Matrix Integral, Hodge Integral, and Integrable System

Si-Qi Liu

based on joint work with
B. Dubrovin, D. Yang, Y. Zhang, and C. Zhou

Department of Mathematical Sciences, Tsinghua University, P. R. China

Feb 28, 2017

References

1. Boris Dubrovin, Si-Qi Liu, Di Yang, Youjin Zhang.
   Hodge integrals and tau-symmetric integrable hierarchies of Hamiltonian evolutionary PDEs.
   Adv. Math. 293 (2016), 382–435.
2. Boris Dubrovin, Si-Qi Liu, Di Yang, Youjin Zhang.
   Hodge-GUE correspondence and the discrete KdV equation.
   Preprint arXiv: 1612.02333.
3. Si-Qi Liu, Youjin Zhang, Chunhui Zhou.
   Fractional Volterra Hierarchy.
   Preprint arXiv: 1702.02840.
4. Coming soon...

What is a matrix integral?

- H_N: a certain submanifold of the space of $N \times N$ matrices with entries in $\mathbb{R}$ or $\mathbb{C}$
- $\mathrm{d}\mu(M)$: a (probability) measure on H_N
- $f : H_N \to \mathbb{R}$ or $\mathbb{C}$: a certain measurable function

Matrix Integral

$$Z_N = \int_{H_N} f(M)\,\mathrm{d}\mu(M)$$

Fundamental problem

What is the limiting behavior of Z_N when $N \to \infty$?

Example (?): Wigner's Semicircle Law

Wigner Matrices

An $N \times N$ random matrix M is called a Wigner matrix, if it is real symmetric, and its entries M_{ij} $(i \leq j)$ are iid, and satisfy

$$E[M_{ij}] = 0, \quad E[M_{ij}^2] = 1,$$

and all higher momemts exist.

Semicircle Law

Let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of a Wigner matrix M, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{\frac{\lambda_i}{\sqrt{N}}} = \begin{cases} \frac{1}{2\pi}\sqrt{4-x^2}\,\mathrm{d}x, & |x| \leq 2, \\ 0, & |x| > 2. \end{cases}$$

Example: Gaussian Unitary Ensemble (GUE)

- H_N: the space of $N \times N$ Hermitian matrices
- $\mathrm{d}M$: the Lebesgue measure on H_N
- $$\mathrm{d}\mu(M) = c_N e^{-\frac{1}{2}\mathrm{tr}(M^2)}\mathrm{d}M$$
- $$\frac{1}{c_N} = \int_{H_N} e^{-\frac{1}{2}\mathrm{tr}(M^2)}\mathrm{d}M = 2^{\frac{N}{2}}\pi^{\frac{N^2}{2}}$$

Problem: how to compute the following integrals?

$$\langle \mathrm{tr}\left(M^{p_1}\right)\cdots\mathrm{tr}\left(M^{p_k}\right)\rangle := \int_{H_N} \mathrm{tr}\left(M^{p_1}\right)\cdots\mathrm{tr}\left(M^{p_k}\right)\mathrm{d}\mu(M)$$

The partition function of GUE

Introduce the generating function $Z_N \in \mathbb{Q}[[g_1, g_2, \ldots]]$

$$Z_N = \sum_{k\geq 0} \sum_{p_1,\ldots,p_k\geq 1} \frac{g_{p_1}\cdots g_{p_k}}{k!} \langle \mathrm{tr}\,(M^{p_1})\cdots \mathrm{tr}\,(M^{p_k})\rangle$$

$$= \int_{H_N} \exp\left(\sum_{p\geq 1} g_p\,\mathrm{tr}\,(M^p)\right) \mathrm{d}\mu(M)$$

then we have

$$\langle \mathrm{tr}\,(M^{p_1})\cdots \mathrm{tr}\,(M^{p_k})\rangle = \left.\frac{\partial^k Z_N}{\partial g_{p_1}\cdots \partial g_{p_k}}\right|_{g_1=g_2=\cdots=0}.$$

GUE and the Toda lattice hierarchy

Introduce the functions:

$$u_N = \frac{\partial}{\partial g_1} \log \frac{Z_{N+1}}{Z_N}, \quad v_N = N \frac{Z_{N+1} \cdot Z_{N-1}}{Z_N \cdot Z_N},$$

and define the operator

$$L = \Lambda + u_N + v_N \Lambda^{-1}, \quad \Lambda \left(\cdots\right)_N = \left(\cdots\right)_{N+1},$$

then L satisfies

$$\frac{\partial L}{\partial g_p} = [\left(L^p\right)_+, L], \quad p = 1, 2, \ldots$$

which is called the Toda lattice hierarchy.

GUE and Virasoro constraints

The partition function Z_N also satisfies the following linear differential equations, which are called Virasoro constraints:

$$\frac{\partial Z_N}{\partial g_1} = \sum_{k\geq 2} k\, g_k \frac{\partial Z_N}{\partial g_{k-1}} + g_1\, N\, Z_N,$$

$$\frac{\partial Z_N}{\partial g_2} = \sum_{k\geq 1} k\, g_k \frac{\partial Z_N}{\partial g_k} + N^2\, Z_N,$$

$$\frac{\partial Z_N}{\partial g_{2+q}} = \sum_{k\geq 1} k\, g_k \frac{\partial Z_N}{\partial g_{k+q}} + 2\, N \frac{\partial Z_N}{\partial g_q} + \sum_{p=1}^{q-1} \frac{\partial^2 Z_N}{\partial g_p \partial g_{q-p}}, \quad q \geq 1.$$

The Toda lattice hierarchy and Virasoro constraints uniquely determine Z_N.

Example: the Kontsevich matrix integral

- H_N: the space of $N \times N$ Hermitian matrices
- $\mathrm{d}M$: the Lebesgue measure on H_N
- X: a positive definite $N \times N$ Hermitian matrix
- $\mathrm{d}\mu_X(M)$: a probability measure on H_N

$$\mathrm{d}\mu_X(M) = c_X e^{-\frac{1}{2}\mathrm{tr}\left(M^2 X\right)}\mathrm{d}M, \qquad \int_{H_N} \mathrm{d}\mu_X(M) = 1$$

- $t_i = -(2i-1)!!\,\mathrm{tr}X^{-1-2i}$, $i = 0, 1, 2, \ldots$
- $Z_{\mathrm{WK}} \in \mathbb{Q}[[t_0, t_1, t_2, \ldots]]$:

$$Z_{\mathrm{WK}} = \lim_{N\to\infty} \int_{H_N} \exp\left(\frac{\sqrt{-1}}{6}\mathrm{tr}\left(M^3\right)\right) \mathrm{d}\mu_X(M)$$

Kontsevich Theorem (1)

Denote by $F_{\mathrm{WK}} = \log Z_{\mathrm{WK}}$, $F_{\mathrm{WK}} = \sum_{g\geq 0} F_{\mathrm{WK},g}$, then

$$\left.\frac{\partial^k F_{\mathrm{WK},g}}{\partial t_{p_1}\cdots\partial t_{p_k}}\right|_{t_0=t_1=t_2=\cdots=0} =: \langle \tau_{p_1}\cdots\tau_{p_k}\rangle_g = \int_{\overline{\mathcal{M}}_{g,k}} \psi_1^{p_1}\cdots\psi_k^{p_k},$$

where

- $\overline{\mathcal{M}}_{g,k}$: the moduli space of k-marked stable curve of genus

$$g = \frac{1}{3}\left(\sum_{i=1}^{k} p_i - k + 3\right) \in \mathbb{N}$$

- ψ_i: c_1 of the i-th tautological line bundle on $\overline{\mathcal{M}}_{g,k}$

Kontsevich Theorem (2)

Define $U = \frac{\partial^2}{\partial t_0^2} F_{\mathrm{WK}}$, then U satisfies the Korteweg-de Vries (KdV) hierarchy

$$\frac{\partial U}{\partial t_p} = \frac{\partial R_{p+1}}{\partial t_0},$$

where $\{R_0, R_1, R_2, \dots\}$ is a series of polynomials of U, $\frac{\partial U}{\partial t_0}$, $\frac{\partial^2 U}{\partial t_0^2}$, ... that is given by $R_0 = 1$ and the recursion relation

$$\left(p + \frac{1}{2}\right) \frac{\partial R_{p+1}}{\partial t_0} = U \frac{\partial R_p}{\partial t_0} + \frac{1}{2} \frac{\partial U}{\partial t_0} R_p + \frac{1}{8} \frac{\partial^3 R_p}{\partial^3 t_0},$$

Witten conjecture (Kontsevich Theorem)

The generating function Z_{WK} for $\langle \tau_{p_1} \cdots \tau_{p_k} \rangle$ is a tau function of the KdV hierarchy.

Virasoro constraints for Z_{WK}

The partition function Z_{WK} is also uniquely determined by

$$L_m Z_{\mathrm{WK}} = 0, \quad m \geq -1,$$

where

$$L_{-1} = -\frac{\partial}{\partial t_0} + \frac{t_0^2}{2} + \sum_{p\geq 0} t_{p+1}\frac{\partial}{\partial t_p},$$

$$L_0 = -\frac{3}{2}\frac{\partial}{\partial t_1} + \frac{1}{16} + \sum_{p\geq 0}\left(p+\frac{1}{2}\right) t_p \frac{\partial}{\partial t_p},$$

$$L_q = -\frac{\Gamma(5/2+q)}{\Gamma(3/2)}\frac{\partial}{\partial t_{q+1}} + \sum_{p\geq 0}\frac{\Gamma(q+p+3/2)}{\Gamma(p+1/2)} t_p \frac{\partial}{\partial t_{q+p}}$$

$$+ \frac{1}{2}\sum_{p=1}^{q-1}\frac{\Gamma(p+3/2)}{\Gamma(1/2)}\frac{\Gamma(q-p+3/2)}{\Gamma(1/2)}\frac{\partial^2}{\partial t_p \partial t_{q-p}}, \quad q \geq 1.$$

What are Hodge integrals?

- $\overline{\mathcal{M}}_{g,k}$: the moduli space of k-marked stable curve of genus g
- ψ_i: c_1 of the i-th tautological line bundle on $\overline{\mathcal{M}}_{g,k}$
- $\mathbb{E}_{g,k}$: the Hodge bundle on $\overline{\mathcal{M}}_{g,k}$
- λ_j: $c_j\left(\mathbb{E}_{g,k}\right)$, where $j = 0, \ldots, g$
- γ_q: $ch_{2q-1}\left(\mathbb{E}_{g,k}\right)$, where $q = 1, 2, \ldots$

$$\langle \tau_{p_1} \cdots \tau_{p_k} \lambda_{q_1} \cdots \lambda_{q_\ell} \rangle_g = \int_{\overline{\mathcal{M}}_{g,k}} \psi_1^{p_1} \cdots \psi_k^{p_k} \lambda_{q_1} \cdots \lambda_{q_\ell},$$

$$\langle \tau_{p_1} \cdots \tau_{p_k} \gamma_{q_1} \cdots \gamma_{q_\ell} \rangle_g = \int_{\overline{\mathcal{M}}_{g,k}} \psi_1^{p_1} \cdots \psi_k^{p_k} \gamma_{q_1} \cdots \gamma_{q_\ell},$$

The generating functions for Hodge integrals

Introduce formal variables $\{s_q \mid q = 1, 2, \dots\}$, then define

$$\mathcal{H}_g = \left\langle \exp\left(\sum_{p\geq 0} t_p \tau_p - \sum_{q\geq 1} \frac{(2q)!}{B_{2q}} s_q \gamma_q \right) \right\rangle_g$$

$$\mathcal{H} = \sum_{g\geq 0} \mathcal{H}_g, \quad \mathcal{Z} = \exp \mathcal{H}.$$

Problem

Find the integrable hierarchies that $\mathcal{H}(t, s)$ or $\mathcal{Z}(t, s)$ satisfies.

Faber-Pandharipande relation

Define operators D_q $(q = 1, 2, \dots)$:

$$D_q = \sum_{p\geq 0} t_p \frac{\partial}{\partial t_{p+2q-1}} - \frac{1}{2}\sum_{p=0}^{2q-2} \frac{\partial^2}{\partial t_p \partial t_{2q-2-p}}.$$

Faber and Pandharipande proved that

$$\frac{\partial \mathcal{Z}}{\partial s_q} = D_q\left(\mathcal{Z}\right), \quad \mathcal{Z}|_{s=0} = Z_{\mathrm{WK}}(t),$$

so we have

$$\mathcal{Z} = \exp\left(\sum_{q=1}^{\infty} s_q D_q\right) Z_{\mathrm{WK}}.$$

How to compute Z_{WK}?

Suppose $V(t_0, t_1, t_2, \dots)$ is the solution to the equation

$$V = t_0 + t_1\, V + t_2\, \frac{V^2}{2} + \cdots .$$

Decompose $F_{\mathrm{WK}} = \log Z_{\mathrm{WK}}$ into $F_{\mathrm{WK}} = \sum_{g\geq 0} F_g$, then

$$F_1 = \frac{1}{24}\log V_1, \quad F_2 = \frac{V_4}{1152\, V_1^2} - \frac{7\, V_3\, V_2}{1920\, V_1^3} + \frac{V_2^3}{360\, V_1^4}, \quad \dots$$

where $V_\ell = \frac{\partial^\ell V}{\partial t_0^\ell}$. They are obtained by using the following theorem.

Quasi-triviality Theorem

There uniquely exist functions $F_g = F_g(V, V_1, \dots, V_{3g-2})$ $(g \geq 1)$ such that the transformation

$$U = V + \frac{\partial^2}{\partial t_0^2}\left(\sum_{g\geq 1} F_g(V, V_1, \dots, V_{3g-2})\right)$$

converts the KdV equation

$$\frac{\partial U}{\partial t_1} = U\frac{\partial U}{\partial t_0} + \frac{1}{12}\frac{\partial^3 U}{\partial t_0^3}$$

into its leading term

$$\frac{\partial V}{\partial t_1} = V\frac{\partial V}{\partial t_0}.$$

How to compute $\mathcal{Z}$?

Theorem (Dubrovin-L-Yang-Zhang, 2014)

There uniquely exist functions $\Delta\mathcal{H}_g(V, V_1, \ldots, V_{3g-2}; s_1, \ldots, s_g)$ such that $\mathcal{H}_g = F_g + \Delta\mathcal{H}_g$.

$$\mathcal{H}_0 = F_0,$$

$$\mathcal{H}_1 = F_1 - \frac{s_1}{2} V,$$

$$\mathcal{H}_2 = F_2 + s_1 \left(\frac{11\, V_2^2}{480\, V_1^2} - \frac{V_3}{40\, V_1} \right) + \frac{7}{40} s_1^2\, V_2 - \left(\frac{s_1^2}{10} + \frac{s_2}{48} \right) V_1^2,$$

$$\mathcal{H}_3 = \cdots\cdots$$

The Hodge hierarchy

Define

$$W = V + \frac{\partial^2}{\partial t_0^2}\left(\sum_{g\geq 1}\mathcal{H}_g(V, V_1, \ldots, V_{3g-2}; s_1, \ldots, s_g)\right),$$

then $\frac{\partial W}{\partial t_p}$ are all differential polynomials. They form a new integrable hierarchy with s_q's as parameters, which is called the Hodge hierarchy.

When $s_q = 0$, the Hodge hierarchy goes back to the KdV hierarchy, so it is a deformation of the KdV hierarchy.

The Hodge hierarchy always possesses a Hamiltonian structure.

Example: Linear case

Take

$$s_q = -\frac{B_{2q}}{2q\,(2q-1)} s^{2q-1}, \quad q \geq 1,$$

then

$$\mathcal{H} = \sum_{g\geq 0} \left\langle \Lambda_g(s) \exp\left(\sum_{p\geq 0} t_p \tau_p\right)\right\rangle_g ,$$

where

$$\Lambda_g(s) = \lambda_0 + s\,\lambda_1 + s^2\,\lambda_2 + \cdots + s^g\,\lambda_g .$$

Theorem (Buryak, 2013)

In this case, the Hodge hierarchy is equivalent to the Intermediate Long Wave (ILW) hierarchy.

Example: Cubic case

Take

$$s_q = (2^{2q} - 1)\frac{B_{2q}}{2q\,(2q-1)}s^{2q-1}, \quad q \geq 1,$$

then

$$\mathcal{H} = \sum_{g\geq 0} \left\langle \Lambda_g(s)\Lambda_g(-2s)\Lambda_g(-2s)\exp\left(\sum_{p\geq 0} t_p\tau_p\right)\right\rangle_g .$$

Conjecture A (Dubrovin-L-Yang-Zhang, 2014)

In this case, the Hodge hierarchy is equivalent to the discrete KdV hierarchy (also called Volterra lattice hierarchy).

Details of Conjecture A

Define

$$U = W + \sum_{g\geq 1} \epsilon^{2g}\, s^g\, \frac{3^{2g+2}-1}{4^{g+1}(2g+2)!} \frac{\partial^{2g} W}{\partial t_0^{2g}},$$

$$\frac{\partial}{\partial T_k} = \sum_{p\geq 0} (2\,k\,s)^p \frac{\partial}{\partial t_p},$$

then

$$4\, s^{3/2}\,\epsilon\, \frac{\partial U(x)}{\partial T_1} = e^{2\,s\,U(x+\sqrt{s}\,\epsilon)} - e^{2\,s\,U(x-\sqrt{s}\,\epsilon)},$$

$$24\, s^{3/2}\,\epsilon\, \frac{\partial U(x)}{\partial T_2} = e^{2\,s\,U^+}\left(e^{2\,s\,U} + e^{2\,s\,U^+} + e^{2\,s\,U^{++}}\right) - e^{2\,s\,U^-}\left(e^{2\,s\,U} + e^{2\,s\,U^-} + e^{2\,s\,U^{--}}\right), \ldots$$

What is the Volterra lattice hierarchy?

Denote $\partial_x = \frac{\partial}{\partial x}$, $\Lambda = e^{\epsilon \partial_x}$, and define the operator

$$L = \Lambda + e^U \Lambda^{-1},$$

then the Volterra lattice hierarchy is given by (up to rescaling)

$$\frac{\partial L}{\partial T_p} = [\left(L^{2p}\right)_+, L].$$

It is actually a reduction of the Toda lattice hierarchy by taking $u_N = 0$, or $g_{2p-1} = 0$.

Question

Both (even) GUE and Hodge are related to the Volterra lattice hierarchy, then is there a relation between them?

Hodge-GUE correspondence (1)

Consider the following even GUE partition function

$$Z_N^{\mathrm{even}}(\mathbf{g};\epsilon) = c_N \int_{H_N} \exp\left(\frac{1}{\epsilon}\mathrm{tr}\left(-\frac{M^2}{2} + \sum_{p\geq 1} g_{2p} M^{2p}\right)\right) \mathrm{d}M.$$

Its asymptotic free energy is defined as

$$F(x,\mathbf{g};\epsilon) = \log Z_N^{\mathrm{even}}(\mathbf{g};\epsilon)\big|_{N=\frac{x}{\epsilon}} - \frac{\log\epsilon}{12} = \sum_{g\geq 0} \epsilon^{2g-2} F_g(x,\mathbf{g}),$$

then define the asymptotic partition function

$$Z_{\mathrm{GUE}}(x,\mathbf{g};\epsilon) = \exp F(x,\mathbf{g};\epsilon).$$

Hodge-GUE correspondence (2)

On the Hodge side, consider the following special cubic Hodge integral

$$\mathcal{H}(\mathbf{t};\epsilon)=\sum_{g\geq 0}\epsilon^{2g-2}\left\langle \Lambda_g\left(\frac{1}{2}\right)\Lambda_g(-1)\Lambda_g(-1)\exp\left(\sum_{p\geq 0}t_p\tau_p\right)\right\rangle_g .$$

Its partition function is

$$\mathcal{Z}_{\mathrm{H}}(\mathbf{t};\epsilon)=\exp\mathcal{H}(\mathbf{t};\epsilon).$$

(We don't lose generality here because of the homogeneous condition on Hodge integrals.)

Hodge-GUE correspondence (3)

Conjecture B (Dubrovin-Yang, 2016)

We have

$$\begin{aligned} & Z_{\mathrm{GUE}}(x, \mathbf{g}; \epsilon) \\ = & e^{\frac{A}{\epsilon^2}+\zeta'(-1)} \mathcal{Z}_{\mathrm{H}}\left(\mathbf{t}\left(x+\frac{\epsilon}{2}, \mathbf{g}\right), \sqrt{2}\epsilon\right) \mathcal{Z}_{\mathrm{H}}\left(\mathbf{t}\left(x-\frac{\epsilon}{2}, \mathbf{g}\right), \sqrt{2}\epsilon\right), \end{aligned}$$

(the explicit expression of A and $\mathbf{t}(x, \mathbf{g})$ are omitted here).

Theorem (Dubrovin-L-Yang-Zhang, 2016)

Both Conjecture A and Conjecture B are correct.

How to prove?

Key point

Virasoro Constraints (VC)!

1. Starting from the VC for Z_{WK}, by using Givental's quantization, obtain VC for $\mathcal{Z}_{\mathrm{H}}$;
2. Starting from the VC for Toda, after the reduction, obtain VC for Z_{GUE};
3. Show that VC for $\mathcal{Z}_{\mathrm{H}}$ and VC for Z_{GUE} are equivalent;
4. Show that the solution of VC is unique.

What is the next?

For $p, q, r \in \mathbb{C}$ satisfying local Calabi-Yau condition

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 0,$$

consider

$$\mathcal{H}_{p,q}(\mathbf{t}; \epsilon) = \sum_{g \geq 0} \left\langle \Lambda_g(-p) \Lambda_g(-q) \Lambda_g(-r) \exp\left(\sum_{\ell \geq 0} t_\ell \tau_\ell \right) \right\rangle_g .$$

Conjecture C

Denote by

$$\alpha = \frac{p}{\sqrt{p+q}}, \quad \beta = \frac{q}{\sqrt{p+q}},$$

and $\Lambda_1 = \Lambda^{\alpha} = e^{\alpha\,\epsilon\,\partial_x}$, $\Lambda_2 = \Lambda^{\beta} = e^{\beta\,\epsilon\,\partial_x}$, $\Lambda_3 = \Lambda_1\Lambda_2$, then define

$$u = (\Lambda_3 - 1)\left(1 - \Lambda_1^{-1}\right)\mathcal{H}_{p,q},$$

and

$$\frac{\partial}{\partial x_k} = \sum_{\ell\geq 0}(k\,p)^{\ell}\frac{\partial}{\partial t_\ell},$$

$$\frac{\partial}{\partial y_k} = \sum_{\ell\geq 0}(k\,q)^{\ell}\frac{\partial}{\partial t_\ell},$$

$$\frac{\partial}{\partial z_k} = \sum_{\ell\geq 0}(k\,r)^{\ell}\frac{\partial}{\partial t_\ell},$$

Conjecture C

then we have

$$\mathit{constant}\,\frac{\partial u}{\partial x_1} = \frac{(\Lambda_3 - 1)\left(1 - \Lambda_1^{-1}\right)}{\Lambda_2 - 1} e^u, \ldots$$

$$\mathit{constant}\,\frac{\partial u}{\partial y_1} = \left(\Lambda_2 - \Lambda_1^{-1}\right) e^{\frac{1-\Lambda_2^{-1}}{1-\Lambda_1^{-1}} u}, \ldots$$

$$\mathit{constant}\,\frac{\partial u}{\partial z_1} = \left(\Lambda_1^{-1} - 1\right) e^{-\frac{\Lambda_2 - 1}{\Lambda_2 - \Lambda_1^{-1}} u}, \ldots$$

Fractional Volterra Hierarchy

Define

$$L = \Lambda_2 + e^u \Lambda_1^{-1} = \Lambda^{\beta} + e^u \Lambda^{-\alpha} = e^{\beta\,\epsilon\,\partial_x} + e^u e^{-\alpha\,\epsilon\,\partial_x},$$

then there exist operators A, B of the following form

$$A = \Lambda_3 + \sum_{i\geq 0} a_i \left(\Lambda_3\right)^{-i},$$
$$B = b_{-1}(\Lambda_3)^{-1} + \sum_{i\geq 0} b_i \left(\Lambda_3\right)^{i},$$

such that $A = L^{\frac{\alpha+\beta}{\beta}}$, $B = L^{\frac{\alpha+\beta}{\alpha}}$.

Fractional Volterra Hierarchy

$$\frac{\partial L}{\partial x_k} = [A_+^k, L], \quad \frac{\partial L}{\partial y_k} = -[B_-^k, L], \quad \frac{\partial L}{\partial z_k} = unknown.$$

Thank you!