On the Saturation Rule for the Stability of Queues

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Abstract

This paper focuses on the stability of open queueing systems under stationary ergodic assumptions. It defines a set of conditions, the *monotone separable framework*, ensuring that the stability region is given by the following saturation rule: 'saturate' the queues which are fed by the external arrival stream; look at the 'intensity' μ of the departure stream in this saturated system; then stability holds whenever the intensity of the arrival process, say λ satisfies the condition $\lambda < \mu$, whereas the network is unstable if $\lambda > \mu$. Whenever the stability condition is satisfied, it is also shown that certain state variables associated with the network admit a finite stationary regime which is constructed pathwise using a Loynes type backward argument. This framework involves two main pathwise properties, external monotonicity and separability, which are satisfied by several classical queueing networks. The main tool for the proof of this rule is sub-additive ergodic theory. It is shown that for various problems, the proposed method provides an alternative to the methods based on Harris recurrence and regeneration; this is particularly true in the Markov case, where we show that the distributional assumptions commonly made on service or arrival times so as to ensure Harris-recurrence can in fact be relaxed.

Keywords: Open queueing network, stationary point processes, monotonicity, separability, first and second-order ergodic properties, subadditive ergodic theorem.

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1 Introduction

A folk theorem of queueing theory states that the stability region of an open queueing systems can be obtained as follows: 'saturate' the queues which are fed by the external arrival stream with an infinite customer population; if μ denotes the 'intensity' of the departure stream in this saturated system, then the system is stable when the intensity of the arrival process, λ, satisfies $\lambda < \mu$. This engineering rule, which we will refer to as the saturation rule, was initially designed for Markovian queueing systems, where μ could be obtained by computing the steady state of a Markov chain of smaller dimension than that of the initial non-saturated system (see for instance [13]).

The aim of the present paper is to set a natural framework in which this rule can be rigorously proved when dropping the Markovian assumptions and replacing them by standard stationary ergodic assumptions, namely the arrival point process is a stationary ergodic marked point process of finite intensity. The conditions required by this framework consist of easy-to-check, pathwise properties of the system under consideration. The two main conditions are *external monotonicity* and *separability*, although two other mild conditions called *causality* and *homogeneity* are also assumed.

For any queueing system within this framework (to be defined below), we will denote X_n the time of the last activity to take place in the system, whenever one starts with n customers, all arrived at time 0 in an empty system. The *maximal dater* X_n turns out to be the adequate way of implementing the saturation idea for non-Markov systems. The main result is then:

Theorem 1 *The sequence $\{X_n\}$ satisfies a SLLN:*

$$\lim_n \frac{X_n}{n} = \gamma(0) \quad \text{a.s.},$$

for some non-negative constant $\gamma(0)$; this constant will also be finite if the input marked process satisfies natural integrability conditions. Whenever the intensity of the input process satisfies the condition $\lambda < \gamma^{-1}(0)$, then the system is stable, whereas it is unstable if $\lambda > \gamma^{-1}(0)$.

Two versions of this theorem are proposed. In the first one, considered in § 2, the monotonicity and separability assumptions which are made, and the results which are obtained only concern the maximal daters $\{X_n\}$. Stability here means that when the system is fed by the restriction of the point process on $(-\infty, t)$, the time to inactivity admits a finite steady state regime.

In the refined version of § 4, monotonicity and separability assumptions are made on all daters, namely the epochs of all events in the network. In that case, a steady regime can be constructed for the number of customers in each queue.

In both versions, this stationary regime is constructed pathwise using a backward argument of the same nature as the argument of Loynes in the $G/G/1$ queue. If $\lambda > \gamma^{-1}$, the system is unstable in the sense that certain variables (either the time to inactivity or the number of customers in some queues) tend to ∞ in probability.

The proposed method provides an alternative to the methods based on regeneration (Harris-recurrence [15] in the Markov-case, Renovating Events in the non-Markov case [8]). Regeneration is replaced by separability. An interesting point when comparing these methods is that regeneration has to be effective (for instance, in the Harris-recurrent case, it is necessary to have an infinite number of regeneration points – forming a recurrent point process – for the method to apply), whereas the method developed here applies even in the case when separation a.s. never takes place.

Several classes of queueing systems for which the stability is an open question (at least in the stationary ergodic case) fall within this framework, like for instance generalized Jackson queueing networks, polling systems and certain non-(max +)-linear Petri nets (see the examples of the final section). So subadditive ergodic theory is central for a much wider class than the class of models based (directly or indirectly) on random graph structures (see for example [11], [2], [1], [5], [6]).

2 The Monotone Separable Framework for Maximal Daters

Arrival Point Process Let N be a marked point process with points $\{T_n\}_{n\in\mathbb{Z}}$ and marks $\{\xi_n\}_{n\in\mathbb{Z}}$, where $\xi_n \in (K, \mathcal{K})$. This point process is *not* assumed to be simple (nor to be stationary at this stage). We only assume that $T_n \leq T_{n+1}$, for all n. We shall use the notations τ_n for $T_{n+1} - T_n$, $c+N$ for the point process $\{T_n + c\}$ and cN for the point process $\{cT_n\}$, where $c \in I\!R$. In what follows, we shall not adopt the usual renumbering rule for point processes, and the n-th point of $N + c$ will be $T_n + c$ by definition.

Maximal Daters For all $m \leq n \in I\!N$, let $X_{[m,n]}(N)$ be the time of the last activity in the network, when this one starts empty and is fed by the $[m, n]$ restriction of N, namely the point process $N_{[m,n]}$, with points $\{T_l\}_{m\leq l\leq n}$. We assume to be given a set of functions $\{f_l\}$, $f_l : I\!R^l \times K^l \to I\!R$, such that:

$$X_{[m,n]}(N) = f_{n-m+1}\{(T_l, \xi_l),\ m \leq l \leq n\}, \tag{1}$$

for all n, m and N. We assume that the functions f_n are such that the following properties hold for all N:

1. **(causality):** For all $m \leq n$,
$$X_{[m,n]}(N) \geq T_n;$$
2. **(external monotonicity):** For all $m \leq n$,
$$X_{[m,n]}(N') \geq X_{[m,n]}(N),$$
whenever $N' \stackrel{\text{def}}{=} \{T'_n\}$ is such that $T'_n \geq T_n$ for all n, a property which we will write $N' \geq N$ for short;
3. **(homogeneity):** $\forall c \in I\!R$, $\forall m \leq n$
$$X_{[m,n]}(c + N) = X_{[m,n]}(N) + c;$$
4. **(separability):** If, for all $m \leq l < n$, $X_{[m,l]}(N) \leq T_{l+1}$, then
$$X_{[m,n]}(N) = X_{[l+1,n]}(N).$$

In words, property (4) simply states that if the arrival of customer $l + 1$ takes place later than the last activity for the arrival process $[m, l]$, then the evolution of the network after time T_{l+1} is the same as in the network which 'starts empty' at this time.

Remark Note that any system satisfying (2) and (3) also satisfies the following *continuity* property: for all n, if $T_k \le T'_k \le T_k + \epsilon$, then

$$X_{[1,n]} \le X'_{[1,n]} \le X_{[1,n]} + \epsilon.$$

A similar property holds true for all daters in the generalized framework of §4.

Properties of Maximal Daters Let

$$Z_{[m,n]}(N) \stackrel{\text{def}}{=} X_{[m,n]}(N) - T_n = X_{[m,n]}(N - T_n).$$

Note that $Z_{[m,n]}(N)$ is a function of $\{\xi_n\}$ and $\{\tau_l\}_{m\le l\le n-1}$ only. In particular, $Z_n(N) \stackrel{\text{def}}{=} Z_{[n,n]}(N)$ is not a function of $\{\tau_n\}$.

Lemma 1 (internal monotonicity of X and Z) *Under the above conditions, the variables $X_{[m,n]}$ and $Z_{[m,n]}$ satisfy the internal monotonicity property: for all N*

$$X_{[m-1,n]}(N) \ge X_{[m,n]}(N), \quad Z_{[m-1,n]}(N) \ge Z_{[m,n]}(N) \quad (m \le n).$$

Proof Consider the point process N' with points:

$$T'_j = \begin{cases} T_j - Z_{m-1}(N) & \text{for } j \le m-1; \\ T_j & \text{for } j \ge m\ . \end{cases}$$

Since the $[m,\infty]$ restrictions of N and N' coincide, $X_{[m,n]}(N) = X_{[m,n]}(N')$. The separability assumption implies that $X_{[m-1,n]}(N') = X_{[m,n]}(N')$. Finally, the external monotonicity implies that $X_{[m-1,n]}(N') \le X_{[m-1,n]}(N)$. □

Lemma 2 (subadditive property of Z.) *Under the above conditions, $\{Z_{[m,n]}\}$ satisfies the following sub-additive property: for all $m \le l < n$, for all N*

$$Z_{[m,n]}(N) \le Z_{[m,l]}(N) + Z_{[l+1,n]}(N).$$

Proof Introduce two auxiliary point processes $N^1 = \{T^1_j\}$ and $N^2 = \{T^2_j\}$ defined by

$$T^1_j = \begin{cases} T_j & \text{for } j \le l; \\ T_j + Z_{[m,l]}(N) & \text{for } j > l\ . \end{cases}$$

and

$$T^2_j = \begin{cases} T_j - Z_{[m,l]}(N) & \text{for } j \le l; \\ T_j & \text{for } j > l\ . \end{cases}$$

So $T^2_j = T^1_j - Z_{[m,l]}(N)$, for all j. Then, using assumptions (1)-(4) of our framework

$$\begin{aligned} X_{[m,n]}(N) &\stackrel{2}{\le} X_{[m,n]}(N^1) \stackrel{4}{=} X_{[l+1,n]}(N^1) \\ &\stackrel{3}{=} X_{[l+1,n]}(N^2) + Z_{[m,l]}(N) = X_{[l+1,n]}(N) + Z_{[m,l]}(N). \end{aligned}$$

Therefore

$$\begin{aligned} Z_{[m,n]}(N) &= X_{[m,n]}(N) - T_n \le X_{[l+1,n]}(N) - T_n + Z_{[m,l]}(N) \\ &= Z_{[l+1,n]}(N) + Z_{[m,l]}(N). \end{aligned}$$

□

Remark The conditions (1)-(4) are given here under their simplest form. They can be relaxed in various ways without altering the conclusions. In particular:

- Causality can be replaced by the weaker conditions that $X_{[1,n]}(N) \geq T_n - U_n$, where U_n is positive and $EU_n \leq cn$ for all n and for some $c < \infty$.
- Separability can be replaced by the following weaker *subseparability* condition:

$$X_{[m,n]}(N) \leq X_{[l+1,n]}(N),$$

  for all $m \leq l < n$, such that $X_{[m,l]}(N) \leq T_{l+1}$.
- Another possible weakening of (4) is as follows: for all $m \leq l < n$, if $X_{[m,l]}(N) + U_{[m,l]}(N) \leq T_{l+1}$, then $X_{[m,n]}(N) = X_{[l+1,n]}(N)$, where $U_{[m,l]}(N)$ is some auxiliary sequence such that
  - $U_{[m,l]}(N) = U_{[m-1,l-1]}(N) \circ \theta$ (see §3);
  - $\{U_{[-n,0]}(N)\}$ is uniformly integrable and $U_{[-n,0]}(N)/n \to 0$ a.s. as $n \to \infty$.

  Concerning the relevant extension of Kingman's subadditive ergodic theorem, see [16].
- Homogeneity can also be weakened and be replaced by some homogeneity in distribution. Concerning the relevant extension of Kingman's subadditive ergodic theorem, see [14].

Several other extensions are possible, like for instance replacing the arrival point process by a random measure.

3 Proof of the Saturation Rule

Stationarity Assumptions Assume the variables $\{\tau_n, \xi_n\}$ are random variables defined on a common probability space $(\Omega, \mathcal{F}, P^0, \theta)$, where θ is a measure-preserving shift transformation, such that $(\tau_n, \xi_n) \circ \theta = (\tau_{n+1}, \xi_{n+1})$. For instance, this space is the Palm space of a stationary ergodic point process. The following integrability assumptions are also assumed to hold:

$$E^0 \tau_n \stackrel{\text{def}}{=} \lambda^{-1} < \infty, \quad E^0 Z_n < \infty.$$

First-Order Ergodic Property Kingman's sub-additive ergodic theorem gives:

Lemma 3 *There exists a finite non-negative constant γ such that the a.s. limits*

$$\lim \frac{Z_{[-n,-1]}}{n} = \lim \frac{E^0 Z_{[-n,-1]}}{n} = \lim \frac{Z_{[1,n]}}{n} = \lim \frac{E^0 Z_{[1,n]}}{n} = \gamma$$

hold P^0-a.s.

Corollary 1 *Under the foregoing assumptions*

$$\lim_n \frac{X_{[1,n]}}{n} = \gamma + \lambda^{-1}.$$

0-1 Law Let A be the event $A = \{\lim Z_{[-n,0]} = \infty\}$.

Theorem 2 *Under the foregoing ergodic assumption, $P^0(A) \in \{0, 1\}$.*

Proof Note that $\theta A = \{\lim Z_{[-n,-1]} = \infty\}$. But owing to the sub-additive property, $Z_{[-n,-1]} \geq Z_{[-n,0]} - Z_0$. This and the integrability of Z_0 imply that $\theta A \supseteq A$. Since θ is ergodic, the proof is concluded. □

Scaling Factor For all $0 \leq c < \infty$, the sequences

$$X_{[m,n]}(cN) \stackrel{\text{def}}{=} f_{n+1-m}\{(c \cdot T_l, \xi_l); m \leq l \leq n\}$$

and

$$Z_{[m,n]}(cN) = X_{[m,n]}(cN) - c \cdot T_n$$

satisfy all the monotonicity and sub-additive properties mentioned above. In addition, for all n
(a) $Z_{[-n,-1]}(cN)$ is decreasing in c;
(b) $X_{[1,n]}(cN)$ is increasing in c.
Thus

Lemma 4 *For all $c \geq 0$, there exists a non-negative constant $\gamma(c)$ such that*

$$\lim \frac{Z_{[-n,-1]}(cN)}{n} = \gamma(c) \quad \text{a.s.;}$$

$\gamma(c)$ is decreasing in c while $\gamma(c) + c\lambda^{-1}$ is increasing in c. In particular, $\gamma + \lambda^{-1} \geq \gamma(0)$, where $\gamma \stackrel{\text{def}}{=} \gamma(1)$.

Second-Order Ergodic Property The main result on the stability region is:

Theorem 3 *If $\lim Z_{[-n,0]}(N) = \infty$ a.s., then $\lambda\gamma(0) \geq 1$. If $\lambda\gamma(0) > 1$, then $\lim Z_{[-n,0]} = \infty$ a.s.*

Proof We first prove the second assertion. We have

$$\lim_n Z_{[-n,0]}(N)/n = \gamma \geq \gamma(0) - \lambda^{-1} > 0$$

a.s. Therefore, if $\lambda\gamma(0) > 1$, then $Z_{[-n,0]}(N) \to \infty$ a.s.

We now prove the first one. Let Q be the point process with all its points equal to 0: $T_n(Q) = 0$ for all n. For each integer $l \geq 1$, let K_l be the random variable

$$K_l = \min\{n \geq 1 : \quad Z_{[-n,0]}(N) \geq T_l - T_0\},$$

which will be P^0 a.s. finite if $Z_{[-n,0]}$ tends to ∞. Owing to the sub-additive property, for all $n, l \geq 1$

$$Z_{[-n,l]} \leq Z_{[-n,0]} + Z_{[1,l]} \leq Z_{[-n,0]} + \sum_{i=1}^{l} Z_i,$$

where the random variables $Z_i = Z_0 \circ \theta^i$ do not depend on the inter-arrival times and are integrable. For all $n \geq 1$, let $\widehat{N}^n$ be the point process with points

$$\widehat{T}_j^n = \begin{cases} T_j - T_0 & \text{for } j \leq 0; \\ Z_{[-n,0]}(N) & \text{for } j \geq 1 \end{cases}$$

and let $\widetilde{N}^n$ be defined by

$$\widetilde{T}_j^n = Z_{[-n,0]}, \quad \text{for all } j.$$

Then

$$(X_{[-n,l]}(N) - T_0) 1_{n \geq K_l} \quad \stackrel{2}{\leq} \quad X_{[-n,l]}(\widehat{N}^n) 1_{n \geq K_l}$$

$$
\begin{aligned}
&\stackrel{4}{=} && X_{[1,l]}(\widehat{N}^n)1_{n\geq K_l} = X_{[1,l]}(\widetilde{N}^n)1_{n\geq K_l} \\
&\stackrel{3}{=} && \left(Z_{[-n,0]}(N) + X_{[1,l]}(Q)\right) 1_{n\geq K_l} \\
&= && \left(Z_{[-n,0]}(N) + Z_{[1,l]}(Q)\right) 1_{n\geq K_l}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
Z_{[-n,l]}(N)1_{n\geq K_l} &= \left(X_{[-n,l]}(N) - T_l\right) 1_{n\geq K_l} \\
&\leq (Z_{[-n,0]}(N) + Z_{[1,l]}(Q) - T_l + T_0)1_{n\geq K_l}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
& Z_{[-n,l]}(N) - Z_{[-n,0]}(N) \\
& \quad = \left(Z_{[-n,l]}(N) - Z_{[-n,0]}(N)\right) [1_{n<K_l} + 1_{n\geq K_l}] \\
& \quad \leq \left(\sum_{i=1}^{l} Z_i\right) 1_{n<K_l} + \left(Z_{[1,l]}(Q) - T_l + T_0\right) 1_{n\geq K_l} \\
& \quad = \Psi_l 1_{n<K_l} + Z_{[1,l]}(Q) - T_l + T_0, \qquad (2)
\end{aligned}
$$

where

$$
\psi_l \stackrel{\text{def}}{=} \sum_{i=1}^{l} Z_i - Z_{[1,l]}(Q) + T_l - T_0
$$

is P^0-integrable. By making use of the relations $Z_{[-n,l]} = Z_{[-n-l,0]} \circ \theta^l$, $Z_{[-n-l,0]} \geq Z_{[-n,0]}$ and $E^0 Z_{[-n-l,0]} < \infty$, we obtain from (2) that

$$
0 \leq E^0 Z_{[-n,l]} - E^0 Z_{[-n,0]} \leq E^0\{\psi_l 1_{n<K_l}\} + E^0 Z_{[1,l]}(Q) - l\lambda^{-1}.
$$

If K_l is a.s. finite for all l, the right-hand side of the last equation tends to $E^0 Z_{[1,l]}(Q) - l\lambda^{-1}$ as $n \to \infty$. Therefore

$$
\frac{E^0 Z_{[1,l]}(Q)}{l} \geq \lambda^{-1},
$$

for all l. Finally, when letting l go to infinity and when making use of Lemma 4, we obtain

$$
\gamma(0) = \lim_l \frac{E^0 Z_{[1,l]}(Q)}{l} \geq \lambda^{-1}.
$$

□

Thus if $\lambda\gamma(0) < 1$, the random variable

$$
Z \stackrel{\text{def}}{=} \lim_n Z_{[-n,0]} \quad \text{a.s.}
$$

is P^0-a.s. finite and it provides a minimal stationary regime for the time to inactivity, which is defined as the time to the last activity in the system when subject to the $[-\infty, 0]$ restriction of N.

4 The Separable-Monotone Framework for Counters and Daters

The network is assumed to be characterized by following two equivalent sets of refined state variables:

- The daters: $X^i_{[m,n]}(k) \in I\!R$, will denote the epoch of the k-th event on node i, when the network has $N_{[m,n]}$ for input point process (here, we take $k \in I\!N$ and $X^i_{[m,n]}(k) = \infty$ if there are less than k events on node i).

- The counters: $\mathcal{X}^i_{[m,n]}(t) \in I\!N$, will denote the number of events (for instance customer departures) which take place on node i before time t (this function will be taken left-continuous with right-hand limits).

These two sets of variables are equivalent since counters and daters are related by

$$\mathcal{X}^i_{[m,n]}(t) = \sum_{k \in I\!N} 1_{\{X^i_{[m,n]}(k) \le t\}} . \tag{3}$$

The refined separable-monotone framework consists of the following set of assumptions:

II. **(external Monotonicity)** If $N_{[m,n]} \le N'_{[m,n]}$, then (with obvious notations)

$$X^i_{[m,n]}(k) \le \{X'\}^i_{[m,n]}(k),$$

for all k and i , or equivalently

$$\mathcal{X}^i_{[m,n]}(t) \ge \{\mathcal{X}'\}^i_{[m,n]}(t),$$

for all t and i.

III. **(homogeneity)** If $T'_l = T_l + c$, then

$$\{X'\}^i_{[m,n]}(k) = X^i_{[m,n]}(k) + c,$$

for all k and i, or equivalently

$$\{\mathcal{X}'\}^i_{[m,n]}(t + c) = \mathcal{X}^i_{[m,n]}(t),$$

for all t and i.

V. **(conservation)** Let[1]

$$\mathcal{X}^i_{[m,n]} \stackrel{\text{def}}{=} \lim_{t \to \infty} \mathcal{X}^i_{[m,n]}(t) . \tag{4}$$

We assume that $\mathcal{X}^i_{[m,n]}$ is *finite* and *independent* of the values taken by the variables T_l, $n \le l \le m$ (provided m, n and $\{T_l\}$ are finite of course).

The *maximal dater* of the network is naturally defined by:

$$X_{[m,n]} = \max_i X^i_{[m,n]}(\mathcal{X}^i_{[m,n]}) . \tag{5}$$

IV. **(separability)** The separability assumption states that if $T_{l+1} \ge X_{[m,l]}$, then

$$\begin{aligned} X^i_{[m,n]}(k) &= X^i_{[m,l]}(k), \quad k \le \mathcal{X}^i[m,l] \\ X^i_{[m,n]}(k + \mathcal{X}^i_{[m,l]}) &= X^i_{[l+1,m]}(k), \quad k \ge 1 \end{aligned}$$

or equivalently

$$\begin{aligned} \mathcal{X}^i_{[m,n]}(t) &= \mathcal{X}^i_{[m,l]}(t), \quad t < T_{l+1} \\ \mathcal{X}^i_{[m,n]}(t) &= \mathcal{X}^i_{[m,l]} + \mathcal{X}^i_{[l+1,m]}(t), \quad t \ge T_{l+1}. \end{aligned}$$

[1] This limit exists since the function $\mathcal{X}^i_{[m,n]}(t)$ is non-decreasing. In words, $\mathcal{X}^i_{[m,n]}$ counts the *total* number of events on node i for $N_{[m,n]}$

Remarks It is easy to check that the separation and the conservation properties imply that for all $m \le l < n$,

$$\mathcal{X}^i_{[m,n]} = \mathcal{X}^i_{[m,l]} + \mathcal{X}^i_{[l+1,n]} \tag{6}$$

regardless of $\{T_l\}$.

Whenever all counters (or daters) of a queueing network satisfy the assumptions II-V, the maximal dater defined by (5) necessarily satisfies Assumptions 2-4 of the framework of Section 2. In what follows, we will also assume that condition (1) of this framework is satisfied.

Let

$$W^i_{[m,n]}(t) \overset{\text{def}}{=} \mathcal{X}^i_{[m,n]} - \mathcal{X}^i_{[m,n]}(T_n + t) \quad t \ge 0 .$$

Theorem 4 (refined internal monotonicity) *Under the above assumptions, for all $t \ge 0$, i and n,*

$$W^i_{[m-1,n]}(t) \ge W^i_{[m,n]}(t),$$

so that

$$\exists \lim_{m \to -\infty} \uparrow W^i_{[m,n]}(t) \overset{\text{def}}{=} W^i_{[-\infty,n]}(t).$$

Proof Let N' be as in the proof of Lemma 1. From the conservation assumption,

$$W^i_{[m,n]}(t) = \{\mathcal{X}'\}^i_{[m,n]} - \{\mathcal{X}'\}^i_{[m,n]}(T_n + t).$$

But in view of (6)

$$\{\mathcal{X}'\}^i_{[m-1,n]} = \{\mathcal{X}'\}^i_{[m-1,m-1]} + \{\mathcal{X}'\}^i_{[m,n]} = \mathcal{X}^i_{[m-1,m-1]} + \mathcal{X}^i_{[m,n]}.$$

Similarly, in view of the separation assumption

$$\{\mathcal{X}'\}^i_{[m-1,n]}(T_n + t) = \{\mathcal{X}'\}^i_{[m-1,m-1]} + \{\mathcal{X}'\}^i_{[m,n]}(T_n + t).$$

When using the last three equations, we obtain

$$\begin{aligned} W^i_{[m,n]}(t) &= \{\mathcal{X}'\}^i_{[m-1,n]} - \{\mathcal{X}'\}^i_{[m-1,n]}(T_n + t) \\ &\le \mathcal{X}^i_{[m-1,n]} - \mathcal{X}^i_{[m-1,n]}(T_n + t) \\ &= W^i_{[m-1,n]}(t), \end{aligned}$$

where the last inequality follows from the (counter) external monotonicity. □

Theorem 5 *Under the stochastic assumptions of the Section 3, if*

$$\{\mathcal{W}_{[-n,0]} \to_{n\to\infty} \infty\} \overset{a.s.}{=} \{\exists i / W^i_{[-n,0]} \to_{n\to\infty} \infty\} \tag{7}$$

then, $\lambda\gamma_0 < 1$, implies $W^i_{[-\infty,n]}(t) < \infty$ for all n, i and t.

Proof In view of Theorem 3 and Assumption (7), the condition $\lambda\gamma(0) < 1$ implies that $W^i_{[-\infty,n]}(t) < \infty$ for all n, i and t. □

Remark If in addition $W^i_{[n,n+k]}(t) = W^i_{[0,k]}(t) \circ \theta^n$ for all n, i and t, then the sequence $W_n = (W^i_{[-\infty,n]}(t),\ t \in [T_n, T_{n+1}),\ i)$ satisfies the compatibility property $W_n = W \circ \theta^n$, and thus provides a way of constructing a stationary version of queueing process. For instance, consider the family of daters $X^{i,j}(k)$ associated with departures from node i to node j, and assume that they satisfy the above assumptions. In this case, it is natural to define the variables:

$$Q^i_{[m,n]}(t) = W^i_{[m,n]}(t) - \sum_j W^{j,i}_{[m,n]}(t) \tag{8}$$

which represent the number of customers in station i at time $t + T_n$ for $N_{[m,n]}$. So, if $\lambda\gamma(0) < 1$, we have constructed a stationary (θ-compatible) version of the $Q^i(t)$ process. For instance, $Q^i_{[-\infty,n]}(t)$ for $t \in [T_n, T_{n+1})$ provides a stationary process Q for the $[-\infty, +\infty]$-restriction of N, namely N.

5 Examples and Counter-Examples

The $G/G/1$ Queue. Here $X_{[m,n]}$ is the departure epoch of customer n, when there are $n+1-m$ customers with arrival epochs T_l and service times σ_l, $m \le l \le n$; $Z_{[m,n]}$ is then the sojourn time of customer n. The computation of $\gamma(0)$ is trivial, by the strong law of large numbers.

The $G/G/s$ Queue. $X_{[m,n]}$ is the last departure time from the queue with customers arriving at T_l, $m \le l \le n$, that is

$$Z_{[m,n]}(N) = \max(W^1_{[m,n]}(N) + \sigma_n, W^s_{[m,n]}(N)),$$

where $W_{[m,n]}(N) = (W^1_{[m,n]}(N), \ldots, W^s_{[m,n]}(N))$ is the ordered workload vector at time T_n-, for this arrival process (we assume that the queue is initially empty). We have

$$\lim_n \frac{Z_{[1,n]}(Q)}{n} = \gamma(0) \quad \text{a.s.},$$

as a consequence of Lemma 4. The computation of the constant $\gamma(0)$ is immediate from the relation

$$\lim \frac{W^j_{[m,n]}(Q)}{n} = \gamma(0) \quad \text{a.s.} \quad (1 \le j \le s). \tag{9}$$

Indeed since

$$\sum_{i=1}^n \sigma_i = \sum_{j=1}^s W^j_{[1,n]}(Q),$$

the relation $\gamma(0) = E^0(\sigma)/s$ follows by an immediate limiting argument.

Proof of (9) For $j = s$, the property follows from the relation $Z_{[1,n]}(Q) = W^s_{[1,n+1]}(Q)$. In order to prove the property for all j, it is enough to show that

$$\frac{W^s_{[1,n]} - W^1_{[1,n]}}{n} \to 0$$

a.s. as $n \to \infty$. Let $U_n = \max(\sigma_n, \ldots, \sigma_{n+s-1})$. By comparing the original queue and the queue with workload $(W^1_{[1,n]}(Q), W^s_{[1,n]}(Q), \ldots, W^s_{[1,n]}(Q))$ at time T_n-, and with constant service time U_n over the interval $n, n+1, \ldots, n+s-1$, we see that

$$W^s_{[1,n+s]}(Q) \le \max(W^s_{[1,n]}(Q), W^1_{[1,n]}(Q) + sU_n) \quad n \ge 0.$$

Let $n = ks + l, \quad 0 \leq l < s$ with l fixed. Since the sequence $W^1_{[1,n]}(Q)$ is non-decreasing in n and since U_n is non-negative, an induction argument leads to the following inequalities:

$$W^s_{[1,n+s]}(Q) \leq \ldots \leq \max(W^s_{[1,l]}(Q), W^1_{[1,ks+l]}(Q)) + s \max_{0 \leq i \leq k} U_{is+l}.$$

Since $\{U_{is+l}\}_{i \geq 0}$ is a stationary sequence with finite mean, then $(\max_{0 \leq i \leq k} U_{is+l})/k \to 0$ a.s. as $k \to \infty$. Obviously, $W^s_{[1,l]}(Q)/k \to 0$ a.s. as $k \to \infty$. Then

$$\begin{aligned}
\gamma(0) &= \lim_{k \to \infty} \frac{W^s_{[1,ks+l]}}{ks} = \liminf(\frac{W^s_{[1,ks+l]}}{ks}) \\
&\leq \liminf \frac{\max(W^s_{[1,l]}(Q), W^1_{[1,ks+l]}(Q))}{ks} + \limsup \frac{\max_{0 \leq i \leq k} U_{is+l}}{k} \\
&= \liminf \frac{W^1_{[1,ks+l]}(Q)}{ks}
\end{aligned}$$

a.s. for all $l = 0, \ldots, s-1$. Therefore,

$$\gamma(0) \leq \liminf \frac{W^1_{[1,n]}(Q)}{n}$$

a.s., and this together with $W^s_{[1,n]}(Q) \geq W^1_{[1,n]}(Q)$ a.s. conclude the proof.

FIFO Kelly-type Networks Customers are given a route through the network upon arrival (namely the route of customer n is a mark associated with point T_n). Then the external monotonicity property is *not* satisfied. Note that the conditions stating that the traffic intensity be less than one at each queue is not sufficient for monotonicity, even in the FIFO case (see [9]).

Kelly-type Networks with Synchronization Constraints These networks were defined in [2]. The routing is mechanism is the same as above, but the service discipline is such that, on a given queue, service requirements brought by the n-th external arrival can in no case be served before all the service requirements brought by the $n-1$-st external arrival have been completed. There, $X_{[m,n]}$ is also the last departure epoch from the system with restricted arrivals as above. The constant $\gamma(0)$ is the $(\max, +)$ Lyapunov exponent of a sequence of stationary random matrices defined from the routing and the service times (see [1] for the notions pertaining to $(\max, +)$-Lyapunov exponents).

Generalized Jackson Networks Whenever the routing decisions and the service times are sequences associated with stations (namely, the n-th customer to reach station j requires the service time σ^j_n, and is routed to station ν^j_n), then the network falls in the refined monotone separable framework, provided the routing sequences satisfy the two stationarity properties described below (see [3]).

Let K be the number of stations.

- We assume that there exist K non-decreasing, integer-valued sequences φ^k_l, $k = 1, \ldots, K$, such that when denoting $F^k_n = \varphi^k_1 + \cdots + \varphi^k_n$, the sequence $\{\{\nu^k_j\}^{F^k_n}_{j=F^k_{n-1}+1}\}^K_{k=1}\}$ satisfies an Euler-type property defined in [3].
- We assume that the driving sequence

$$\xi_n \stackrel{\text{def}}{=} \{\nu^0_n, \{\{\sigma^k_j, \nu^k_j\}^{F^k_n}_{j=F^k_{n-1}+1}\}^K_{k=1}\}_{n \geq 1} \tag{10}$$

is stationary ergodic.

Both conditions are for instance satisfies in the i.i.d. case provided the routing matrix is without capture.

In this case, the constant $\gamma(0)$ is equal to $\max_k \rho^k$, where $\rho^k = E(\sum_{j=1}^{\varphi_1} \sigma_j^k)$.

Stochastic Petri Nets Open FIFO stochastic event graphs fall in our framework, namely they satisfy the four conditions of the framework (see [1]). The constant $\gamma(0)$ is also the $(\max, +)$ Lyapunov exponent of a sequence of random matrices defined from the topology of the network and the service times. Tandem networks (open or closed, with finite or infinite capacity), synchronized Kelly Networks (and in particular Fork-Join networks) all fall in that class of systems.

For general open stochastic Petri nets with switching, also defined in [1], take $X_{[m,n]}$ to be the epoch of the last firing time of whole net for the $[m, n]$ restriction of the arrival point process. Conditions 1-3 of the framework are always satisfied. The validity of condition 4 of the framework is discussed in [4]. This class contains generalized Jackson networks, although it is much more general.

Polling Systems For a wide class of polling models, it is possible to prove stability under general assumptions (e.g. stationary ergodic input, routing and walking mechanism with regenerative structure). The properties (1)-(4) are still valid, but (3)-(4) only hold in distribution. The constant $\gamma(0)$ coincides with the well-known constant (see [10]).

A Non-separable Network for which the Rule Does Not Hold Consider an assembly queue with two independent Poisson arrival streams with the same intensity $\lambda/2$. The system starts empty. Whenever there are customers of both classes in the queue, service is provided at rate μ. The completion of a service consumes one customer of each class. Whenever the queue has no customers of either class, it is blocked. Let consider as input stream the superposition of the two Poisson processes properly marked. If one saturates the system with an infinite customer population, the (Markov) departure rate is μ. Similarly, if one takes the viewpoint of letting n customers of this input stream arrive at time 0, the last activity of the system takes place at time $\mu^{-1}n/2 + o(n)$. A naive application of the saturation rule would suggest that if $\lambda < \mu$, the system is stable. However, such queues are always unstable (see [1]), whatever the values of λ and μ. Note that this system does not satisfy the separability property.

Remark It is not true in general that systems satisfying the properties of the monotone separable framework admit a unique stationary regime. The existence of multiple stationary regimes is well known for $G/G/s$ queues. The same multiplicity arises in generalized Jackson networks (see the last section of [4]), depending on the initial condition which is chosen. However, one can always define a *maximal solution* following the ideas of Remark 30, §6 in [3].

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