Geometry of third-order homogeneous Hamiltonian operators and applications to integrable systems

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First-order Dubrovin–Novikov (homogeneous) operators

Dubrovin–Novikov (homogeneous) operators were introduced in 1983 for the Hamiltonian formalism of hydrodynamic-type equations

$$u^i_t = v^i_j(\mathbf{u})u^j_x = P_1^{ij}\frac{\delta \mathcal{H}_1}{\delta u^j} \quad \mathcal{H}_1 = \int h(\mathbf{u})dx$$

$\mathbf{u} = (u^i(t,x))$, $i, j = 1,\ldots,n$ (n-components). The operators have the form

$$P_1^{ij} = g^{ij}(\mathbf{u})\partial_x + b^{ij}_k(\mathbf{u})u^k_x$$

Homogeneity: $\deg \partial_x = 1$.

Geometry of 1st-order Dubrovin–Novikov operators

Any change of coordinates of the type $\bar{u}^i = \bar{u}^i(u^j)$ will not change the 'nature' of the above operator. g^{ij} transforms as a contravariant 2-tensor; usually it is required that g^{ij} is non-degenerate; $\Gamma^j_{ik} = -g_{is}b^{sj}_k$ transforms as a linear connection.

Conditions for P_1 to define a Poisson bracket:

- $P_1^* = -P_1$ is equivalent to: symmetry of g^{ij}, $\nabla[\Gamma]g = 0$;
- $[P_1, P_1] = 0$ is equivalent to: g_{ij} flat pseudo-Riemannian metric and $\Gamma^j_{ik} = \Gamma^j_{ki}$, or Γ is the Levi-Civita connection of g.

Third-order Dubrovin–Novikov operators

Dubrovin–Novikov operators were defined for higher orders too. In particular

$$\begin{aligned} R_3^{ij} =& g^{ij}(\mathbf{u})\partial_x^3 + b_k^{ij}(\mathbf{u})u_x^k\partial_x^2 \\ &+ [c_k^{ij}(\mathbf{u})u_{xx}^k + c_{km}^{ij}(\mathbf{u})u_x^k u_x^m]\partial_x \\ &+ d_k^{ij}(\mathbf{u})u_{xxx}^k + d_{km}^{ij}(\mathbf{u})u_x^k u_{xx}^m + d_{kmn}^{ij}(\mathbf{u})u_x^k u_x^m u_x^n, \end{aligned}$$

Examples of Hamiltonian equations of the form

$$u_t^i = R_3^{ij}\left(\frac{\delta\mathcal{H}_2}{\delta u^j}\right)$$

are in the 2-component case the Chaplygin gas equation (Mokhov DrSc thesis, '96) and the 3-component case WDVV equation (Ferapontov, Galvao, Mokhov, Nutku CMP '95).

Example: 3-component WDVV equation

The simplest associativity (WDVV) equation:

$$f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xtt}$$

can be presented by $a = f_{xxx}$, $b = f_{xxt}$, $c = f_{xtt}$ as

$$a_t = b_x, \quad b_t = c_x, \quad c_t = (b^2 - ac)_x.$$

From Ferapontov, Galvao, Mokhov, Nutku, CMP (1997), there are two local Dubrovin-Novikov Hamiltonian operators, first-order P_1 and third-order R_3,

$$R_3 = \begin{pmatrix} 0 & 0 & \partial_x^3 \\ 0 & \partial_x^3 & -\partial_x^2 a \partial_x \\ \partial_x^3 & -\partial_x a \partial_x^2 & (\partial_x^2 b \partial_x + \partial_x b \partial_x^2 + \partial_x a \partial_x a \partial_x) \end{pmatrix}$$

First results

Non-degenerate ($\det(g^{ij}) \neq 0$) third-order homogeneous Hamiltonian operators have the canonical form (Potemin '86, '97; Potemin–Balandin, '01; Doyle '95):

$$R_3 = \partial_x \circ (g^{ij}\partial_x + c^{ij}_k u^k_x) \circ \partial_x,$$

where (Ferapontov, Pavlov, V., JGP 2014)

$$c_{nkm} = \frac{1}{3}(g_{nm,k} - g_{nk,m}),$$
$$g_{mk,n} + g_{kn,m} + g_{mn,k} = 0,$$
$$c_{mnk,l} = -g^{pq} c_{pml} c_{qnk}.$$

Monge metric

Projective-geometric interpretation: g_{ij} is the Monge form of a **quadratic line complex**, c_{ijk} is the corresponding **tangential line complex**;

Differential-geometric interpretation: $c^i_{jk} = g^{is}c_{sjk}$ is a **flat metric connection** with torsion of the **first Cartan type**.

Monge metrics

Example: $n = 3$

$$
\begin{aligned}
g_{11} &= -[R_{12}(u^2)^2 + R_{13}(u^3)^2 + 2B_{12}u^2u^3 + 2H_{12}u^2 + 2H_{13}u^3 + D_1],\\
g_{22} &= -[R_{12}(u^1)^2 + R_{23}(u^3)^2 + 2B_{22}u^1u^3 + 2H_{21}u^1 + 2H_{23}u^3 + D_2],\\
g_{33} &= -[R_{23}(u^2)^2 + R_{13}(u^1)^2 + 2B_{32}u^1u^2 + 2H_{31}u^1 + 2H_{32}u^2 + D_3],
\end{aligned}
$$

$$
\begin{aligned}
g_{12} &= R_{12}u^1u^2 + B_{12}u^1u^3 + B_{22}u^2u^3 - B_{32}(u^3)^2 + H_{12}u^1 + H_{21}u^2 + (E_2 - E_1)u^3 + F_{12},\\
g_{13} &= R_{13}u^1u^3 + B_{12}u^1u^2 - B_{22}(u^2)^2 + B_{32}u^2u^3 + H_{13}u^1 + H_{31}u^3 + (E_1 - E_3)u^2 + F_{13},\\
g_{23} &= R_{23}u^2u^3 - B_{12}(u^1)^2 + B_{22}u^1u^2 + B_{32}u^1u^3 + H_{23}u^2 + H_{32}u^3 + (E_3 - E_2)u^1 + F_{23},
\end{aligned}
$$

WDVV example revisited

The operator:

$$R_3 = \begin{pmatrix} 0 & 0 & \partial_x^3 \\ 0 & \partial_x^3 & -\partial_x^2 a \partial_x \\ \partial_x^3 & -\partial_x a \partial_x^2 & (\partial_x^2 b \partial_x + \partial_x b \partial_x^2 + \partial_x a \partial_x a \partial_x) \end{pmatrix}$$

The operator is completely determined by its metric:

$$g_{ij} = \begin{pmatrix} -2b & a & 1 \\ a & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Classification results for third-order operators

n = 1: The 1-component case was described by Gel'fand-Dorfman – point-equivalent to ∂_x^3.
n = 2: three normal forms of homogeneous third-order Hamiltonian operators up to **point transformations** (Ferapontov, Pavlov, V, JGP 2014)

$$R_3^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x^3, \qquad R_3^{(2)} = \partial_x \begin{pmatrix} 0 & \partial_x \frac{1}{u^1} \\ \frac{1}{u^1}\partial_x & \frac{u^2}{(u^1)^2}\partial_x + \partial_x \frac{u^2}{(u^1)^2} \end{pmatrix} \partial_x,$$

$$R_3^{(3)} = \partial_x \begin{pmatrix} \partial_x & \partial_x \frac{u^2}{u^1} \\ \frac{u^2}{u^1}\partial_x & \frac{(u^2)^2+1}{2(u^1)^2}\partial_x + \partial_x \frac{(u^2)^2+1}{2(u^1)^2} \end{pmatrix} \partial_x.$$

Classification results for third-order operators

$\mathbf{n = 3}$: six normal forms of homogeneous third-order Hamiltonian operators up to **reciprocal transformations of projective type** (Ferapontov, Pavlov, V, JGP 2014)

$$g^{(1)} = \begin{pmatrix} (u^2)^2 + c & -u^1u^2 - u^3 & 2u^2 \\ -u^1u^2 - u^3 & (u^1)^2 + c(u^3)^2 & -cu^2u^3 - u^1 \\ 2u^2 & -cu^2u^3 - u^1 & c(u^2)^2 + 1 \end{pmatrix},$$

$$g^{(2)} = \begin{pmatrix} (u^2)^2 + 1 & -u^1u^2 - u^3 & 2u^2 \\ -u^1u^2 - u^3 & (u^1)^2 & -u^1 \\ 2u^2 & -u^1 & 1 \end{pmatrix},$$

$$g^{(3)} = \begin{pmatrix} (u^2)^2 + 1 & -u^1u^2 & 0 \\ -u^1u^2 & (u^1)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$g^{(4)} = \begin{pmatrix} -2u^2 & u^1 & 0 \\ u^1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g^{(5)} = \begin{pmatrix} -2u^2 & u^1 & 1 \\ u^1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad g^{(6)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Classification results for third-order operators

$\mathbf{n=4}$: Ferapontov, Pavlov, V., IMRN (2016).
Any Monge metric of a third-order homogeneous Hamiltonian operator admits the following decomposition:

$$g_{ij} = \varphi_{\alpha\beta}\psi_i^\alpha\psi_j^\beta$$

where $\psi_i^\alpha du^i$ are *linear line complexes*, $\varphi_{\alpha\beta}$ is a non-degenerate bilinear form and

$$\varphi_{\alpha\beta}\psi_{(i}^\alpha\psi_{j,k)}^\beta = 0.$$

The above condition can always be fulfilled for any Monge metric (*generalized Clebsch normal form*). From the projective classification of metabelian Lie algebrae (Galitski-Timashev 1999) we have a classification of 4-frames of linear line complexes $\psi_i^\alpha du^i$ and $\varphi_{\alpha\beta}$ with 32 classes. $\mathbf{n\geqslant 5}$ **wild!**

Applications to integrable systems

EV Ferapontov, MV Pavlov, V. (2017 - work in progress).

Theorem. Each third-order homogeneous operator determines a family of hydrodynamic-type systems which are *linearly degenerate* and in the *Temple class*. The system is equivalent to a *linear line congruence.*

Classification results under the action of reciprocal transformations of projective type (*also involving* t!) are available in $n = 2$ (Chaplygin gas eq.), $n = 3$ (WDVV system); work in progress on $n = 4$ ...

More applications: trios of compatible operators

Lorenzoni, Savoldi, V. `arxiv:1607.07020` (2016)

It was observed (Olver and Rosenau, 1996) that many PDEs admit a bi-Hamiltonian structure which is defined by a trio of mutually compatible Hamiltonian operators.

Examples: the scalar case

$$P_1 = \partial_x, \qquad Q_1 = 2u\partial_x + u_x, \quad R_3 = \partial_x^3.$$

Poisson pencil of KdV hierarchy (Magri (1978)):

$$\Pi_\lambda = Q_1 + \epsilon^2 R_3 - \lambda P_1 = 2u\partial_x + u_x - \lambda\partial_x + \epsilon^2\partial_x^3$$

Poisson pencil of Camassa–Holm hierarchy:

$$\tilde{\Pi}_\lambda = Q_1 - \lambda(P_1 + \epsilon^2 R_3) = 2u\partial_x + u_x - \lambda(\partial_x + \epsilon^2\partial_x^3).$$

Examples: the 2-component case

$$P_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} 2u\partial_x + u_x & v\partial_x \\ \partial_x v & -2\partial_x \end{pmatrix},$$

$$R_2 = \begin{pmatrix} 0 & -\partial_x^2 \\ \partial_x^2 & 0 \end{pmatrix}$$

- $\Pi_\lambda = Q_1 + \epsilon^2 R_3 - \lambda P_1$ AKNS (or two-boson) hierarchy;
- $\tilde{\Pi}_\lambda = Q_1 - \lambda(P_1 + \epsilon^2 R_3)$ two-component Camassa-Holm hierarchy.

We say the pencils of the type of Π_λ (or $\tilde{\Pi}_\lambda$) to be **bi-Hamiltonian structures of KdV-type**.

Classification of bi-Hamiltonian structures of KdV type

The problem: classify compatible trios of Hamiltonian operators P_1, Q_1, R_m where P_1 and Q_1 are homogeneous first-order Hamiltonian operators (Dubrovin and Novikov, 1983)

$$P_1 = g^{ij}\partial_x + \Gamma^{ij}_k u^k_x, \qquad Q_1 = h^{ij}\partial_x + \Xi^{ij}_k u^k_x,$$

and R_m is a homogeneous Hamiltonian operator of degree $m > 1$.

A strategy for the classification

The above pencils can be thought as a deformation of a Poisson pencil of hydrodynamic type.

Due to the general theory of deformations the only interesting cases are $m = 2$ and $m = 3$. In the remaining case the deformations can always be eliminated by Miura type transformations (Liu and Zhang, 2005).

Our strategy: knowing the normal forms of R_2 and R_3 we find all possible compatible first-order Poisson pencils of hydrodynamic type $P_1 - \lambda Q_1$. This yields bi-Hamiltonian structures of KdV type with $n = 2$ (or $n = 3$).

Homogeneous Hamiltonian operators, degree 2

Second-order operators R_2 have been completely described in the non degenerate case $\det(\ell^{ij}) \neq 0$ (Potemin 1987, 1991, 1997; Doyle 1993):

$$R_2 = \partial_x \ell^{ij} \partial_x,$$

where $\ell_{ij} = T_{ijk}u^k + T^0_{ij}$, and T_{ijk}, T^0_{ij} are constant and completely skew-symmetric, without further conditions.

When $n = 2$ there is only one homogeneous second-order Hamiltonian operator (up to point transformations):

$$R_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_x^2.$$

Results: trios P_1, Q_1, R_2

$\mathbf{n = 1}$: nothing new, KdV and Camassa-Holm hierarchies.

We focus on the $\mathbf{n = 2}$**-component** case.
In what follows c_i are constants, Levi-Civita conditions:

$$g^{is}\Gamma^{jk}_s = g^{js}\Gamma^{ik}_s$$
$$\Gamma^{ij}_k + \Gamma^{ji}_k = \partial_k g^{ij}$$

Theorem: P_1 is compatible with R_2 if and only if

$$g^{11} = c_1 u^1 + c_2, \tag{1a}$$
$$g^{12} = \frac{1}{2}c_3 u^1 + \frac{1}{2}c_1 u^2 + c_5 \tag{1b}$$
$$g^{22} = c_3 u^2 + c_4. \tag{1c}$$

The above metric is flat for every value of the parameters. Any Q_1 with a metric h^{ij} of the above form makes a trio P_1, Q_1, R_2.

Results: trios P_1, Q_1, $R_3^{(1)}$

Theorem: P_1 is a Hamiltonian operator compatible with $R_3^{(1)}$ if and only if

$$\begin{aligned} g^{11} =& c_1u^1 + c_2u^2 + c_3, \\ g^{12} =& c_4u^1 + c_1u^2 + c_5 \\ g^{22} =& c_6u^1 + c_4u^2 + c_7 \end{aligned} \tag{2}$$

together with the Levi-Civita conditions

$$c_1c_4 - c_2c_6 = 0, \quad c_3c_4 - c_7c_2 = 0, \quad c_3c_6 - c_1c_7 = 0. \tag{3}$$

The above conditions imply the flatness of g.
There is a 5 parameter family of mutually commuting pairs P_1, Q_1 that commute with $R_3^{(1)}$.

Results: trios P_1, Q_1, $R_3^{(2)}$

Theorem: P_1 is a Hamiltonian operator compatible with $R_3^{(2)}$ if and only if

$$g^{11} = c_1 u^1 + c_2 u^2, \tag{4a}$$

$$g^{12} = c_4 u^1 + \frac{c_3}{u^1} + \frac{c_2 (u^2)^2}{2u^1}, \tag{4b}$$

$$g^{22} = 2c_4 u^2 + \frac{c_6}{u^1} - \frac{c_1 (u^2)^2}{u^1} + c_5, \tag{4c}$$

together with the Levi-Civita conditions

$$c_2 c_6 + 2c_1 c_3 = 0, \quad c_2 c_5 = 0, \quad c_1 c_5 = 0. \tag{5}$$

The above conditions imply the flatness of g.
There exists a 4 parameter family of mutually commuting pairs P_1, Q_1 that commute with $R_3^{(2)}$.

Results: trios P_1, Q_1, $R_3^{(3)}$

Theorem: P_1 is a Hamiltonian operator compatible with $R_3^{(3)}$ if and only if

$$g^{11} = c_1 u^1 + c_2 u^2 + c_3, \tag{6a}$$

$$g^{12} = c_4 u^1 - \frac{c_2}{2u^1} + \frac{c_3 u^2}{u^1} + \frac{c_2 (u^2)^2}{2u^1}, \tag{6b}$$

$$g^{22} = 2c_4 u^2 + \frac{c_1}{u^1} + \frac{c_5 u^2}{u^1} - \frac{c_1 (u^2)^2}{u^1} + c_6, \tag{6c}$$

together with the Levi-Civita conditions

$$c_2 c_5 + 2c_1 c_3 = 0, \quad c_2 c_6 - 2c_3 c_4 = 0, \quad c_1 c_6 + c_4 c_5 = 0, \tag{7}$$

The above conditions imply the flatness of g.
There exists a 4 parameter family of mutually commuting pairs P_1, Q_1 that commute with $R_3^{(3)}$.

Known examples with R_2

- The Kaup–Broer system (Kupershmidt 1985):

$$\begin{cases} u^1_t = ((u^1)^2/2 + u^2 + \beta u^1_x)_x, \\ u^2_t = (u^1 u^2 + \alpha u^1_{xx} - \beta u^2_x)_x, \end{cases} \tag{8}$$

- In De Sole, Kac, Turhan 2014, a six-parameter family of pairwise compatible Hamiltonian operators defined by the cohomology spaces of curves is considered. A subset of these operator belongs to our class, with R_2.

Known examples with $R_3^{(1)}$

- A version of the Dispersive Water Waves system (Antonowicz-Fordy, 1989):

$$
\begin{aligned}
u^1_t =& \frac{1}{4}u^2_{xxx} + \frac{1}{2}u^2u^1_x + u^1u^2_x,\\
u^2_t =& u^1_x + \frac{3}{2}u^2u^2_x
\end{aligned}
$$

- Coupled Harry-Dym hierarchy (Antonowicz-Fordy, 1988):

$$
\begin{aligned}
u^1_1 =& \left(\frac{1}{4(u^2)^{1/2}}\right)_{xxx} - \alpha\left(\frac{1}{(u^2)^{1/2}}\right)_x\\
u^2_t =& u^1\left(\frac{1}{(u^2)^{1/2}}\right)_x + \frac{u^1_x}{2(u^2)^{1/2}}
\end{aligned}
$$

New example with $R_3^{(2)}$

Two identical copies of the metric which solves the compatibility problem with $R_3^{(2)}$, g and h.
Metric g of P_1 parametrized by c_i.
Metric h of Q_1 parametrized by d_i.

Choosing $c_3 = 0$, $d_3 = 1$, $c_2 = 2$, $c_4 = 1$, $d_4 = 0$, $d_5 = 0$ we obtain the bi-Hamiltonian system

$$
\begin{aligned}
u^1_{t_2} &= 2u^2u^1_x + u^1u^2_x \\
u^2_{t_2} &= u^1u^1_x + 2u^2u^2_x - \frac{u^1_x u^1_{xx}}{(u^1)^2} + \frac{u^1_{xxx}}{u^1},
\end{aligned}
$$

Another new example with $R_3^{(2)}$

Choosing $c_4 = 0$, $c_1 = -1$, $c_6 = -1$, $c_2 = 0$, $d_2 = 0$, $d_1 = 0$ we obtain the bi-Hamiltonian system

$$
\begin{aligned}
u^1_{t_2} =& \frac{3}{2}\frac{u^2_x}{u^1} - \frac{3}{2}\frac{u^2 u^1_x}{(u^1)^2} - \frac{u^1_{xxx}}{(u^1)^3} + 9\frac{u^1_x u^1_{xx}}{(u^1)^4} - 12\frac{(u^1_x)^3}{(u^1)^5} \\
u^2_{t_2} =& \frac{3}{2}\frac{(1-(u^2)^2)u^1_x}{(u^1)^3} + \frac{3}{2}\frac{u^2 u^2_x}{(u^1)^2} - \frac{30u^2(u^1_x)^3}{(u^1)^6} + 10\frac{u^2_x(u^1_x)^2}{(u^1)^5} \\
& + 12\frac{u^2_x(u^1)^2_x}{(u^1)^5} + -\frac{3u^2_x u^1_{xx}}{(u^1)^4} - 2\frac{u^2 u^1_{xxx}}{(u^1)^4} - \frac{u^2_{xx}u^1_x}{(u^1)^4}.
\end{aligned}
$$

New examples with $R_3^{(3)}$

Choosing

$$c_1 = 1, \quad c_2 = -1, \quad d_3 = 1, \quad c_3 = 0, \quad c_4 = 0$$

one easily gets the first non trivial flows of the associated bi-Hamiltonian hierarchy. Too big to be shown.

The multiparametric families of solutions allow for a great variety of bi-Hamiltonian systems.

Dubrovin and Zhang's perturbative approach

Our pencils can be regarded as deformations of a Poisson pencil of hydrodynamic type. The classification of deformations with respect to the Miura group

$$\tilde{u}^i = f^i(u^1, \ldots, u^n) + \sum_{k\geq 1} \epsilon^k F_k^i(u, u_x, \ldots, u_{(k)}), \tag{9}$$

has been obtained in recent years in the semisimple case (see Liu and Zhang (2005).

Deformations are uniquely determined by their dispersionless limit and by n functions of one variable, the central invariants. Deformations with vanishing central invariants can be transformed to their dispersionless limit, and are *trivial*.

Central invariants of the examples with $R_3^{(2)}$

First example, canonical coordinates:

$$\lambda^1 = (u^1 + u^2)^2, \qquad \lambda^2 = (u^1 - u^2)^2,$$

central invariants:

$$s_1 = -\frac{1}{8\sqrt{\lambda^1}}, \qquad s_2 = \frac{1}{8\sqrt{\lambda^2}}.$$

Second example, canonical coordinates:

$$\lambda^1 = \frac{u^2 + 1}{u^1}, \qquad \lambda^2 = \frac{u^2 - 1}{u^1}$$

central invariants:

$$s_1 = \frac{1}{2}, \qquad s_2 = -\frac{1}{2}.$$

Central invariants of the example with $R_3^{(3)}$

In the example with $R_3^{(3)}$ (not shown), canonical coordinates:

$$\lambda^1 = -\frac{1}{2}\,\frac{(u^2)^2-1}{u^2}, \qquad \lambda^2 = \frac{1}{2}\,\frac{4\,(u^1)^2-4\,u^1u^2+(u^2)^2-1}{2u^1-u^2},$$

central invariants:

$$s_1 = \frac{1}{2}\,\frac{\lambda^1\sqrt{(\lambda^1)^2+1}-(\lambda^1)^2-1}{(\lambda^1)^2+1},$$

$$s_2 = -\,\frac{1}{2}\,\frac{\lambda^2\sqrt{(\lambda^2)^2+1}+(\lambda^2)^2+1}{(\lambda^2)^2+1}.$$

This means that all the new examples of Poisson pencils obtained in the previous Section are not Miura-trivial.

Symbolic computations

Within the REDUCE CAS (now free software) we use the packages CDIFF and CDE, freely available at `http://gdeq.org`.

Forthcoming book:
JS Krasil'shchik, AM Verbovetsky, RF Vitolo: The symbolic computation of integrability structures for PDEs, book, to appear in the Springer series "Texts and Monographs in Symbolic Computations" (2017).

Cooperation with AC Norman (Trinity College, Cambridge) to improvements and documentation of REDUCE's kernel.

Open problems

- Classification of trios of compatible Hamiltonian operators for $n \geqslant 3$. Many new integrable systems already found in $n = 3$, $n = 4$. Projective classification?
- Geometry of WDVV equations. All of them have a third-order H.o., and many have a first-order H.o.
- Non-local Hamiltonian operators of second and third order.

Thank you!

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Example: 2-component Chaplygin gas equation

(O. Mokhov, '96) The Monge–Ampère equation $u_{tt}u_{xx} - u_{xt}^2 = -1$ can be reduced to hydrodynamic form

$$a_t = b_x, \quad b_t = \left(\frac{b^2 - 1}{a}\right)_x,$$

via the change of variables $a = u_{xx}$, $b = u_{xt}$. It possesses the Hamiltonian formulation

$$\begin{pmatrix} a \\ b \end{pmatrix}_t = \partial_x \begin{pmatrix} 0 & \partial_x \frac{1}{a} \\ \frac{1}{a}\partial_x & \frac{b}{a^2}\partial_x + \partial_x \frac{b}{a^2} \end{pmatrix} \partial_x \begin{pmatrix} \delta H/\delta a \\ \delta H/\delta b \end{pmatrix},$$

and the nonlocal Hamiltonian,

$$H = -\int \left(\frac{1}{2} a(\partial_x^{-1} b)^2 + \partial_x^{-2} a\right) dx.$$