Compatibility of different classes of almost Hermitian 6-manifolds

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Almost Hermitian manifolds

Let (M, g, J, ω) is an almost Hermitian manifold with an almost complex structure J and J–invariant Riemannian metric

$$g(J\cdot, J\cdot) = g(\cdot, \cdot), \omega(\cdot, \cdot) = g(J\cdot, \cdot)$$

Almost Hermitian structures with the same metric g

g - orthogonal almost complex structures

$\mathcal{AO}_g^+$ is space of positively oriented g - orthogonal a. c. s. I on M.

$$g(I\cdot, I\cdot) = g(\cdot, \cdot)$$

$\mathcal{H}_g = \{(g, I, \omega_I) : I \in \mathcal{AO}_g^+\}$ - the space of all a.H.s. with the same metric g

Almost Hermitian structures with the same a.c.s I

I-invariant Riemannian metrics

$\mathcal{M}_{\mathrm{I}}$ is space of I-invariant Riemannian metrics g on M.

$\mathcal{H}_{\mathrm{I}} = \{(\mathrm{g}, \mathrm{I}, \omega_{\mathrm{g}}) : \mathrm{g} \in \mathcal{M}_{\mathrm{I}}\}$ – the space of all a.H.s. with the same a.c.s I on M.

$$\omega_{\mathrm{g}}(\cdot, \cdot) = \mathrm{g}(\mathrm{I}\cdot, \cdot)$$

Almost Hermitian structures with the same 2-form ω

Positively ω-tamed a.c.s

$\mathcal{A}_\omega^+$ is space of positively ω-tamed a.c.s. on M:
1. $\omega(\mathrm{I}\cdot, \mathrm{I}\cdot) = \omega(\cdot, \cdot)$;
2. $\mathrm{g_I}(\cdot, \cdot) = \omega(\cdot, \mathrm{I}\cdot)$ is positively definite.

ω-positively associated metrics

$$\mathcal{AM}_\omega^+ = \{\mathrm{g_J} : \mathrm{g_J}(\cdot, \cdot) = \omega(\cdot, \mathrm{J}\cdot), \quad \mathrm{J} \in \mathcal{A}_\omega^+\}$$

$\mathcal{H}_\omega = \{(\mathrm{g_I}, \mathrm{I}, \omega) : \mathrm{I} \in \mathcal{A}_\omega^+\}$ – the space of all a.H.s. with the same 2-form ω on M.

Theorem [Smolentsev N.K.] (2001)

The space $\mathcal{M}$ of all Riemannian metrics on almost Hermitian manifold $(\mathrm{M}, \mathrm{g}, \mathrm{J}, \omega)$ is a smooth trivial bundle over $\mathcal{A}\mathrm{M}_\omega$ with $\mathcal{M}_\mathrm{I}$ as a fiber over $\mathrm{g}_\mathrm{I} \in \mathcal{A}\mathrm{M}_\omega$.

Theorem [Daurtseva N.A.] (2005)

The space $\mathcal{A}^+$ of all almost complex structures, which define the same orientation as J on almost Hermitian manifold $(\mathrm{M}, \mathrm{g}, \mathrm{J}, \omega)$ is a smooth local trivial bundle over $\mathcal{A}\mathrm{O}^+_\mathrm{g}$ with $\mathcal{A}^+_{\omega_\mathrm{I}}$ as a fiber over $\mathrm{I} \in \mathcal{A}\mathrm{O}^+_\mathrm{g}$.

Gray-Herwella classification

Class $\mathcal{W}$ of almost Hermitian manifolds (M, g, J, ω) is decomposed into sum of classes

$$\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$$

- $\mathcal{W}_1 = \mathcal{NK}$ is class of nearly Kähler manifolds, $\nabla_X(\omega)(X, \cdot) = 0$;
- $\mathcal{W}_2 = \mathcal{AK}$ is class of almost Kähler manifolds, $d\omega = 0$;
- $\mathcal{W}_3$ is class of special Hermitian manifolds, $\delta\omega = N = 0$
- $\mathcal{W}_4$ is class defined by condition $\nabla_X(\omega)(Y, Z) = \frac{-1}{2(n-1)}\{g(X, Y)\delta\omega(Z) - g(X, Z)\delta\omega(Y) - g(X, JY)\delta\omega(JZ) + g(X, JZ)\delta\omega(JY)\}$
- The class of Kähler manifolds $\mathcal{K}$ is in any $\mathcal{W}_i$ and satisfy to $\nabla\omega = 0$.

Lejmi M. (2006)

- Whether or not a given almost complex structure J on M is ω-tamed by some symplectic form ω?
- He showed that the almost complex structure underlying a non-Kähler, nearly Kähler 6-manifold cannot be compatible with any symplectic form, even localy.

Lejmi result in terms of Gray-Herwella

If $(\mathrm{M}^6, \mathrm{g}, \mathrm{J}, \omega) \in \mathcal{W}_1$, then for any metric $\mathrm{g}_\mathrm{J} \in \mathcal{M}_\mathrm{J}$ manifold $(\mathrm{M}^6, \mathrm{g}_\mathrm{J}, \mathrm{J}, \omega_\mathrm{J}) \notin \mathcal{W}_2$, even locally

Question

As largest class, which intersect with $\mathcal{W}_2$ at $\mathcal{K}$ is $\mathcal{G}_1 = \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, then it is logical to ask whether or not the fact that $(\mathrm{M}^6, \mathrm{g}, \mathrm{J}, \omega) \in \mathcal{G}_1$ is obstruction to existence of almost Kähler structures among $(\mathrm{g}_\mathrm{J}, \mathrm{J})$, for any $\mathrm{g}_\mathrm{J} \in \mathcal{M}_\mathrm{J}$?

Theorem 1

If $(\mathrm{M}^6, \mathrm{g}, \mathrm{J}, \omega) \in \mathcal{G}_1$, and J is not integrable, then $(\mathrm{M}^6, \mathrm{g}_\mathrm{J}, \mathrm{J}, \omega_\mathrm{J})$ can not be $\mathcal{G}_2$-manifold for any $\mathrm{g}_\mathrm{J} \in \mathcal{M}_\mathrm{J}$, even locally.
$(\mathcal{G}_2 = \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4)$

Compatibility on $\mathcal{H}_J$

Theorem 2

Let (M, g, J, ω) is almost Hermitian 6-manifold.

- If $(M, g, J, \omega) \in \mathcal{W}_1, \mathcal{W}_1 \oplus \mathcal{W}_3, \mathcal{W}_1 \oplus \mathcal{W}_4, \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ then $(M, g_J, J, \omega_J) \notin \mathcal{G}_2$ for any $g_J \in \mathcal{M}_J$, even locally;
- If $(M, g, J, \omega) \in \mathcal{W}_3, \mathcal{W}_4, \mathcal{W}_3 \oplus \mathcal{W}_4$ then J is locally tamed by some symplectic form, but globally $(M, g_J, J, \omega_J) \in \mathcal{W}_1 \oplus \mathcal{W}_2$ just in case when $(M, g_J, J, \omega_J) \in \mathcal{K}$;
- If $(M, g, J, \omega) \in \mathcal{W}_1 \oplus \mathcal{W}_2$ then $(M, g_J, J, \omega_J) \notin \mathcal{W}_3 \oplus \mathcal{W}_4$ for any $g_J \in \mathcal{M}_J$;
- If $(M, g, J, \omega) \in \mathcal{W}_2, \mathcal{W}_2 \oplus \mathcal{W}_3, \mathcal{W}_2 \oplus \mathcal{W}_4, \mathcal{G}_2$, then $(M, g_J, J, \omega_J) \notin \mathcal{G}_1$ for any $g_J \in \mathcal{M}_J$, even locally;
- If $(M, g, J, \omega) \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$, then $(M, g_J, J, \omega_J) \notin \mathcal{W}_3 \oplus \mathcal{W}_4$ for any $g_J \in \mathcal{M}_J$
- If $(M, g, J, \omega) \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$, then $(M, g_J, J, \omega_J) \notin \mathcal{W}_3 \oplus \mathcal{W}_4$ for any $g_J \in \mathcal{M}_J$

Compatibility for $\mathcal{H}_\omega$

Theorem 3. [Daurtseva N.A. (2014)]

Let $(\mathrm{M}^6, \mathrm{g}, \mathrm{J}, \omega) \in \mathcal{NK}$, then $(\mathrm{M}^6, \mathrm{g_I}, \mathrm{I}, \omega)$ can't be $\mathcal{G}_2 = \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ – manifold for any $\mathrm{I} \in \mathcal{A}_\omega^+$. In particular, I is not integrable.

Compatibility for $\mathcal{H}_\omega$

Theorem 4

Let $(\mathrm{M}, \mathrm{g}, \mathrm{J}, \omega)$ is almost Hermitian 6-manifold.
1). If $(\mathrm{M}, \mathrm{g}, \mathrm{J}, \omega) \in \mathcal{W}_1$ strictly, then $(\mathrm{M}, \mathrm{g_I}, \mathrm{I}, \omega) \notin \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ for any $\mathrm{I} \in \mathcal{A}^+_\omega$;
2). If $(\mathrm{M}, \mathrm{g}, \mathrm{J}, \omega) \in \mathcal{W}_1 \oplus \mathcal{W}_2$ strictly, then $(\mathrm{M}, \mathrm{g_I}, \mathrm{I}, \omega) \notin \mathcal{W}_3 \oplus \mathcal{W}_4$ for any $\mathrm{I} \in \mathcal{A}^+_\omega$;
3). If $(\mathrm{M}, \mathrm{g}, \mathrm{J}, \omega) \in \mathcal{W}_1 \oplus \mathcal{W}_3$ strictly, then $(\mathrm{M}, \mathrm{g_I}, \mathrm{I}, \omega) \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$, but not in $\mathcal{W}_2$ for any $\mathrm{I} \in \mathcal{A}^+_\omega$;
4). If $(\mathrm{M}, \mathrm{g}, \mathrm{J}, \omega) \in \mathcal{W}_1 \oplus \mathcal{W}_4, \mathcal{W}_2 \oplus \mathcal{W}_4$ or $\mathcal{W}_4$ strictly, then $(\mathrm{M}, \mathrm{g_I}, \mathrm{I}, \omega) \notin \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ for any $\mathrm{I} \in \mathcal{A}^+_\omega$;
5). If $(\mathrm{M}, \mathrm{g}, \mathrm{J}, \omega) \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$, then $(\mathrm{M}, \mathrm{g_I}, \mathrm{I}, \omega) \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ for any $\mathrm{I} \in \mathcal{A}^+_\omega$;
6). If $(\mathrm{M}, \mathrm{g}, \mathrm{J}, \omega) \in \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ strictly, then $(\mathrm{M}, \mathrm{g_I}, \mathrm{I}, \omega) \notin \mathcal{W}_2$ for any $\mathrm{I} \in \mathcal{A}^+_\omega$;
7). If $(\mathrm{M}, \mathrm{g}, \mathrm{J}, \omega) \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$ strictly, then $(\mathrm{M}, \mathrm{g_I}, \mathrm{I}, \omega) \notin \mathcal{W}_3$ for any $\mathrm{I} \in \mathcal{A}^+_\omega$;

Compatibility for $\mathcal{H}_\omega$

Theorem 4

8). If $(\mathrm{M}, \mathrm{g}, \mathrm{J}, \omega) \in \mathcal{W}_2$, then $(\mathrm{M}, \mathrm{g_I}, \mathrm{I}, \omega) \in \mathcal{W}_2$ for any $\mathrm{I} \in \mathcal{A}_\omega^+$;
9). If $(\mathrm{M}, \mathrm{g}, \mathrm{J}, \omega) \in \mathcal{W}_2 \oplus \mathcal{W}_3$ strictly, then $(\mathrm{M}, \mathrm{g_I}, \mathrm{I}, \omega) \notin \mathcal{W}_1 \oplus \mathcal{W}_4$ for any $\mathrm{I} \in \mathcal{A}_\omega^+$;
10). If $(\mathrm{M}, \mathrm{g}, \mathrm{J}, \omega) \in \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ strictly, then $(\mathrm{M}, \mathrm{g_I}, \mathrm{I}, \omega) \notin \mathcal{W}_1$ for any $\mathrm{I} \in \mathcal{A}_\omega^+$;
11). If $(\mathrm{M}, \mathrm{g}, \mathrm{J}, \omega) \in \mathcal{W}_3$ strictly, then $(\mathrm{M}, \mathrm{g_I}, \mathrm{I}, \omega) \notin \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$ for any $\mathrm{I} \in \mathcal{A}_\omega^+$;
12). If $(\mathrm{M}, \mathrm{g}, \mathrm{J}, \omega) \in \mathcal{W}_3 \oplus \mathcal{W}_4$ strictly, then $(\mathrm{M}, \mathrm{g_I}, \mathrm{I}, \omega) \notin \mathcal{W}_1 \oplus \mathcal{W}_2$ for any $\mathrm{I} \in \mathcal{A}_\omega^+$.