Non-commutative normal forms and inverse spectral problems.

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The main object is the one-dimensional Schrödinger operator:

\[ \hat{H} = \frac{\hat{p}^2}{2} + V(x), \quad \hat{p} = -i\hbar \frac{d}{dx} \]

where the potential \( V(x) \) is an even degree polynomial s.t. \( V(x) > 0 \) for \( x > 0 \), \( V(0) = V'(0) = 0 \) and \( V''(0) = 1 \). Its spectrum is discrete, its 'low lying eigenvalues' (i.e. lying in \([0, Ch]\)) are

\[ \hat{H}\psi_\nu = E_\nu \psi_\nu, \quad E_\nu \sim \hbar (\nu + 1/2) + \sum_{k=2}^{\infty} \mathcal{E}_\nu^k \hbar^k \]

as \( \hbar \to 0 \). Number \( \nu \in \mathbb{Z}_+ \) is fixed.
The goal

\[ \hat{H}\psi_\nu = E_\nu \psi_\nu, \quad E_\nu \sim h(\nu + 1/2) + \sum_{k=2}^{\infty} \mathcal{E}_k^\nu h^k \]

1. New method for calculating \( \mathcal{E}_k^\nu \) via Taylor coefficients of \( V \) based on an analog of Birkhoff normal forms in a non-commutative algebra.

2. New results about the growth of coefficients \( \mathcal{E}_k^\nu \).

3. New inverse results: how to recover \( V \) from the knowledge of \( \mathcal{E}_k^\nu \)?
Formal graded Heisenberg algebra

**Definition**
The formal graded Heisenberg algebra $\mathbb{H}$ is an algebra of formal non-commutative series $F = \sum_{z=z_1\ldots z_k} f_z z_j$, $z_j \in \{p, x\}$, $f_z \in \mathbb{C}$, with identities $rp = pr$, $rx = xr$, where $r = px - xp$.

**Remark**
The grading: $\mathbb{H} = \prod_{k=0}^{\infty} \mathbb{H}_k$ (or informally $\mathbb{H} = \bigoplus_{k=0}^{\infty} \mathbb{H}_k$). Here $\mathbb{H}_k$ is the space of degree $k$ homogeneous forms. E.g. $\deg p = \deg x = 1$, $\deg r = 2$.

**Remark**
In quantum mechanics $p = -i\hbar \frac{d}{dx}$, $r = -i\hbar$. 
Proposition

For any \( F, G \) there is a unique \( W \), s.t. \([F, G] \equiv FG - GF = rW\).

Definition

(Quantum) Poisson bracket: \( \{F, G\} := W \) (or informally \( \{F, G\} = \frac{FG - GF}{r} \)).

Remark

deg \( F = n \), deg \( G = m \) \( \Rightarrow \) deg \( \{F, G\} = n + m - 2 \).

Definition

Involution: \((\sum f_z z_1 \cdots z_k)^* = \sum \overline{f_z} z_k \cdots z_1 \). \( F \) is Hermitian, if \( F = F^* \).

Remark

There is a 'homomorphism into classics' \( \pi : \mathbb{H} \rightarrow \mathbb{O} \). Here \( \mathbb{O} \) is an algebra of formal Taylor series with standard product and formal Poisson bracket.
Normal form in $\mathbb{H}$

Definition
$$e^{\text{ad}_W} H := \sum_{k=0}^{\infty} \frac{\text{ad}_W^k}{k!} H \equiv H + \{W, H\} + \frac{1}{2}\{W, \{W, H\}\} + \ldots,$$

Proposition (D. V. Treschev, 2005)

If $W \in \prod_{k=3}^{\infty} \mathbb{H}_k$, then $e^{\text{ad}_W} : \mathbb{H} \to \mathbb{H}$ is an automorphism of non-commutative algebra and Lie algebra.

Proposition (A., 2009)

Assume that
$$H = H^* = H_2 + \sum_{k=3}^{\infty} H_k, \quad H_2 = \frac{1}{2}(p^2 + x^2), \quad \deg H_k = k. \quad \text{There is a Hermitian } W = \sum_{k=3}^{2N+1} W_k \text{ s.t.}$$

$$e^{\text{ad}_W} H = H_2 + G + R, \quad \{G, H_2\} = 0, \quad R \in \prod_{k=2N+2}^{\infty} \mathbb{H}_k.$$
Let us think that we deal with operators.

\[ e^{\text{ad}_W \mathbf{H}} = U^* \mathbf{N} U, \quad \mathbf{N} = \mathbf{H}_2 + \mathbf{G} + \mathbf{R}, \quad U = e^{-i \frac{\mathbf{W}}{\hbar}} \]

\( U \) is unitary \( \Rightarrow \) \( \mathbf{N} \) is self-adjoint with the same spectrum as \( \mathbf{H} \).

Let us truncate: \( \tilde{\mathbf{N}} = \mathbf{H}_2 + \mathbf{G} \). Since \( [\tilde{\mathbf{N}}, \mathbf{H}_2] = 0 \), the eigenfunctions \( \psi_\nu \) of \( \mathbf{H}_2 \) are eigenfunctions of \( \tilde{\mathbf{N}} \).

**Lemma (Easy exercise)**

*If* \( \mathbf{R}_k \in \mathbb{H}_k \), *then* \( \| \mathbf{R} \psi_n u \|_{L^2(\mathbb{R})} = O(h^{k/2}) \).

Let \( \mathbf{G} = \sum_{k=2}^{N} \mathbf{G}_{2k} \), then \( \mathbf{G}_{2k} \psi_\nu = a_k(\nu) h^k \psi_\nu \), and we arrive at

\[ E_\nu = h(\nu + 1/2) + \sum_{k=2}^{N} a_k(\nu) h^k + O(h^{N+1}). \]
Theorem

Let \( \hat{H} = \frac{\hat{p}^2}{2} + V(\hat{x}) \), where \( V(x) \geq 0 \) is an even degree polynomial s.t. \( V(0) = V'(0) = 0 \), and \( V''(0) = 1 \). Let \( \hat{G}(\hat{p}, \hat{x}) \) be the differential operator of degree \( 2N \) that corresponds to the normal form of \( \hat{H} \). Then \( \hat{H} \) has a series of eigenvalues \( E_\nu = \lambda_\nu + O(h^{N+1}) \), where \( \hat{G}\psi_\nu = \lambda_\nu \psi_\nu u \), and \( \psi_\nu \) are eigenfunctions of \( \hat{H}_2 = \frac{\hat{p}^2}{2} + \frac{\hat{x}^2}{2} \).

The idea of proof (suggested by V.E. Nazaikinskii) is based on lifting a formal normal form into a pseudo-differential algebra of ordered symbols.

Remark

In a different language: S. vu Ngoc, L. Charles, 2008.
Explicit formulas

Let us expand $\mathbb{H}_k = \bigoplus_{s=0}^{k} \mathbb{H}_{(s,k-s)}$ ($\mathbb{H}_{(s,k-s)}$ is generated by words, where $p$ appears $s$ times, and $x$ appears $k - s$ times).

**Proposition**

*The following system is a basis in $\mathbb{H}_{(s,r)}$:*

\[ p^s x^r, p^{s-1} x^r p, \ldots, px^r p^{s-1}, x^r p^s, \text{ if } s \leq r. \]

\[ x^r p^s, x^{r-1} p^s x, \ldots, xp^s x^{r-1}, p^s x^r \text{ if } r \leq s. \]

**Definition**

This basis is called primitive.

**Remark**

A traditional approach of $h^k$ expansions and symbol maps means using the basis: $p^s x^r, rp^{s-1} x^{r-1}, \ldots, r^{s-1} px^{r-s+1}, r^s x^{r-s}$.
Birkhoff normal form: reminder

To reduce a classical Hamiltonian $H = \sum_{k=2}^{\infty} H_k$, $H_2 = \frac{p^2 + x^2}{2}$ to a normal form, we use steps as follows:

1. Linear change of variables: $p = \frac{a_- + ia_+}{\sqrt{2}}$, $x = \frac{ia_- + a_+}{\sqrt{2}}$.

2. A sequence of near-identity maps: $e^{adW_3}, e^{adW_4}, \ldots$:
   
   $e^{adW_k} : a_- = \tilde{a}_- + O(\tilde{a}_-, \tilde{a}_+)^{k-1}, a_+ = \tilde{a}_+ + O(\tilde{a}_-, \tilde{a}_+)^{k-1}$

3. Inverse change of variables to step (1): $(a_-, a_+) \rightarrow (p, x)$.

$$H(p, x) = \frac{p^2 + x^2}{2} + \sum_{k=3}^{\infty} H_k \xrightarrow{(1)} ia_- a_+ + \sum_{k=3}^{\infty} \tilde{H}_k \xrightarrow{(2)} ia_- a_+ + \sum_{k=3}^{N} \tilde{G}_k + O(a_-, a_+)^{N+1} \xrightarrow{(3)} \frac{p^2 + x^2}{2} + \sum_{k=3}^{N} G_k + O(a_-, a_+)^{N+1},$$

$$G_{2k} = g_{2k}(p^2 + x^2)^k, \quad \tilde{G}_{2k} = \tilde{g}_{2k}(a_- a_+)^k, \quad G_{2k+1} = \tilde{G}_{2k+1} = 0.$$
Non-commutative analog

After the changes of coordinates reducing $H \in \mathbb{H}$ to a normal form: $(p, x) \rightarrow (a_-, a_+) \rightarrow (\tilde{a}_-, a_+) \ H$ maps to the form:

$$\tilde{H} = \frac{i}{2}(a_- a_+ + a_+ a_-) + \sum_{k=2}^{N} G_{2k}(a_-, a_+) + O(a_-, a_+)^{2N+2}$$

, where $G_{2k} = \sum_{s=0}^{2k} \alpha_s^{2k} a_+^s a_-^{2k-s} a_+^2$.

Theorem

*The low lying eigenvalues of $\hat{H} = \hat{p}^2 + V(x)$ are*

$$E_\nu = h(\nu + 1/2) + \sum_{k=2}^{N} \mathcal{E}_k^\nu h^k + O(h^{N+1})$$, where

$$\mathcal{E}_k^\nu = k!(-i)^k \sum_{s=0}^{k} \binom{\nu + k - s}{k} \alpha_s^{2k}.$$
**Proposition (D. V. Treschev, 2005)**

For any monomial \( z \) of type \((r, k)\) the linear combination

\[
z = \sum_{s=0}^{k} \alpha_s x^s p^r x^{k-s}
\]

is convex, i.e. \( \alpha_s \geq 0 \) and \( \sum_{s=0}^{k} \alpha_s = 1 \).

**Remark**

The expansions in \( h^k \) are much worse. E.g. in the sum

\[
x^k p^k = \sum_{s=0}^{k} \beta_s x^s p^{r-s} x^{k-s}
\]

also \( \sum_{s=0}^{k} \beta_s = 1 \) but \( \beta_s \) may be \( \sim k! \).

**Definition**

A norm in \( \mathbb{H}_{(r,k)} \) is \( \| F \| := \inf_{z_s} \sum_{s=0}^{k} |\beta_s| \), where \( F = \sum_{s=0}^{k} \beta_s z_s \).

**Remark**

\[
\| F \| := \sum_{s=0}^{k} |\alpha_s|
\]
Application 1: Growth of coefficients

**Definition**
An element $\mathbf{F} = \sum_{r,k=0}^{\infty} \mathbf{F}_{(r,k)}$, where $\mathbf{F}_{(r,k)} \in \mathbb{H}_{(r,k)}$ is analytic, if

$$\|\mathbf{F}_{(r,k)}\| \leq \beta \gamma^{r+k}$$

for some constants $\beta, \gamma > 0$.

**Remark**
If $\mathbf{F}$ is analytic, then $\pi \mathbf{F}$ is a convergent Taylor series.

**Theorem (A., 2009)**

*In one degree of freedom a normal form of an analytic element is analytic.*

**Corollary**

*Coefficients of the normal form in primitive basis: $|\alpha_s^{2k}| \leq \beta \gamma^k$*
Application 1: Growth of coefficients

Recall that

\[ E_\nu(h) \sim \sum_{k=1}^{\infty} h^k \mathcal{E}_k^\nu, \quad \mathcal{E}_k^\nu = k!(\text{e}^{-i})^k \sum_{s=0}^{k} \binom{\nu + k - s}{k} \alpha_s^{2k} \]

Corollary

\[ |\mathcal{E}_k^\nu| \leq (\nu + 1) \ldots (\nu + k) \beta \gamma^k \leq k! \beta \gamma^k. \]

Example: \( \mathcal{E}_k^0 = k!(\text{e}^{-i})^k \alpha_k^{2k}. \)

Question

What can be said about the remainder \(|E_\nu(h) - \sum_{k=1}^{N} h^k \mathcal{E}_k^\nu|\)?
Recall that $V(x)$ is an even degree polynomial s.t. $V(0) = V'(0) = 0$ and $V''(0) = 1$.

**Proposition**

Denote $\gamma_{k,s}^{\nu} = \binom{\nu+k-s}{k} \binom{k}{s}$ and $\tau_{s}^{k} = 2k^{2} + 7k + 16s(k-s)$, then

$$\mathcal{E}_{k}^{\nu} = \frac{(2k)!}{2^{2k}k!} \left[ u_{2k} \sum_{s=0}^{k} \gamma_{k,s}^{\nu} - \left( \sum_{s=0}^{k} \gamma_{k,s}^{\nu} \tau_{s}^{k} \right) \frac{u_{3}u_{2k-1}}{3k} \right] + \tilde{\mathcal{E}}_{k}^{\nu}$$

where $\tilde{\mathcal{E}}_{k}^{\nu}$ depends only on $u_{3}, \ldots, u_{2k-2}$.

**Proposition**

$$\begin{vmatrix}
\sum_{s=0}^{k} \gamma_{k,s}^{\nu} & \sum_{s=0}^{k} \gamma_{k,s}^{\nu} \tau_{s}^{k} \\
\sum_{s=0}^{k} \gamma_{k,s}^{\mu} & \sum_{s=0}^{k} \gamma_{k,s}^{\mu} \tau_{s}^{k}
\end{vmatrix} \neq 0 \text{ for all } \mu \neq \nu \text{ and } k \geq 2.$$
Application 2: Inverse spectral problem

Theorem
Fix $\nu \neq \mu$. In the class of potentials s.t. $V^{(3)}(0) \neq 0$, sequences $\mathcal{E}_k^\mu$ and $\mathcal{E}_k^\nu$ determine $V(x)$ uniquely up to the $x \to -x$ symmetry.

Remark
Known results: $V(x)$ is determined by $\mathcal{E}_k^\mu$ for all $\mu \geq 0$ and $k \geq 2$ (Y. Colin de Verdiere, H. Hezari, 2008). (The assumption $V^{(3)}(0) \neq 0$ is also imposed).

Remark
In the degenerate case $V^{(3)}(0) = 0$ one should consider first $V^{(5)}(0) \neq 0$. This leads also to a $2 \times 2$ determinant but more complicated.