Lefschetz trace formulas for flows on foliated manifolds

Yuri A. Kordyukov
joint work in progress with J. Álvarez López and E. Leichtnam

Institute of Mathematics, Ufa Science Center RAS, Ufa, Russia

Dynamics in Siberia, 2017
The setting

- $M$ a closed manifold, $\dim M = n$.
- $\mathcal{F}$ a codimension one foliation on $M$.
- $\phi^t : M \to M, t \in \mathbb{R}$ a foliated flow (i.e., $\phi^t$ takes each leaf to a leaf).

A Lefschetz number of the flow $\phi$:

$$L(\phi) = \sum_{j=0}^{n-1} (-1)^j \text{Tr} \left( \phi^* : H^j \to H^j \right)$$

$H^j$ is some cohomology theory associated to $\mathcal{F}$, $\text{Tr}$ is some trace.

The corresponding Lefschetz trace formula:

$$L(\phi) = \text{a contribution of closed orbits and fixed points of the flow}.$$
Simple flows

**Definition**
A closed orbit $c$ of period $l$ (not necessarily minimal) of the flow $\phi$ is called *simple*, if

$$\det(id - \phi_*^l : T_x\mathcal{F} \to T_x\mathcal{F}) \neq 0, \quad x \in c.$$ 

**Definition**
A fixed point $x$ of the flow $\phi$ is called *simple* if

$$\det(id - \phi_*^t : T_xM \to T_xM) \neq 0, \quad t \neq 0.$$
Simple flows

- $\text{Fix}(\phi)$ the fixed point set of $\phi$ (closed in $M$).
- $M^0$ the $\mathcal{F}$-saturation of $\text{Fix}(\phi)$ (the union of leaves with fixed points).
  Observe that $M^0$ is $\phi$-invariant.
- $M^1 = M \setminus M^0$ the transitive point set.

**Definition**

The foliated flow $\phi$ is **simple**, i.e.:
- all of its fixed points and closed orbits are simple,
- its orbits in $M^1$ are transverse to the leaves:

\[ T_x M = \mathbb{R} Z(x) \oplus T_x \mathcal{F}, \quad x \in M^1, \]

where $Z$ is the infinitesimal generator of $\phi$ (a vector field on $M$).
Guillemin-Sternberg formula

There is a canonical expression for the right-hand side of the Lefschetz trace formula, which follows from the Guillemin-Sternberg formula.

In $\mathcal{D}'(\mathbb{R}^+)$,

$$L(\phi) = \sum_c l(c) \sum_{k=1}^{\infty} \varepsilon_{kl}(c)(c) \delta_{kl}(c) + \sum_p \varepsilon_p |1 - e^{\kappa_p t}|^{-1},$$

$c$ runs over all closed orbits and $p$ over all fixed points of $\phi$:

- $l(c)$ the minimal period of $c$,
- $\varepsilon_l(c) := \text{sign det } (\text{id} - \phi^l_\ast : T_x\mathcal{F} \to T_x\mathcal{F})$, $x \in c$.
- $\varepsilon_p := \text{sign det } (\text{id} - \phi^t_\ast : T_p\mathcal{F} \to T_p\mathcal{F})$, $t > 0$.
- $\kappa_p \neq 0$ is a real number such that
  $$\bar{\phi}^t_\ast : T_pM/T_p\mathcal{F} \to T_pM/T_p\mathcal{F}, \quad x \mapsto e^{\kappa_p t} x.$$
Introduction

Problems

Problem
To define a Lefschetz number of the flow $\phi$:

$$L(\phi) = \sum_{j=0}^{n-1} (-1)^j \text{Tr} (\phi^* : H^j \rightarrow H^j)$$

- $H^j$ is some cohomology theory associated with $\mathcal{F}$,
- $\text{Tr}$ is a trace,

in such a way that the above Guillemin-Sternberg formula holds.

Motivation:
Deninger’s program to study zeta- and L-functions for algebraic schemes over the integers, in particular, the Riemann zeta-function (Berlin, ICM, 1998).
ASSUMPTIONS:

- $M$ a closed manifold, $\dim M = n$.
- $\mathcal{F}$ a codimension one foliation on $M$.
- $\phi^t : M \to M$, $t \in \mathbb{R}$ a simple foliated flow.
- $\phi$ has no fixed points:
  - all the closed orbits are simple,
  - all the orbits in $M$ are transverse to the leaves.
Leafwise de Rham complex

\((\Omega(\mathcal{F}), d_{\mathcal{F}})\) the leafwise de Rham complex of \(\mathcal{F}\):

- \(\Omega^\cdot(\mathcal{F}) = \mathcal{C}^\infty(M, \wedge \cdot T^*\mathcal{F})\) smooth leafwise differential forms;
- \(d_{\mathcal{F}} : \Omega^\cdot(\mathcal{F}) \to \Omega^{\cdot+1}(\mathcal{F})\) the leafwise de Rham differential.

In a foliated chart with coordinates \((x_1, \ldots, x_{n-1}, y) \in \mathbb{R}^{n-1} \times \mathbb{R}\) such that leaves are given by \(y = c\), a \(p\)-form \(\omega \in \Omega^p(\mathcal{F})\) is written as

\[
\omega = \sum_{\alpha_1 < \alpha_2 < \ldots < \alpha_p} a_{\alpha}(x, y) dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_p}
\]

and \(d_{\mathcal{F}}\omega \in \Omega^{p+1}(\mathcal{F})\) is given by

\[
d_{\mathcal{F}}\omega = \sum_{j=1}^{n-1} \sum_{\alpha_1 < \alpha_2 < \ldots < \alpha_p} \frac{\partial a_{\alpha}}{\partial x_j}(x, y) dx_j \wedge dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_p}
\]
Leafwise de Rham cohomology

- The reduced leafwise de Rham cohomology of $\mathcal{F}$:
  $$\overline{H}(\mathcal{F}) = \ker d_{\mathcal{F}} / \text{im } d_{\mathcal{F}},$$
  the closure is in $C^\infty$-topology.
- $\phi$ is a foliated flow $\iff d_{\mathcal{F}} \circ \phi^t = \phi^t \circ d_{\mathcal{F}}$.
The induced action:
  $$\phi^{t*} : \overline{H}(\mathcal{F}) \to \overline{H}(\mathcal{F}).$$

Question

The trace of $\phi^{t*} : \overline{H}(\mathcal{F}) \to \overline{H}(\mathcal{F})$?
The leafwise Hodge decomposition

- $g$ the Riemannian metric on $M$ such that the infinitesimal generator $Z$ of the flow $\phi$ is of length one and is orthogonal to the leaves — a bundle-like metric (so $\mathcal{F}$ is a Riemannian foliation.).

- $\Delta_{\mathcal{F}} = d_{\mathcal{F}}\delta_{\mathcal{F}} + \delta_{\mathcal{F}}d_{\mathcal{F}}$ the leafwise Laplacian on $\Omega(\mathcal{F})$ (a second order tangentially elliptic differential operator on $M$).

- $\mathcal{H}(\mathcal{F})$ the space of leafwise harmonic forms on $M$:

$$\mathcal{H}(\mathcal{F}) = \{\omega \in \Omega(\mathcal{F}) : \Delta_{\mathcal{F}}\omega = 0\}.$$

**Theorem (Alvarez Lopez - Yu. K)**

*The Hodge isomorphism*

$$\overline{\mathcal{H}(\mathcal{F})} \cong \mathcal{H}(\mathcal{F}).$$
The Lefschetz distribution

For any \( f \in C_c^\infty(\mathbb{R}) \), define

\[
A_f = \int_{\mathbb{R}} \phi^t \cdot f(t) \, dt \circ \Pi : L^2 \Omega(\mathcal{F}) \to L^2 \Omega(\mathcal{F}),
\]

where \( \Pi : L^2 \Omega(\mathcal{F}) \to L^2 \mathcal{H}(\mathcal{F}) \) is the orthogonal projection.

\( A_f \) is a smoothing operator:

The Schwartz kernel \( K_{A_f} = K_{A_f}(x, y) \, |dy| \) of \( A_f \) is smooth:

\[
A_f u(x) = \int_M K_{A_f}(x, y) u(y) \, |dy|.
\]

In particular, \( A_f \) is of trace class and

\[
\text{Tr} \, A_f = \int_M \text{tr} \, K_{A_f}(x, x) \, |dx|.
\]
The Lefschetz distribution

For any \( f \in C_c^\infty(\mathbb{R}) \),

\[
A_f = \int_\mathbb{R} \phi^t \cdot f(t) \, dt \circ \Pi : L^2 \Omega(F) \to L^2 \Omega(F),
\]

where \( \Pi : L^2 \Omega(F) \to L^2 \mathcal{H}(F) \) is the orthogonal projection.

The Lefschetz distribution \( L(\phi) \in \mathcal{D}'(\mathbb{R}) \):

\[
< L(\phi), f > = \text{Tr}^s A_f := \sum_{j=1}^{n-1} (-1)^j \text{Tr} A_f^{(i)}, \quad f \in C_c^\infty(\mathbb{R}),
\]

where \( A_f^{(i)} \) is the restriction of \( A_f \) to \( \Omega^i(F) \).
The Lefschetz formula

Theorem (Alvarez Lopez - Y.K.)

Assume that $\phi$ is simple and has no fixed points.

1. On $\mathbb{R} \setminus \{0\}$

$$L(\phi) = \sum_c l(c) \sum_{k \neq 0} \varepsilon_{kl}(c)(c) \delta_{kl}(c),$$

when $c$ runs over all closed orbits of $\phi$ and $l(c)$ denotes the minimal period of $c$.

2. In some neighborhood of 0 in $\mathbb{R}$:

$$L(\phi) = \chi_{\Lambda}(\mathcal{F}) \cdot \delta_0.$$

$\chi_{\Lambda}(\mathcal{F})$ the $\Lambda$-Euler characteristic of $\mathcal{F}$ given by the holonomy invariant transverse measure $\Lambda$ (Connes, 1979).
The setting

**ASSUMPTION:**

- $M$ a closed manifold, $\dim M = n$.
- $\mathcal{F}$ a codimension one foliation on $M$.
- $\phi^t : M \to M$, $t \in \mathbb{R}$ a simple foliated flow.

- $\text{Fix}(\phi)$ the fixed point set of $\phi$ (closed in $M$).
- $M^0$ the $\mathcal{F}$-saturation of $\text{Fix}(\phi)$.
- $M^1 = M \setminus M^0$ the transitive point set.

**Definition**

The foliated flow $\phi$ is **simple**, i.e.:

- all of its fixed points and closed orbits are simple,
- its orbits in $M^1$ are transverse to the leaves.
Difficulties

\( \mathcal{F} \) is a foliation almost without holonomy:

If \( \phi \) is simple, then:

- \( M^0 \) is a finite union of compact leaves,
- only the leaves in \( M^0 \) may have non-trivial holonomy groups.

In particular, \( \mathcal{F} \) is not a Riemannian foliation.

- The leafwise Laplacian \( \Delta_{\mathcal{F}} \) is transversally elliptic only on the transitive point set \( M^1 \), not on \( M^0 \).
- As a consequence, the operator

\[
A_f = \int_{\mathbb{R}} \phi^* \cdot f(t) \, dt \circ \Pi : L^2\Omega(\mathcal{F}) \to L^2\Omega(\mathcal{F})
\]

is not a smoothing operator. Its Schwartz kernel is smooth on \( M^1 \times M^1 \) and singular near \( M^0 \times M^0 \).

So its trace is not well-defined.
The transitive point set and its blow-up

- $M^1_i$, $i = 1, \ldots, r$, the connected components of $M^1(= M \setminus M^0)$:

$$
(M^1, \mathcal{F}^1) = \bigsqcup_i (M^1_i, \mathcal{F}^1_i).
$$

- $M^l$ is the closure of $M^1_i$:

$$
M^l = \overline{M^1_i}.
$$

Thus, $M^l_i$ is a connected compact manifold with boundary, endowed with a smooth foliation $\mathcal{F}^l_i$ tangent to the boundary.

- Put

$$
M^c := \bigsqcup_i M^l_i, \quad \mathcal{F}^c := \bigsqcup_i \mathcal{F}^l_i.
$$

- The flow lifts to a simple foliated flow $\phi^{c,t}$ of $\mathcal{F}^c$ tangent to $\partial M^c$. 

Yuri A. Kordyukov (Ufa, Russia)
There exists a Riemannian metric $g^1$ on $M^1$:

- $M^1_i$ equipped with $g^1_i := g^1|_{M^1_i}$ is a manifold of bounded geometry;
- $g^1$ is bundle-like for $\mathcal{F}^1$;
- $\mathcal{F}^1_i$ a Riemannian foliation of bounded geometry;
- $\phi^t_i$ a flow of bounded geometry.

Remarks:

- $g^1$ is singular at $M^0$.
- Each $(M^1_i, g^1_i)$ is a Riemannian manifold with cylindrical ends.
Local model for $g^1$ near a compact leaf

Take a compact leaf $L \subset M^0$. Then, by the local stability theorem,

- a tubular nbhd $V$ of $L$ in $M$ is diffeomorphic to a tubular nbhd $V_L$ of $L$ in the suspension foliated manifold $(M_L = \tilde{L} \times \Gamma \mathbb{R}, \mathcal{F}_L)$:

$$V \subset M \equiv V_L \subset M_L = \tilde{L} \times \Gamma \mathbb{R},$$

- the flow $\phi^t$ on $V \equiv V_L$:

$$\phi^t([\tilde{y}, x]) = [\phi^t_x(\tilde{y}), e^{\kappa_L t} x], \quad [\tilde{y}, x] \in V_L \subset M_L = \tilde{L} \times \Gamma \mathbb{R},$$

- the Riemannian metric $g^1$ on $M^1 \equiv M_L \setminus L = \tilde{L} \times \Gamma (\mathbb{R} \setminus \{0\})$:

$$g^1 = g_{\mathcal{F}_L} + \frac{dx^2}{x^2}, \quad [\tilde{y}, x] \in \tilde{L} \times \Gamma (\mathbb{R} \setminus \{0\}),$$

where $g_{\mathcal{F}_L}$ is a leafwise Riemannian metric on $(M_L, \mathcal{F}_L)$. 
Differential operators on the blow-up

The blow up of the transitive point set $M^1$:  

$$M^c = \bigsqcup_l M_l, \quad F^c = \bigsqcup_l F_l,$$

$M_l$ a connected compact manifold with boundary, $F_l$ a smooth foliation tangent to the boundary:

$$\hat{M}_l \equiv M^1_l, \quad \hat{F}_l \equiv F^1_l.$$

We transfer the Riemannian metric $g^1$ to $\hat{M}_l$. Then $\hat{M}_l$ is a manifold of bounded geometry and $\hat{F}_l$ is a Riemannian foliation of bounded geometry.

- $d_{\hat{F}_l}$ the leafwise de Rham differential on $\Omega(\hat{F}_l)$.
- $\delta_{\hat{F}_l}$ the leafwise de Rham codifferential on $\Omega(\hat{F}_l)$.
- $D_{\hat{F}_l} = d_{\hat{F}_l} + \delta_{\hat{F}_l}$. 

\[\text{Yuri A. Kordyukov (Ufa, Russia)}\]
Singular flows

Smoothing operators

Definition

Let $\mathcal{A}$ be the Fréchet algebra of functions $\psi : \mathbb{R} \to \mathbb{C}$ such that the Fourier transform $\hat{\psi}$ satisfies that, for every $k \in \mathbb{N}$, there is some $A_k > 0$ such that, for all $\xi \in \mathbb{R}$,

$$|\hat{\psi}(\xi)| \leq A_k e^{-k|\xi|}.$$

$\mathcal{A}$ contains all functions with compactly supported Fourier transform, as well as the Gaussians $x \mapsto e^{-tx^2}$ with $t > 0$.

Definition

For any $\psi \in \mathcal{A}$, $f \in C^\infty_c(\mathbb{R})$ and $l$, the operator

$$\hat{P}_l = \int_{-\infty}^{\infty} \phi^* \cdot f(t) \, dt \circ \psi(D_{\hat{f}_l})$$

is a smoothing operator on $\hat{M}_l$, but its kernel is singular near $\partial\hat{M}_l$. 
Theorem (Alvarez Lopez, Yu.K., Leichtnam)

\[ \mathcal{P}_l = \int_{-\infty}^{\infty} \phi^t \cdot f(t) \, dt \circ \psi(D_{\mathcal{F}_l}) \text{ gives rise to } P_l \in \Psi_b^{-\infty}(M_l; \bigwedge T\mathcal{F}_l^*). \]

- The Schwartz kernel \( K_{P_l} \) is smooth in the interior \( \mathring{M}_l \times \mathring{M}_l \).
- \( K_{P_l} \) has a \( C^\infty \) extension to \( M_l \times M_l \setminus \partial M_l \times \partial M_l \) that vanishes to all orders at \( (\partial M_l \times M_l) \cup (M_l \times \partial M_l) \).
- Consider a tubular neighborhood of \( L \subset \pi_0(\partial M_l) \) with coordinates \( (\rho, y), \rho \in (0, \infty), y \in L \).

Then \( K_{P_l} = K_{P_l}(\rho, y, \rho', y') u(\rho', y') \, d\rho' \, |dy'| \) has the form

\[ K_{P_l}(\rho, y, \rho', y') = \frac{1}{\rho'} \kappa_{P_l}(\rho, y, \frac{\rho'}{\rho}, y'), \]

where \( \kappa_{P_l}(\rho, y, s, y') \) is smooth up to \( L \) (that is, up to \( \rho = 0 \)).
b-trace

In a tubular neighborhood of $L$ with coordinates $\rho \in (0, \epsilon_0)$, $y \in L$,

$$P_l u(\rho, y) = \int K_{P_l}(\rho, y, \rho', y') u(\rho', y') |d\rho'| |dy'|,$$

$$K_{P_l}(\rho, y, \rho', y') = \frac{1}{\rho'} \kappa_{P_l}(\rho, y, \frac{\rho'}{\rho}, y'),$$

and $\kappa_{P_l}(\rho, y, s, y')$ is smooth up to $L$ (that is, up to $\rho = 0$).

Definition

$$b^{\text{Tr}} (P_l) = \lim_{\epsilon \to 0} \left( \int_{\rho > \epsilon} K_{P_l}(\rho, y, \rho, y) |d\rho| |dy| + \ln \epsilon \int \kappa_{P_l}(0, y, 1, y) |dy| \right).$$

Key fact

The functional $b^{\text{Tr}}$ doesn’t have trace property, but $b^{\text{Tr}} [P, P']$ is expressed in terms of traces of some explicit integral operators on $\partial M_l$. 
Operators on the transitive point set

Since $M^c = \bigsqcup_i M_i$, $\mathcal{F}^c = \bigsqcup_i \mathcal{F}_i$, we get the operator

$$P \equiv \bigoplus_i P_i = \int_{-\infty}^{\infty} \phi^t \cdot f(t) \, dt \circ \psi(D\mathcal{F}^c)$$

$$\in \psi^{-\infty}_b(M^c; \bigwedge T\mathcal{F}^c^*) \equiv \bigoplus_i \psi^{-\infty}_b(M_i; \bigwedge T\mathcal{F}_i^*).$$

In particular, its b-trace $b\mathrm{Tr} (P)$ is well-defined. The b-supertrace of $P$:

$$b\mathrm{Tr}^s(P) = \sum_{j=1}^{n-1} (-1)^j b\mathrm{Tr} (P^{(j)}),$$

where $P^{(j)}$ is the restriction to $j$-forms.
Singular flows

Derivative of the $b$-supertrace

We follow the heat kernel approach to index theory:

Fix an even $\psi \in \mathcal{A}$ and $f \in C^\infty_c(\mathbb{R})$. For $u > 0$, let

$$P_{\psi u, f} = \int_{-\infty}^{\infty} \phi^t \ast f(t) \, dt \circ \psi(uD_{\mathcal{F}c})$$

Since the $b$-trace is not a trace, $\frac{d}{du} b\text{Tr}^s(P_{\psi u, f}) \neq 0$.

Theorem

$$\frac{d}{du} b\text{Tr}^s(P_{\psi u, f}) = \sum_{L \in \pi_0(M^0)} \frac{2}{|\kappa_L|} \sum_{\gamma \in \Gamma_L} \text{Tr}^s_{\Gamma_L} \left( T^*_{\gamma} \tilde{R}_{L,u,t_{L,\gamma}} \right) f(t_{L,\gamma})$$
Singular flows

Notation

Theorem

\[
\frac{d}{du} \text{bTr}^s(P_{\psi u}, f) = \sum_{L \in \pi_0(M^0)} \frac{2}{|\kappa_L|} \sum_{\gamma \in \Gamma_L} \text{Tr}^s \Gamma_L \left( T^*_\gamma \tilde{R}_{L, u, t_L, \gamma} \right) f(t_L, \gamma),
\]

- \(\tilde{L}\) the universal covering of \(L\), \(\Gamma_L := \pi_1 \tilde{L}\).
- \(T^*_\gamma\) the induced action of \(\gamma \in \Gamma_L\) on \(\Gamma_L\)-invariant operators on \(\tilde{L}\).
- \(\text{Tr}^s \Gamma_L\) the \(\Gamma_L\)-trace on \(\Gamma_L\)-invariant operators on \(\tilde{L}\).
- \(\tilde{R}_{L, u, t} = u \tilde{\eta} \wedge \phi^t_L \psi' (uD_L)\).
- \(\tilde{\eta}\) a closed one-form on \(\tilde{L}\), the lift of a closed one-form \(\eta\) on \(L\).

If we consider \(\eta\) as a closed leafwise 1-form on the suspension manifold \(M_L = L \times_\Gamma \mathbb{R}\), then there exists a 1-form \(\omega\) on \(M_L\) satisfying \(T \mathcal{F}_L = \ker \omega\) such that \(d \omega = \eta \wedge \omega\).

- \(\phi^t_L : L \to L\) the restriction of the flow to \(L\).
More notation

**Theorem**

\[
\frac{d}{du} b\text{Tr}^s(P_{\psi u,f}) = \sum_{L \in \pi_0(M^0)} \sum_{\gamma \in \Gamma_L} \text{Tr}^s \left( T_{\gamma}^* \tilde{R}_{L,u,t_L,\gamma} \right) f(t_L,\gamma),
\]

- \( \kappa_L \neq 0 \) a real number such that, for \( p \in L \),

\[
\bar{\phi}^t : N_p F \to N_p F, \quad x \to e^{\kappa_L t} x.
\]

- \( t_{L,\gamma} = -\kappa_L^{-1} \log a_{L,\gamma} \) relative periods, where a homomorphism \( \gamma \in \Gamma_L \mapsto a_{L,\gamma} \in \mathbb{R}^+ \) is given by the holonomy homomorphism

\[
\gamma \in \Gamma_L \mapsto \bar{h}_{L,\gamma} \in \text{Diffeo}_+(\mathbb{R},0), \quad \bar{h}_{L,\gamma}(x) = a_{L,\gamma} x.
\]
Variation of the b-supertrace and Lefschetz distribution

For $u, v > 0$,

$$\text{bTr}^s(P_{\psi_v,f}) - \text{bTr}^s(P_{\psi_u,f}) = \sum_{L \in \pi_0(M^0)} \frac{2}{|\mathcal{X}_L|} \sum_{\gamma \in \Gamma_L} \text{Tr}^s_{\Gamma_L} \left( T^*_\gamma S_{\tilde{L},u,v,t_L,\gamma} \right) f(t_L, \gamma),$$

$$\tilde{S}_{\tilde{L},u,v,t} = \int_u^v \tilde{R}_{\tilde{L},w,t} \, dw = \tilde{\eta} \wedge \phi^*_{\tilde{L}} \frac{\psi(vD_{\tilde{L}}) - \psi(uD_{\tilde{L}})}{D_{\tilde{L}}}. $$

Definition

The Lefschetz distribution

$$\langle L(\phi), f \rangle = \text{bTr}^s(P_{\psi_v,f}) - \lim_{u \to 0} \sum_{L \in \pi_0(M^0)} \frac{2}{|\mathcal{X}_L|} \sum_{\gamma \in \Gamma_L} \text{Tr}^s_{\Gamma_L} \left( T^*_\gamma S_{\tilde{L},u,v,t_L,\gamma} \right) f(t_L, \gamma).$$

Here the right-hand side is independent of $v$.  

Yuri A. Kordyukov (Ufa, Russia)  
Lefschetz trace formulas for flows  
Dynamics in Siberia, 2017  
27 / 29
Theorem

$L(\phi)$ is a well-defined distribution on $\mathbb{R}_+$ given by

$$L(\phi) = \sum_c l(c) \sum_{k=1}^{\infty} \varepsilon_{kl}(c)(c) \cdot \delta_{kl}(c)$$

on $\mathbb{R}_+$, where $c$ runs over all closed orbits of $\phi^t$, $l(c)$ denotes the minimal period of $c$, and $x$ is an arbitrary point of $c$. 
Concluding remarks

Remark

The next problem is to give a cohomological interpretation of the limit as $v \to +\infty$ of

$$\text{bTr}^s(P_{\psi^v, f}) - \lim_{u \to 0} \sum_{L \in \pi_0(M^0)} \frac{2}{|\kappa_L|} \sum_{\gamma \in \Gamma_L} \text{Tr}^s_{\Gamma_L} \left( T^*_\gamma \tilde{S}_{L,u,v,t_L,\gamma} \right) f(t_L, \gamma).$$

Remark

Contribution of fixed points as in the Guillemin-Sternberg formula.