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REGENERATION AND RENOVATION IN QUEUES

S.G. FOSS

Novosibirsk State University, Pirogova 2, 630090 Novosibirsk, USSR

V.V. KALASHNIKOV

Institute for Systems Studies, 9, Prospect 60 let Oktjabrja, 117312 Moscow, USSR

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Regenerative events for different queueing models are considered. The aim of this paper is to construct these events for continuous-time processes if they are given for the corresponding discrete-time model. The construction uses so-called renovative events revealing the property of the state at time n of the discrete-time model to be independent (in an algebraic sense) of the states referring to epochs not later than $n - L$ (where L is some constant) given that there are some restrictions on the "governing sequence". Different types of multi-server and multi-phase queues are considered.

Keywords: Regeneration, renovation, stopping time, queueing system, multi-server system, multi-phase system.

1. Introduction

The notion of regeneration is a very important one in probability theory in general and in queueing theory in particular. It is used both for qualitative analysis of queueing processes (their ergodicity, boundedness, stability, convergence to a stationary regime) and quantitative estimates including simulation (stationary characteristics, estimates of stability, convergence rates etc.). Being introduced by W.L. Smith this notion was generalized by Thorisson [13] and Asmussen [1] in such a way that they permitted regeneration cycles to be dependent given that any cycle is independent of the preceding regeneration times. This generalization is very important for queueing theory as it preserves all important results and leads to the possibility of considering a rather wide class of queues (in comparison with Smith's regeneration). This generalization is very natural. In fact, different examples of it were considered independently by other authors; see e.g. Kalashnikov [9].

Another (but not totally different) approach was suggested by Borovkov ([3] for so-called stochastic recursive sequences. Its essence is the construction of so-called "renovative events" with the following property: the process is independent (in an algebraic sense but not necessarily a probabilistic one!) of the preceding terms of the governing sequence after the time when some renovative event does occur.

It turned out that "coupling" of these two approaches permits us to construct regenerative events in some new situations (e.g. for multi-server and multi-phase queues).

The aim of this paper is to suggest a general construction of regeneration points for continuous-time models given that they exist for some imbedded discrete-time process. Now the construction of regeneration points for discrete-time processes is well-known (see Thorisson [13], Asmussen [1], Kalashnikov [9] and references in these papers). As a rule it exploits the Harris-recurrency of the underlying Markov processes, see e.g. Nummelin [12]. But its generalization on continuous time is not self-evident. Here we propose it for rather general processes occurring in queueing. So, it can be used for many queueing models. We do it step by step, starting from the discrete case, using such notions as regeneration and renovation.

Given that regeneration points are constructed it is possible to apply any known result for regenerative processes (convergence rate estimates, stability estimates, estimates of distribution functions of the first-occurrence times, etc.) in order to study the corresponding queueing model. An example of such a study is contained in Asmussen and Foss [2] where the proposed general construction is used for obtaining ergodicity results.

The idea of this paper arose after fruitful discussions with S. Asmussen and H. Thorisson during the meeting on queueing theory and point processes in Karpacz, Poland (January, 1990).

2. Regeneration and renovation in discrete time

Regeneration events in queueing are often connected with emptiness of queues, or arrival of customers to an empty system.

Example 1: The single-server queue

Let us consider the $GI/GI/1/\infty$ queue which satisfies Lindley's equation:

$$w_{n+1} = (w_n + s_n - e_n)_+, \quad n \geqslant 0, \tag{2.1}$$

where $(\cdot)_+ = \max(0, \cdot)$, and $\{x_n\}$, $\{e_n\}$ are sequences of service and interarrival times respectively consisting of i.i.d.r.v.'s, and $\{w_n\}$ is a sequence of actual waiting times. Here and below, interarrival and service times are numbered from 0: the zeroth customer having service time s_0 arrives at time e_0 and so on. Then the event

$$A_n = \{w_n = 0\} \tag{2.2}$$

is a regenerative one. Let us denote $S(k)$ the kth occurrence time of the regenerative event, $\theta_k = S(k) - S(k-1)$, $k \geqslant 1$, $S(0) = 0$. If

$$Es_0 < Ee_0 \tag{2.3}$$

then the events $\{A_n\}$ are positive recurrent. This means that under the condition (2.3) the following relation is true

$$E\theta_k \leqslant c < \infty, \quad k \geqslant 1, \tag{2.4}$$

where the constant c depends on the distribution functions (d.f.) of s_0 and e_0 in general. Let us notice that the inequality (2.3) follows from

$$P(s_0 < e_0) > 0. \tag{2.5}$$

Sometimes (e.g. in stability analysis, see Borovkov [4], Kalashnikov [8], Kalashnikov and Rachev [11]) we need this constant to be the same for some class of regenerative processes. Then we have to demand the restrictions to be uniform over this class (in some natural sense).

This example shows that the process $\{w_n\}$ is a regenerative one in the sense of Smith, i.e. the "fragments" of this process belonging to different cycles are independent. The "condition" of regeneration is expressed here in pure algebraic form (2.2).

Example 2: Multi-server queues

Consider a multi-server system (with N servers) and use for its description Kiefer–Wolfowitz equations:

$$w_{n+1} = R(w_n + \delta s_n - Ie_n)_+, \quad n \geqslant 0, \tag{2.6}$$

where again $\{s_n\}$ and $\{e_n\}$ are sequences of service and interarrival times, $w_n = (w_{n1}, \dots, w_{nN})$ is a waiting time vector referring to the nth customer, $w_{n1} \leqslant \dots \leqslant w_{nN}$ and $R(\cdot)$ is an operator which orders the components of $(\cdot)$ in a non-decreasing way, $\delta = (1, 0, \dots, 0)$, $I = (1, 1, \dots, 1)$. If $\{s_n\}$ and $\{e_n\}$ consist of i.i.d.r.v.'s then it is possible to define regenerative events for this system. Very often one tries to expand the previous construction to the multi-server case in the following way. Let

$$A_n = \{w_n = (0, 0, \dots, 0)\}. \tag{2.7}$$

Of course, this is a regenerative event in the sense of Smith. In order for these events to be positive recurrent we need to impose the ergodicity condition:

$$Es_0 < NEe_0. \tag{2.8}$$

But this condition is not sufficient in general. We have to demand additionally that

$$P(s_0 < e_0) > 0. \tag{2.9}$$

It is easy to prove that under conditions (2.8) and (2.9) the events $\{A_n\}$ constructed by formula (2.7) are positive recurrent.

This construction is not good because it demands an additional restriction (2.9). A similar restriction (2.5) for a single-server queue was implied by ergodicity condition (2.3) and so it was not an additional one. It is well-known (see e.g. Borovkov [3], Kalashnikov [9], Foss [6], Kalashnikov and Rachev [11]) that it is possible to eliminate the restriction (2.9) if we use regeneration in the sense of S. Asmussen and H. Thorisson. Namely, let us fix an integer $L > 0$ and consider the events

$$B_n(\Delta, \sigma, \varepsilon) = \{s_j - Ne_j \leqslant -\Delta,\ s_j \leqslant \sigma,\ e_j \geqslant \varepsilon,\ n - L \leqslant j < n\}, \tag{2.10}$$

$$C_n(W) = \{w_{nN} \leqslant W\}, \tag{2.11}$$

where Δ, σ, ε, W are some positive constants. It is possible to prove that there exists such integer $L > 0$ that the values w_n $(n \geqslant L)$ calculated with the help of eq. (2.6) do not depend on $w_0, \ldots, w_{n-L}$, (but only on $e_{n-L}, \ldots, e_{n-1}$ and $s_{n-L}, \ldots, s_{n-1}$) in an algebraic sense given that the event

$$A_n = C_{n-L}(W) \cap B_n(\Delta, \sigma, \varepsilon) \tag{2.12}$$

does occur. In fact, this is a consequence of the Kiefer–Wolfowitz equation. Of course, the constant L depends on W, Δ, σ, ε and it is possible to give a corresponding estimate, see Kalashnikov and Rachev [11]. It is reasonable to name A_n a "renovative event". Let us write $I(A) = 1$ if an event A does occur and $I(A) = 0$ otherwise. If we denote $S(0) = 0$,

$$S(1) = \min\{k\colon I(A_k) = 1\}, \tag{2.13}$$

$$S(n+1) = \min\{k\colon k > S(n) + L,\ I(A_k) = 1\}, \tag{2.14}$$

then the sequence $\{S(n)\}_{n \geqslant 1}$ of "renovation epochs" is a renewal one, i.e. r.v.'s $\theta_n = S(n) - S(n-1)$ are i.i.d. $(n > 1)$. Besides, the ergodicity condition (2.8) implies that $E\theta_1 < \infty$. So, we managed to eliminate condition (2.9) and obtained a regenerative process without it. Though inter-regeneration times are i.i.d.r.v.'s, the cycles are dependent in general. Namely, the behaviour of the process $\{w_n\}$ beginning from some epoch $S(k)$ can depend on the values $s_{S(k)-L}, \ldots, s_{S(k)-1}$ and $e_{S(k)-L}, \ldots, e_{S(k)-1}$ and, hence, on the values $w_{S(k)-L+1}, \ldots, w_{S(k)-1}$. Of course, the distribution of the process $\{w_k^{(k)}\}_{n \geqslant 0} \equiv \{w_{n+S(k)}\}_{n \geqslant 0}$ does not depend on $\{S(0), \ldots, S(k)\}$ for any k. This means that the process $\{w_n\}$ is a regenerative one in the sense of S. Asmussen and H. Thorisson. It is useful to note that the epochs $\{S(n)\}$ are regeneration ones also for other processes such as queue-length.

Let us generalize this construction.

Let $\{Y_n\}$ be a sequence taking values in a measurable state space $\mathscr{Y}$ and defined by the recursive relation

$$Y_{n+1} = f(Y_n, X_n), \quad n \geqslant 0, \tag{2.15}$$

where X_n are elements of the "governing sequence" taking values from some measurable space $\mathscr{X}$ and the mapping $f\colon \mathscr{Y} \times \mathscr{X} \to \mathscr{Y}$ is supposed to be measurable too. So, if $Y_n = y$ is fixed then we are able to find values Y_{n+m} for any $m > 0$ in terms of y, $X_n, \ldots, X_{n+m-1}$ with the help of (2.15):

$$Y_{n+m} = f_m(y, X_n, \ldots, X_{n+m-1}). \tag{2.16}$$

We suppose that there exist such an integer $L > 0$, a measurable subset $B(L) \subset \mathscr{X}^L \equiv \mathscr{X} \times \ldots \times \mathscr{X}$, and $C \subset \mathscr{D}$ that for any $y_1 \in C$, $y_2 \in C$ and $(x_1, \ldots, x_L) \in B(L)$ the following relation is true

$$f_L(y_1, x_1, \ldots, x_L) = f_L(y_2, x_1, \ldots, x_L). \tag{2.17}$$

If C contains only one point then the above equality is true for any L and $B(L)$. Otherwise, it demands some algebraic independence of the state of the process $\{Y_n\}$ after L steps given that the state of the process is in the set C at the beginning and that L successive "governing" variables X_n take values from the set $B(L)$.

We now introduce probabilistic notions in the above constructions. Suppose that the sequence $\{X_n\}$ consists of i.i.d.r.v.'s defined on some probability space $(\Omega, \mathscr{F}, P)$, let $\mathscr{F}_n$ be the σ-algebra generated by $\{Y_0, X_0, \ldots, X_n\}$, $\mathscr{F}_n \in \mathscr{F}$. Hence, r.v. Y_{n+1} is $\mathscr{F}_n$-measurable. Define events

$$C_n = \{Y_n \in C\}, \tag{2.18}$$

$$B_n = \{(X_{n-L}, \ldots, X_{n-1} \in B(L)\}, \tag{2.19}$$

$$A_n = C_{n-L} \cap B_n(L). \tag{2.20}$$

Then the sequence $\{S(n)\}$ of occurrence times of $\{A_n\}$ (renovation times) as defined in (2.13) and (2.14) is a renewal process. This follows from (2.17) and the fact that each $S(n)$ is a stopping time of the family $\{\mathscr{F}_n\}$. Besides, the distribution of the "shifted" process $\{Y_n^{(k)}\}_{n \geqslant 0} \equiv \{Y_{n+S(k)}\}_{n \geqslant 0}$ does not depend on $\{S(0), \ldots, S(k)\}$ for any k. Hence, the process $\{Y_n\}$ is a regenerative one in the sense of S. Asmussen and H. Thorisson and the behaviour of the process on each cycle depends on the L last values of the state belonging to the preceding cycle.

In order for this construction to be useful the events $\{A_n\}$ are to be positive recurrent. This imposes some restrictions on the choice of subsets C and $B(L)$ defining regenerative events. In queueing the following standard situation takes place: ergodic conditions give the possibility to choose necessary subsets. This possibility has already been illustrated for the multi-server system. Below, we consider other examples of queueing systems having the same property.

3. More examples

Example 3: Multi-phase systems

Let us consider now the model $GI/GI/N_1 \to GI/N_2 \to \ldots \to GI/N_m$ which consists of m phases, each of them representing a multi-server system (see Kalashnikov and Rachev [11], Kalashnikov [10]). Again, let $\{e_n\}$ be the i.i.d.

interarrival times and $s_n(j)$ the $(N_1+\ldots+N_{j-1}+n)$th service time at the jth phase, all these times being mutually independent. Let

$$N^0=0, \quad N^k=N_1+\ldots+N_k, \quad 1\leqslant k\leqslant m, \; N\equiv N^m.$$

Define the waiting time vector $w_n=(w_n(1),\ldots,w_n(N))$ in a formal way as satisfying the following recurrent relation (the sense of this vector and the reasoning behind the following relations can be found in Kalashnikov [10]):

$$\begin{aligned} w_{n+1}=\big(&R\big((w_n(1))_+ + s_n(1),\, w_n(2),\ldots,w_n(N^1)\big),\ldots,\\ &R\big(\max\big(w_n(N^k+1),\, w_n(N^{k-1}+1)\big)+s_n(k+1),\, w_n(N^k+2),\ldots,\\ &\quad w_n(N^{k+1})\big),\ldots,R\big(\max\big(w_n(N^{m-1}+1),\, w_n(N^{m-2}+1)\big)+s_n(m),\\ &\quad w_n(N^{m-1}+2),\ldots,w_n(N)\big)-e_nI. \end{aligned} \tag{3.1}$$

It is proved in Kalashnikov and Rachev ([11], section 5.7.3) that renovative events A_n can be chosen as in (2.18)–(2.20) and one can choose

$$C_n=\Big\{\max_{1\leqslant i\leqslant N} w_n(i)\leqslant W\Big\}, \tag{3.2}$$

$$B_n=\{e_k\geqslant\varepsilon,\, s_k(i)\leqslant\sigma(i),\, 1\leqslant i\leqslant m,\; n-L\leqslant k\leqslant n-1\}, \tag{3.3}$$

where constants ε and $\sigma(i)$ meet the inequality

$$\min_{1\leqslant i\leqslant m}(N_i\varepsilon-\sigma(i))\geqslant\Delta>0. \tag{3.4}$$

If the ergodicity condition

$$\min_{1\leqslant i\leqslant m} E(N_ie_0-s_0)>0 \tag{3.5}$$

is satisfied then the renovative events A_n having been chosen with the help of relation (2.20) (where C_n and B_n are from (3.2)–(3.4)) are positive recurrent ones.

A similar construction can be used for multi-phase models consisting of a sequence of multi-server systems with some restrictions (e.g. with finite waiting rooms). Moreover, it is possible to construct similar renovative events for queueing networks without feedback or for Jackson-type networks, see Borovkov [4], Foss [7].

Example 4: Infinite-server queues

Consider the model $GI/GI/\infty$. Its description is similar to that of the $GI/GI/N/\infty$ model. Namely, let us consider the waiting-time vector $w_n=(w_{n1},w_{n2},\ldots)$ where $w_{n1}\geqslant w_{n2}\geqslant\ldots$ and

$$w_{n+1}=D(s_n-e_n,\, w_{n1}-e_n,\, w_{n2}-e_n,\ldots)_+, \quad n\geqslant 0. \tag{3.6}$$

In eq. (3.6) the operator $D(\cdot)$ orders the components of the vector $(\cdot)$ in a non-increasing way, so the sense of the component of the vector w_n is quite clear. Of course, it is possible to use the operator D when describing models from examples 2 and 3, but we preferred the more traditional description.

Let us fix an integer $L > 0$ and consider events

$$B_n(\varepsilon) = \{e_j \geqslant \varepsilon,\ n - L \leqslant j < n\}, \tag{3.7}$$

$$C_n(W) = \{w_{n1} \leqslant W\}, \tag{3.8}$$

$$A_n = C_{n-L}(W) \cap B_n(\varepsilon), \tag{3.9}$$

where ε and W are some positive constants. It is easy to prove that events A_n are renovative ones if we choose $L \geqslant W/\varepsilon$. So, the relations $Es_0 < \infty$, $P(e_0 > 0) > 0$ guarantee that the events A_n are positive recurrent for some W, ε, L.

Example 5: Multi-server queues with finite waiting room

Consider the $GI/GI/N/k$ model with waiting room of capacity k. Let q_n be the number of customers in the system at the nth arrival epoch. If w_n is the waiting-time vector, which is quite similar to that from example 2, then

$$w_{n+1} = R(w_n + \delta s_n I(q_n < k) - Ie_n)_+, \quad n \geqslant 0. \tag{3.10}$$

The sequence of pairs $\{w_n, q_n\}$ is clearly defined by some recursive relation (as q_{n+1} depends only on w_n, e_n, s_n and q_n). Here we again consider renovative events A_n or type (2.20):

$$A_n = \{w_{n-L,N} \leqslant W\} \cap \{s_j \leqslant c \leqslant e_j,\ n - L \leqslant j \leqslant n-1\}. \tag{3.11}$$

We are ready to prove that there exist values W, c, L such that these events are really renovative and positive recurrent, given that

$$P(s_0 \leqslant Ne_0) > 0. \tag{3.12}$$

Similar constructions of renovative events can be proposed for other multi-server queues with different types of restrictions (see Foss [6]). Besides, we can consider other queueing characteristics (rather than the waiting-time vector). For example, denote $T_n = e_0 + \dots + e_n$, $n \geqslant 0$, and let $Q_n(u)$ be the total number of customers who arrived to the system not later than T_n and departed after $T_n + u$, $Q_n \equiv \{Q_n(u),\ u \geqslant 0\}$. Then $\{Q_n\}_{n \geqslant 0}$ is a stochastic recursive sequence. Moreover, this sequence defines the sequence $\{w_n\}_{n \geqslant 0}$. For example, for the $GI/GI/N/\infty$ model

$$w_{ni} = \inf\{u \geqslant 0\colon Q_n(u) \leqslant N - i\}, \quad 1 \leqslant i \leqslant N. \tag{3.13}$$

It is interesting to note that the suggested events A_n are renovative ones for the sequence $\{Q_n\}$ too and so the epochs $\{S(n)\}$ are regeneration ones for $\{Q_n\}$.

Other examples can be found in Borovkov [3], Foss [5], and Kalashnikov and Rachev [11]. It is important to mention that the property (2.17) is of an algebraic nature and probabilistic arguments are necessary only for proving the recurrency of A_n.

4. Regeneration in continuous time

It is very attractive to use the above regenerative events for studying continuous-time processes. We shall do this assuming that the above sequence $\{Y_n\}$ is now imbedded in some continuous-time process $Z=\{Z(t)\}_{t\geqslant 0}$ with the measurable state space $\mathscr{E}$ and the sequence $\{X_n\}$ (consisting again of i.i.d.r.v.'s and taking values from the space $\mathscr{X}$) governs the dynamic of Z. Let us proceed to correct definitions.

We construct the process Z recurrently and represent it as consisting of cycles which are in fact semi-regeneration ones. Let us consider a "cycle of semi-regeneration" defined on some probability space $(\Omega^*, \mathscr{F}^*, P^*)$

$$Z^* = Z^*(t, \omega^*, y, x),\ 0 \leqslant t < \xi(\omega^*, x),\ \omega^* \in \Omega^*,\ x \in \mathscr{X},\ y \in \mathscr{D}, \tag{4.1}$$

where ξ is the length of the cycle and Z^* is the random process over the cycle, ξ depending on some parameter $x \in \mathscr{X}$ and Z^* depending on $x \in \mathscr{X}$ and $y \in \mathscr{D}$.

Briefly, the sense of the parameters defining the cycle is the following: x is a value of a member of a "governing sequence" defining the dynamic of the cycle; y is a value containing all necessary information about the "prehistory" (in fact, y is a value of some imbedded discrete-time regenerative process); ω^* is an element permitting the cycle to be "random". If the pair (Z^*, ξ) treated as a function of ω^* is constant P^*-a.s. then the cycle (Z^*, ξ) is in fact a deterministic pair given that x and y are fixed. We shall refer to this case as (Z^*, ξ) being "conditionally deterministic". Such situations are typical in queueing models. In this case all "randomness" is contained in the values of the governing sequence.

We define the governing r.v. $X = X(\omega^*)$ also on $(\Omega^*, \mathscr{F}^*, P^*)$. Besides, let $Y = Y(\omega^*, Z^*, x)$ be some functional (random in general and depending on $x \in \mathscr{X}$) defined on the above cycle.

Now let $(\Omega, \mathscr{F}, P)$ be a Cartesian product of a denumerable number of copies $(\Omega^*, \mathscr{F}^*, P^*)$:

$$(\Omega, \mathscr{F}, P) = (\Omega^*, \mathscr{F}^*, P^*) \times (\Omega^*, \mathscr{F}^*, P^*) \times \dots. \tag{4.2}$$

Let us denote an element of Ω as $\omega = (\omega_0, \omega_1, \dots)$. Define random element $Y_0(\omega) = Y_0(\omega_0)$. Below we define r.v.'s $Y_n(\omega)$, $n \geqslant 1$, recursively. Denote for $n \geqslant 1$

$$X_{n-1} = X_{n-1}(\omega) = X(\omega_n); \tag{4.3}$$

$$\xi_n = \xi_n(\omega) = \xi(\omega_n, X(\omega_n)); \tag{4.4}$$

$$T_n = \xi_1 + \dots + \xi_n, \quad T_0 = 0; \tag{4.5}$$

$$Z_n = Z_n(t, \omega) = Z^*(t, \omega_n, Y_{n-1}(\omega), X_{n-1}(\omega)), \quad 0 \leqslant t < \xi_n. \tag{4.6}$$

Put

$$Z(t, \omega) = \begin{cases} Z_1(t, \omega) & \text{for } 0 \leqslant t < T_1, \\ Z_{n+1}(t - T_n, \omega) & \text{for } T_n \leqslant t < T_{n+1},\ n \geqslant 1, \end{cases} \tag{4.7}$$

and

$$Y_{n+1}(\omega) = Y(\omega_{n+1},\ Z_{n+1},\ X_n), \quad n \geqslant 0. \tag{4.8}$$

Then relations (4.7), (4.8) together with (4.3)–(4.6) define the process Z completely.

We shall view the sequence $\{Y_n\}$ as imbedded in the process Z and consider the $\{T_n\}$ as "imbedded epochs". In fact, $\{Y_n\}$ is a Markov chain and $\{X_n\}$ consists of i.i.d.r.v.'s. The sequence $\{Y_n\}$ meets the following stochastic recursive equation

$$Y_{n+1} = f(\omega_{n+1},\ Y_n,\ X_n), \tag{4.9}$$

where the mapping f can be recovered easily from relations (4.6) and (4.8). Equation (4.9) is a sort of generalization of relation (2.15). Namely, in (2.15) the value Y_{n+1} is completely determined by Y_n and X_n, and in (4.9) the value Y_{n+1} is a random one given that Y_n and X_n are fixed. If (Z^*, ξ) is conditionally deterministic and the functional Y does not depend on the first argument ω^* then relation (4.9) is reduced to (2.15). Of course, denoting $X_n' = (\omega_{n+1},\ X_n)$, we can carry out this reduction in the general case, but sometimes it is important to distinguish between governing r.v.'s and "random" elements ω_n.

As in (2.15) we are able to find functions f_m, $m \geqslant 1$, defining values Y_{n+m} in terms of $Y_n = y,\ X_n, \ldots, X_{n+m-1},\ \omega_{n+1}, \ldots, \omega_{n+m}$:

$$Y_{n+m} = f_m(\omega_{n+1}, \ldots, \omega_{n+m},\ y,\ X_n, \ldots, X_{n+m-1}). \tag{4.10}$$

Suppose that there exists an integer $L > 0$, a measurable set $\mathscr{B}(L) \subset (\Omega^*)^L \equiv \Omega^* \times \ldots \times \Omega^*$ (L times) and $C \subset \mathscr{D}$ such that for any $y_1 \in C$, $y_2 \in C$ and $(\omega_{n+1}, \ldots, \omega_{n+L}) \in \mathscr{B}(L)$ the following relation is true:

$$\begin{aligned} &f_L(\omega_{n+1}, \ldots, \omega_{n+L},\ y_1,\ X(\omega_{n+1}), \ldots, X(\omega_{n+L})) \\ &\quad = f_L(\omega_{n+1}, \ldots, \omega_{n+L},\ y_2,\ X(\omega_{n+1}), \ldots, X(\omega_{n+L})). \end{aligned} \tag{4.11}$$

Define the events

$$C_n = \{Y_n \in C\}, \tag{4.12}$$

$$B_n(L) = \{(\omega_{n-L+1}, \ldots, \omega_n) \in \mathscr{B}(L)\}, \tag{4.13}$$

$$A_n = C_{n-L} \cap B_n(L). \tag{4.14}$$

Then $\{A_n\}$ is a sequence of regenerative events for the imbedded process $\{Y_n\}$ in the sense of S. Asmussen and H. Thorisson.

We consider the following particular case which is important in queueing theory especially when the cycle (Z^*, ξ) is conditionally deterministic. In this case (cf. section 2) we construct the set $\mathscr{B}(L)$ through some set $B(L) \subset \mathscr{X}^L$. Namely, we demand the existence of a set $B(L) \subset \mathscr{X}^L$ such that (4.11) is true for

$$\mathscr{B}(L) = \{(\omega_{n+1}, \ldots, \omega_{n+L}) \colon (X(\omega_{n+1}), \ldots, X(\omega_{n+L}) \in B(L)\}. \tag{4.15}$$

The above construction is a generalization of the construction in section 2. If (Z^*, ξ) is conditionally deterministic then formulas (4.12)–(4.14) are completely the same as in section 2.

Now we would like to use $\{A_n\}$ to construct regenerative events for a continuous time process Z. Let $R(k)$ be the kth regeneration epoch for $\{Y_n\}$ and let $S(k)$ be the kth occurrence time of events $\{C_n\}$ (see (2.18)), both $R(k)$ and $S(k)$ being integers. Denote $\rho_k = T_{R(k)}$, $\sigma_k = T_{S(k)}$. It is tempting to claim the epochs $\{\rho_k\}$ as regeneration ones for Z. Unfortunately, they may not have the "regenerative property" even in the sense of S. Asmussen and H. Thorisson. Really, though the values of $Z(s)$, $s \geqslant \rho_k$, depend on only Y_k, Y_k can depend on the value $\rho_k - \sigma_k^*$ where $\sigma_k^* = \max\{\sigma_j\colon \sigma_j < \rho_k\}$. Hence, the values $Z(s)$, $s \geqslant \rho_k$, can depend on the length of the previous inter-regeneration times. This contradicts the definition of the regenerative process in any sense mentioned above.

So, in order to define regeneration times let us fix some constant $\lambda > 0$ and define the following events using in fact "discrete-time constructions":

$$B_n(\lambda, L) = B_n(L) \cap \{\xi(\omega_{n-L+1}, X_{n-L}) + \dots + \xi(\omega_n, X_{n-1}) < \lambda\}, \tag{4.16}$$

$$A(n, \lambda) = C_{n-L} \cap B_n(\lambda, L). \tag{4.17}$$

Now we are ready to define epochs which are "candidates" for regeneration times:

$$\tau_1 = \min\{\tau = T_k + \lambda\colon I(A(k, \lambda)) = 1\}, \tag{4.18}$$

$$\tau_{n+1} = \min\{\tau = T_k + \lambda\colon I(A(k, \lambda)) = 1,\ \tau > \tau_n + \lambda\}, \quad n \geqslant 1. \tag{4.19}$$

These times are really regeneration ones due to the following:

Let $\tau_n = T_{\nu(n)} + \lambda$. Then the sequence $\{T_{\nu(n)}\}$ is a sequence of stopping times of the family $\{\mathscr{F}^{(n)}\}$ where $\mathscr{F}^{(n)}$ is a σ-algebra generated by $\{(Y_0, X_0), (Y_1, X_1, \xi_1), (Y_2, X_2, \xi_2), \dots, (Y_n, X_n, \xi_n)\}$. Let κ be the ordering number of the semi-regeneration cycles (of the process Z) starting at time $T_{\nu(n)}$. Then both the inter-regeneration time $\tau_{n+1} - \tau_n$ and the "shifted" process $Z^{(n)} = \{Z(t + \tau_n, \omega)\}_{t \geqslant 0}$ depend only on $(\omega_\kappa, \omega_{\kappa+1}, \dots)$. Of course, the initial L elements of this sequence are not arbitrary but belong to the set $\mathscr{B}(L)$. It is important to note that the preceding inter-regeneration time $\tau_n - \tau_{n-1}$ does not depend on these L elements as $\tau_n - T_{\nu(n)} = \lambda$. Hence, successive inter-regeneration times are independent and $Z(n)$ does not depend on $\{\tau_0, \dots, \tau_n\}$ for any n though successive "regeneration cycles" can be dependent. This means that $\{\tau_n\}$ are regeneration epochs for Z in the sense of S. Asmussen and H. Thorisson.

From the definition of the process Z follows that the constructed regenerative events (4.17) are positive recurrent if

(i) the set C is positive recurrent for the Markov chain $\{Y_n\}$;

(ii) the constant λ and events $B_n(\lambda, L)$ are such that $P(B_n(\lambda, L)) > 0$ (this probability does not depend on n);

(iii) $0 < E\xi_n < \infty$.

It should be mentioned that there is another way of constructing regenerative events. Namely, one can use the so-called "splitting technique" (see Nummelin [12]) in the following way. Let us add a binary coordinate ψ to the underlying process Z so that this coordinate can be changed only at times T_k and $T_k + \lambda$: $\psi(t) = 1$ if and only if $T_k \leqslant t < T_k + \lambda$ and $I(C_k \cap B_{k+L}(\lambda, L)) = 1$. Then the sequence $\{\beta_j\}$ defined by the equalities

$$\beta_1 = \min\{T_k \colon \psi(T_k) = 1\}, \tag{4.20}$$

$$\beta_{j+1} = \min\{T_k \colon T_k > \beta_j, \psi(T_k) = 1\}, \tag{4.21}$$

is a sequence of regeneration times (in the sense of S. Asmussen and H. Thorisson) for the "expanded" process (Z, ψ).

5. Examples: multi-server queues in continuous time

Let us start with the $GI/GI/N/\infty$ queue. The construction of a discrete-time regenerative process for it with the help of the Kiefer–Wolfowitz relations was given in example 2. In order to describe this model with a continuous-time regenerative process we introduce some additional notations. Let $Q(t)$ be the queue-length (the number of customers presented to the system) at time t, $r(t) = (r_1(t), \dots, r_Q(t))$ be a vector of dimension $Q = Q(t)$ consisting of residual service times for all customers at time t. It follows that $r_j(t)$ decreases with unit velocity if $1 \leqslant j \leqslant N$ and $r_j(t)$ stays constant if $j > N$.

Suppose for a while that no customer enter the system after time t, i.e. we consider only the service process. Define the process $V(u) = (Q(u), r(u))$, $u \geqslant t$, having the following trajectories:

(i) If $Q(t) = 0$ then $V(u) \equiv V(t) = 0$ for all $u \geqslant t$ – in this situation the component $r(u)$ is not defined.

(ii) If $0 < Q(t) = Q \leqslant N$ then all $r_j(u)$, $1 \leqslant j \leqslant Q$, decrease with unit velocity and $Q(u) = Q$ over the time-interval $t \leqslant u < t + \chi$ where $\chi = \min\{r_j(t) \colon 1 \leqslant j \leqslant Q\}$. Define $Q(t + \chi) = Q - 1$ and construct $r(t + \chi)$ from the vector $r(t + \chi - 0)$ deleting from the latter "zero"-components. Further, the dynamics of $V(u)$, $u \geqslant t + \chi$, is evident (if $Q(t + \chi) > 0$ then it has been described just above, if $Q(t + \chi) = 0$ then see (i)).

(iii) If $Q(t) = Q > N$ then again all $r_j(u)$, $1 \leqslant j \leqslant N$, decrease with unit velocity, $r_j(u) = r_j(t)$, $j > N$, and $Q(u) = Q$ over the time-interval $t \leqslant u < t + \chi$ where $\chi = \min\{r_j(t) \colon 1 \leqslant j \leqslant N\}$. Define $Q(t + \chi) = Q - 1$ and construct $r(t + \chi)$ from the vector $r(t + \chi - 0)$ removing the component $r_{N+1}(t)$ to the place of the "zero"-component (if there are several "zero"-components in $r(t + \chi - 0)$ then several components $r_j(t)$ beginning from $j = N + 1$ are removed to new places). After these steps the dynamics of $V(u)$, $u \geqslant t + \chi$, is evident.

So, $Q(u)$ is a non-increasing function of time $u \geqslant t$ and hence the dimension of the vector $r(u)$ is non-increasing too. If we know $V(t)$ then we are able to construct the Kiefer–Wolfowitz waiting time vector $w(t) = (w_1(t), \dots, w_N(t))$ (referring to the time t) in the following way:

$$w_j(t) = \min\{u: Q(t+u) \leqslant N - j\}. \tag{5.1}$$

We use the process $V(u)$ in order to construct the regenerative cycle of the underlying process. In order to do this we put for $n \geqslant 0$ (cf. eqs. (4.3)–(4.8))

$$X_n = (e_n, s_n); \tag{5.2}$$

$$\xi_{n+1} = e_n; \tag{5.3}$$

$$T_0 = 0, \quad T_{n+1} = e_0 + \dots + e_n; \tag{5.14}$$

$$Y_n = (Q_n, r_n, w_n), \tag{5.5}$$

where $Q_n = Q(T_n)$, $r_n = r(T_n)$, $w_n = w(T_n)$;

$$Z_{n+1} = V(u), \quad 0 \leqslant u < \xi_{n+1}, \tag{5.6}$$

where $V(0) = (Q_n, r_n)$.

We can see that the constructions (5.2)–(5.6) are particular cases of (4.3)–(4.7) and the cycles defined by (5.6) are conditionally deterministic, the sequence $\{Y_n\}$ being defined by the recursive relation of type (2.15). From relation (5.5) and example 2 it follows that there exist constants Δ, σ, ε, W and L such that events A_n defined by relation (2.12) are renovative ones for the imbedded process $\{Y_n\}$. This means that relation (2.17) is true for this process. Besides, $\{A_n\}$ are positive recurrent events given that the ergodicity condition (2.8) holds. Hence, in order to construct positive recurrent regeneration epochs $\{\tau_n\}$ in accordance with relations (4.16)–(4.19) we need to assume additionally that there exists a constant ε' such that

$$P(e_0 < \varepsilon') > 0 \tag{5.7}$$

(we need this relation for the events $B_n(\lambda, L)$ from (4.16) to have positive probability). However, the relation (5.7) is true for any non-degenerate r.v. e_0. This means that relation (2.8) guarantees that the sequence $\{\tau_n\}$ constructed by eqs. (4.16)–(4.19) is a sequence of positive recurrent regenerative times (in the sense of S. Asmussen and H. Thorisson) for the initial process $Z(t) = (Q(t), r(t))$ defining all main characteristics of the $GI/GI/N/\infty$ queueing model.

Similar constructions remain valid for $GI/GI/\infty$ and $GI/GI/N/k$ queueing models (see examples 4 and 5), the underlying process $Z(t)$ being the same as above. Regenerative events for it are constructed with the help of relations (4.16)–(4.19) and corresponding regenerative events for imbedded discrete-time processes $\{Y_n\}$, their construction being described in the above examples.

For the multi-phase model $GI/GI/N_1 \to GI/N_2 \to \dots \to GI/N_m$ (see example 3) we can consider the process $Z(t) = (Q^{(1)}(t), r^{(1)}(t), \dots, Q^{(m)}(t), r^{(m)}(t))$ where

$Q^{(i)}(t)$ is the queue-length at the ith phase at time t and $r^{(i)}(t)$ is the corresponding vector of residual service times. Then, using the construction of regenerative events for the discrete-time imbedded process (3.1), we can obtain regenerative events for the process $Z(t)$ with the help of eqs. (4.16)–(4.19) and the ergodicity condition (3.4) is sufficient for these events to be positive recurrent.

References

[1] S. Asmussen, *Applied Probability and Queues* (Wiley, 1987).

[2] S. Asmussen and S. Foss, Renovation, regeneration and coupling in multi-server queues in continuous time, Preprint No. 1990-2, Dept. of Math., Chalmers Univ. of Technology, The University of Göteborg (1990), to appear in Ann. Appl. Prob.

[3] A. Borovkov, *Asymptotic Methods in Queueing Theory* (Nauka, Moscow, 1980) (in Russian). (English translation: Wiley, 1984).

[4] A. Borovkov, Limit theorems for queueing networks I, Theory Prob. Appl. 31 (1986) 474–490.

[5] S. Foss, On conditions of ergodicity for multi-server queues, Sibirsk. Math. Zhurnal 24 (1983) 168–175 (in Russian).

[6] S. Foss, The method of renovation events and its applications in queueing theory, *Semi-Markov Models. Theory and Application, Proc. 1-st Int. Symp. on Semi-Markov Processes*, Brussel (1984) (Plenum, 1986) pp. 337–350.

[7] S. Foss, On some properties of open queueing networks, Probl. Inf. Transmission 25 (1989) 90–97.

[8] V. Kalashnikov, *Qualitative Analysis of Complex Systems Behaviour by Test Functions Method* (Nauka, Moscow, 1978) (in Russian).

[9] V. Kalashnikov, Stability estimates for renovative processes, Eng. Cybern. 17 (1980) 85–89.

[10] V. Kalashnikov, Regenerative queueing processes and their qualitative and quantitative analysis, Queueing Systems 6 (1990) 113–136.

[11] V. Kalashnikov and S. Rachev, *Mathematical Methods for Construction of Queueing Models* (Wadsworth and Brooks/Cole, 1990).

[12] E. Nummelin, *General Irreducible Markov Chains and Non-Negative Operators* (Cambridge Univ. Press, 1984).

[13] H. Thorisson, The coupling of regenerative processes, Adv. Appl. Prob. 15 (1983) 531–561.