Hence, for $x \neq y$,

$$\lim_{m\to\infty} \mathscr{L}V_m(t,x,y)=\exp\left(-\frac{1}{2}\int_0^t K_4(s,R)\,ds\right)$$
$$\times\left\{-K_4(t,R)|x-y|+|x-y|^{-1}\left[2(x-y,a(t,x)-a(t,y))\right.\right.$$
$$\left.\left.+\sum_{j=1}^{l}|\sigma_j(t,x)-\sigma_j(t,y)|^2-|x-y|^{-2}\sum_{j=1}^{l}(x-y,\sigma_j(t,x)-\sigma_j(t,y))^2\right]\right\}\leqq 0$$

by (6). Moreover (assuming that $K_4(t,R)\geqq 0$) we also have as a consequence of the conditions $\int_0^z \varphi_m(z_1,R)\,dz_1\leqq 1$, (7) and (8) that

$$\mathscr{L}V_m(t,x,y)\leqq \exp\left(-\frac{1}{2}\int_0^t K_4(s,R)\,ds\right)\left(K_4(t,R)|x-y|+\frac{K_6(R)|x-y|K_5(t,R)\rho_R^2(|x-y|^2)}{\rho_R^2(|x-y|^2)}\right)$$
$$\leqq 2R\exp\left(-\frac{1}{2}\int_0^t K_4(s,R)\,ds\right)(K_4(t,R)+K_5(t,R)K_6(R)).$$

Condition (2) holds by Theorem 2, as desired.

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RECURRENCY OF AN OSCILLATING RANDOM WALK

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(*Translated by K. Durr*)

1. By an oscillating random walk (see [17]) we mean a homogeneous Markov chain $Y=\{y_n, n=0,\cdots\}$ with state space $Z=\{0,\pm1,\pm2,\cdots\}$, for which $y_0=x$, $x\in Z$,

$$\mathbf{P}\{y_{n+1}=k+l|y_n=l\}=\begin{cases}\mathbf{P}\{\xi_1'=k\} & \text{if } l<0,\\ \mathbf{P}\{\xi_1''=k\} & \text{if } l>0,\\ p\mathbf{P}\{\xi_1'=k\}+q\mathbf{P}\{\xi_1''=k\} & \text{if } l=0,\end{cases}$$

where $k \in Z, l \in Z, p+q=1, p, q \geqq 0$, and $\{\xi'_n\}_{n=1}^{\infty}$ and $\{\xi''_n\}_{n=1}^{\infty}$ are independent sequences of independent and identically (in each sequence) distributed random variables with values in Z. We shall assume that the greatest common divisor of the k for which $\mathbf{P}\{\xi'_1 = k\} > 0$ is equal to 1 and that the same holds for ξ''_1.

The recurrency of the chain Y (here and below by the recurrency or non-recurrency of Y we mean the recurrency or non-recurrency of the state 0 of Y), as shown by examples, is not expressed in terms of the recurrency of the homogeneous random walks $S' = \{S'_n, n=0, 1, \cdots\}$ and $S''=\{S''_n, n=0,1,\cdots\}$,

$$S'_n = \sum_{k=1}^{n} \xi'_k, \qquad S''_n = \sum_{k=1}^{n} \xi''_k,\ n=1,2,\cdots, \qquad S'_0 = S''_0 = 0.$$

In Theorem 1 of this paper we give conditions for recurrency of Y in terms of the distributions of the ladder heights of the random walks S' and S''. Use of Theorem 1 makes it possible to find conditions for the recurrency of Y if the distributions of ξ'_1 and ξ''_1 belong to the region of attraction of stable laws (Theorem 2). In Section 4 we give examples illustrating that Y can be non-recurrent (transient) even in the case when $\mathbf{E}\xi'_1 = \mathbf{E}\xi''_1 = 0$.

The results of this work are based on the following lemma (see [1]).

Lemma 1. *The chain Y is recurrent if and only if*

$$\sum_{h=0}^{\infty} C(h)C(-h) = \infty, \tag{1}$$

where $C(0)=1$ and, for $h=1,2,\cdots$,

$$C(h) = \sum_{n=1}^{\infty} \mathbf{P}\{\min_{1\leqq i \leqq n} S'_i > 0, S'_n = h\},$$

$$C(-h) = \sum_{n=1}^{\infty} \mathbf{P}\{\max_{1\leqq i \leqq n} S''_i < 0, S''_n = -h\}.$$

Let us define on the event $A_+ = \{\sup_{1\leqq n<\infty} S'_n > 0\}$ the ladder random variables (see [7]) $T_+ = \min\{k: S'_k > 0\}$ and $H_+ = S'_{T_+}$, and on the event $A_- = \{\inf_{1\leqq n<\infty} S''_n < 0\}$ the ladder random variables $T_- = \min\{k: S''_k < 0\}$ and $H_- = S''_{T_-}$. For the random variable η and the event A set $\mathbf{E}\{\eta; A\} = \int_A \eta\, d\mathbf{P}$.

Lemma 1 may be reformulated with the aid of the next assertion.

Lemma 2. *Condition* (1) *is equivalent to*

$$\lim_{t\uparrow 1} \int_{-\pi}^{\pi} \operatorname{Re}\left((1-\mathbf{E}\{e^{i\lambda H_+}t^{T_+}; A_+\})^{-1}(1-\mathbf{E}\{e^{i\lambda H_-}t^{T_-}; A_-\})^{-1}\right) d\lambda = \infty.$$

PROOF. For $|t|<1$ and $\operatorname{Im}\lambda = 0$ let

$$C_+(t,\lambda) = 1 + \sum_{n=1}^{\infty} t^n \sum_{h=1}^{\infty} e^{i\lambda h}\mathbf{P}\{\min_{1\leqq i\leqq n} S'_i > 0, S'_n = h\}, \tag{2}$$

$$C_-(t,\lambda) = 1 + \sum_{n=1}^{\infty} t^n \sum_{h=-\infty}^{-1} e^{i\lambda h}\mathbf{P}\{\max_{1\leqq i\leqq n} S''_i < 0, S''_n = h\}. \tag{3}$$

Since, for every t, $|t|<1$, the series (2) and (3) are absolutely convergent,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} C_+(t,\lambda)C_-(t,\lambda)\,d\lambda = \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Re}(C_+(t,\lambda)C_-(t,\lambda))\,d\lambda$$

$$= 1 + \sum_{h=1}^{\infty}\left(\sum_{n=1}^{\infty} t^n \mathbf{P}\{\min_{1\leqq i\leqq n} S'_i > 0, S'_n = h\}\right)\left(\sum_{n=1}^{\infty} t^n\mathbf{P}\{\max_{1\leqq i\leqq n} S''_i < 0, S''_n = -h\}\right)$$

for $0<t<1$. From the equalities (see, for example, [2], p. 416)

$$C_+(t,\lambda)=1+\sum_{n=1}^{\infty} t^n \mathbf{E}\{e^{i\lambda S_n'};\ \min_{1\leqq i\leqq n} S_i'>0\} = (1-\mathbf{E}\{e^{i\lambda H_+}t^{T_+};A_+\})^{-1} \quad \text{for} \quad \operatorname{Im}\lambda\geqq 0, |t|<1, \tag{4}$$

$$C_-(t,\lambda)=1+\sum_{n=1}^{\infty} t^n \mathbf{E}\{e^{i\lambda S_n''};\ \max_{1\leqq i\leqq n} S_i''<0\} = (1-\mathbf{E}\{e^{i\lambda H_-}t^{T_-};A_-\})^{-1} \quad \text{for} \quad \operatorname{Im}\lambda\leqq 0, |t|<1, \tag{5}$$

we obtain the lemma.

From (4) and (5) it follows that

$$\sum_{h=0}^{\infty} C(h)\,e^{i\lambda h}=(1-h_+(\lambda))^{-1} \quad \text{for } \operatorname{Im}\lambda>0, \tag{6}$$

$$\sum_{h=-\infty}^{0} C(h)\,e^{i\lambda h}=(1-h_-(\lambda))^{-1} \quad \text{for } \operatorname{Im}\lambda<0, \tag{7}$$

where

$$h_+(\lambda)=\mathbf{E}\{e^{i\lambda H_+};A_+\} \quad \text{for} \quad \operatorname{Im}\lambda\geqq 0,\ h_-(\lambda)=\mathbf{E}\{e^{i\lambda H_-};A_-\} \quad \text{for} \quad \operatorname{Im}\lambda\leqq 0,$$

hence, for $h=1,2,\cdots$,

$$C(h)=\sum_{k=1}^{h} p_k(h), \qquad C(-h)=\sum_{k=1}^{h} p_k(-h),$$

where, for $h=1,2,\cdots$, $p_k(h)$ are defined by the relations

$$(h_+(\lambda))^k=\sum_{h=k}^{\infty} p_k(h)\,e^{i\lambda h}, \qquad (h_-(\lambda))^k=\sum_{h=-\infty}^{-k} p_k(h)\,e^{i\lambda h}.$$

In a fashion similar to that in which Lemma 2 was proved we see from the relations

$$(1-th_+(\lambda))^{-1}=1+\sum_{k=1}^{\infty} t^k(h_+(\lambda))^k=1+\sum_{h=1}^{\infty} e^{i\lambda h}\Big(\sum_{k=1}^{h} t^k p_k(h)\Big),$$

$$(1-th_-(\lambda))^{-1}=1+\sum_{h=-\infty}^{-1} e^{i\lambda h}\Big(\sum_{k=1}^{-h} t^k p_k(h)\Big),$$

which are valid for $|t|<1$ and $\operatorname{Im}\lambda=0$, that condition (1) is equivalent to

$$\lim_{t\uparrow 1}\int_{-\pi}^{\pi} \operatorname{Re}\,((1-th_+(\lambda))^{-1}(1-th_-(\lambda))^{-1})\,d\lambda=\infty, \tag{8}$$

or to

$$\lim_{t\uparrow 1}\int_{-\pi}^{\pi} \operatorname{Re}\,((1-th_+(\lambda))^{-1})\operatorname{Re}\,((1-th_-(\lambda))^{-1})\,d\lambda=\infty. \tag{9}$$

The expressions for $C(h)$ given by (6) and (7) permit us to derive the following conditions for recurrency of Y.

Corollary 1. *If* $\mathbf{P}\{A_+\}<1$ *or* $\mathbf{P}\{A_-\}<1$, *then* Y *is non-recurrent. If* $\mathbf{P}\{A_+\}=\mathbf{P}\{A_-\}=1$ *and* $\mathbf{E}H_+<\infty$ *or* $-\mathbf{E}H_-<\infty$, *then* Y *is recurrent.*

PROOF. If $\mathbf{P}\{A_+\}<1$, then from (6) we obtain $\sum_{h=0}^{\infty} C(h)<\infty$, while since $C(-h)\leqq C<\infty$ for $h=1,2,\cdots$, the chain Y is non-recurrent by (1).

In view of the condition $\mathbf{E}H_+<\infty$ and the renewal theorem, $\lim_{h\to\infty} C(h)=1/\mathbf{E}H_+$, while since $\mathbf{P}\{A_-\}=1$, we have $\sum_{h=0}^{\infty} C(-h)=\infty$. Hence $\sum_{h=0}^{\infty} C(h)C(-h)=\infty$; thus under the condition that $\mathbf{P}\{A_+\}=\mathbf{P}\{A_-\}=1$, $\mathbf{E}H_+<\infty$, the chain Y is recurrent by Lemma 1.

Note that $\mathbf{E}H_+<\infty$, if $0<\mathbf{E}\xi_1'<\infty$ or $\mathbf{E}\xi_1'=0$ and $\mathbf{E}(\max(0,\xi_1'))^2<\infty$ (see [3]).

2. Theorem 1. *If for some* $\delta>0$

$$\int_{-\delta}^{\delta}|1-h_+(\lambda)|^{-1}|1-h_-(\lambda)|^{-1}\,d\lambda<\infty, \tag{10}$$

then the random walk Y *is non-recurrent.*

If

$$\begin{gathered}\operatorname{Re}((1-h_+(\lambda))(1-h_-(\lambda)))\geqq 0 \quad \textit{for} \quad |\lambda|<\delta \quad \textit{for some} \quad \delta>0 \quad \textit{and} \\ \int_{-\delta}^{\delta}\operatorname{Re}((1-h_+(\lambda))^{-1}(1-h_-(\lambda))^{-1})\,d\lambda=\infty,\end{gathered} \tag{11}$$

then the random walk Y *is recurrent.*

PROOF. For $\operatorname{Im}\lambda=0$, $0<t<1$,

$$0\leqq\operatorname{Re}((1-th_+(\lambda))^{-1})\leqq|1-th_+(\lambda)|^{-1}\leqq t^{-1}|1-h_+(\lambda)|^{-1},$$

since

$$\operatorname{Re}(1-th_+(\lambda))=1-t+t\operatorname{Re}(1-h_+(\lambda))\geqq t\operatorname{Re}(1-h_+(\lambda))\geqq 0,$$
$$\operatorname{Im}(1-th_+(\lambda))=t\operatorname{Im}(1-h_+(\lambda)).$$

Similarly,

$$0\leqq\operatorname{Re}((1-th_-(\lambda))^{-1})\leqq t^{-1}|1-h_-(\lambda)|^{-1}.$$

Therefore

$$\lim_{t\uparrow 1}\int_{-\pi}^{\pi}\operatorname{Re}\frac{1}{1-th_+(\lambda)}\cdot\operatorname{Re}\frac{1}{1-th_-(\lambda)}\,d\lambda\leqq\int_{-\pi}^{\pi}\frac{d\lambda}{|1-h_+(\lambda)||1-h_-(\lambda)|}.$$

From this we obtain, in view of (9), that under the conditions of the first half of the theorem the random walk Y is non-recurrent.

If the conditions of the second half of the theorem hold, then, in view of the condition

$$\operatorname{Re}((1-th_+(\lambda))^{-1}(1-th_-(\lambda))^{-1})\geqq 0,$$

for $|\lambda|\leqq\delta$,

$$\lim_{t\uparrow 1}\int_{-\delta}^{\delta}\operatorname{Re}\frac{1}{(1-th_+(\lambda))(1-th_-(\lambda))}\,d\lambda\geqq\int_{-\delta}^{\delta}\operatorname{Re}\frac{1}{(1-h_+(\lambda))(1-h_-(\lambda))}\,d\lambda$$

by Fatou's lemma. Here (since $|1-h_\pm(\lambda)|>\varepsilon>0$ for $\delta\leqq|\lambda|\leqq\pi$)

$$\lim_{t\uparrow 1}\int_{-\pi}^{\pi}\operatorname{Re}((1-th_+(\lambda))^{-1}(1-th_-(\lambda))^{-1})\,d\lambda=\infty,$$

i.e., (8) holds, and thus Y is recurrent.

Suppose that ξ_1' and ξ_1'' are identically distributed. In this case the oscillating random walk Y is a homogeneous walk on the line, and conditions of recurrency for Y coincide with the known conditions for a homogeneous walk. Indeed, since in this case (see [3])

$$(1-h_+(\lambda))(1-h_-(\lambda))B=1-\mathbf{E}\,e^{i\lambda\xi_1'}, \tag{12}$$

where

$$B=\exp\left\{-\sum_{n=1}^{\infty}\mathbf{P}\{S_n'=0\}/n\right\},$$

conditions (10) and (11) become, respectively,

$$\int_{-\delta}^{\delta}|1-\mathbf{E}\,e^{i\lambda\xi_1'}|^{-1}\,d\lambda<\infty \quad\text{and}\quad \int_{-\delta}^{\delta}\operatorname{Re}\frac{1}{1-\mathbf{E}\,e^{i\lambda\xi_1'}}\,d\lambda=\infty.$$

Turning again to an oscillating random walk, consider the case when ξ'_1 and $-\xi''_1$ are identically distributed. We shall call such oscillating random walks *symmetric*. In this case the following assertion holds.

Corollary 2. *A symmetric oscillating random walk is recurrent if and only if*

$$\int_{-\pi}^{\pi} |1-h_+(\lambda)|^{-2}\, d\lambda = \infty.$$

PROOF. Here H_+ and $-H_-$ are identically distributed; thus $h_+(\lambda) = \overline{h_-(\lambda)}$, where $\bar{z}$ is the complex conjugate of z, and therefore

$$(1-h_+(\lambda))(1-h_-(\lambda)) = |1-h_+(\lambda)|^2.$$

Consequently, if

$$\int_{-\pi}^{\pi} |1-h_+(\lambda)|^{-2}\, d\lambda < \infty,$$

then, by the first part of Theorem 1, Y is non-recurrent while if

$$\int_{-\pi}^{\pi} |1-h_+(\lambda)|^{-2}\, d\lambda = \infty,$$

then, by the second part of Theorem 1, Y is recurrent.

3. Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of identically distributed independent random variables, $S_n = \sum_{k=1}^{n} \xi_k$, $S_0 = 0$. Let the random walk $S = \{S_n, n = 0, 1, 2, \cdots\}$ be strongly attracted to the stable law F, i.e., there is a sequence $\{a_n\}_{n=1}^{\infty}$ of non-negative numbers such that $F(x) = \lim_{n\to\infty} \mathbf{P}\{S_n < a_n x\}$; in this case set $a = \alpha(1-F(0))$, where α, $0<\alpha \leqq 2$, is the index of stability of F. Set $a = 1$ if S is relatively stable, i.e., there is a sequence of non-negative numbers $\{a_n\}_{n=1}^{\infty}$ such that $S_n/a_n \to 1$ as $n\to\infty$ in probability, and $a = 0$ if $\{-S_n\}$ is relatively stable. In all these cases we shall say that the random walk S is *stable*, and the number a is called the *index of stability* of S. Note that if $\{S_n\}$ is stable with index of stability a, then $\{-S_n\}$ is stable and its index of stability equals $\alpha - a = \alpha F(0)$ if S is strongly attracted to the stable law F, and equals $1-a$ otherwise.

Theorem 2. *If the homogeneous walks $\{S'_n\}_{n=1}^{\infty}$ and $\{-S''_n\}_{n=1}^{\infty}$ are stable with indices a_1 and a_2, then, for $a_1+a_2<1$, Y is non-recurrent, while, for $a_1+a_2>1$, Y is recurrent.*

To prove the theorem we need several assertions in which it is assumed that ξ'_1, ξ''_1 and ξ_1 are identically distributed and for S we use the notation introduced for S' and S''.

1. $0 \leqq a \leqq 1$ (see [4]).
2. If $0<a<1$, then the homogeneous random walk with jump distribution coinciding with that of H_+ is strongly attracted to the stable spectrally positive law with index a (see [4]).
3. If $a = 1$, then the homogeneous random walk with jump distribution coinciding with that of H_+ is relatively stable (see [4]).
4. If S is strongly attracted to the stable law F with index α, $0<\alpha<1$, then for any $\varepsilon>0$ there is a $\delta>0$ such that $|1-\mathbf{E}\, e^{i\lambda\xi_1}| \geqq \lambda^{\alpha+\varepsilon}$ for $0 \leqq \lambda \leqq \delta$.

This assertion follows from Theorem 2.6.5 in [5].

5. If S is relatively stable, then, for any $\varepsilon>0$ and sufficiently small $\delta>0$,

$$|1-\mathbf{E}\, e^{i\lambda\xi_1}| \geqq \lambda^{1+\varepsilon} \quad \text{for} \quad 0 \leqq \lambda \leqq \delta. \tag{13}$$

For relatively stable walks it is known that (see [6])

$$\nu(t) = \mathbf{E}\{\xi_1;\, |\xi_1|<t\}$$

is positive for all sufficiently large t, and varies slowly at infinity, and

$$\lim_{t\to\infty} t\mathbf{P}\{|\xi_1| \geqq t\}/\nu(t) = 0.$$

For $\lambda > 0$, consider

$$\operatorname{Im} \mathbf{E}\, e^{i\lambda\xi_1} = \int_{-\infty}^{\infty} \sin x\lambda \, d\mathbf{P}\{\xi_1 < x\} = \lambda \int_{-\pi/\lambda}^{\pi/\lambda} \frac{\sin x\lambda}{x\lambda} x \, d\mathbf{P}\{\xi_1 < x\} + \int_{|x|>\pi/\lambda} \sin x\lambda \, d\mathbf{P}\{\xi_1 < x\}.$$

We have

$$\left| \int_{|x|>\pi/\lambda} \sin x\lambda \, d\mathbf{P}\{\xi_1 < x\} \right| \leqq \mathbf{P}\{|\xi_1| > \pi(\lambda)\} = o(\lambda\nu(1/\lambda))$$

as $\lambda \to 0$. Further, for $f(x) = x^{-1} \sin x$,

$$I = \int_{-\pi/\lambda}^{\pi/\lambda} x f(x\lambda)\, d\mathbf{P}\{\xi_1 < x\} = \int_0^{\pi/\lambda} f(x\lambda)\, d\nu(x) = \lambda - \int_0^{\pi/\lambda} f'(x\lambda)\nu(x)\, dx$$

$$= -\int_0^{\pi} f'(x)\nu(x/\lambda)\, dx = -\int_0^{t_0\lambda} - \int_{t_0\lambda}^{\eta} - \int_{\eta}^{\pi} = I_1 + I_2 + I_3,$$

where η is some fixed scalar and $\lambda t_0 < \eta < 1$. Note that $Cx \geqq -f'(x) \geqq 0$ for $0 \leqq x \leqq \pi$.

Let us take t_0 large enough so that $\nu(t) \geqq 0$ for $t \geqq t_0$ and $\nu(x\lambda^{-1})/\nu(\lambda^{-1}) \leqq 1/\sqrt{x}$ for $\lambda t_0 \leqq x \leqq \eta$. The validity of the last inequality for sufficiently large t_0 follows immediately from the Karamata representation for the slowly varying function $\nu(t)$ (see [7], p. 281). Thus, as $\lambda \to 0$,

$$|I_2| \leqq \nu(1/\lambda) \int_{\lambda t_0}^{\eta} |f'(x)| \frac{\nu(x\lambda^{-1})}{\nu(\lambda^{-1})}\, dx \leqq C\nu(1/\lambda) \int_0^{\eta} \sqrt{x}\, dx \leqq C\eta\nu(1/\lambda),$$

$$|I_1| \leqq \int_0^{\lambda t_0} |f'(x)|\, |\nu(x/\lambda)|\, dx = O(\lambda^2),$$

since $|\nu(y)|$ is bounded for $0 \leqq y \leqq t$.

Further, use of the Karamata representation for $\nu(t)$ makes it possible to see without difficulty that $\lim_{\lambda\to 0} \nu(x\lambda^{-1})/\nu(\lambda^{-1}) = 1$ uniformly in x, $\eta \leqq x \leqq \pi$. Thus

$$|I_3 - \nu(1/\lambda)| \leqq \left| \int_{\eta}^{\pi} f'(x)(\nu(x/\lambda) - \nu(1/\lambda))\, dx \right| + \nu(1/\lambda) \left| \int_0^{\eta} f'(x)\, dx \right|$$

$$\leqq \varphi_1(\lambda)\nu(1/\lambda) + C\eta\nu(1/\lambda)$$

and

$$\varphi_1(\lambda) = C\pi \int_{\eta}^{\pi} |\nu(x\lambda^{-1})/\nu(\lambda^{-1}) - 1|\, dx \to 0 \quad \text{as} \quad \lambda \to 0.$$

Combining the estimates for I_1, I_2 and I_3 we obtain

$$\overline{\lim_{\lambda\to 0}}\, |I/\nu(\lambda^{-1}) - 1| \leqq 2C\eta,$$

while, since η is arbitrary, $\lim_{\lambda\to 0} I/\nu(\lambda^{-1}) = 1$, whence there immediately follows the relation

$$\lim_{\lambda\to 0} \operatorname{Im} \mathbf{E}\, e^{i\lambda\xi_1}/\lambda\nu(\lambda^{-1}) = 1, \tag{14}$$

and hence inequality (13) as well.

6. If the random walk S is stable with $a = 1$, then, for any $\varepsilon > 0$ and all sufficiently small $\lambda > 0$,

$$\lambda^{1+\varepsilon} \leqq -\operatorname{Im}(1 - h_+(\lambda)) \leqq \lambda^{1-\varepsilon}, \qquad \operatorname{Re}(1 - h_+(\lambda)) \leqq \lambda^{1-\varepsilon}.$$

The first relation follows immediately from Assertion 3 and relation (14). The inequality for $\operatorname{Re}(1-h_+(\lambda))$ follows from that fact that

$$\int_0^\infty (1-\cos x\lambda)\, d\mathbf{P}\{H_+<x\} \leqq 2\int_0^\infty \left(\sin\frac{\lambda x}{2}\right)^2 d\mathbf{P}\{H_+<x\}$$

$$\leqq 2\int_0^{2\pi/\lambda} \sin\frac{\lambda x}{2}\, d\mathbf{P}\{H_+<x\} + 2\mathbf{P}\left\{H_+>\frac{2\pi}{\lambda}\right\}$$

$$\leqq 2\operatorname{Im} h_+(\lambda/2) + \varphi_2(\lambda)\lambda\nu(1/\lambda),$$

where

$$\varphi_2(\lambda) = 2\mathbf{P}\{H_+>2\pi/\lambda\}/\lambda\nu(\lambda^{-1}) \to 0 \quad \text{as} \quad \lambda \to 0.$$

7. If the random walk S is stable and $0<a<1$, then, for any $\varepsilon>0$ and all sufficiently small $\lambda>0$,

$$\lambda^{a+\varepsilon} \leqq \operatorname{Re}(1-h_+(\lambda)) \leqq \lambda^{a-\varepsilon}, \qquad \lambda^{a+\varepsilon} \leqq -\operatorname{Im}(1-h_+(\lambda)) \leqq \lambda^{a-\varepsilon}.$$

This assertion follows from Assertion 2 and arguments in [5], Chapter 2 § 6.

8. If the random walk S is stable and $a=0$, then $|1-h_+(\lambda)| \geqq \lambda^{\varepsilon}$ for any $\varepsilon>0$ and all sufficiently small $\lambda>0$.

Indeed, let $a=0$. If $\{S_n\}$ is strongly attracted to the stable law F with index α, then since $\alpha(1-F(0))=0$ we have $F(0)=1$, whence it follows that $0<\alpha<1$. Hence $\{-S_n\}$ is stable with index α, and therefore applying Assertion 7 to $h_-(\lambda)$ we obtain for all sufficiently small $\lambda>0$ that $|1-h_-(\lambda)| \leqq \lambda^{\alpha-\varepsilon/3}$.

Further, in view of the factorization identity (12) and Assertion 4, for all sufficiently small $\lambda>0$,

$$|1-h_+(\lambda)| = \frac{|1-\mathbf{E}\, e^{i\lambda\xi_1}|}{B|1-h_-(\lambda)|} \geqq \frac{\lambda^{\alpha+\varepsilon/3}}{B\lambda^{\alpha-\varepsilon/3}} \geqq \lambda^{\varepsilon}.$$

If $a=0$ *and* $\{-S_n\}$ is relatively stable, then applying Assertion 6 to $h_-(\lambda)$ we see that $|1-h_-(\lambda)| \leqq \lambda^{1-\varepsilon/3}$ for all sufficiently small $\lambda>0$. Use of Assertion 5 and the identity (12), as in the preceding case, yields the estimate $|1-h_+(\lambda)| \geqq \lambda^{\varepsilon}$.

Let us turn directly to the proof of Theorem 2.

Let $a_1+a_2<1$, take $\varepsilon>0$ so that $a_1+a_2+2\varepsilon<1$. Then

$$\int_{-\delta}^{\delta} |1-h_+(\lambda)|^{-1}|1-h_-(\lambda)|^{-1}\, d\lambda \leqq 2\int_0^{\delta} \lambda^{-a_1-\varepsilon}\lambda^{-a_2-\varepsilon}\, d\lambda < \infty.$$

Here for $|1-h_+(\lambda)|$ and $|1-h_-(\lambda)|$ we have used the lower estimates contained in assertions 7 and 8. Whence, using Theorem 1, we have the non-recurrency of Y.

Let $a_1+a_2>1$. Take $\varepsilon>0$ so that $a_1+a_2-6\varepsilon>1$. Then

$$\int_{-\delta}^{\delta} \operatorname{Re}((1-h_+(\lambda))^{-1}(1-h_-(\lambda))^{-1})\, d\lambda \geqq 2\int_0^{\delta} \frac{-\operatorname{Im} h_+(\lambda)\operatorname{Im} h_-(\lambda)\, d\lambda}{(|1-h_+(\lambda)|\,|1-h_-(\lambda)|)^2}$$

$$\geqq 2\int_0^{\delta} \frac{\lambda^{a_1+\varepsilon}\lambda^{a_2+\varepsilon}}{\lambda^{2(a_1-\varepsilon)}\lambda^{2(a_2-\varepsilon)}}\, d\lambda$$

$$= 2\int_0^{\delta} \lambda^{-a_1-a_2+6\varepsilon}\, d\lambda = \infty.$$

Here for $|1-h_+(\lambda)|$ and $|1-h_-(\lambda)|$ we have used the upper estimates, and for $\operatorname{Im} h_+(\lambda)$ and $-\operatorname{Im} h_-(\lambda)$ the lower estimates contained in assertions 6 and 7. Theorem 1 yields, also in this case, the desired assertion as to the recurrency of Y.

4. In conclusion we shall give examples of non-recurrent random walks Y with $\mathbf{E}\xi_1' = \mathbf{E}\xi_1'' = 0$. Set, for $2 > \alpha > 1$,

$$\mathbf{P}\{\xi_1' = k\} = k^{-\alpha-1}/(\zeta(\alpha)+\zeta(\alpha+1)), \qquad k = 1, 2, \cdots,$$
$$\mathbf{P}\{\xi_1' = -1\} = \zeta(\alpha)/(\zeta(\alpha)+\zeta(\alpha+1)),$$
$$\mathbf{P}\{\xi_1' = 0\} = \mathbf{P}\{\xi_1' < -1\} = 0,$$

where $\zeta(\beta) = \sum_{k=1}^{\infty} k^{-\beta}$, $\beta > 1$. The distribution of ξ_1'' coincides with that of $-\xi_1'$.

It is obvious that $\mathbf{E}\xi_1' = \mathbf{E}\xi_1'' = 0$ and that S' is strongly attracted to the stable spectrally positive law F_α for which, for $\mu > 0$,

$$\int_{-\infty}^{\infty} e^{-\mu x}\, dF_\alpha(x) = \exp\{\mu^\alpha\}.$$

Let us show that $a_1 = \alpha - 1$. Indeed, the random variable H equal to the first positive sum in the sequence $\{-S_n'\}_1^\infty$ has finite expectation since in this case $\mathbf{E}\xi_1' = 0$, $\mathbf{E}(\max(0, -\xi_1'))^2 < \infty$ (see [3]). Further, this yields, by Theorem 9 in [4], that $\alpha F_\alpha(0) = 1$ since H is relatively stable. From this it follows that S' is stable with index $a_1 = \alpha(1 - F_\alpha(0)) = \alpha - 1$. Obviously, $a_2 = a_1$, and thus for $\alpha < 3/2$ the walk will be non-recurrent by Theorem 2, while for $\alpha > 3/2$ it will be recurrent. Use of Corollary 2 and more precise estimates for $|1 - h_+(\lambda)|$ allow one to conclude that for $\alpha = 3/2$ the walk is recurrent.

Now assume that ξ_1' is distributed as in the preceding example, while for $\beta > 1$ we have

$$\mathbf{P}\{\xi_1'' = -2^n\} = C_\beta 2^{-n} n^{-\beta}, \qquad n = 1, 2, \cdots, \qquad C_\beta = \left(2 \sum_{n=1}^{\infty} 2^{-n} n^{-\beta}\right)^{-1},$$

$$\mathbf{P}\{\xi_1'' = [2C_\beta\zeta(\beta)] + i\} = \frac{C_\beta\zeta(\beta)}{3([2C_\beta\zeta(\beta)]+1)}, \qquad i = 0, 1, 2,$$

$$\mathbf{P}\{\xi_1'' = 0\} = \frac{1}{2} - \frac{C_\beta\zeta(\beta)}{[2C_\beta\zeta(\beta)]+1},$$

and $\mathbf{P}\{\xi_1'' = k\} = 0$ for the other values of k. Thus in this case $\mathbf{E}\xi_1' = \mathbf{E}\xi_1'' = 0$.

From the results of [6] it follows immediately that $\{S_n''\}$ is relatively stable and therefore $a_2 = 0$, $a_1 = \alpha - 1$, and consequently for $\alpha < 2$ Y is non-recurrent by Theorem 2, while for $\alpha > 2$ (since $\mathbf{E}H_+ < \infty$ and $\mathbf{P}\{A_-\} = \mathbf{P}\{A_+\} = 1$) Y is recurrent by Corollary 1. In this example the homogeneous random walk S'' is stable, while the distribution of the random variable ξ_1'' does not belong to the region of attraction of any stable law whatsoever.

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