}x^3 - \frac{135289}{576}y^2)\hbar^6 \\ &+ (y^2 - x^3)(\frac{2277275}{576}x^3 - \frac{1844843}{576}y^2)x\hbar^8 + \cdots \end{aligned}$$

Poisson Structure on the Plane Induced by Rankin-Cohen Brackets

We are interested in the Poisson structure for this star product. By Ramanujan's identities, it is easy to see that

$$[E_4, E_6]_1 = 2E_6^2 - 2E_4^3 = 2 \cdot 1728\Delta,$$

where $\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}$ is called the discriminant.

It follows that the induced Poisson bracket on $\mathbb{C}^2$ is given by:

$$[x, y] = 2y^2 - 2x^3.$$

The corresponding bivector field is $2(y^2 - x^3)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$.

This is of type A_2 in the classification of Poisson structures on the plane by Arnold.

Symplectic Structure on the Plane Induced by Rankin-Cohen Brackets

The above Poisson structure corresponds to the following symplectic structure:

$$\omega = \frac{1}{2(y^2 - x^3)} dx \wedge dy,$$

which is singular along the curve

$$y^2 = x^3.$$

This is an algebraic curve with a cusp singularity.

The singularity is of type A_2.

Hamiltonian Vector Field Corresponding to Serre Derivation

Recall the Serre derivation ∇ acts on M_k by $\nabla = \frac{\partial}{\partial\tau} - \frac{k}{12}E_2(\tau)\cdot$.
And so by Ramanujan identities again

$$\nabla E_4 = -\frac{1}{3}E_6, \qquad\qquad \nabla E_6 = -\frac{1}{2}E_4^2.$$

So it corresponds to the following vector field on $\mathbb{C}^2$:

$$X^{\nabla} = -\frac{1}{3}y\frac{\partial}{\partial x} - \frac{1}{2}x^2\frac{\partial}{\partial y},$$

This vector field is "Hamiltonian":

$$\iota(X^{\nabla})\omega = -\frac{y}{6(y^2-x^3)}dy + \frac{x^2}{4(y^2-x^3)}dx = -d\frac{\log(y^2-x^3)}{12}.$$

Hamiltonian System Associated with the Serre Derivation

Consider the following system associated with the Serre derivation:

$$\frac{d}{dt}x(t) = -\frac{1}{3}y(t), \qquad \frac{d}{dt}y(t) = -\frac{1}{2}x(t)^2.$$

It has the following solution

$$x(t) = 36\mathcal{P}(t + c_1; 0, g_3),$$
$$y(t) = -108\mathcal{P}'(t + c_1; 0, g_3).$$

Recall the Weierstrass $\mathcal{P}$-function $\mathcal{P}(t; g_2, g_3)$ satisfies the differential equation:

$$\mathcal{P}'(t; g_2, g_3)^2 = 4\mathcal{P}(t; g_2, g_3)^3 - g_2\mathcal{P}(t; g_2, g_3) - g_3.$$

So the solution curves are given by the algebraic equations:

$$y(t)^2 = x(t)^3 - 108^2 g_3.$$

Moyal Bracket on the Plane Induced by Rankin-Cohen Brackets

The Moyal bracket is defined by the star product:

$$\{P, Q\}_\hbar := \frac{P *_\hbar Q - Q *_\hbar P}{2\hbar}.$$

This defines a deformation of the Poisson bracket above:

$$\begin{aligned}\{x, y\}_\hbar &= 2(y^2 - x^3) + \frac{245}{18}(y^2 - x^3)x\hbar^2 + \frac{154}{3}(y^2 - x^3)x^2\hbar^4 \\ &+ (y^2 - x^3)(\frac{6125}{24}x^3 - \frac{2645}{24}y^2)\hbar^6 + \cdots\end{aligned}$$

Deformation of the Vector Field Corresponding to Serre Derivation

Recall the vector field corresponding to Serre derivation

$$X^{\nabla} = -\frac{1}{3}y\frac{\partial}{\partial x} - \frac{1}{2}x^2\frac{\partial}{\partial y},$$

is Hamiltonian with Hamiltonian $H = -\frac{\log(y^2 - x^3)}{12}$:

With the same Hamiltonian, the Hamiltonian vector field with respect to $\{\cdot,\cdot\}_\hbar$ is

$$\begin{aligned} X_\hbar^{\nabla} &= (1 + \frac{245}{36}x\hbar^2 + \frac{154}{6}x^2\hbar^4 + (\frac{6125}{48}x^3 - \frac{2645}{48}y^2)\hbar^6 + \cdots) \\ &\cdot(-\frac{1}{3}y\frac{\partial}{\partial x} - \frac{1}{2}x^2\frac{\partial}{\partial y}). \end{aligned}$$

Summary and Outlook

E_4 and E_6 can be understood as parameters for family of elliptic curves in Weierstrass form:

$$Y^2 = 4X^3 - 60E_4X - 140E_6.$$

The theory of modular forms leads to the constructions of some nice geometric structures familiar in mathematical physics on the space of (E_4, E_6):

1. A Poisson structure
2. A star-product that induces a deformation of the Poisson structure
3. A Hamiltonian vector field together with a deformation

Summary and Outlook

In the above we have seen the algebraic curve $y^2 = x^3$ naturally appears as the singular locus of the Poisson structure on the (x, y)-plane. This curve appears also in another setting: Emergent geometry of spectral curve for Witten-Kontsevich tau-function.

We will combine the (x, y)-plane and the (X, Y)-plane and define a Poisson structure again by considerations from invariant theory.

We will also apply the same construction to the invariant theory of binary polyhedral groups.

An example of emergent phenomenon: Witten Conjecture/Kontsevich Theorem

For topological 2D gravity, the n-point correlators are defined by

$$\langle \tau_{m_1} \cdots \tau_{m_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{m_1} \cdots \psi_n^{m_n}.$$

$$F_g(\mathbf{t}) = \sum_{n \geq 0} \frac{1}{n!} \sum_{a_1,\ldots,a_n \geq 0} t_{a_1} \cdots t_{a_n} \langle \tau_{a_1}, \cdots , \tau_{a_n} \rangle_g.$$

$$F(\mathbf{t}; \lambda) = \sum_{g \geq 0} \lambda^{2g-2} F_g(\mathbf{t}).$$

$$Z_{WK} = \exp F(\mathbf{t}; \lambda).$$

This is called the Witten-Kontsevich tau-function .

Special deformation of the Airy curve

Theorem (Z.) Consider the following series ($t_k = (2k+1)!!u_k$):

$$Y = f - \sum_{n\geq 0}(2n+1)u_n X^{n-1/2} - \sum_{n\geq 0}\frac{\partial F_0}{\partial u_n}(\mathbf{u})\cdot X^{-n-3/2}.$$

Then one has

$$\begin{aligned} Y^2 =& X\Big(1-\sum_{n\geq 1}(2n+1)u_n X^{n-1}\Big)^2 \\ &-2u_0\Big(1-\sum_{n\geq 1}(2n+1)u_n X^{n-1}\Big) \\ &+2\sum_{n\geq 0}\sum_{k\geq n+2}(2k+1)u_k\cdot\frac{\partial F_0}{\partial u_n}\cdot X^{k-n-2}. \end{aligned}$$

Emergence of the spectral curve for topological 2D gravity

Consider the Puiseux series:

$$Y = X^{1/2} - \frac{u_0}{X^{1/2}} - \sum_{n \geq 0} \frac{\partial F_0}{\partial u_n}(u_0, 0, \ldots) \cdot X^{-n-3/2},$$

then one has

$$X = Y^2 + 2u_0.$$

When $u_0 = 0$, this gives us the Airy curve:

$$Y^2 = X.$$

It is the spectral curve of the topological 2D gravity.

This is an emergent phenomenon: You have to go through the infinite-dimensional big phase space to see the spectral curve.

Examples of the special deformation

When $t_j = 0$ for $j \geq 3$, the curve is deformed to:

$$\begin{aligned} Y^2 &= -2u_0(1-3u_1) + 10u_2 \frac{\partial F_0}{\partial u_0}(u_0, u_1, u_2) \\ &+ ((1-3u_1)^2 + 10u_0u_2)X \\ &- 10u_2(1-3u_1)X^2 + 25u_2^2X^3. \end{aligned}$$

When $t_1 = 1$ i.e., $u_1 = 1/3$,

$$Y^2 = 10u_2 \frac{\partial F_0}{\partial u_0}(u_0, 1/3, u_2) + 10u_0u_2 \cdot X + 25u_2^2 \cdot X^3.$$

The spectral curve undergoes a phase change from a rational curve to a family of elliptic curves in the (X, Y)-plane parameterized by (u_0, u_2)!

Elliptic functions

An elliptic function is a meromorphic function f with two periods u and v:

$$f(z + mu + nv; u, v) = f(z; u, v), \quad m, n \in \mathbb{Z},$$

where $\tau = \frac{v}{u}$ lies in the upper half plane $\mathcal{H}$.

One can consider a $\mathbb{Z}^2$-action:

$$(m, n) \cdot (z; u, v) = (z + mu + nv; u, v),$$

then elliptic functions are invariant under this action.

Derivatives of elliptic functions

Taking ∂_z, ∂_u and ∂_v respectively,

$$\partial_z f(z+mu+nv;u,v) = \partial_z f(z;u,v),$$
$$\partial_u f(z+mu+nv;u,v) + m\partial_z f(z+mu+nv;u,v) = \partial_u f(z;u,v),$$
$$\partial_v f(z+mu+nv;u,v) + n\partial_z f(z+mu+nv;u,v) = \partial_v f(z;u,v).$$

Therefore, given an elliptic function f, $\partial_z f$ is still an elliptic function, but it is not the case for $\partial_u f$ and $\partial_v f$ in general.

A Poisson bracket for elliptic functions

We are in the same situation as in the case of modular forms: The space of elliptic functions may not be close under derivatives, but it may be close under some bracket operation.

For two elliptic functions f and g with periods u and v, one can check that

$$\{f, g\} = \frac{2\pi i}{u}\frac{\partial f}{\partial v}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{2\pi i}{u}\frac{\partial g}{\partial v}$$

is an elliptic function with period u, v.

In other words, we consider the Poisson structure on the space of (z, u, v) defined by

$$\frac{2\pi i}{u}\frac{\partial}{\partial v} \wedge \frac{\partial}{\partial z},$$

and consider its restriction to the space of elliptic functions.

Weierstrass function

Weierstrass function with period u and v are defined by:

$$\mathcal{P}(z;\tau) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \left(\frac{1}{(z+mu+nv)^2} - \frac{1}{(mu+nv)^2} \right),$$

Weierstrass $\mathcal{P}$-function has the following Laurent expansion at $z = 0$:

$$\mathcal{P}(z;\tau) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1) e_{2n+2}(u,v) z^{2n},$$

where the coefficients $e_{2k}(\tau)$ are the Eisenstein series:

$$e_{2k}(u,v) = \sum_{m,n}{}' \frac{1}{(mu+nv)^{2k}}.$$

Weierstrass function and its derivatives

By a result of Eisenstein,

$$\frac{2\pi i}{u}\frac{\partial e_2}{\partial v} = 5e_4 - e_2^2,$$
$$\frac{2\pi i}{u}\frac{\partial e_4}{\partial v} = 14e_6 - 4e_2e_4,$$
$$\frac{2\pi i}{u}\frac{\partial e_6}{\partial v} = \frac{60}{7}e_4^2 - 6e_2e_6,$$
$$\frac{2\pi i}{u}\frac{\partial \mathcal{P}}{\partial v} = E_1\mathcal{P}' + 2\mathcal{P}^2 - 2e_2\mathcal{P} - 20e_4,$$
$$\frac{2\pi i}{u}\frac{\partial \mathcal{P}'}{\partial v} = E_1(6\mathcal{P}^2 - 30e_4) + 3\mathcal{P}\mathcal{P}' - 3e_2\mathcal{P}'.$$

where

$$E_1 = \frac{1}{z} - \sum_{n=1}^{\infty} e_{2n+2}(u,v)z^{2n+1}.$$

Poisson bracket of Weierstrass function and its z-derivative

One finds:

$$\{\mathcal{P}, \mathcal{P}'\} \;=\; (420e_6 - 120e_2e_4)P + 600e_4^2 - 420e_2e_6.$$

Recall

$$(\mathcal{P}')^2 = 4\mathcal{P}^3 - 60e_4\mathcal{P} - 140e_6,$$

we can rewrite the above Poisson bracket in the following form:

$$\{\mathcal{P}, \mathcal{P}'\} \;=\; \frac{1}{2}\frac{2\pi i}{u}\frac{\partial}{\partial v}(Y^2 - 4X^3 + 60e_4X + 140e_6)|_{X=\mathcal{P}, Y=\mathcal{P}'}.$$

Induced Poisson bracket

Similar to the case of modular forms, we get an induced Poisson structure on the space of (X, Y, u, v) defined by:

$$\frac{1}{2}\frac{2\pi i}{u}\frac{\partial}{\partial v}(Y^2 - 4X^3 + 60e_4X + 140e_6) \cdot \frac{\partial}{\partial X} \wedge \frac{\partial}{\partial Y}.$$

It is easy to see that $f = Y^2 - 4X^3 + 60e_4X + 140e_6$ is a Casimir function for this Poisson structure, i.e. $\{f, g\} = 0$ for all g.

Future directions

Recall Dubrovin has defined a structure of twisted Frobenius manifold on the locus

$$f = 0$$

In our future investigations we will examine how these two kinds of structures interact with each other.

Another direction of research is to understand these structures from the point of view of emergent geometry, i.e., to see their relationship with GRomov-Witten theory.

Poisson structures arising from classical invariant theory

The above discussions in the settings of modular forms and elliptic functions motivate us to carry out the same constructions in the setting of polynomial invariants.

A particularly interesting case is the binary polyhedral groups.

They are related to the Platonic solids and simple Lie algebras of type ADE.

Induced Poisson bracket and McKay correspondence

Let $\Gamma \subset SU(2)$ be a finite subgroup. Then

$$\mathbb{C}[x,y]^\Gamma = \mathbb{C}[X,Y,Z]/\langle F(X,Y,Z)\rangle,$$

where X, Y, Z are homogeneous invariant polynomials in x, y, and $F(X,Y,Z)$ is a weighted homogeneous polynomial in X, Y, Z.

For $f, g \in \mathbb{C}[x,y]$,

$$\{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \in \mathbb{C}[x,y]^\Gamma,$$

so by computing $\{X,Y\}$, $\{Y,Z\}$, $\{Z,X\}$, one defines a Poisson structure on the space of (X,Y,Z), such that

$$\{X,Y\} = \frac{\partial F}{\partial Z}, \ \ \{Y,Z\} = \frac{\partial F}{\partial X}, \ \ \{Z,X\} = \frac{\partial F}{\partial Y},$$

and F is a Casimir.

Type A

The group is the cyclic group of order n, generated by

$$\begin{pmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{pmatrix}$$

The invariants are generated by

$$X = x^n, \qquad Y = xy, \qquad Z = y^n.$$

We have

$$\{X, Y\} = nX, \qquad \{Y, Z\} = nZ \qquad \{Z, X\} = -n^2 Y^{n-1}.$$

We have $F = n(XZ - Y^n)$ as the Casimir. One can check that

$$XZ - Y^n = 0.$$

Type D

The group is the binary dihedral group of order $4n$, generated by

$$g_1 = \begin{pmatrix} e^{2\pi i/(2n)} & 0 \\ 0 & e^{-2\pi i/(2n)} \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

When n is odd, the invariants are generated by

$$X = (xy)^2, \qquad Y = x^{2n} - y^{2n}, \qquad Z = xy(x^n - y^n)^2.$$

$$\{X, Y\} = -n(8X^{(n+1)/2} + 4Z),$$
$$\{Y, Z\} = -n(4(n+1)X^{(n-1)/2}Z - 2Y^2),$$
$$\{Z, X\} = 4nXY.$$

We have $F = 2n(XY^2 - Z^2 - 4X^{(n+1)/2}Z)$ as the Casimir. One has:

$$XY^2 - Z^2 - 4X^{(n+1)/2}Z = 0.$$

Type D

When n is even, the invariants are generated by

$$X = x^2y^2, \qquad Y = (x^n - y^n)^2, \qquad Z = xy(x^{2n} - y^{2n}).$$

We have

$$\begin{aligned}
\{X, Y\} &= -4nZ, \\
\{Y, Z\} &= n(4(n+2)X^{n/2}Y + 2Y^2), \\
\{Z, X\} &= 4n(XY + 2X^{n/2+1}).
\end{aligned}$$

We have $F = 2n(XY^2 - Z^2 + 4X^{n/2+1}Y)$ as the Casimir, and we have

$$XY^2 - Z^2 + 4X^{n/2+1}Y = 0.$$

Type E_6

Γ is the binary tetrahedral group of order 24, generated by

$$g_1 = \begin{pmatrix} e^{2\pi i/4} & 0 \\ 0 & e^{-2\pi i/4} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad g_3 = \frac{1}{1-i} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

The invariants are generated by

$$\begin{aligned} X &= xy(x^4 - y^4), \\ Y &= x^8 + 14x^4y^4 + y^8, \\ Z &= x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}. \end{aligned}$$

$$\{X, Y\} = -8Z, \qquad \{Y, Z\} = 12y^2, \qquad \{Z, X\} = -1728x^3.$$

We have $F = -4(108X^4 - Y^3 + Z^2)$ as the Casimir, and we have

$$108X^4 - Y^3 + Z^2 = 0.$$

Type E_7

Γ is the binary octahedral group of order 48, generated by

$$g_1 = \begin{pmatrix} e^{2\pi i/8} & 0 \\ 0 & e^{-2\pi i/8} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad g_3 = \frac{1}{1-i}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

$$\begin{aligned} X &= x^2y^2(x^4-y^4)^2, \\ Y &= x^8+14x^4y^4+y^8, \\ Z &= xy(x^{16}-34x^{12}y^8+34x^8y^{12}-y^{16}). \end{aligned}$$

We have

$$\{X,Y\} = -16Z, \quad \{Y,Z\} = 8Y^4 - 2592X^3, \quad \{Z,X\} = 24XY^2.$$

We have $F = -8(81X^4 - XY^4 + Z^2)$ as the Casimir, and we have

$$81X^4 - XY^4 + Z^2 = 0.$$

Type E_8

Γ is the binary icosahedral group of order 120, generated by

$$g_1 = \begin{pmatrix} \xi_{10} & 0 \\ 0 & \xi_{10}^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad g_3 = \frac{1}{\sqrt{5}} \begin{pmatrix} \xi_5 - \xi_5^4 & \xi_5^2 - \xi_5^3 \\ \xi_5^2 - \xi_5^3 & -\xi_5 + \xi_5^4 \end{pmatrix}$$

where $\xi_n = e^{2\pi i/n}$. The invariants are generated by

$$\begin{aligned} X &= xy(x^{10} + 11x^5y^5 - y^{10}), \\ Y &= -(x^{20} + y^{20}) + 228(x^{15}y^5 - x^5y^{15}) - 494x^{10}y^{10}, \\ Z &= x^{30} + y^{30} + 522(x^{25}y^5 - x^5y^{25}) - 10005(x^{20}y^{10} + x^{10}y^{20}). \end{aligned}$$

$$\{X, Y\} = 20Z, \qquad \{Y, Z\} = -86400X^4, \qquad \{Z, X\} = 30Y^2.$$

We have $F = -10(1728X^5 - Y^3 - Z^2)$ as the Casimir, and we have

$$1728X^5 - Y^3 - Z^2 = 0.$$

Future study of the induced Poisson bracket

It turns out that a natural deformation of the above Poisson structure has been introduced in the setting of transversal Poisson structure of nilpotent orbits.

The parameter space is the parameter space of the universal unfolding of the corresponding simple singularity, hence it has a natural structure of a Frobenius manifold and it is related to the FJRW theory.

Again we are facing with the problems of investigating the interactions between the Poisson structure and Frobenius manifold structure, and understanding them from the point of view of emergent geometry.

A Conjecture

Since the Poisson structures we have considered are induced from a symplectic structure on $\mathbb{C}^2$, the Moyal products induce deformation quantizations as in the case of modular forms.

We conjecture all such deformation quantizations coincide with the deformation quantizations constructed by Kontsevich.

Thank you very much for your attentions!